THE GENERALIZED CAYLEY-HAMILTON THEOREM IN N DIMENSIONS

John S. Lew

Brown University
Providence, Rhode Island

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The Generalized Cayley-Hamilton Theorem in n Dimensions

John S. Lew
Brown University

In 1945, Reiner\(^{(1)}\), by means of the Cayley-Hamilton theorem, obtained a canonical form for a polynomial relation between a stress matrix and a strain-velocity matrix; since that time an extensive theory has been developed for canonical forms of non-linear constitutive equations. More recently, for a polynomial relation between \(n\) tensor and a number of other tensors, the problem of finding the restrictions imposed by a symmetry group was reduced by Smith and Rivlin\(^{(2)}\), and by Pipkin and Rivlin\(^{(3)}\), to that of finding an integrity basis for a set of such tensors; and then, for the full (or proper) orthogonal group in Euclidean 2-space or 3-space, such a basis was determined by Rivlin\(^{(4)}\), Spencer and Rivlin\(^{(5)}\), and Spencer\(^{(6)}\), and its irreducibility proven by Smith\(^{(7)}\).

In this development, an important tool has been a generalization of the Cayley-Hamilton theorem, in 2-space or 3-space, from one to several matrix variables\(^{(8)}\). During this time, it has been clear that the corresponding identity in \(n\)-space, for any particular \(n\), could be obtained in a finite but discouraging number of steps; however the form of this relation for an arbitrary \(n\) has not been given. We shall obtain this form, which is the intuitive generalization of the results in two and three

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(1) See Reference 4.
(2) See Reference 7.
(3) See Reference 3.
(4) See Reference 5.
(5) See References 9,10,11.
(6) See Reference 8.
(7) See Reference 6.
(8) See Reference 5.
dimensions, and note that, properly expressed, it is a polynomial relation in the given matrices and their traces with all coefficients \pm 1.

For an arbitrary real or complex \( n \times n \) matrix \( A \) the Cayley-Hamilton theorem states that

\[
\sum_{i=0}^{n} (-1)^i A^{n-i} s_i(A) = 0
\]

where \( A^0 = I \), and \( s_i(A) \) is the \( i \)th symmetric polynomial in the characteristic roots of \( A \). If we let \( t_j(A) = \text{tr} A^j \) for \( j = 1, 2, \ldots \) then the well-known relations\(^{(9)}\)

\[
s_1 = t_1
\]

\[
2s_2 = s_1 t_1 - t_2
\]

\[
3s_3 = s_2 t_1 - s_1 t_2 + t_3
\]

and so forth can be solved recursively for each \( s_i \) in terms of \( t_1, \ldots, t_i \) to yield

\[
1!s_1 = t_1
\]

\[
2!s_2 = t_1^2 - t_2
\]

\[
3!s_3 = t_1^3 - 3t_1^2 t_2 + 2t_3
\]

and so forth. Thus the Cayley-Hamilton theorem can be expressed as a relation in \( A \) and the \( t_j(A) \).

In one dimension this process gives

\[
A - I \text{ tr } A = 0
\]

which is trivial; and in two dimensions it gives

\[
A^2 - a \text{ tr } A + b = 0,
\]

where

\[
a = \frac{1}{2} \text{ tr} [(\text{ tr } A)^2 - \text{ tr } A^2]
\]

\[(9)\) See p. 9 of Reference 12.]
If we apply to this equation the polarization operator \( d_{BA} \), that is, if we replace \( A \) by \( A + xB \), for a real variable \( x \), and evaluate the derivative in \( x \) at the point \( x = 0 \), then we obtain

6) \[
AB + BA - A \text{ tr } B - B \text{ tr } A + I[\text{tr } A \text{ tr } B - \text{tr } AB] = 0
\]

which is the generalized identity in two dimensions\(^{(10)}\). Note in this result that all permutations of \( A \) and \( B \) appear, since \( A \) and \( B \) need not commute, but that the fraction \( \frac{1}{2} \) disappears, since scalars commute and traces have cyclical symmetry.

In three dimensions equations (1) and (3) give

7) \[
A^3 - A^2 \text{tr } A + \frac{1}{2} A[(\text{tr } A)^2 - \text{tr } A^2]
- \frac{1}{6} I[(\text{tr } A)^3 - 3 \text{tr } A \text{tr } A^2 + 2 \text{tr } A^3] = 0
\]

applied to which the polarization operator \( d_{BA} \) again yields a relation in two variables. However, we desire a completely polarized relation, in which no matrix variable has degree more than unity. Thus if we also apply \( d_{CA} \) for another \( 3 \times 3 \) matrix \( C \), and let \( \Sigma \) denote the sum over all permutations of \( (A,B,C) \), then we obtain

8) \[
0 = \Sigma ABC - \Sigma AB \text{ tr } C + \Sigma A[\text{tr } B \text{ tr } C - \text{tr } BC]
- I[\text{tr } A \text{ tr } B \text{ tr } C - \text{tr } A \text{ tr } BC - \text{tr } B \text{ tr } CA - \text{tr } C \text{ tr } AB
+ \text{tr } ABC + \text{tr } CBA]
\]

which is the generalized identity in three dimensions\(^{(11)}\). Note here again that all fractions disappear by the properties of scalars and traces.

\(^{(10)}\) See Reference 5.
\(^{(11)}\) See Reference 5.
Now in the derived expression for each \( s_i \), the terms correspond to partitions of \( i \), that is, to sequences \( \mu = (m_1, m_2, \ldots) \) of non-negative integers with \( q(\mu) = i \), where

\[
p(\mu) = \sum_{j=1}^{\infty} (j-1)m_j \quad \text{and} \quad q(\mu) = \sum_{j=1}^{\infty} jm_j .
\]

Each \( m_j \) is interpreted as the number of subsets containing precisely \( j \) elements in the corresponding subdivision of a set containing precisely \( i \) elements, so that clearly \( m_j = 0 \) for \( j > i \) and thus such sequences have all entries but a finite number equal to zero. If we let \( \tau \) denote the sequence \((t_1, t_2, \ldots)\) with \( t_j = \text{tr} A^j \) as before, then we may let

\[
\tau^\mu = t_1^{m_1} t_2^{m_2} \ldots ,
\]

a product which thus has all factors but a finite number equal to unity.

Each partition \( \mu \) with \( q(\mu) = q \) labels a conjugate class \( C \) in the group \( S_q \) containing all permutations of \( q \) elements, namely that class in which all permutations may be factored into disjoint cycles of which \( m_1 \) have length 1, \( m_2 \) have length 2, and so forth. The parity of all elements in \( C_{\mu} \) is easily shown to be \( \text{sgn}(\mu) = (-1)^{p(\mu)} \), and the number of elements in \( C_{\mu} \) is well-known to be

\[
c(\mu) = \frac{q!}{m_1!m_2! \ldots 1^{m_1} 2^{m_2} \ldots}
\]

a quotient whose denominator has all factors but a finite number equal to unity. However, the general expression of the set (3) can then be written

\[
11s_i = \sum_{q(\mu)=i} \text{sgn}(\mu)c(\mu)\tau^\mu
\]

(12) See 3.6 of Reference 1 or IV.4 of Reference 2.
(13) See 6.2 of Reference 1 or (4.24) of Reference 2.
and the form (1) of the Cayley-Hamilton theorem can be rewritten

\[ \sum_{i=0}^{n} (-1)^i A^{n-1} \sum_{q(\mu)=1} \text{sgn}(\mu)c(\mu)tr(A)_{i1} = C. \]

Now we need only completely polarize this equation, noting that it is homogeneous of degree \( n \) in \( A \); that is, we need only replace the \( n \) equal variables \( A \) by all permutations of \( n \) distinct variables \( A_1, \ldots, A_n \), and put the sum of all such expressions equal to zero. For each \( i \) the corresponding term in (13) then yields \( n! \) terms, of which we may collect all those terms such that \( A_{\pi(1)}, \ldots, A_{\pi(i)} \), in any order, appear in the inner sum, and \( A_{\pi(i+1)}, \ldots, A_{\pi(n)} \), in any order, appear in the outer sum. Thus for each \( i \) we obtain a sum over the \( \binom{n}{i} \) ways to select a subset of \( i \) elements from \( \{A_1, \ldots, A_n\} \), with each summand of the form

\[ (-1)^i [\sum \text{all permutations of } A_{\pi(1)} \ldots A_{\pi(u)}] \text{coeff.} (A_{\pi(1)}, \ldots, A_{\pi(1)}) \]

and to find the coefficient for each selection we need only replace the \( i \) equal variables \( A \) by all permutations of \( i \) distinct variables \( A_{\pi(1)}, \ldots, A_{\pi(i)} \) in

\[ \sum_{q(\mu)=1} \text{sgn}(\mu)(\text{tr } A)^{m_1} \ldots (\text{tr } A^i)^{m_1/m_1} \ldots m_i^{m_1} \ldots m_i^{m_1} \]

and take the sum of all such expressions.

But for each \( \mu \) in the sum (15) many of the resulting \( i! \) terms are equal; in particular, for each \( j \) the \( m_j \) traces of products containing \( j \) factors may be permuted in all \( m_j! \) ways without changing the result, and in each of these \( m_j \) traces the \( j \) factors may be cycled in \( j \) ways without changing the result. Since the various terms are otherwise distinct, the order of their
degeneracy is precisely the denominator associated with the
given \( \mu \) in the sum (15), and thus the coefficient is precisely
the sum, over all partitions \( \mu \) of \( i \) and all essentially distinct
permutations of the \( A_{\pi(j)} \), of terms

\[
\text{sgn}(\mu) \prod_{j=1}^{m_1} \text{tr}(A_{\pi(j)}) \prod_{j=1}^{m_2} \text{tr}(A_{\pi(m_1+2j-1)} A_{\pi(m_1+2j)}) \ldots
\]

In summary, the generalized Cayley-Hamilton theorem in \( n \)
dimensions asserts the vanishing of the sum for \( i = 0, \ldots, n \) of
all essentially distinct terms of the form (14), in which the
coefficient is the sum for \( q(\mu) = i \) of all essentially distinct
terms of the form (16). Furthermore, the coefficients are
independent of \( n \), and have the forms given in equation (8) for
\( i = 1, 2, 3 \); finally, by the principle just stated, the
coefficient for \( i = 4 \) has the form

\[
\text{tr}A \text{tr}B \text{tr}C \text{tr}D - \text{tr}A \text{tr}B \text{tr}C \text{tr}BD - \text{tr}A \text{tr}B \text{tr}CD - \text{tr}B \text{tr}C \text{tr}AD
- \text{tr}B \text{tr}D \text{tr}AC - \text{tr}C \text{tr}D \text{tr}AB + \text{tr}A(\text{tr}BCD + \text{tr}DCB) + \text{tr}B(\text{tr}ACD + \text{tr}DCA)
+ \text{tr}C(\text{tr}ABD + \text{tr}DBA) + \text{tr}D(\text{tr}ABC + \text{tr}CBA) + \text{tr}AB \text{tr}CD + \text{tr}AC \text{tr}BD
+ \text{tr}AD \text{tr}BC - \text{tr}AB \text{tr}CD + \text{tr}AC \text{tr}BD - \text{tr}AD \text{tr}BC - \text{tr}ABC - \text{tr}AC \text{tr}BD
\]

Since the polarization process \( \delta_{BA} \) can be defined over any
field of characteristic zero\(^{(14)} \), these results are all valid
over any such field. Indeed since the coefficients are simpler
in the completely polarized equations, these results may well be
provable directly in \( n \) variables, rather than through (13).
However, this discussion indicates the explicit form of the
desired relation, and thus reduces the labor of deriving it to
merely that of writing it down.

\(^{(14)} \text{See p. 4 of Reference 12.} \)
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References


2. F. D. Murnaghan, The Theory of Group Representations, Johns Hopkins 1938


The Generalized Cayley-Hamilton Theorem in n Dimensions

For any positive integer n, this paper derives the explicit form of the identity in n matrices, each nxn, which is obtained by complete polarization of the usual Cayley/Hamilton theorem, and which is used repeatedly, for n = 2 or 3, in the determination of an integrity basis for symmetric tensors.
**Cayley-Hamilton theorem**

- matrix variables
- integrity basis for tensors

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