Technical Note

An Efficient Technique for the Calculation of Velocity-Acceleration Periodograms

18 May 1966

Prepared for the Advanced Research Projects Agency under Electronic Systems Division Contract AF 19(628)-5167 by Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts
The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. This research is a part of Project DEFENDER, which is sponsored by the U.S. Advanced Research Projects Agency of the Department of Defense; it is supported by ARPA under Air Force Contract AF 19(628)-5167 (ARPA Order 498).

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AN EFFICIENT TECHNIQUE FOR THE CALCULATION OF VELOCITY-ACCELERATION PERIODOGRAMS

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Group 42

TECHNICAL NOTE 1966-31

18 MAY 1966
ABSTRACT

The Cooley-Tukey method for greatly reducing the number of computations required to evaluate a velocity periodogram has been extended to the evaluation of velocity-acceleration periodograms. For N data points, this method requires approximately a factor of $2/3$ fewer computations than would be required by straightforward evaluation of the periodogram.

Accepted for the Air Force
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An Efficient Technique for the Calculation of Velocity — Acceleration Periodograms

The velocity — acceleration periodogram associated with the (complex) data samples \( r_0, \ldots, r_{N-1} \) is defined by

\[
P(f, \alpha) = \sum_{k=0}^{N-1} r_k e^{j2\pi kA} e^{j2\pi \alpha (k\Delta)^2}
\]

where \( \Delta \) denotes the (uniform) time separation between successive data points. \( P \) is periodic in \( f \) with period \( \Delta^{-1} \) and periodic in \( \alpha \) with period \( \Delta^{-2} \) so that \( P \) need only be evaluated over the \((f, \alpha)\) region defined by \( 0 \leq f < \Delta^{-1}, \)
\( 0 \leq \alpha < \Delta^{-2} \). Furthermore, since the velocity and acceleration resolutions of the periodogram are given (approximately) by \((N\Delta)^{-1}\) and \((N\Delta)^{-2}\) respectively, it is usually sufficient to evaluate \( P \) at the discrete points given by \( f = n(N\Delta)^{-1}, \alpha = m(N\Delta)^{-2} \) where \( n = 0, 1, \ldots, N-1 \) and \( m = 0, 1, \ldots, N^2-1 \). These considerations transform the original periodogram problem to the evaluation of the expression:

\[
P(n, m) = \sum_{k=0}^{N-1} r_k W^{nk} V^{nk^2}
\]

where \( n = 0, 1, \ldots, N-1; \ m = 0, 1, \ldots, N^2-1; \ W = \exp(j 2\pi/N), \ V = \exp(j 2\pi/N^2). \)
Following Cooley and Tukey*, we assume that $N = 2^p$ and proceed to express the integers $k$, $n$, $m$ in binary form as follows:

$$k = k_{\ell-1}2^{p-1} + \ldots + k_12 + k_0$$

$$n = n_{\ell-1}2^{p-1} + \ldots + n_12 + n_0$$

$$m = m_{2\ell-1}2^{2p-1} + \ldots + m_12 + m_0$$

where $k_1$, $n_1$ and $m_1$ take on the values 0 and 1. In addition, it will be convenient to express $k^2$ in the form

$$k^2 = (k^2)_{\ell-1} + \ldots + (k^2)_0$$

where $(k^2)_{\ell-1}$ = those terms in $k^2$ that depend on $k_{\ell-1}$ but not on $k_{\ell-1+1}$, ..., $k_{\ell-1}$. Thus,

$$(k^2)_{\ell-1} = k_{\ell-1}2^{p-\ell+1} \sum_{g=\ell+1}^p k_{p-g}2^{p-g} + k_{p-\ell}2^{2(p-\ell)}$$

The derivation of this last formula is straightforward exercise. Note that

* Cooley and Tukey, An Algorithm for the Machine Calculation of Complex Fourier Series, Math. of Comp. 12; April, 1965.
(k^2)_{p-\ell} contains a factor 2^{p-\ell+1} except when \ell = p.

Next we note that

\[ w^nk = W^{(n_o + ... + n_{p-1} 2^{p-1})(k_o + ... + k_{p-1} 2^{p-1})} \]

\[ = W^{k_{p-1} 2^{p-1} n_o} W^{k_{p-2} 2^{p-2}(n_o + n_{12})} \]

\[ ... W^{k_o(n_o + ... + n_{p-1} 2^{p-1})} \]

and

\[ v^{m(k^2)} = v^{(m_o + ... + m_{2p-1} 2^{2p-1})[(k^2)_{o} + ... + (k^2)_{p-1}]} \]

\[ = v^{(k^2)_{p-1}(m_o + ... + m_{p-1} 2^{p-1}) ...} \]

\[ v^{(k^2)_{1}(m_o + ... + m_{2p-3} 2^{2p-3})} v^{(k^2)_{o}(m_o + ... + m_{2p-1} 2^{2p-1})} \]

because the exponent of W need only be computed modulo \( N = 2^p \) and the exponent of V need only be computed modulo \( N^2 = 2^{2p} \).

With some obvious changes of notation, equation (2) now can be written in the form

\[ P(n_o, ... n_{p-1}, m_o, ... m_{2p-1}) = \]
For computational purposes, it is convenient to think of equation (3) as a sequence of \( p \) calculations as follows: First compute

\[
P_1(k_o, \ldots, k_{p-2}, n_o, m_o, \ldots, m_{p-1})
\]

\[
= \sum_{k_0} \frac{k_0(n_0 + \cdots + n_{p-1} 2^{p-1})}{v} \left( k^2 \right)_o(m_0 + \cdots + m_{2p-1} 2^{2p-1})
\]

\[
\cdots \sum_{k_{p-1}} \left( k_{p-1} \right) \frac{k_{p-1} 2^{p-1} n_o}{v} \left( k^2 \right)_{p-1}(m_o + \cdots + m_{p-1} 2^{p-1})
\]

(3)

then successively compute \( P_\ell \) from \( P_{\ell-1} \), \( \ell = \ell, \ldots, p-1 \), according to the formula

\[
P_\ell(k_o, \ldots, k_{p-\ell-1}, n_o, \ldots, n_{\ell-1}, m_o, \ldots, m_{p+\ell-2}) = \]

\[
\sum_{k_{p-\ell-1}} \left( k_{p-\ell-1} \right) \frac{k_{p-\ell-1} 2^{p-\ell-1} n_o}{v} \left( k^2 \right)_{p-\ell}(m_o + \cdots + m_{p-\ell} 2^{p-\ell})
\]

(4)
Finally, $P_p$ is computed from the formula,

$$P_p(n_0, \ldots n_{p-1}, m_0, \ldots m_{2p-1})$$

$$= \sum_{k_0} P_{p-1}(k_0, n_0, \ldots n_{p-2}, m_0, \ldots m_{2p-3})$$

$$= k_0(n_0 + \ldots n_{p-1} 2^{p-1}) (k_2)^2 (m_0 + \ldots + m_{2p-1} 2^{2p-1})$$

The last computed function $P_p$ is the desired function $P$ given by equation (3).

A straightforward computation of the periodogram from equation (2) would require $(N-1)N^3$ computations. (A computation is defined as being the performance of two complex multiplications followed by an addition. Thus, each evaluation of the sum in equation (2) requires $N-1$ computations and, since there are $N\cdot N^2 = N^3$ values of $n$ and $m$ for which the sum must be evaluated, the resulting number of computations is $(N-1)N^3$.) The computation method just proposed requires many fewer computations as will now be demonstrated.

The calculation of $P_1$ requires $2^{p-1} 2^2 2^p = 2^{2p}$ computations and the calculation of $P_l$, from $P_{l-1}$, $l = 2, \ldots p-1$, requires $2^{p-l} 2^l 2^{p+l-1} = 2^{2p+l-1}$ computations. Finally, the calculation of $P_p$ from $P_{p-1}$ requires $2^p 2^{2p} = 2^{3p}$
computations. Thus, the total number of computations is given by

\[ C = \sum_{\ell=1}^{p-1} 2^{2p+\ell-1} + 2^{3p} = \frac{1}{2} N^2 (3N-2) \]

For large \( N \), this figure is roughly a factor of \( \frac{2}{3} N \) smaller than the number of computations required by straightforward evaluation of equation (2).

A further reduction in the number of computations can be effected if \( P \) need not be evaluated for all possible values of its arguments. For example, assume that \( P \) is to be evaluated for all velocity resolution cells but only for the \( M \) smallest acceleration cells where \( M \) is of the form \( M = 2^{p+g} \), \( 0 \leq g < p \). (The reason for assuming \( M \) to be of this form will become apparent in a moment.) In this case, the binary expansion for \( m \) requires only \( p + g \) instead of \( 2p \) binary digits; i.e. \( m = m_0 + \ldots + m_{p+g-1} 2^{p+g-1} \). Examination of equations (1), (5), and (6) now reveals that the number of computations required for \( P_\ell \) is equal to \( 2^{2p+\ell} \) for \( \ell = 1, \ldots, g \) and equal to \( 2^{2p+g-1} \) for \( \ell = g+1, \ldots, p \). It follows that the total number of computations \( C_M \) is given by

\[ C_M = \sum_{\ell=1}^{g} 2^{2p+\ell-1} + (p-g) 2^{2p+g-1} \]

\[ = N(M-N) + \frac{NM}{2} \log_2 \left( \frac{N^2}{M} \right) \quad (7) \]
It is interesting to compare the value of $C_M$ given by equation (6) with the number of computations required by (two other methods) for evaluating $P$ for $N$ velocity resolution cells and $M$ acceleration resolution cells. Straightforward evaluation of equation (2) requires $NM(N-1)$ computations; thus the efficiency of the above proposed method can be assessed by evaluating the ratio

$$\frac{C_M}{NM(N-1)} = \frac{1 - \frac{N}{M}}{N-1} + \frac{1}{2(N-1)} \log_2 \left( \frac{N^2}{M} \right)$$

(8)

As a numerical example, consider the numbers $N = M = 32$ for which equation (8) yields $\frac{C_M}{NM(N-1)} = 0.08$. This illustrates the considerable computational advantage the proposed method has over straightforward evaluation of equation (2).

Another way of calculating $P$ for $N$ velocity resolution cells and $M$ acceleration resolution cells is to combine the acceleration factor $v^{mk^2}$ with the data $r_k$ in equation (2) and then apply the Cooley-Tukey method for a pure velocity periodogram for each desired value of $m$. This approach results in a total of $NM \log_2 N$ computations which when compared with $C_M$ yields

$$\frac{C_M}{NM \log_2 N} = \frac{(1 - \frac{N}{M}) + \frac{1}{2} \log_2 \left( \frac{N^2}{M} \right)}{\log_2 N}$$

(9)

Substituting $N = M = 32$ in equation (8) results in $C_M/NM \log_2 N = 1/2$ which means that, in this case, our method is only a factor of two more efficient than the modified Cooley-Tukey method.
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1. ORIGINATING ACTIVITY (Corporate author)
Lincoln Laboratory, M.I.T.

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP
None

3. REPORT TITLE
An Efficient Technique for the Calculation of Velocity-Acceleration Periodograms

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Note

5. AUTHOR(S) (Last name, first name, initial)
Hofstetter, Edward M.

6. REPORT DATE
18 May 1966

7a. TOTAL NO. OF PAGES
14

7b. NO. OF REFS
1

9a. ORIGINATOR'S REPORT NUMBER(S)
Technical Note 1966-31

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
ESD-TR-66-208

10. AVAILABILITY/LIMITATION NOTICES
Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES
None

12. SPONSORING MILITARY ACTIVITY
Advanced Research Projects Agency,
Department of Defense

13. ABSTRACT
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14. KEY WORDS
periodograms