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William Larimer Mellon, Founder

Management Sciences Research Report No. 63

COMPUTATION OF EXPANSION RATES FOR THE GENERALIZED VON NEUMANN MODEL OF AN EXPANDING ECONOMY

by

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I. Introduction and Terminology

In recent years a great deal of interest has developed in the study of economic growth development. For instance, many economists have been quite concerned over whether the U. S. economy is expanding faster than the Russian economy. The traditional measure of the growth rate of a country's economy is the rate of change of the country's G.N.P. However, it has been pointed out by J. K. Galbraith [5, p.368] , among others, that G.N.P. is not a very useful measure since it does not measure what we want it to--a country's economic capability. From considerations such as these it therefore seems desirable to search for alternative models and measures of a less summary nature. With this in mind we have been considering alternative routes and one such alternative that deserves exploration is the von Neumann model [14] of balanced growth. Here we report one such piece of research by reference to the generalized von Neumann model as developed and reported by Kemeny, Morgenstern, and Thompson [8] .

The von Neumann model has taken on additional significance since the turnpike conjecture of Dorfman, Samuelson, and Solow [3] . McKenzie [12] , Morishima [13] , and Radner [16] have proved various versions of the turnpike theorem. Koopman [9] gives a lucid discussion of the significance of the von Neumann model for turnpike theorems and then of turnpike theorems for growth theory.

There will be nothing in this report concerned with turnpike theory. This paper summarizes theoretical results of our study of the generalized von Neumann model. We present a computationally feasible method for finding expansion rates and the other variables of interest in a von Neumann

model. We explore briefly what kind of data is needed for von Neumann analysis. Our analysis is applied to input-output data for the United States in 1919, 1929, and 1939 compiled by Leontief [10 , 11] and in 1947 compiled by Evans and Hoffenberg [4] .

This paper is divided into two relatively independent parts. The first part develops additional theory necessary to construct a search technique to find all pertinent expansion rates. We compare our search technique with another method, that of Weil [17] , devised for the same purpose and show how the two methods can be combined.

The second section discusses the data requirements of the von Neumann model and the limitations of the data we use in our examples. We offer an extension of the von Neumann analysis to help solve a possible planning problem. Finally we argue that the von Neumann model and related analyses can be made into tools not only for theoretical analysis and qualitative characterizations but also for quantitative-numerical results and guides for planning and evaluation.

TERMINOLOGY

The terminology which we will use is the same as the terminology used by Kemeny, Morgenstern and Thompson [8] . For the reader unfamiliar with this terminology we now provide a review as follows.

The following vectors and matrices characterize the structure of the economy.

The A - matrix

The (mxn) A - matrix denotes the input requirements of the various industries in the economy -- that is a_{ij} denotes the amount of product j necessary in order that industry i may be operated at unit intensity.

The B - Matrix

The (mxn) B - matrix denotes the products produced by the various industries. That is, b_{ij} denotes the amount of good j produced by industry i when it is operated at unit intensity.

The x - vector

The x - vector is a row vector and denotes the intensity with which each industry should be operated. The intensities are normalized so that the ith process operates at intensity x_i where $0 \leq x_i$ and $\sum_{i=1}^{m} x_i = 1$. Thus the intensity vector (a row vector) is an n-dimensional probability vector. The components of the (1xn) row vector xA denote the amounts of inputs used up in production and the components of the (1xn) row vector xB denote the amounts produced.

The y - vector

The y - vector denotes the prices which should be assigned to the goods. Prices are also normalized so that y (a column vector) is an n-dimensional probability vector.

Finally we have the scalars α and β which are the expansion factor and the interest factor respectively for the economy. Let the (mxn) matrix $M_\alpha = B - \alpha A$. It is then shown that to find the expansion

rate with associated price and intensity vectors, given the A matrix and B matrix, is equivalent to finding vectors x and y and the scalar α which satisfy the following conditions.

(1) $x(M_\alpha) \geq 0$

(2) $(M_\alpha)y \leq 0$

(3) $xBy > 0$

If M_α is interpreted as a matrix game we see that we are actually looking for an α so that the value of the game is zero. We can also see that the x - vector which we are seeking is the optimal strategy vector for the maximizing (row) player and the y - vector the optimal strategy vector for the minimizing (column) player.

The von Neumann model consists of an input matrix and an output matrix from which are calculated an expansion rate with associated price and intensity vector. As we shall discuss later, there may be more than one expansion rate for a given model. The maximum von Neumann expansion rate is the maximum rate of growth sustainable in the economy over time. This scalar which describes the potential growth rate is entirely different from the rate of growth of G.N.P. which describes observed performance. The price and intensity vectors can provide further information useful for making comparisons between economies.

To see the latter point consider a new vector $s = xB \neq 0$. Divide each component of s, s_i, by the quantity $\sum_i s_i \neq 0$ to form a new vector s'. This new vector s' might be called a <u>stock-ratio</u> vector for it shows the relative importance of each good in the economy. For example we might find a technology with growth rate 25% in which the stock-ratios

were 50% guns and 50% butter. Another technology might exist with growth rate 6% in which the stock-ratios were 10% guns and 90% butter. Rather than chosing between the two technologies on the basis of the growth rate alone, decisions can include consideration of the goods ratios. 2/

II THE SEARCH TECHNIQUE

This section is divided into two parts. The first part deals with the description of the function $v(M\alpha)$ and the second part discusses a computer program that will find the maximum and/or minimum expansion factor (the maximum and minimum roots of the function $v(M\alpha)$) and the optimal strategies associated with these values of α.

A. Description of the Function $v(M\alpha)$

An optimal search technique was wanted for these solutions. Thus it was first necessary to learn as much as possible about the function with which we had to deal. In [9] it was proved that if α' and α'' ($< \alpha'$) are two distinct allowable values of α(i.e. $V(M_{\alpha'}) = v(M_{\alpha''}) = 0$) then $v(M_\alpha) = 0$ for all α such that $\alpha' \leq \alpha \leq \alpha''$. Moreover, if x' is

2/ It should be noted that s' is not independent of the units of measurement. Care should be taken that in the two economies being compared the units of measurement are the same. The stock-ratios can be made dimensionless by taking prices into account. If a good previously measured in units u has price p has its units changed to ku, then its price will be p/k so that the product $up = ku(p/k)$ remains unchanged. Rather than using $s = xB$, we can use $s = xB \otimes y$, where s_i is then the product of the ith element of xB times the ith element of y. The elements of s' formed from s, so defined, will be independent of the units of measurement.

optimal in $M_{\alpha'}$ and y'' is optimal in $M_{\alpha''}$ then the pair (x',y'') is optimal in M_α for all α in the same interval. It is also not hard to see that $v(M_\alpha)$ is a continuous function of α. It had also been stated [5] but not proved rigorously that over the domain of all non negative real numbers the function $v(M_\alpha)$ is non-increasing. The two proceeding statements about the function $v(M_\alpha)$ lead to the following picture:

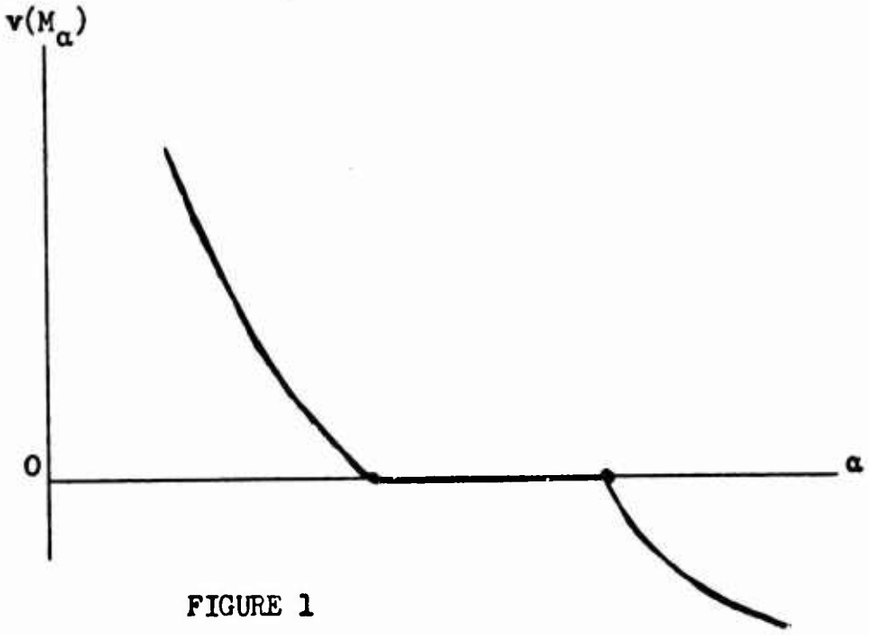

FIGURE 1

More can be said about the function and the next three theorems show that the graph of the function must be as in Figure 2

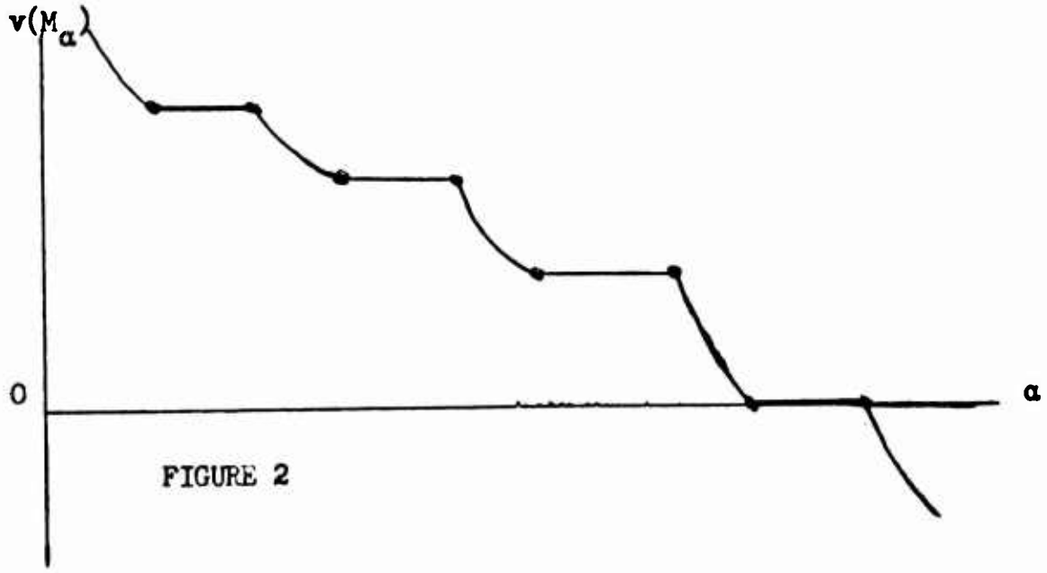

FIGURE 2

Theorem I

The function $v = v(M_\alpha)$ is non increasing

Proof

For any two α such that

$$\alpha' > \alpha''$$

We have that

(a) $M_{\alpha'} = M_{\alpha''} - (\alpha' - \alpha'') A \leq M_{\alpha''} - (\alpha' - \alpha'') a_m E \leq M_{\alpha''}$

where

(4) a_m = the value of the minimum entry of $A(a_m \geq 0)$.

(5) E is a matrix such that $e_{ij} = 1$ for all i and j .

We then have that

(b) $xM_{\alpha'} \leq x M_{\alpha''}$ for every x where x represents a strategy vector of the maximizing player in M_α. In particular (b) is also true for x' which is the row player's optimal strategy in $M_{a'}$

thus

$$x' M_{\alpha'} \leq x' M_{\alpha''}$$

also

$$v' = \min_j \; x' M_{\alpha'}$$

and

$$v'' \geq \min_j \; x' M_{\alpha''}$$

where v' and v'' are the values of the games $M_{\alpha'}$ and $M_{\alpha''}$ respectively. Thus $v' = \min_j x' M_{\alpha'} \leq \min_j x' M_{\alpha''} \leq v''$ which completes the proof.

Theorem II

If $A > 0$ then the function $v = v(M_\alpha)$ is strictly decreasing

Proof

The proof of this theorem is similar to that of Theorem I. The major source of difference is that a_m (the minimum entry of A) is now greater than zero. Thus equations (a) and (b) of Theorem I now become strict inequalities. Thus

(a') $M_{\alpha'} < M_{\alpha''}$

(b') $xM_{a'} < xM_{a''}$

where x and x' have the same meaning as they did in Theorem I. Therefore we have that

$$x' M_{\alpha'} < x' M_{\alpha''}$$

$$v' = \min_j x' M_{\alpha'}$$

$$v'' \geq \min_j x' M_{\alpha''}$$

thus

$$v' = \min_j x' M_{\alpha'} < \min_j x' M_{\alpha''} \leq v''$$

Theorem III

If $A \geq 0$ then if α' and α'' $(<\alpha')$ are two distinct values of α such that

$$v(M_{\alpha'}) = v(M_{\alpha''}) = V$$

then

(a) $V = v(\bar{B})$ where $\bar{B}$ is a submatrix of B

(b) $\bar{A} = 0$ where $\bar{A}$ is the corresponding submatrix in A

(c) $V \geq 0$

(d) $v(M_\alpha) = V$ for all $\alpha \in [\alpha'', \alpha']$

(e) If x' is optimal in $M_{\alpha'}$ and y'' is optimal in $M_{\alpha''}$ then the pair (x', y'') is optimal for all $\alpha \in [\alpha'', \alpha']$.

Proof

Let x' be an optimal strategy for the maximizing player in the game $M_{\alpha'}$; then $x' M_{\alpha'} \geq Ve$, where e is a row vector all of whose components are 1. Then for any $a^* \in [\alpha'', \alpha']$ we have

$$x' M_{\alpha^*} = x'(B - \alpha^* A) = x'(B - \alpha' A) + x'(\alpha' - \alpha^*) A \geq x' M_{\alpha'} \geq Ve$$

hence

$$v(M_{\alpha^*}) \geq V . \tag{6}$$

Similarly, let y'' be optimal for the minimizing player in the game $M_{\alpha''}$; then $M_{\alpha''} y'' \leq Ve$.

We then have

$$M_{\alpha^*} y'' = (B - \alpha^* A)\, y'' = (B - \alpha'' A)\, y'' + (\alpha'' - \alpha^*)\, A\, y'' \leq Ve$$

hence

$$v(M_{\alpha^*}) \leq V \,. \tag{7}$$

The two inequalities (6) and (7) show that $v(M_\alpha) = V$ and also show that (x', Y'') are optimal strategies in the game M_α for all $\alpha \in [\alpha'', \alpha']$. This proves part (d) and (e). From the above we have that

$$x' M_{\alpha''} y'' = V = x' M_{\alpha'} y''$$

$$x' B y'' - x' A \alpha'' y'' = x' B y'' - x' A\alpha' y''$$

$$x' A \alpha' y'' - x' A\alpha'' y'' = 0$$

$$x' (\alpha' - \alpha'') A y'' = 0$$

Since $(\alpha' - \alpha'') \neq 0, x' A y'' = 0$. Therefore $\bar{A}$ the submatrix of A to which the pair of strategies (x', y'') assign positive weight must be identically equal to zero. This proves part (b). (When in addition $V = 0$ it was shown in (4) that $\bar{B} = 0$ also.)

We then have that

$$V = v(M_{\alpha^*}) = x' (\bar{B} - \alpha^* \bar{A})\, y' = x' \bar{B}\, y'$$

which proves part (a).

Finally since $B \geq 0$, $x' \geq 0$, $y'' \geq 0$ we have that $V \geq 0$.

B. Development of an Algorithm for Finding Expansion Rates.

Having described the function $v(M_\alpha)$ in its most general form the next step is to devise a technique for searching over values of α to find where the function v is zero. For any α, $v(M_\alpha)$ is the value of the matrix game M_α. It is well-known that the value of a two-person, zero-sum game can be found by linear programming. 3/

Thus, for any α, $v(M_\alpha)$ is found by solving a linear programming problem. In order to find the maximum α so that $v(M_\alpha) = 0$, the following search procedure is used:

1. Choose a nonnegative number L so that $v(M_L)$ is known to be greater than zero ($L = 0$ will always work); choose a number R so that $v(M_R)$ is known to be negative (it is easy to show that such a number exists). In other words L and R are such that whenever $v(M_\alpha) = 0$ then $L < \alpha < R$.

3/ To find the value of a matrix game G carry out the following:

1. Add the same amount d to each element of G to form the game G^*. The quantity d must be large enough to insure that $v(G^*) > 0$. For this it is sufficient to make $g^*_{ij} > 0$ for all i and j.
2. Solve the linear programming problem

$$\text{Minimize} \quad z = \sum_i w_i$$

$$\text{Subject to} \quad G^* \geq e$$

where e is a row vector of all 1's.

3. The value of the game is $v = 1/z - d$ and the components of the optimal strategies for the maximizing player are $x_i = w_i/z$.

2. Let $\alpha = (L + R)/2$.

3. Solve for $v(M_\alpha)$.

   a. If $v(M_\alpha) \geq 0$, then replace L by α ; go to step 4.

   b. If $v(M_\alpha) < 0$, then replace R by α ; go to 4.

4. If $(R - L) < \epsilon$, for some prescribed ϵ , then halt. Otherwise go to 2.

By moving the equality test from step 3a to 3b (so that 3a uses ">" and 3b uses "$\leq$") the algorithm will find the minimum α such that $v(M_\alpha) = 0$.

In [8] it was shown that when there is more than one α for which $v(M_\alpha) = 0$, the economy can break up into self-sufficient subeconomies. The commodities in such a subeconomy can grow independently of the others. These subeconomies are not necessarily disconnected: that is, some goods and/or activities may belong to more than one subeconomy.

The procedure for finding balanced growth rates outlined above will find only the maximal and minimal allowable growth rates. If there is a subeconomy -- say one that produces solely the good "rabbits" -- which can grow independently of the rest of the economy and whose potential growth rate is higher than that of any other subeconomy, then it is the "rabbit rate" which the procedure will find.[4/] The economist or planner may be more interested in some other subeconomy that grows at a smaller rate and which produces other more desirable goods.

4/ Gale [6, page 288] credits his referee with pointing out the possible existence of "rabbit rates."

In [8] it was shown that there are at most a finite number of growth rates satisfying the axioms of the economy. It is possible to find these intermediate growth rates via the technique of constrained games (see Charnes [1], or Charnes and Cooper [2,Chapter XX]). We describe this technique next.

The constrained game version of the expanding economy model is defined in terms of the matrices A, B, and M_α as defined previously, and consists of the following set of simultaneous inequalities:

(8) $xM_\alpha \geq 0$

(9) $M_\alpha y \leq 0$

(10) $xBy > 0$

(11) $xB \geq d_1$

(12) $xA \leq d_2$

(13) $x \geq b_1$

(14) $y \geq b_2$

(15) $x \leq b_3$

(16) $y \leq b_4$

(The question as to the dimensions of the various quantities can be inferred from context.) Inequalities (8), (9), and (10) are the same as (1), (2), and (3) previously defined. (11) is a lower bound constraint on outputs which can be used to insure that certain goods are produced in at least desired minimal quantities; (12) is an upper bound constraints on inputs that insures that certain goods not be used up too fast;

(13) is a lower bound constraint on intensities that makes certain that certain processes be utilized (this is equivalent to subsidies); (14) is a lower bound constraint on prices that insures that prices of desired goods do not drop below certain levels (i.e., price supports); (15) is an upper bound constraint on intensities which prevents too heavy use of them; and (16) is an upper bound on prices which prevents too high prices on certain goods. Other kinds of constraints might be added, if necessary to achieve other kinds of goals. And not all of the stated constraints are needed in every application of the model.

The algorithm for solving the constrained game expanding economy model is a simple modification of the algorithm presented above. It can be described as follows:

1. Select upper and lower bounds on intensities, prices, inputs, and outputs and set up the linear programming problem corresponding to the constrained game in (8) - (16). This is just the linear programming problem given in footnote 3 with constraints corresponding to (11) through (16) added.

2. Determine L and R as before so that the value of the game is positive at L and negative at R.

3. Use the search procedure to vary alpha and determine the largest (or smallest) alpha so that the value of the constrained game is zero.

Of course, care must be taken that the constraints are not so stringent that no feasible alpha can be found. Observe that the problem of finding solutions to constrained games is no more difficult than to the unconstrained game, since it corresponds to solving linear programming problems with some additional constraints.

III. Comparison with another Algorithm

Weil [17] has proposed an alternate algorithm for finding solutions to the expanding economy model. His method is based on a theorem of Gale [6] which asserts that at most as many activities are necessary to achieve the largest growth rate as there are goods in the economy. The method involves selecting square subsets of activities and goods in such a way that the submatrix of either the A matrix or the B matrix is non-singular. An eigenvalue problem is derived from the subset, and solved using one of the standard eigenvalue codes. The resulting eigenvalues and vectors are tested to see if they yield economic solutions to the model. For economies with more activities than goods it requires considerably more calculation than the constrained game search technique outlined above. It is shown in [17] that the method will require finding eigenvalues and eigenvectors for, on the average, <u>at least</u> as many as $w(z-w)/(2z)$ matrices of dimension w by w, where $w = \text{Min}\ (m,n)$ and $z = \text{Max}\ (m,n)$.

The constrained game search technique picks a range, limited by the numbers L and R, such that α is known to lie in the interval $[L,R]$.

At each iteration the allowable range for α is cut in half so that in k iterations the range in which it must lie is less than or equal to $(R - L)/2^k$. We have found in practice that α is accurate to three significant digits if the search procedure is terminated when $|v(M_\alpha)| \leq .0001$. This kind of accuracy can be determined in about 14 iterations, if the starting range, $R - L$, is about 1, and this involves solving 14 linear programming problems of size $m \times n$. For moderate m and n this is not a prohibitive amount of calculation.

A combination of the two techniques can also be used. Namely the search technique can be employed until the correct set of processes and goods has been selected by the linear programming problem. The eigenvalue technique can then be applied to that subset to obtain final answers.

III. Application of the Algorithm

The data necessary for von Neumann growth model analyses are not generally available in the requisite form and for the relevant periods. We shall discuss these data requirements and since our recourse is to already available input-output data we shall also examine the limitations that this imposes on the kinds of results we may expect.

We have used input-output data for 1919, 1929, and 1939 compiled by Leontief [1951, 1953] and for 1947 compiled by Evans and Hoffenberg [1952]. We have aggregated the data to nine-industry models. Leontief's data were synthesized and presented in a 41-industry model. Evans and Hoffenberg used a 45-industry model. Aggregation of the data to a 9-industry

model has mixed benefits. [8] While the comparability of industry classifications over time increases as aggregation increases, detail is lost. The following discussion will show that we do expect much from the data; even aggregating as much as we have, there remain classification incompatibilities. The other difficulties will be explained shortly. In the appendix where our aggregated models are given we also show the correspondence between our industry breakdown and those of the original compilers.

The Data Problem

The problem of data availability is, of course, a perplexing one in almost all parts of empirical research in economics. Ideally, we should have up-to-date technological data specifying the output capabilities and input requirements of U. S. industries in pertinent detail. Thanks to previous input-output analyses we are able to obtain input coefficients. However, this data measures inputs in dollars of output of other industries and what we really want for our von Neumann analysis is inputs in units of products used.

For input-output analyses the original data are generally in terms of dollar values for all transactions. Division by the total value of each industry's outputs is then used to obtain the "physical" coefficients and possible further adjusted by index number techniques. This, in principle, is not adequate for a von Neumann model which should

[8] There are other known aggregation difficulties which we do not discuss, however. See, e.g., Hatanaka [1952].

be measured in physical units as obtained directly from, say, engineering-technological considerations. It seems unlikely to us that the data will ever be available in physical units. There are thousands of goods in an economy and we do not expect suitable accurate data to be collected for so large a matrix. The data on input and output units will, of necessity, be aggregated and the best medium for aggregation will probably continue to be dollars and the existing prices of goods.

The y vector of a balanced growth solution as a vector of prices optimally imputed to the units of each good. Since these prices need not—and, in general, will not—coincide with the prices used for the extant input-output data, there are two sets of prices to be considered. The output matrix of input-output analysis is an identity matrix so that the output of each industry is one dollar's worth of output of the given industry. Presumably, the price of the dollar's worth of output is one dollar or, normalizing so that the sum of prices is one, 1/n for an n-industry model. The price vector of the von Neumann solution gives prices that support a balanced growth equilibrium. The "distance" between the vector Y and the vector (1/n, 1/n, ..., 1/n) will in a sense, measure how far the economy from which the input-output data were gathered is from being the balanced growth economy supportable by the given technology. Charnes and Cooper [1961, pp. 154-167] discuss various measures of distances between vectors.

Further problems arise because the classifications of data for the years we studied are not the same. We have tried to overcome this

difficulty by a high degree of aggregation. Rather than dealing with 40-odd industry models--the data for which are not comparable--we have chosen for the detailed study 9-industry models in the belief that this might yield increased comparability. But this, too, cannot be asserted with complete assurance and, in addition, certain further unwanted consequences may also be encountered. Thus, for instance, Kemeny, et al. [1956, Section 9] prove that the aggregation method we use has unpredictable effect on the results when used for von Neumann growth analyses. They show that if only goods are aggregated, the growth rate may increase and can never decrease and if only processes are aggregated, the growth rate may decrease and never increase. When both goods and processes are aggregated, as we have done, the growth rate may rise or fall in ways that are not easy to analyze or predict.

The data that we have used are rough estimates. This problem would hardly have to be mentioned; however, as Kemeny, et al. [1956, Section 4] showed, small changes in the entries of the input and output matrices can have very large effects on the components of the intensity and price vectors.

Possibly the greatest difficulty in using input-output analysis data are the technical problems in the treatment of capital and labor.

There are neither capital nor labor problems in von Neumann's closed model. Capital goods and various kinds of labor are no different from any other commodity in the economy. There is a separate row in both the input and output matrices for every kind of capital good and every kind

of labor good. There are activities for producing each of these goods. Depreciation is handled by labeling capital goods of different ages with different names. Thus, a process can input a unit of g_1 and output a unit of g_2 where a g_2 is nothing but a one-period-old g_1.

Input-output data show the purchases of one industry from another in a given year (net of capital). The implicit assumption is that all of the purchased goods are used for production, broadly defined to include inventory changes, foreign trade, etc., during the year in which they are purchased. The input coefficients do not show the amount of a product that <u>must be on hand</u> for production to be carried on. As Leontief [1951, p.211] has pointed out, input-output data in closed models are concerned with flows and do not show the stocks necessary for production. The omission of capital coefficients will introduce a downward bias into the intensities for the capital goods industries.

As more and better capital goods are used in the economy over time, the production requirements in a closed input-output model are increasingly understated. Thus we may predict that over the past 50 years in the U. S. growth rates estimated from closed model, input-output data will have a tendency to increase if for no other reason than because the measured flow of goods becomes relatively less important for production than the unmeasured stock of goods. In the same vein, we can predict that the price ratio between those industries which use relatively more capital to those which use relatively less will fall over time. That is, as more and more of the inputs for production of a good are not explicitly stated, the good seems cheaper to produce.

Input-output analyses sometimes show a final row in the input matrix of the wages paid per dollar of output. This method treats labor

as an exogenous supply for there is no activity or industry for producing the labor. Furthermore, it does not differentiate between different kinds (or skills) of labor.

Because the data show neither capital nor labor requirements, the growth rates found to be possible by von Neumann analysis are unreasonably high. In the cases we studied, all growth rates are at least 1.5 and some were higher than 2.5.

Finally, the data requirements of closed input-output analysis are, in some ways, much more restrictive than those of von Neumann analysis. Input-output analysis forbids both joint production and alternate means of production and thus requires both square matrices and an identity matrix for the output matrix. 9/ The von Neumann model is more general -- allowing both joint production and alternate activities. By applying von Neumann analysis to input output data, we are not confronting the model with as complex an environment as it could handle.

Uses of the Model

The previous section has outlined at least a half-dozen reasons why the numbers we find for expansion rates, intensity and price vectors should not be taken literally and may, in fact, require serous qualification. From one standpoint, then, we may regard these as only illustrative of possible applications. On the other hand, it at least provides a

9/ Samuelson's substitution theorem [1951] shows that all solutions between alternate processes have been carried out. There may have been alternate means of production but the best processes were selected by the market mechanism. The other, less profitable processes can be ignored and are ignored by input-output analysis. One of the problems solved in the von Neuman model is the selection of the processes to be used.

start--albeit a very limited one--toward the kind of research which is necessary if the von Neumann type models of growth are to be accorded the kinds of uses that might be pertinent for purposes of positive economics and policy uses.

At a minimum and with the warnings that have already been noted, we at least have data for more than one period which is a prime requisite for any real use of such growth models. Of course, this implies at a minimum that the classification of input output data are constant enough so that some comparisons over time can be made. On this supposition we can then apply our suggested algorithms to solve von Neumann economies for two or more different years and observe how patterns change. In this fashion we can at least give a concrete illustration of the results that can be obtained via these solution procedures both with reference to growth rates and the accompanying intensity vectors.

Thus refer to Table VII in the appendix where, for each year, we exhibit the growth rate as well as the normalized price and intensity vectors for the nine-industry models exhibited in Tables III, IV, V, and VI.

Note that the growth rate rises during the period 1919-1929-1939 but falls for 1939-1947. The first result is consistent with what we might expect because capital is not included in the technology. There are three possible explanations for the fall in growth rate from 1939 to 1947:

(1) our theory is incorrect;

(2) the increase in rate caused by the extra hidden capital was more than offset by technological regress;

(3) the 1947 data, compiled by a different group, are not comparable enough to the data of earlier years to show the trend.

For the present, at least, we think that (3) is the most likely explanation.

In Table VIII we show prototype calculations of the kind we think may eventually be the most rewarding applications of the von Neumann model--selecting from alternative processes. We have considered, for example, the processes of 1929 and 1939 to comprise a single technology. Thus we have an economy with 9 goods and 18 industries, two methods for producing each good. We use the algorithms to select that subset of processes which will sustain the highest growth rate. The resulting rate will of course be at least as high as for the maximum rate for the economies considered separately and will be higher if some processes from both years are used. [10] We have also run and exhibit in Table VIII a combined-technology model for 1939-1947, 1929-1939-1947, and 1919-1929-1939-1947. Since the output matrix is an identity matrix, $xB = xI = x$ and the intensity vector is the same as the stock-ratio vector.

[10] Weil's [1964, Appendix II] "Improvement Theorem" provides a proof of this intuitive statement.

The algorithms select those of the alternate means of production to be used for optimal balanced growth. In these calculations results are particularly vulnerable to changes in classifications of the industries and quality changes in the goods. The units and quality of the same good should be the same for different processes and thus for different periods. For example, the combined 1929-1939-1947 technology calls for the use of 1939 autos and metal industry but the 1947 transportation industry. In the combined technology the assumption must be that one dollar of 1939 bus supplied to the 1947 transportation industry provides the same productive service as one dollar of 1947 bus. 11/

Perhaps the best argument for presenting some applications of the algorithms to data is to show to what uses the von Neumann model might be put. If these uses appear attractive to the potential users, then it is more probably that the needed data will be gathered in the future. Until the use for the data is demonstrated, there is little incentive to gather the data in the first place.

Von Neumann analysis can be used to select between alternate means of production. Thus, under suitable restrictive conditions, von Neumann analysis can serve as a planning tool.

We suggest that von Neumann analysis will prove useful in international economic analysis. By combining the technologies of two (or more)

11/ Deflating prices will not solve the problem of quality change when the processes represent technologies at different times. Multiplying the 1947 matrix entries by a price deflator to equate the purchasing power of 1939 and 1947 dollars will not alter the relative efficiencies of processes for the two years. The 1947 processes after deflation will merely be operated more intensely than before deflation to make up for the change in units.

separate countries into one and applying the algorithms to the single technology, we can ascertain the areas of production in which each economy should, ideally, specialize. Implicitly the von Neumann model assumes that all resources are mobile between sectors. Consequently the solution to the model might well indicate that one particular sector (or country) should produce everything. Traditionally international economic analysis assumes that at least some of the resources are not mobile so that we expect von Neumann analysis to yield international equilibria somewhat different from the standard approach. We shall summarize the results of our application of the von Neumann model to international economics in a future report.

APPENDIX

In this appendix we present the data, data aggregation key, and the results of our computations.

Table I shows which industries of the economy comprise each of our nine industries.

Table II shows the correspondence between our industry classifications and those of Leontief [1951, 1953] and Evans and Hoffenberg [1952] from which our data were aggregated.

For each of the years 1919, 1929, 1939, and 1947 we have used the algorithms to calculate the expansion rate for the nine-industry model. In Table VII we have shown the expansion factor, price vector, and intensity vector for each of these years.

In Table VIII we show the results of our prototype "combined technology" calculations. We constructed combined technologies for 1929-1939, 1939-1947, 1929-1939-1947, and 1919-1929-1939-1947. For each of the combined technologies we show the expansion rate. Further we show, in each model, for each good:

a) the equilibrium price for that good;

b) the intensity of the activity to produce that good, and

c) the year of the technology from which the activity to produce that good was selected.

TABLE I

THE INDUSTRIAL CLASSIFICATIONS USED FOR THE NINE-INDUSTRY MODELS

Industry Number	1919-1929 Industries Composed of	1939 Industries Composed of	1947 Industries Composed of
1	Agriculture and Food Processing	Agriculture and Food Processing	Agriculture and Food Processing
2	Ferrous Metals, Nonferrous Metals, Non-Metalic Minerals	Ferrous Metals, Nonferrous Metals, Non-Metalic Minerals	Ferrous Metals, Nonferrous Metals, Non-Metalic Minerals
3	Automobiles, Fabricated Metal Products	Automobiles, Fabricated Metal Products	Automobiles, Fabricated Metal Products
4	Petroleum; Coal and Coke; Manufactured Gas and Electric Utilities	Petroleum; Coal and Coke; Manufactured Gas and Electric Power	Products of Petroleum and Coal; Coal, Gas, and Electric Power; Communication
5	Chemicals	Chemicals	Chemicals
6	Wood, Timber, and Paper Products	Wood, Timber, and Paper Products	Wood, Timber, and Paper Products
7	Leather, Textiles, and Rubber	Leather, Textiles, and Rubber	Leather, Textiles, and Rubber
8	Construction	Construction	New Construction and Maintenance
9	Steam Railroad Transportation	Transportation and Trade	Transportation and Trade

TABLE II

CORRESPONDENCE BETWEEN INDUSTRY CLASSIFICATIONS

Our Industry Number	Leontief's Industry Numbers, 1919-1939 a/	Evans and Hoffenberg Industry Numbers, 1947 b/
1	1 - 10	1 - 3
2	11 - 19	14 - 16
3	14, 15	17 - 29
4	20 - 25	11, 30, 35
5	26	10
6	27 - 31	6 - 9
7	32 - 38	4, 5, 12, 13
8	40	45
9	41	31 - 34

a/ Note that industry 39, Industry, not elsewhere classified has been omitted from our aggregated model.

b/ Note that industries 36 through 44 have been omitted. These are the industries for Finance and Insurance, Rental, Business Services, Personal and Repair Services, Medical, Educational, and Nonprofit Organizations, Amusements, Scrap and Miscellaneous Industries, Undistributed, and Eating and Drinking Places.

TABLE III

THE A-MATRIX FOR 1919

Industries	Products								
	1	2	3	4	5	6	7	8	9
1	.514	.003	.021	.004	.015	.008	.004		.028
2		.168	.003	.043	.001	.001			.029
3		.128	.135	.016	.001	.011	.013		.012
4		.025	.061	.221	.005	.011			.123
5	.125	.031	.003	.026	.063	.016	.003		.003
6	.001	.001	.018	.015	.005	.254	.012		.051
7	.100		.012	.009	.008	.002	.259		
8		.241	.256	.002	.009	.085			
9		.021	.261	.094		.023			.001

Source: Reference Leontief [1951]

TABLE IV

THE A-MATRIX FOR 1929

Industries	Products								
	1	2	3	4	5	6	7	8	9
1	.420	.004	.023	.006	.018	.009	.002	.002	.037
2		.321	.010	.064	.009	.001		.012	.064
3		.165	.192	.015	.005	.011	.018	.006	.017
4		.025	.061	.218	.003	.006		.029	.114
5	.066	.038	.001	.036	.072	.003	.006	.005	.028
6	.014	.003	.017	.015	.008	.224	.008	.006	.042
7	.055	.001	.012	.011	.016	.001	.256	.004	.002
8		.187	.209	.005	.008	.047			
9		.035	.238	.061		.028		.009	

Source: Reference Leontief [1951]

TABLE V

THE A-MATRIX FOR 1939

Industries	Products								
	1	2	3	4	5	6	7	8	9
1	.255	.006	.031	.020	.015	.011	.005	.012	.210
2		.310	.008	.062	.007	.006	.000	.007	
3		.118	.179	.017	.008	.006	.019	.004	.179
4		.004	.023	.171	.002	.000	.000	.062	.284
5	.045	.045	.009	.040	.170	.014	.003	.004	.206
6	.022	.013	.018	.024	.013	.239	.010	.005	.153
7	.054	.001	.010	.013	.026	.004	.237	.002	.241
8	.008	.193	.153	.012	.030	.080	.001		
9		.003	.026	.065	.001	.011	.002	.039	.042

Source: Reference Leontief [1951]

TABLE VI

THE A-MATRIX FOR 1947

Industries	Products								
	1	2	3	4	5	6	7	8	9
1	.413	.004	.010	.010	.027	.010	.005	.004	.047
2	.001	.326	.009	.065	.013	.011	.002	.006	.060
3	.001	.146	.244	.011	.014	.017	.023	.004	.030
4		.005	.008	.286	.011	.009		.019	.043
5	.137	.033	.013	.039	.192	.036	.003	.003	.043
6	.010	.009	.014	.017	.016	.263	.018	.003	.049
7	.092	.002	.013	.009	.058	.008	.308	.003	.038
8		.102	.152	.024	.022	.094	.004		.132
9	.003	.004	.011	.040	.002	.013	.004	.024	.036

Source: Evans and Hoffenberg [1952]

TABLE VII

Individual Technologies

Industry	Year							
	$\alpha = 1.92$		$\alpha = 2.24$		$\alpha = 2.39$		$\alpha = 2.22$	
	1919		1929		1939		1947	
	Intensity[a]	Price	Intensity	Price	Intensity	Price	Intensity	Price
1 Ag. and Food	.10	.56	.11	.30	.02	.13	.38	.23
2 Minerals	.05	.00	.35	.10	.25	.12	.13	.07
3 Autos and Metal	.08	.01	.18	.09	.10	.12	.06	.10
4 Fuel and Power	.04	.01	.15	.09	.16	.10	.13	.03
5 Chemicals	.02	.16	.02	.08	.03	.11	.07	.16
6 Wood and Paper	.03	.02	.04	.06	.06	.11	.07	.07
7 Leather, etc.	.02	.22	.02	.11	.02	.13	.04	.24
8 Construction	.00	.01	.02	.09	.06	.13	.02	.08
9 Transportation	.06	.01	.11	.07	.29	.04	.10	.02

[a] Note that since the output matrix is an identity matrix, $xB = xI = x$ so that the intensity vector is also the stock ratio vector.

TABLE VIII

Combined Technologies

Industry	1929-1939 $\alpha = 2.46$ Year	Intensity[a]	Price	1939-1947 $\alpha = 2.58$ Year	Intensity	Price	1929-1939-1947 $\alpha = 2.65$ Year	Intensity	Price	1919-1929-1939-1947 $\alpha = 3.09$ Year	Intensity	Price
1 Ag. and Food	1939	.02	.15	1939	.04	.14	1939	.04	.15	1939	.09	.24
2 Minerals	1939	.28	.15	1939	.21	.14	1939	.24	.17	1919	.18	.04
3 Autos and Metal	1939	.11	.15	1939	.07	.13	1939	.07	.16	1919	.13	.06
4 Fuel and Power	1939	.17	.12	1939	.28	.08	1947	.30	.09	1919	.22	.10
5 Chemicals	1929	.02	.07	1947	.04	.12	1929	.03	.07	1929	.02	.09
6 Wood and Paper	1929	.06	.06	1939	.09	.10	1929	.07	.07	1929	.02	.11
7 Leather, etc.	1929	.02	.09	1947	.03	.14	1929	.03	.11	1929	.09	.25
8 Construction	1939	.06	.15	1947	.03	.14	1947	.04	.15	1929	.06	.08
9 Transportation	1939	.27	.05	1947	.21	.03	1947	.19	.03	1947	.20	.04

[a] See footnote a for Table VII

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13 ABSTRACT

The first part of the paper is a theoretical study of $v(B - \alpha A)$ considered as a function of α, where A and B are the input and output matrices of the model. Then a simple search algorithm is presented for computing the maximum and minimum expansion rates of the model. The algorithm uses the simplex method of linear programming as its basic subroutine. For a typical set of data, an expansion rate can be found to sufficient accuracy by solving 14 or fewer linear programming problems. With modern computers this is not a difficult task.

The last part of the paper discusses the available data, in this case, input-output data, and gives the computational results found using it. Limitations of the data and conclusions drawn from it are discussed. Finally, the use of the model for economic planning and evaluation is considered.

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