EVALUATION OF THE ALBEDO INTEGRAL FOR MARK I

Cord H. Link, Jr.
ARO, Inc.

February 1966

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FOREWORD

The work reported herein was done at the request of Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC), under Program Element 65402234.

The results of the research were obtained by ARO, Inc. (a subsidiary of Sverdrup and Parcel, Inc.), under Contract AF 40(600)-1200. The work was performed from February to August, 1964, under ARO Project No. SM3105, and the manuscript was submitted for publication on August 31, 1965.

This report is an extension of the work reported in AEDC-TDR-63-206 (February 1964).

This technical report has been reviewed and is approved.

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This report is concerned with the development of a fast computer method for evaluating the albedo integral. This integral defines the illumination on an arbitrarily oriented surface element at any point in space about a diffusely reflecting sphere. It enters the calculation of simulation control parameters in the Arnold Engineering Development Center Aerospace Environmental Chamber (Mark I). The semi-numerical method developed here is faster than ordinary numerical integration by a factor of about ten. A typical computer program, which formerly required about thirty minutes, now produces the same results in under four minutes.
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NOMENCLATURE

\( A_e \)  \hspace{1em} \text{Albedo, fraction of solar radiation reflected by earth}

\( C_y \)  \hspace{1em} \text{Horizon circle}

\( C_\theta \)  \hspace{1em} \text{Terminator curve}

\( C_\xi \)  \hspace{1em} "Target plane cut" curve

\( \Delta A_i \)  \hspace{1em} \text{Surface element on albedo source}

\( \Delta A_t \)  \hspace{1em} \text{Target surface element}

\( h \)  \hspace{1em} \text{Altitude}

\( I_s \)  \hspace{1em} \text{Solar constant, intensity}

\( I_t \)  \hspace{1em} \text{Intensity of target illumination}

\( L \)  \hspace{1em} \text{Orbit angular momentum vector}

\( N \)  \hspace{1em} \text{Vector normal to target surface}

\( N_s \)  \hspace{1em} \text{Solar node vector}

\( R \)  \hspace{1em} \text{Target position vector}

\( r_e \)  \hspace{1em} \text{Radius of the albedo source sphere}

\( S \)  \hspace{1em} \text{Sun position (unit)vector}

\( a \)  \hspace{1em} \text{Azimuth angle}

\( a_i \)  \hspace{1em} \( (i = 1, 2 \ldots) \) Boundary values in azimuth

\( a_N \)  \hspace{1em} \text{Azimuth of target normal vector}

\( \overset{\wedge}{a}, \overset{\natural}{a} \)  \hspace{1em} \text{Minimum, maximum values of azimuth angle}

\( \beta \)  \hspace{1em} \text{Nadir angle}

\( \beta_N \)  \hspace{1em} \text{Nadir angle of target normal vector}

\( \overset{\wedge}{\beta}, \overset{\natural}{\beta} \)  \hspace{1em} \text{Minimum, maximum values of nadir angle}

\( \gamma \)  \hspace{1em} \text{Relative altitude parameter, nadir angle of horizon}

\( \theta_e \)  \hspace{1em} \text{Angular distance from} \ S \ \text{to arbitrary point} \ (dA_i) \ \text{on albedo sphere; source to sun view angle}

\( \theta_s \)  \hspace{1em} \text{Angular distance from} \ R \ \text{to} \ S \text{ }

\( \theta_v \)  \hspace{1em} \text{Inclination of plane of} \ N_s \ \text{and} \ R \ \text{to} \ S \text{ }

\( \xi \)  \hspace{1em} \text{Angle at} \ dA_i \ \text{between} \ N \ \text{and} \ dA_i \ ; \ \text{target view angle} \)
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SECTION I
INTRODUCTION

This report extends one of the problems discussed in an earlier report\(^1\): the development of a method for evaluating the "albedo integral". The aim of this study is to improve the speed at which certain quantities are computed for the control of simulation parameters in the Aerospace Environmental Chamber (Mark I). In earlier study programs, the albedo integral was evaluated by strictly numerical integration techniques. The present seminumerical method is faster, by nearly an order of magnitude, than the numerical methods formerly used. This method has been incorporated into a Fortran language computer subroutine.

A derivation of the albedo integral, for illumination intensity, is reproduced in Appendix I, under assumptions that the albedo source is a homogeneous sphere with a diffusely scattering (Lambert) surface, so that the albedo is otherwise independent of surface and atmospheric conditions.

SECTION II
THE ALBEDO PROBLEM

In order to properly control the simulation of secondary radiation (albedo and planet radiance) in Mark I, it is necessary to determine the illumination on an arbitrarily oriented surface element at arbitrary altitude and at any position in a trajectory or orbital flight near a reflecting celestial body.

A derivation of the albedo integral, which expresses the illumination intensity, is given in the previous report\(^2\) under assumptions that the albedo source is a sphere having a homogeneous, diffusely scattering surface so that the albedo is otherwise independent of surface and atmospheric conditions. Then a different primary body may be distinguished


\(^2\)Ibid.
by a solar constant suitable for the distance from the sun, its mean albedo, and its radius. This last factor enters all secondary illumination calculations since they depend on relative altitude.

\[
\frac{L_2}{\Delta A_2}(\theta_s, \gamma, a_N, \beta_N) = 
\int_{\alpha(\theta_s, \gamma, a_N, \beta_N)}^{\beta(\theta_s, \gamma, a_N, \beta_N)} \int_{\alpha(\theta_s, \gamma, a_N, \beta_N)}^{\beta(\alpha, \theta_s, \gamma, a_N, \beta_N)} \cos \theta(a, \beta; \theta_s, \gamma) \cos \xi(a, \beta; a_N, \gamma_N) \sin \beta \, d\beta \, da
\]

The integral contains four parameters which determine the configuration of boundaries of the surface over which the integration is to be carried out, namely, the albedo source region.

The limits of integration are functions of these same parameters as well as of the second integration variable. The parameters establish the limits for the second integration, as well as controlling the functional form of the integration limits.

In principle either of the integration variables may be selected for the first integration. The azimuth angle \(a\) provides simple first integral forms, but the function limits, involving the nadir angle, are sometimes double valued functions, \(a(\beta)\).

On the other hand, the first integration taken relative to the nadir angle \(\beta\) leads to more complex expressions for the first integral, but the functional limits, involving the azimuth angle \(a\), are single valued functions, \(\beta(a)\).

This second alternative is chosen. Having once found all the "anti-derivatives" of the integrand functions of \(\beta\), a simple differencing of function values for maximum and minimum values of \(\beta\) (at a particular value of \(a\)) provides first definite integral numerical values, which are now functions of the four configuration parameters and \(a\). Integration over \(a\) involves summation of first definite integral values.

Regardless of which variable, \(a\) or \(\beta\), is first used, when the function limits (\(a(\beta)\) or \(\beta(a)\)) are inserted, some of the expressions become rather formidable, and analytical evaluations of the second
integrals for many of these have not been found. It is reasonable to use simple numerical integration in the remaining variable \(a\).

The region of integration is bounded by curves beyond which one or more of the integrand factors become negative. There are three such curves (see sketch above). The ever-present horizon circle \(C_y\) is determined by the relative altitude of the target above the albedo source. The terminator \(C_\theta\), the sunlight-shadow line, is determined both by altitude and by the angular distance of the target position from the subsolar point on the albedo source. Finally, the "target plane cut" \(C_\xi\), the intersection of the plane of the target with the albedo source, depends on the specific orientation of the target and altitude. The curve \(C_\theta\) may fall outside the circle, and the curve \(C_\xi\) does not exist outside the horizon circle. So, depending on the four parameters, the region of integration may be bounded by one curve \(C_y\), by two curves (\(C_y\) with \(C_\theta\), \(C_\theta\) with \(C_\xi\), or \(C_y\) with \(C_\xi\)), or finally by portions of all three curves.

Not only are the boundary curves defined by the four integration parameters, but their intersections are also, and there may be as many as six intersections (see above). From this arises part of the complexity of the problem, since the \(C_\xi\) curve may have any azimuthal relation to the \(C_\theta\) curve, or within the \(C_y\) circle. The logical sorting involved in determining the boundary curves and their limits, for arbitrary parameters, is rather involved in the number of decisions to be made. Yet for a given configuration, only one sequence of a few decisions serves to provide all the information required.
From the standpoint of computer programming, the method described here leads to a large program, of which only a small part is executed for one given set of parameters. In practice all the parameters may be continually varying.

In the following sections, the analysis will be developed, leading to the computer program displayed herein as a subroutine. A logical flow chart and Fortran II listing of the major routine is given as well as a Fortran II listing of the supporting subroutines. This method turns out to be approximately ten times as fast in computing as a corresponding purely numerical integration method.

SECTION III
THE ALBEDO INTEGRAL

The albedo integral, in its complete form, provides an expression for the intensity of illumination $I_2$ on an arbitrarily oriented and positioned target surface element $\Delta A$, attributable to albedo $A_e$ of a homogeneous diffusely scattering sphere exposed to solar radiation intensity $I_s$. We begin with the definitions

$$I_2 = \frac{I_s A_e}{\pi} \int \int \cos \theta_e \cos \xi \sin \beta \, d\beta \, d\alpha$$

$$\cos \theta_e = \cos \theta_s \cos \phi_e + \sin \theta_s \sin \phi_e \cos \alpha$$

$$\cos \xi = \cos \beta \cos \beta_N + \sin \beta \sin \beta_N \cos (\alpha - a_N)$$

$$\sin \gamma = r_e / (h + r_e)$$

$$\phi_e = \psi - \beta$$

$$\sin \psi = \sin \beta / \sin \gamma$$

The integration is over all $\alpha, \beta$ within the region where the integrand factors are all positive. The parameters $a_N, \beta_N$ define the orientation of a target surface element to the particular albedo source configuration defined by $\theta_s, \gamma$.

\[3Ibid.\]
For present purposes, the factor $I_s \Lambda_e$ is taken as unity, leaving the integral
\[
\frac{1}{\pi} \int \int \cos \theta_e \cos \xi \sin \beta \, d\beta \, d\alpha
\]
which may be called the "albedo view factor". It is a measure of efficiency of conversion of collimated illumination into scattered illumination on an arbitrarily oriented surface element at any point in space about a perfect diffusely reflecting sphere.

The integration relative to nadir angle $\beta$ is given in Appendix II.

![Model Geometry](image)
SECTION IV
PARAMETERS OF ALBEDO INTEGRAL

If $S$ is a unit vector indicating the sun direction, a unit orbital angular momentum vector, and $R$ the target position vector, then a node vector $N_S$ may be constructed from $S \times L$. Then the orbital angular position $\phi_V$ may be defined as the angle between $N_S$ and $R$. The plane of the orbit is inclined at angle $\theta_V$ from the plane of $N_S$ and $S$. Then the angular distance $\theta_n$ of $R$ from $S$ is defined by (Fig. 1a)

$$\cos \theta_n = \sin \phi_V \cos \theta_V$$

b. Geometry Defining $\alpha, \beta, \gamma, C_\gamma$ and $C_\theta$

Fig. 1 Continued
The \( \alpha, \beta \) coordinate system is defined at the vehicle position by polar coordinates, taking \( \beta = 0 \) as the \((-R)\) direction, \( \alpha = 0 \) in the plane of \( R \) and \( S \), \( \alpha \) positive by a right-hand rotation about \((+R)\) (Fig. 1b).

Then \( \alpha_N, \beta_N \) are defined as the coordinates of the normal to the albedo target surface element \( \Delta A_2 \) (Fig. 1c).

The relative altitude parameter \( \gamma \) is defined by Eq. (4). The relations of Eqs. (4) and (5) are shown in Fig. 1d.

---

c. Geometry Defining \( \alpha_N, \beta_N, \xi \) and \( C_\xi \)

Fig. 1 Continued
SECTION V
BOUNDARY CURVES AND INTERSECTIONS

Since the boundary curves separate the region in \((\alpha, \beta)\) for which the integrand factors of Eq. (1) are positive from the region where any factor is negative, we may write boundary equations as follows:

\(C_y\) (Horizon Circle):
\[ \beta = \gamma \]  \hspace{1cm} (7)
for all \(\alpha\).

\(C_\theta\) (Terminator, from Eq. (2)):
\[ \cos \theta_e = 0, \]
hence
\[ \cos \alpha = - \ctn \phi_e \ctn \theta_s \]  \hspace{1cm} (8)
in which Eqs. (4), (5), and (6) are used to obtain expressions for \(\alpha(\beta)\) or \(\beta(\alpha)\).

\(C_\xi\) (Target Plane Cut, from Eq. (3)):
\[ \cos \xi = 0, \]
hence
\[ \cos (\alpha - \alpha_N) = - \ctn \beta_N \ctn \beta \]  \hspace{1cm} (9)
The intersections of \(C_\xi\) with \(C_y\) are obtained from Eqs. (7) and (9) by letting \(\Delta \alpha = \alpha - \alpha_N\).

Then
\[ \cos \Delta \alpha = - \ctn \beta_N \ctn \gamma \]
\[ a_i = \alpha_N + \Delta \alpha \]
\[ a_2 = \alpha_N - \Delta \alpha \]  \hspace{1cm} (MODULO 2\pi)  \hspace{1cm} (10)
The intersection of \(C_\theta\) with \(C_y\) is found by using Eqs. (6), (5), and (7) with Eq. (8) as follows

\[ \beta = \gamma \]
\[ \psi = \pi/2 \]
\[ \phi_e = \pi/2 - \gamma \]
\[ a_3 = \cos^{-1} (- \ctn \theta_s \tan \gamma) \]
\[ a_4 = 2\pi - a_3 \]  \hspace{1cm} (11)
The intersections of \( C_0 \) with \( C_\xi \) are not needed in the present method, but will be essential if it is desired to attempt purely formal second-stage integration in the future. The calculation of this intersection is given in Appendix III; transformations between the angles \( \psi, \phi_e, \) and \( B \) implied by Eqs. (5) and (6) are given in Appendix IV.
It is desirable to ensure that all the angles of intersections \((a_1, a_2, a_3,\) and \(a_4\)) of Eqs. (10) and (11) are expressed as positive angles within \((0, 2\pi)\) to eliminate ambiguities that otherwise occur in determining when the variable \(\alpha\) is in the range of definition of one of the curves, \(C_\theta\) or \(C_\xi\).

It is apparent that the limit points \(a_1\) and \(a_2\) are symmetrically placed with respect to \(\alpha_N\) at positions determined by \(\gamma\) and \(\beta_N\). Correspondingly, \(a_3\) and \(a_4\) are symmetric relative to \(\alpha = 0\) (located from expressions involving \(y\) and \(\theta_s\)). The four integration parameters \(\gamma\), \(\theta_s\), \(\alpha_N\), \(\beta_N\) remain arbitrary, subject to limitations

\[
0 \leq \gamma \leq \pi/2 \\
0 \leq \theta_s \leq \pi \\
0 \leq \alpha_N \leq 2\pi \\
0 \leq \beta_N \leq \pi
\]

SECTION VI
MAJOR DIVISIONS OF PARAMETER RANGES

All required quantities are now defined, and we shall examine the meaning of the values of the four parameters \(a_N, \beta_N, \theta_s,\) and \(\gamma\). From the definition of \(\gamma\) (Eq. (4)), we find that \(\gamma\) approaches \(\pi/2\) as the altitude vanishes, and \(\gamma\) approaches zero as altitude grows large.
We confine our attention to the horizon circle and its interior, \( \beta \leq \gamma \). In the \((a, \beta)\) coordinate system about the origin (the vehicle location), a unit radius sphere is erected. The horizon circle is a small circle, \( \beta = \gamma \), which entirely encompasses the albedo source region, of which no part exists outside the horizon, \( \beta > \gamma \). The curve \( C_{\xi} \) is a great circle on the \( a, \beta \) sphere and hence passes through the origin and maps across the interior of the horizon circle as a straight line. The curve \( C_{\theta} \), a great circle on the albedo source sphere, maps into the horizon circle as a part of an ellipse tangent to the horizon circle. The values of \( \theta_s \) and \( \beta_N \) control the existence of \( C_{\theta} \) and \( C_{\xi} \) within \( C_\gamma \), and the points of closest approach of these curves to \( \beta = 0 \). The missing ellipse branch of the \( C_{\theta} \) curve, the continuation of the terminator, is not defined in \((a, \beta)\), since it is physically outside the horizon circle or "behind" it (see sketch on page 10).

Figure 2a illustrates the major divisions of characteristics imposed by \( \theta_s \), \( \beta_N \), and \( \gamma \) and by typical patterns of the integration region. For this illustration, \( a_N \) is arbitrarily set at \( \pi/2 \). Later, we shall examine the influence of \( a_N \) on the problem. Figure 2a illustrates schematically some typical boundary patterns.

Fig. 2 Major Divisions of Parameter Ranges in the Horizon Circle \( C_\gamma \)
The major ranges are noted in Fig. 2a by use of paired numbers \((n, m)\), the first referring to the \(\theta_s\) range, the second to the \(\beta_N\) range. In range \((0, 0)\), only \(C_\gamma\) bounds the region, and for integration we have \(0 \leq \beta \leq \gamma, 0 \leq \alpha \leq 2\pi\). This corresponds to a point on the vehicle nearest the source sphere, near \(\beta = 0\) and a location of the vehicle not far from the subsolar point \((\theta_s = 0)\) on the albedo source sphere.

With no other changes, as \(\beta_N\) increases we move from \((0, 0)\) to \((0, 1)\) where \(C_\zeta\) comes into the horizon circle. The vehicle itself begins to mask part of the source. The point \(\beta = 0\) is still within the source so the \(\alpha\) limits are 0 and \(2\pi\); \(\beta = 0\) is the minimum \(\beta\) value \((\hat{\beta})\), and the maximum \((\check{\beta})\) is either \(\gamma\) or dependent on \(\alpha\) through the equation for curve \(C_\xi\).

When \(\beta_N = \pi/2\), the target plane (or its equivalent \(C_\xi\)) bisects the horizon circle. Now \(\alpha\) has a range of \(\pi/2\) either side of \(\alpha_N\); \(\check{\gamma}\) is zero, and \(\hat{\gamma}\) is \(\gamma\) only. Or we may allow \(\alpha\) its full \((0, 2\pi)\) range, but during half of this range the curve \(C_\xi\) provides \(\check{\beta} = 0\) and in the other half \(C_\gamma\) gives \(\hat{\beta} = \gamma\) while \(\hat{\beta} = 0\).

As \(\beta_N\) grows, \(C_\xi\) moves into \((0, 2)\), on past the nadir point \(\beta = 0\), and provides \(\hat{\beta}\) while \(\check{\gamma}\) is \(\gamma\). The range of \(\alpha\) is now \((\alpha_1, \alpha_2)\), the region where \(C_\xi\) is defined. Finally, \(\beta_N\) increases so far that it is "on top" of the vehicle, \(C_\xi\) has swept completely across the interior of the horizon circle, and the integral value becomes zero. The target now completely masks itself from the albedo source.

It must be noted that the integrand contains \(\cos \theta_e\), \(\theta_e\) being measured from the subsolar point on the albedo sphere. Thus, generally, the source intensity is not symmetric in any way unless \(\beta_N = 0\) or \(\pi\), and these two instances are not equivalent since one of them includes areas nearer the subsolar point, and the other is directly opposite. But \(\alpha_N\) is arbitrary, in practice a function of \(\beta_N\) determined by the vehicle geometry and orientation.

The dependence of the \(\theta_s\) and \(\beta_N\) ranges on the altitude parameter, \(\gamma\), is illustrated in Fig. 2b.

We now return to case \((0, 0)\) and allow \(\theta_s\) to vary. As we enter \((1, 0)\), the terminator \(C_\theta\) appears in the horizon circle at \(\alpha = \pi\). The nadir point is still within the integration region so \(\alpha\) ranges \((0, 2\pi)\), \(\check{\beta} = 0\), and \(\beta\) are determined from either \(C_\theta\) or \(C\).
Regions Identified by Numbered Ranges ($\theta_s, \beta_n$)

- Shadow Side

$0 \leq \gamma < \frac{\pi}{4}$
(High Altitude)

$\frac{\pi}{4} < \gamma < \frac{\pi}{2}$
(Low Altitude)

b. Effect of $\gamma$ on Boundaries

Fig. 2 Concluded
Allowing $\theta_s$ to increase to $\pi/2$ (as the vehicle crosses over the terminator), $C_\theta$ bisects the source field, and we move on into $(2, 0)$. Here $C_\theta$ provides $\beta$, $\beta = \gamma$, and $a$ ranges only over $a_1$, $a_4$.

Finally $\theta_s$ increases so far that $C_\theta$ leaves the horizon circle at $a = 0$, and we have the eclipse condition where no part of the illuminated albedo sphere is visible at the vehicle. The integral vanishes. We note that $\cos \xi$ in the integrand also destroys the apparent symmetry in $a$ of the source function except in special cases. Instances where symmetry occurs were treated in Appendix II of the previous report\textsuperscript{4} as special cases in which the albedo integral can be obtained in closed form.

The parameter ranges labeled $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$ are superpositions of those just described. The bounds are dependent on all four parameters. When only one of the curves $C_\xi$ or $C_\theta$ establishes the $\beta$, then the $a$ range necessarily lies within the corresponding end points $(a_1, a_2)$ or $(a_3, a_4)$. More precise statements are developed in subsequent sections as we go more deeply into the logic of sorting out the various cases.

SECTION VII
CONFIGURATIONS OF ALBEDO SOURCE BOUNDARIES

In this section, we display 43 distinct configurations of boundaries covering all useful values of the four integration parameters. All of these must be examined for the purpose of establishing exact integration ranges in $a$, ranges in which the boundaries are different functions $\beta(a)$. As earlier indicated, we shall eventually integrate over $(a, \beta)$ by using exact expressions for the first definite integral in $\beta$, which contains functions $\beta(a)$ and $\alpha(a)$, then numerically integrating in $a$.

We recall that all end points $(a_1, a_2, a_3, a_4)$ are defined to have values in $(0, 2\pi)$, that the curve $C_\gamma$ is defined by $\beta = \gamma$ for all $a$, that $C_\xi$ is defined (Eq. (8)) in $(a_1, a_2)$, and $C_\theta$ is defined in $(a_3, a_4)$. We do not require the intercepts of $C_\theta$ with $C_\xi$, which would be denoted $(a_5, a_6)$, because results based on this knowledge are readily obtainable by an artifice which we shall use in the numerical $a$ integration. These points would be

\textsuperscript{4}Ibid.
required if one were to attempt to find a complete analytical expression for the albedo integral.

We return to the notation of the previous section for discussion of the major divisions. Viewed from the origin of the \((a, \beta)\) coordinate system, boundary curve \(C_y\) is a circle whose interior contains the regions of interest. Curve \(C_\xi\) is a straight line segment, and \(C_\theta\), although an ellipse section tangent to \(C_y\), is indicated as a circular arc for clarity and ease of drawing. The variously numbered points are simply numbered in sketches. The sides of \(C_\xi\) and \(C_\theta\) on which the corresponding integrand factors are positive are indicated by a small arrow, pointing toward \(a_N\) on \(C_\xi\), and toward \(a = 0\) on \(C_\theta\), or to the "interior" of these curves. Then the region of interest is just that part of the pattern which is common to the interiors of all three curves.

**Case (0, 0):** \(\beta_N \leq \pi - \gamma, \ \theta_5 \leq \gamma\)

\[
0 < a < 2\pi, \ \beta = 0, \ \beta = \gamma
\]

**Case (1, 0):** \(\beta_N \leq \pi - \gamma, \ \gamma \leq \theta_5 \leq \pi\)

\[
0 < a < 2\pi, \ \beta = 0, \ \beta = \gamma
\]

**Case (2, 0):** \(\theta_N \leq \pi - \gamma, \ \pi \leq \theta_5 \leq \pi - \gamma\)

\[
0 < a < a_3 \quad \text{split} \quad \beta = \beta(C_\theta) \quad \beta = \gamma
\]

**Case (0, 1):** \(\theta_5 \leq \gamma, \ \pi - \gamma \leq \beta_N \leq \pi\)

\[
0 < a < 2\pi, \ \beta = 0, \ \beta = \gamma
\]

**Case (0, 2):** \(\theta_5 \leq \gamma, \ \pi \leq \beta_N \leq \pi + \gamma\)

\[
\hat{\beta} = \gamma
\]

**Fig. 3** Configurations of One and Two Boundary Curves
Cases (0, 0), (0, 1), (0, 2), (1, 0), and (2, 0), shown in Fig. 3, are largely self-explanatory. Cases (0, 1), (0, 2), and (2, 0), however, introduce the problem of the "split range". Although the two orientations shown in (0, 1) and (0, 2) are geometrically equivalent, they are logically distinct since all intersection values are defined in (0, 2π). For example, in case (1, 0) when the order of end points is \( a_1 < a_2 \), the \( C_e \) curve is defined in \((a_1, a_2)\), but when the order is \( a_2 < a_1 \), \( C_e \) is defined in the split range \((0, a_2)\) and \((a_1, 2\pi)\). Thus, the order of end points is essential to the orderly determination of the range in which \( a \) may be during numerical integration and for the selection of the proper boundaries \((\beta, \beta)\) for a given \( a \).

The split range is also used in the initiation and advance of \( a \) during numerical integration. If the range is split, \( a \) is scanned over the two parts successively.

Case (1, 1) is the most complicated group of configurations because the \( a \) range is \( 0, 2\pi \) and both \( C_e \) and \( C_0 \) are present, Fig. 4. Each configuration is labeled by letter referring to the corresponding permutation.
of end points, given in the table at the bottom of the figure. Beside each permutation appears the order of subscripts in the boundary curves from which $\hat{\beta}$ is to be found. For this case, the point $\beta = 0$, lying interior to the integration region, is also $\hat{\beta} = 0$. Where a double subscript occurs, we make use of the artifice (previously mentioned) to determine whether to use $C_{\theta}$ or $C_{\xi}$. Here, when $\hat{\beta}$ is to be found in a range of $\alpha$ where both $C_{\theta}$ and $C_{\xi}$ are defined, we use the $C_{\theta}$ and $C_{\xi}$ definitions to determine both values of $\hat{\beta}$, i.e., $\hat{\beta}(C_{\theta})$ and $\hat{\beta}(C_{\xi})$; we then select the least of these to be $\hat{\beta}$.

We give an example of interpretation of the table of Fig. 4. Select configuration B. Initiate $\alpha$ at 0, change the value by the (fixed) step size, and test $\alpha$ to see when it is in each subsequent range. In $(0, \alpha_1)$, $\hat{\beta} = \gamma$; in $(\alpha_1, \alpha_2)$, $\hat{\beta} = \beta(C_{\xi})$. In $(\alpha_2, \alpha_3)$, $\hat{\beta} = \text{min} \{ \beta(C_{\theta}), \beta(C_{\xi}) \}$; then in $(\alpha_3, \alpha_4)$, $\hat{\beta} = \beta(C_{\theta})$. Finally, in $(\alpha_4, 2\pi)$, $\hat{\beta} = \gamma$.

Note that in any configuration in which $\alpha_2 < \alpha_1$, the range test must be split, as noted earlier.

Case (1, 2) in Fig. 5 has $\hat{\beta}$ defined by the curve $C_{\xi}$, and, hence, $\alpha$ is scanned only over the range of definition of $C_{\xi}$; that is, over $(\alpha_1, \alpha_2)$ if $\alpha_1 < \alpha_2$, or over $(0, \alpha_3)$ and $(\alpha_4, 2\pi)$ if $\alpha_2 < \alpha_1$. In the table, the symbol $[\theta]$ means that we use $\hat{\beta} = \beta(C_{\theta})$ only if it is greater than $\hat{\beta} = \beta(C_{\xi})$; otherwise, there is no contribution to the integral for the current value of $\alpha$.

Case (2, 1) in Fig. 6 has $\hat{\beta}$ defined by $C_{\theta}$, so $\alpha$ is scanned over $(0, \alpha_3)$ and $(\alpha_4, 2\pi)$, a split scan. The notation $\xi^*$ in the table means that $\hat{\beta} = \beta(C_{\xi})$ if only $\hat{\beta} > \hat{\beta}$; otherwise, the current value at $\alpha$ contributes nothing to the integral.

Finally, case (2, 2) in Fig. 7 has $\hat{\beta}$ defined by $\beta = \gamma$, and we select $\hat{\beta}$ as the greatest of $\beta(C_{\xi})$ and $\beta(C_{\theta})$, which is the meaning of the symbol $(\theta, \xi)$ in the table. Note that configuration G illustrates that nonoverlapping of the ranges of $(\alpha_1, \alpha_2)$ and $(\alpha_3, \alpha_4)$ leads to zero value of the integral. For case (2, 2) we let $\alpha$ scan only the least of the spans of $C_{\theta}$ or $C_{\xi}$; if this is $C_{\theta}$ then the $\alpha$ scan is split, but if $C_{\xi}$, the $\alpha$ scan may or may not be split. Notations of split scan test appear on the figures.

This completes the details of the logical procedures for doing the numerical integration in $\alpha$. From the tables on the figures, the logic flow chart in Appendix V was derived; the problem was then programmed for computer directly from the flow chart. The Fortran listing is shown in Appendix VI.
### Three Boundary Curves

\[ \frac{\pi}{2} < \beta_N < \frac{\pi}{2} + \gamma, \quad \gamma \leq \beta_E \leq \frac{\pi}{2}, \quad \hat{\beta} = \beta(C_E) \]

If \( a_2 < a_1 \), then \( a_2 \leq a \leq a_1 \); but if \( a_1 < a_2 \),

then \( 0 \leq a < a_1 \) and \( a_2 < a < 2\pi \).

Split Scan: \( 0 < a < a_3 \) and \( a_4 < a < 2\pi \)

---

<table>
<thead>
<tr>
<th>Configuration</th>
<th>( \alpha ) Order</th>
<th>Boundary Curves for ( \beta )</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(0) 1 3 2 4 (2\pi)</td>
<td>( \gamma ) (-) ( \beta ) ( \gamma ) *</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1 3 4 2</td>
<td>( \gamma ) (-) ( \gamma ) *</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>2 1 3 4</td>
<td>( \gamma ) (-) ( \gamma ) *</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>2 3 1 4</td>
<td>( \gamma ) (-) ( \beta ) ( \gamma ) *</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>3 1 4 2</td>
<td>( \gamma ) (-) ( \gamma )</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>3 2 1 4</td>
<td>( \gamma ) (-) ( \gamma )</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>3 2 4 1</td>
<td>( \gamma ) (-) ( \beta ) ( \gamma )</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>3 4 2 1</td>
<td>( \gamma ) (-) ( \gamma )</td>
<td></td>
</tr>
</tbody>
</table>

Symbol Meaning

\( \beta \) = \( \beta(C_E) \) only if \( \hat{\beta} > \beta \), Otherwise No Contribution to Integral

- Means No Contribution to Integral

* Split Scan in \( \alpha \)

---

<table>
<thead>
<tr>
<th>Configuration</th>
<th>( \alpha ) Order</th>
<th>Boundary Curves for ( \beta )</th>
<th>Notes</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>(0) 1 2 3 4 (2\pi)</td>
<td>( \gamma ) ( \gamma ) ( \gamma )</td>
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<tr>
<td>G</td>
<td>3 2 4 1</td>
<td>( \gamma ) ( \beta ) ( \gamma )</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>3 4 2 1</td>
<td>( \gamma ) ( \gamma )</td>
<td></td>
</tr>
</tbody>
</table>

Symbol Meaning

\( \epsilon \) = \( \beta(C_E) \) if \( \hat{\beta} > \beta \), Otherwise No Contribution

* Split Scan

** Identically Zero Integral

---

**Fig. 5 Case (1, 2)**

**Fig. 6 Case (2, 1)**
Three Boundary Curves \( \frac{\pi}{2} < \beta_N < \frac{\pi}{2} + \gamma, \ \frac{\pi}{2} < \theta_S < \pi - \gamma, \ \hat{\beta} = \gamma \)

\( \alpha \) Ranges over Least Span, Covering Range of \( C_\xi  \) if \( \Delta \alpha \leq \alpha_3 \).
Otherwise over \( C_\theta \). Span of \( C_\theta \) Always Split \( 0 \leq \alpha \leq \alpha_3 \) and \( \alpha_4 \leq \alpha \leq 2\pi \)

<table>
<thead>
<tr>
<th>Configuration</th>
<th>( \alpha ) Order</th>
<th>Boundary Curves for ( \hat{\beta} )</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(0) 1 3 2 4 (2\pi)</td>
<td>(( \theta, \xi ))</td>
<td>- - - (( \theta, \xi )) *</td>
</tr>
<tr>
<td>B</td>
<td>1 3 4 2</td>
<td>(( \theta, \xi ))</td>
<td>- - - (( \theta, \xi )) *</td>
</tr>
<tr>
<td>C</td>
<td>2 1 3 4</td>
<td>-</td>
<td>(( \theta, \xi )) - - -</td>
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<tr>
<td>D</td>
<td>2 3 1 4</td>
<td>-</td>
<td>(( \theta, \xi )) - - -</td>
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<tr>
<td>E</td>
<td>3 1 2 4</td>
<td>( \theta )</td>
<td>- - - ( \theta ) *</td>
</tr>
<tr>
<td>F</td>
<td>3 1 4 2</td>
<td>(( \theta, \xi ))</td>
<td>- - - (( \theta, \xi )) *</td>
</tr>
<tr>
<td>G</td>
<td>3 2 1 4</td>
<td>-</td>
<td>- - - - **</td>
</tr>
<tr>
<td>H</td>
<td>3 2 4 1</td>
<td>-</td>
<td>- - - (( \theta, \xi )) -</td>
</tr>
<tr>
<td>I</td>
<td>3 4 2 1</td>
<td>-</td>
<td>- - - (( \theta, \xi )) -</td>
</tr>
</tbody>
</table>

Symbol

\( \xi = \max \left[ \beta(C_\theta), \beta(C_\xi) \right] \)

* Split Test if \( \alpha \) Ranges over \( C_\xi \)

** Identically Zero Integral

Fig. 7 Case (2, 2)
SECTION VIII
CONCLUSIONS

In an attempt to gain computing speed in the evaluation of the albedo integral, a double integral, the problem has been changed from a straightforward numerical integration to a much faster but more complex semi-numerical integration. For example, whereas formerly the $\alpha$, $\beta$ range was covered by a mesh of 72 X 36 points, the present method requires somewhat more computation per point through a more complex logic network at only 72 points, and the accuracy is improved by the formal first integration. The time improvement is approximately one order of magnitude.

In application, a particular Mark I control program formerly required from 25 to 35 min (IBM 7074) to generate simulation parameters for a 90-min orbit with a simulation interval of two minutes. The same results are now produced in approximately four minutes.

It appears unlikely that significant gains in computing speeds can be made by using a purely formal solution to this problem. Second integrals will contain many more terms, some quite complex, and much of the gains made by having a single evaluation to perform will be lost in the sheer bulk of the expressions involved. Most of the logic of the present method would still apply for selecting integration limits and function groups to be evaluated. Some gain may result in changing the variable of first integration, and this will be studied in the future. It may also be possible to develop rapidly computing empirical approximating functions, especially over limited ranges of the integral parameters.
APPENDIX I
DERIVATION OF THE ALBEDO INTEGRAL

In the following discussion, the albedo source body is taken to be the earth. Substitution of appropriate values for radius, albedo, and solar constant allows extension to any source body.

To compute earth albedo and radiance integrals for a surface element having arbitrary orientation and position, we make the following assumptions:

1. that Albedo is a uniform property of the earth's surface,
2. that the earth is a sphere,
3. that the earth's surface is diffusely reflecting, and
4. that the earth has no atmosphere.

$I_s$ Solar constant, intensity of solar radiation at earth
$A_e$ Albedo, fraction of solar constant reflected, a surface property.

The solar radiation incident on an area element $\Delta A_1$ having its normal inclined at angle $\theta_e$ to sun direction is

$$\Delta I_e = \begin{cases} I_s \cos \theta_e \Delta A_1 & \text{for } \cos \theta_e \geq 0 \\ 0 & \text{for } \cos \theta_e < 0 \end{cases}$$

Of this a fraction $A_e$ is reflected diffusely by $\Delta A_1$; hence, the intensity per unit solid angle $\Delta I_\psi$ in a direction inclined at angle $\psi$ to the surface normal is

$$\Delta I_\psi = \frac{A_e I_s}{\pi} \cos \theta_e \cos \psi \Delta A_1$$

The intensity included in solid angle $\Delta \omega$ is

$$\Delta I_\omega = \frac{A_e I_s}{\pi} \cos \theta_e \cos \psi \Delta A_1 \Delta \omega$$

An area element $\Delta A_2$, at distance $\rho_e$ from $\Delta A_1$, having its normal inclined at angle $\xi$ to the direction of $\rho_e$, intercepts a solid angle (Fig. I-1).

$$\Delta \omega = \frac{\Delta A_2 \cos \xi}{\rho_e^2}$$
Hence the intensity arriving at $\Delta A_2$ is

$$\Delta I_2 = \frac{A_e \cdot I_s}{\pi \rho^2} \cos \theta_e \cos \psi \cos \xi \Delta A_1 \Delta A_2$$

![Diagram showing solid angle geometry for albedo and Earth radiance calculation.](image)

**Fig. 1-1** Solid Angle Geometry for Albedo and Earth Radiance Calculation

Let the area element $\Delta A_1$ be located at altitude $h$ above earth of radius $r_e$. From this point, the portion of the earth that can be seen is confined within a horizon circle. The angle $\gamma$ between the direction of earth center and the horizon circle is defined by

$$\sin \gamma = r_e / (r_e + h) \quad (0 < \gamma < \frac{\pi}{2})$$

At the earth center, let $\theta_s$ be the angle between the direction to $\Delta A_2$ and the sun direction; let $\theta_e$ be the angle between the area element $\Delta A_1$ on earth and the sun; let $\phi_e$ be the angle between the directions of $\Delta A_1$ and $\Delta A_2$. 
At the element $\Delta A_1$, $\psi$ is the angle between the normal to $A_1$ and the direction of $\Delta A_2$ as before.

At the element $\Delta A_2$, let $\beta$ be the angle between the direction to earth center and $\Delta A_1$. Angle $\beta$ is a "nadir" angle.

Then the following relations hold (Fig. I-2):

$$\psi = \beta + \phi$$

$$\sin \psi = \sin \beta / \sin \gamma$$

$$\cos \theta_e = \cos \theta_s \cos \phi + \sin \theta_s \sin \phi_e \cos \alpha$$

where $\alpha$ is the angle about the line from $\Delta A_1$ to earth center measured from the plane including this line and the sun. Angle $\alpha$ is the azimuth angle (Fig. I-3).
Now the element \( \Delta A_1 \) is described in spherical coordinates having polar axis along the earth-to-\( \Delta A_2 \) line, longitude \( a \), and co-latitude \( \phi_e \): \[
\Delta A_1 = r_c^2 \sin \phi_e \, d\alpha \, d\phi_e
\]

The \( \alpha, \beta \) Coordinate System at \( \Delta A_2 \)

![Diagram showing the \( \alpha, \beta \) coordinate system at \( \Delta A_2 \).]

Similarly, we may describe a spherical area element in terms of \( \alpha, \beta, \) and \( \rho_e \) from \( \Delta A_2 \):
\[
\Delta A_\rho = \rho_e^2 \sin \beta \, d\beta \, d\alpha
\]

Hence any point may be described by either \( (r_c, \phi_e, a) \) or \( (\rho_e, \beta, a) \) at \( \Delta A_1 \), and we find that \( \Delta A_\rho \) and \( \Delta A_1 \) are related by simple projective properties:
\[
\Delta A_1 \cos \psi = \Delta A_\rho
\]
so that
\[
\Delta A_1 = \frac{\rho_e^2 \sin \beta \, d\beta \, d\alpha}{\cos \psi}
\]
We may now write the intensity at \( \Lambda A_2 \) caused by reflection from \( \Lambda A_1 \):

\[
\frac{\Lambda I_2}{\Lambda A_2} = \frac{A_0 I_s}{\pi} \cos \theta_c \cos \psi \frac{\cos \xi}{\rho_e^2} \frac{\rho_e^2 \sin \beta}{\cos \psi} \quad da \, d\beta
\]

Integrating over the interior of the horizon circle, we obtain

\[
\frac{I_2}{\Lambda A_2} (\Lambda A_2, \theta_s, \gamma, \xi) = \frac{A_0 I_s}{\pi} \int_0^\gamma \int_{a_{\min}}^{a_{\max}} \cos \theta_c (d, \beta, \theta_s, \gamma) \cos \xi \sin \beta \, d\beta \, da
\]

where \( a_{\min}, a_{\max} \) may be functions of \( \beta \), and there may be several distinct regions or functions.

In the spherical coordinates \( a, \beta \), any orientation of surface element may be described by the direction angles of its normal, \( a_N, \beta_N \). Then the angle \( \xi \) is found from

\[
\cos \xi = \cos \beta \cos \beta_N + \sin \beta \sin \beta_N \cos (a-a_N)
\]

Since

\[
\cos \theta_c = \psi - \beta, \quad \text{and} \quad \sin \psi = \sin \beta / \sin \gamma
\]

\[
\cos \theta_c = \sin \psi (\cos \theta_s \sin \beta + \sin \theta_s \cos \beta \cos a)
\]

\[
+ \cos \psi (\cos \theta_s \cos \beta - \sin \theta_s \sin \beta \cos a)
\]

The intensity integrand is completely expressible in the two variables \( a, \beta \), and the configuration parameters \( \theta_s, \gamma, a_N, \beta_N \).

At this stage it is possible to integrate numerically by letting \( a \) range from 0 to \( 2\pi \) and \( \beta \) range from 0 to \( \gamma \), provided that

\[
\cos \xi \geq 0
\]

\[
\cos \theta_c \geq 0
\]

\[
\cos \psi \geq 0
\]

and using \( (=0) \) for any \( (a, \beta) \) violating these conditions.

Ignoring for the moment the constant \( A_0 I_s / \pi \), we have the following terms to be integrated over \( a \) and \( \beta \):

1. \( \cos \beta_N \cos \theta_s \cos^2 \beta \sin \beta \cos \psi \)
2. \( -\cos \beta_N \sin \theta_s \cos \beta \sin^2 \beta \cos \psi \cos a \)
3. \( \cos \beta_N \cos \theta_s \cos \beta \sin \beta \sin \psi \)
4. \( \cos \beta_N \sin \theta_s \cos^2 \beta \sin \beta \sin \psi \cos a \)
As long as there are no boundaries of earth surface for which \( a = a(\beta) \), so that \( a \) can range from 0 to \( 2\pi \), we may integrate relative to \( a \) and obtain simple results. Integrals (1, 3) do not contain \( a \) so the integration results in a factor \( 2\pi \). Integrals (2, 4, 5, 6, 8, 9, 10, 12) contain only \( \sin a \) or \( \cos a \) and vanish. Integrals (7, 11) contain \( \cos^2 a \) or \( \cos a \sin a \) and result in a factor of \( \pi \).

These conditions are satisfied as long as we have both

\[
\begin{align*}
\beta_N &\leq \frac{\pi}{2} - \gamma \\
\theta_s &\leq \gamma
\end{align*}
\]

If \( \beta_N \geq \pi/2 + \gamma \) or \( \theta_s \geq \pi - \gamma \), the earlier conditions on \( \cos \xi \) or \( \cos \theta_e \) are violated and the entire integral (1/AV) vanishes.

For

\[
\frac{\pi}{2} - \gamma < \beta_N < \frac{\pi}{2} + \gamma
\]

and/or

\[
\gamma < \theta_s < \pi - \gamma
\]

there exist boundaries of form \( a(\beta) \), and the integration becomes complicated. The integration is bounded by arcs of one, two, or three curves of \( a(\beta) \), whose intersections are generally given by implicit functions. A first integration may be done formally; expressions result for which the integrals are not available in closed form.

Numerical integration may be accomplished in an easily comprehended manner by referring to the earlier integral expression. The product \( \cos \xi \cos \theta_e \sin \beta \) may be calculated term by term, and in addition, the expression \( \cos \psi \) can be evaluated to ensure that the conditions

\[
\begin{bmatrix}
\cos \xi \\
\cos \theta_e \\
\cos \psi
\end{bmatrix} \geq 0
\]

are satisfied. For some value combinations of \( a_N, \beta_N, \theta_s \) the integrations can be carried out.
APPENDIX II
FIRST INTEGRATION IN NADIR ANGLE $\beta$

Table II-I displays the twelve possible integrands, with their parameter coefficients. The last eight of these may be grouped in pairs and combined by use of the identity

$$\cos a \cos a_N + \sin a \sin a_N = \cos (a - a_N)$$

Further grouping can then be performed based on the formal similarity of integrands. We use the numbering of Table II-I to identify integrands and the following definitions of parameter functions.

$$A_1 = \sin \beta_N \sin \theta_s \cos a \cos (a - a_N)$$

$$A_2 = \cos \beta_N \cos \theta_s$$

$$A_3 = \sin \beta_N \cos \theta_s \cos (a - a_N)$$

$$A_4 = \cos \beta_N \sin \theta_s \cos a$$

(1) \( A_2 \int \cos^2 \beta \sin \beta \left(1 - \sin^2 \beta / \sin^2 \gamma\right)^{1/2} \, d\beta \)

(2) \(-A_4 \int \cos \beta \sin^2 \beta \left(1 - \sin^2 \beta / \sin^2 \gamma\right)^{1/2} \, d\beta \)

(3) \( A_2 \int \cos \beta \left(\sin^3 \beta / \sin \gamma\right) \, d\beta \)

(4) \( A_4 \int \cos^2 \beta \left(\sin^2 \beta / \sin \gamma\right) \, d\beta \)

(5, 6) \( A_1 \int \cos \beta \sin^2 \beta \left(1 - \sin^2 \beta / \sin^2 \gamma\right)^{1/2} \, d\beta \)

(7, 8) \(-A_4 \int \sin^3 \beta \left(1 - \sin^2 \beta / \sin^2 \gamma\right)^{1/2} \, d\beta \)

(9, 10) \( A_3 \int \left(\sin^4 \beta / \sin \gamma\right) \, d\beta \)

(11, 12) \( A_4 \int \cos \beta \left(\sin^3 \beta / \sin \gamma\right) \, d\beta \)
TABLE II-1
INTEGRAND FORMS FOR ALBEDO

1. \( \cos \beta_N \cos \theta_S \) \( \cos^2 \beta \sin \beta \cos \psi \)
2. \(-\cos \beta_N \sin \theta_S \) \( \cos \beta \sin^2 \beta \cos \psi \cos \alpha \)
3. \( \cos \beta_N \cos \theta_S \) \( \cos \beta \sin^2 \beta \sin \psi \)
4. \( \cos \beta_N \sin \theta_S \) \( \cos^2 \beta \sin \beta \sin \psi \cos \alpha \)
5. \( \sin \beta_N \cos \theta_S \cos \alpha_N \sin^2 \beta \cos \beta \cos \psi \cos \alpha \)
6. \( \sin \beta_N \cos \theta_S \sin \alpha_N \sin^2 \beta \cos \beta \cos \psi \sin \alpha \)
7. \(-\sin \beta_N \sin \theta_S \cos \alpha_N \sin^3 \beta \cos \psi \cos^2 \alpha \)
8. \(-\sin \beta_N \sin \theta_S \sin \alpha_N \sin^3 \beta \cos \psi \sin \alpha \cos \alpha \)
9. \( \sin \beta_N \cos \theta_S \cos \alpha_N \sin^3 \beta \sin \psi \cos \alpha \)
10. \( \sin \beta_N \cos \theta_S \sin \alpha_N \sin^3 \beta \sin \psi \sin \alpha \)
11. \( \sin \beta_N \sin \theta_S \cos \alpha_N \sin^2 \beta \cos \beta \sin \psi \cos^2 \alpha \)
12. \( \sin \beta_N \sin \theta_S \sin \alpha_N \sin^2 \beta \cos \beta \sin \psi \sin \alpha \cos \alpha \)

Note: \( \sin \psi = \sin \beta / \sin \gamma \)
Regrouping for formal similarity

\[(1, 7, 8) \quad (A_2 + A_1) \int \cos^2 \beta \left(1 - \sin^2 \beta / \sin^2 \gamma \right)^{\frac{1}{2}} \sin \beta \, d\beta \]
\[- A_1 \int \left(1 - \sin^2 \beta / \sin^2 \gamma \right)^{\frac{1}{2}} \sin \beta \, d\beta \]

\[(2, 5, 6) \quad (A_3 - A_4) \int \sin^2 \beta \left(1 - \sin^2 \beta / \sin^2 \gamma \right)^{\frac{1}{2}} \cos \beta \, d\beta \]

\[(3, 11, 12) \quad (A_1 + A_2) \int \left(\sin^2 \beta / \sin \gamma \right) \cos \beta \, d\beta \]

\[(4, 9, 10) \quad A_4 \int \left(\sin^2 \beta / \sin \gamma \right) \sin \beta \, d\beta + (A_1 - A_4) \int \left(\sin^4 \beta / \sin \gamma \right) \, d\beta \]

We proceed to integrate, first making the following substitutions

\[(1, 7, 8) \text{ Let } \sin \gamma = G \quad \cos \gamma = B \quad \cos \beta = x \]

and note that

\[\sin^2 \gamma - \sin^2 \beta = \cos^2 \beta - \cos^2 \gamma \]
\[\sin \beta \, d\beta = -d \cos \beta = -dx \]

Then we obtain

\[- \frac{(A_2 + A_1)}{G} \int x^2 (x^2 - B^2)^{\frac{1}{2}} \, dx + \frac{A_4}{G} \int (x^2 - B^2)^{\frac{1}{2}} \, dx \]
\[= \frac{(A_2 + A_1)}{G} \left[ \frac{x}{4} (x^2 - B^2)^{\frac{1}{2}} + \frac{B^2}{8} x (x^2 - B^2)^{\frac{1}{2}} - \frac{B^4}{8} \ln (x + (x^2 - B^2)^{\frac{1}{2}}) \right] \]
\[+ \frac{A_4}{G} \left[ \frac{x}{2} (x^2 - B^2)^{\frac{1}{2}} - \frac{B^2}{2} \ln (x + (x^2 - B^2)^{\frac{1}{2}}) \right] \]
\[= - \frac{(A_2 + A_1)}{G} \frac{x}{4} (x^2 - B^2)^{\frac{1}{2}} \]
\[+ \left[ \frac{A_1}{2G} - \frac{A_1 + A_2}{G} \frac{B^2}{8} \right] \left\{ x (x^2 - B^2)^{\frac{1}{2}} - B^2 \ln (x + (x^2 - B^2)^{\frac{1}{2}}) \right\} \]

Finally

\[(1, 7, 8) \quad \frac{1}{4G} \left[2A_1 - \frac{1}{2} (A_1 + A_2) B^2 \right] \left\{ x (x^2 - B^2)^{\frac{1}{2}} - B^2 \ln (x + (x^2 - B^2)^{\frac{1}{2}}) \right\} \]
\[- (A_1 + A_2) x (x^2 - B^2)^{\frac{1}{2}} \]
(2, 5, 6) Let
\[
\sin \gamma = G \quad \sin \beta = y \quad \cos \psi = z
\]
where
\[
\sin \psi = \sin \beta / \sin \gamma
\]
and note
\[
\cos \beta \, d\beta = d \sin \beta = dy
\]
Then we obtain
\[
\frac{(A_3 - A_4)}{G} \int y^2 (G^2 - y^2)^{1/2} \, dy
\]
\[
= \frac{(A_3 - A_4)}{G} \left[ -\frac{y}{4} (G^2 - y^2)^{3/2} + \frac{G^2}{8} \left\{ y (G^2 - y^2)^{1/2} + G^2 \sin^{-1} \frac{y}{G} \right\} \right]
\]
\[
= \frac{A_4 - A_3}{4G} \left[ y (G^2 - y^2)^{3/2} - \frac{G^2}{2} \left\{ y (G^2 - y^2)^{1/2} + G^2 \psi \right\} \right]
\]
\[
= \frac{A_4 - A_3}{4G} \left[ y z^3 - \frac{1}{2} (yz + G\psi) \right] G^3
\]

(3, 11, 12) Let
\[
G = \sin \gamma
\]
then
\[
(A_3 + A_4) \int \frac{\sin^3 \beta \cos \beta \, d\beta}{\sin \gamma} = \frac{(A_2 + A_1)}{4G} \sin^4 \beta
\]

(4, 9, 10) Let
\[
G = \sin \gamma
\]
then
\[
\frac{A_4}{G} \int \sin^2 \beta \, d\beta + \frac{(A_3 - A_4)}{G} \int \sin^4 \beta \, d\beta
\]
\[
= \frac{A_4}{G} \left[ \frac{1}{2} (\beta - \sin \beta \cos \beta) \right]
\]
\[
+ \frac{(A_2 - A_1)}{G} \left[ -\frac{\sin^3 \beta \cos \beta}{4} + \frac{3}{4} \left\{ \frac{1}{2} (\beta - \sin \beta \cos \beta) \right\} \right]
\]
\[
= \frac{1}{4G} \left[ (A_4 - A_3) \sin^3 \beta \cos \beta \right.
\]
\[
+ \frac{1}{2} (A_4 + 3A_4) (\beta - \sin \beta \cos \beta) \]
Now all groups have a common factor $1/(4G)$; this factor is ignored in practice until final calculation of the albedo view factor, which is the calculated value of the integral multiplied by

$$\frac{1}{(4\pi \sin \gamma)}$$

The actual albedo illumination intensity is then gotten by multiplying by the albedo $A_e$ and solar constant $I_s$. 
APPENDIX III
INTERIOR INTERSECTIONS OF BOUNDARY CURVES

Intercepts of Cθ with Cξ are defined by the system of equations
\[ \cos a = -\frac{1}{(\tan \theta_s \tan \phi_e)} \quad (\text{III-1}) \]
\[ \cos (a - \alpha_N) = -\frac{1}{(\tan \beta_N \tan \beta)} \quad (\text{III-2}) \]
with the conditions
\[ \phi_e = \psi - \beta \quad (\text{III-3}) \]
\[ \sin \psi = \sin \beta \gamma \sin \gamma \quad (\text{III-4}) \]
thus all four parameters θ_s, γ, α_N, and β_N are involved. From the conditions of Eqs. (III-3) and (III-4) we derive the relation
\[ \tan \beta = \frac{\sin \phi_e \sin \gamma}{1 - \sin \gamma \cos \phi_e} \quad (\text{III-5}) \]
We expand the left side of Eq. (III-2) and use Eq. (III-5) on the right of Eq. (III-2) to obtain
\[ C \cos a + S \sin a = \frac{(G \cos \phi_e - 1)}{BG \sin \phi_e} \quad (\text{III-6}) \]
where
\[ B = \tan \beta_N \]
\[ C = \cos \alpha_N, \; G = \sin \gamma, \; S = \sin \alpha_N \]
Let
\[ T = \tan \theta_s \]
and substitute Eq. (III-1) on the left of Eq. (III-6) to obtain
\[ -\frac{C \cos \phi_e}{T \sin \phi_e} + S \left(1 - \frac{\cos^2 \phi_e}{T^2 \sin^2 \phi_e}\right)^{1/2} = \frac{G \cos \phi_e - 1}{BG \sin \phi_e} \]
Now write the middle term as
\[ \frac{S}{T \sin \phi_e} \left(T^2 \sin^2 \phi_e - \cos^2 \phi_e\right)^{1/2} = \frac{S}{T \sin \phi_e} \left[T^2 - (T^2 + 1) \cos^2 \phi_e\right]^{1/2} \]
and factor out \( \sin \phi_e \) assuming \( \phi_e \neq 0 \). Isolate the radical on the left side and obtain, by squaring and rearranging,
\[ G^2 \left[(B^2 + 1)(T^2 + 1) - (1 - BCT)^2\right] \cos \phi_e - 2T (BC + T) G \cos \phi_e + T^2 (1 - S^2 B^2 G^2) = 0 \quad (\text{III-7}) \]
Solving this quadratic for \( \cos \phi_e \), we find

\[
\cos \phi_e = \frac{T}{C} \cdot \frac{(BC + T) \pm \left[ (BC + T)^2 - \left( B^2 + 1 \right) \left( T^2 + 1 \right) - (1 - BCT)^2 \left( 1 - S^2 B^2 C^2 \right) \right]^{1/2}}{(B^2 + 1)(T^2 + 1) - (1 - BCT)^2}
\]

(III-8)

From these two \((\pm)\) values of \( \cos \phi_e \), we may find corresponding values for \( \sin \phi_e \), noting that \( \phi_e \leq \frac{\pi}{2} - \gamma \) always, by definition. Use the values of \( \cos \phi_e \) and \( \sin \phi_e \) in Eq. (III-5) to obtain two corresponding values of \( \tan \beta \), hence \( \beta \), noting that for this purpose \( \beta \leq \gamma \). At the same time, Eq. (III-1) allows evaluation of the two values of \( a \), and the intersection points for \( C_\theta \) and \( C_\xi \) can be found. These points are labeled \((a_s, \beta_s)\) and \((a_s, \beta_s)\).

Now the discriminant of Eq. (III-8) provides indications of the presence of two, one, or no intersections of \( C_\theta \) and \( C_\xi \).

From the fact that \((a_s, a_s)\) contains \( a \) values of intersection for two curves defined between both pairs \((a_1, a_2)\) and \((a_3, a_4)\), of necessity \((a_s, a_s)\) values lie inside both ranges \((a_1, a_2)\) and \((a_3, a_4)\). That is, \((a_s, a_s)\) values are within the \( a \) ranges of definition which are common to both \( C_\xi \) and \( C_\theta \). This is the region where, in practice, we may compute both \( \beta(C_\theta) \) and \( \beta(C_\xi) \) to select which shall be used. In this way we avoid calculation of \((a_s, a_s)\) and avoid further complication of the selection logic.
APPENDIX IV
TRANSFORMATIONS FOR $\beta, \phi_e$, AND $\psi$

Transformations for $\psi, \phi_e$, and $\beta$ are based on the relations

$$\psi = \phi_e + \beta$$
$$\sin \psi = \sin \beta / \sin \gamma$$
$$F = (1 + \sin^2 \gamma - 2 \sin \gamma \cos \phi_e)^{1/2}$$

The transformations are

$$\sin \psi = \sin \beta / \sin \gamma = \sin \phi_e / F$$
$$\cos \psi = (\sin^2 \gamma - \sin^2 \beta)^{1/2} / \sin \gamma$$
$$= (\cos \phi_e - \sin \gamma) / F$$

$$\sin \beta = \sin \gamma \sin \phi / F = \sin \gamma \sin \psi$$
$$\cos \beta = (1 - \sin \gamma \cos \phi) / F = (1 - \sin^2 \gamma \sin^2 \psi)^{1/2}$$

$$\sin \phi_e = \left[ \cos \beta - (\sin^2 \gamma - \sin^2 \beta)^{1/2} \right] \sin \beta / \sin \gamma$$
$$= \sin \psi \left[ (1 - \sin^2 \gamma \sin^2 \psi)^{1/2} - \sin \gamma \cos \psi \right]$$

$$\cos \phi_e = \left[ \sin^2 \beta + \cos \beta (\sin^2 \gamma - \sin^2 \beta)^{1/2} \right] / \sin \gamma$$
$$= \cos \psi (1 - \sin^2 \gamma \sin^2 \psi)^{1/2} + \sin \gamma \sin^2 \psi$$

$$\tan \psi = \sin \beta / (\sin^2 \gamma - \sin^2 \beta)^{1/2} = \sin \phi_e / (\cos \phi_e - \sin \gamma)$$

$$\tan \beta = \sin \gamma \sin \phi / (1 - \sin \gamma \cos \phi_e) = \frac{\sin \gamma \sin \psi}{(1 - \sin^2 \gamma \sin^2 \psi)^{1/2}}$$

$$\tan \phi_e = \frac{\left[ \cos \beta - (\sin^2 \gamma - \sin^2 \beta)^{1/2} \right] \sin \beta}{\left[ \sin^2 \beta + \cos \beta (\sin^2 \gamma - \sin^2 \beta)^{1/2} \right]^{\sin \beta}}$$

$$= \sin \psi \frac{\left[ (1 - \sin^2 \gamma \sin^2 \psi)^{1/2} - \sin \gamma \cos \psi \right]}{\left[ \cos \psi (1 - \sin^2 \gamma \sin^2 \psi)^{1/2} + \sin \gamma \sin^2 \psi \right]}$$
APPENDIX V
FLOW CHART
FOR
SUBROUTINE ALBEDO (ALBDO)
APPENDIX V
FLOW CHART FOR COMPUTER PROGRAM

Case Select in $\beta_S$

To Case Select in $\beta_N$

Eclipse, No Albedo - Planet Radiance Only

* $a_3 = \frac{\pi}{2} + \sin^{-1} \left( \frac{\tan \gamma}{\tan \theta_5} \right)$
* $a_4 = 2\pi - a_3$

Subroutine "ALBDO" Arguments:
$\omega_N$, $\beta_N$, $\theta_5$, $\gamma$: $\delta\alpha$
To Adjust, Case Switch

Exit

To a Adjust, Case Switch

Topside, No Albedo, No Planet Radiance

\[ \alpha_1 = \alpha_N + \Delta \alpha \]
\[ \alpha_2 = \alpha_N - \Delta \alpha \]

\[ * \Delta \alpha = \frac{\pi}{2} \sin^{-1} \left( \frac{1}{\tan \beta_N \tan \gamma} \right) \]
Subroutine BDI, with Arguments $\alpha, \alpha_N, \hat{\beta}, \hat{\beta}, \gamma, \theta_s; \text{Sum, } \delta\alpha$
Evaluates the First Integral Between $\hat{\beta}, \hat{\beta}$
Adding Results to Sum, Advancing $\alpha$ by $\delta\alpha$
Subroutine CXI, Arguments $\alpha$, $\alpha_N$, $\beta_N$
Evaluates $\beta$ on the Curve $C_5(\alpha, \beta)$
defined by

$$
\cos (\alpha - \alpha_N) = -\cot \beta \cot \beta_N
$$
or

$$
\beta = \tan^{-1} \left[ \frac{-1}{\tan \beta_N \cos (\alpha - \alpha_N)} \right]
$$
Case 0, 2

$$\hat{\beta} = \gamma$$
$$\alpha = \alpha_2 + \delta \alpha/2$$

Is Range Split?

Yes: $(\alpha_1 - \alpha_2)$

No:

$$\alpha = \delta \alpha/2$$

$$\alpha_{\text{lim}} = 2\pi$$
$$K = 1$$

Call Subroutine CX1 ($\hat{\beta}$)

$$\alpha_{\text{lim}} = \alpha_1$$
$$K = 2$$

Call Subroutine BDI

$K = 1$

Exit
Case 1, 0

Subroutine CTH, Arguments $\alpha$, $\theta_5$, $\gamma$
Evaluates $\beta$ on the Curve $C_6(\alpha, \beta)$
defined by the Set of Equations

\[
\begin{align*}
\cos \alpha &= -\cotn \theta_5 \cotn \varphi \\
\varphi &= \psi - \beta \\
\sin \psi &= \sin \beta / \sin \gamma
\end{align*}
\]
Case 2, 0

\[ PFC-3 \]

\[ K = 1 \]
\[ \alpha = \delta \alpha / 2 \]
\[ \alpha_{\text{lim}} = \alpha_3 \]
\[ \beta = \gamma \]

\[ \alpha - \alpha_{\text{lim}} \]

Call Subroutine \( \text{THET} \) (\( \beta \))

Call Subroutine \( \text{BDI} \)

\[ K = 2 \]
\[ \alpha = \alpha_4 + \delta \alpha / 2 \]
\[ \alpha_{\text{lim}} = 2\pi \]

\[ K = 1 \]

Exit
Case 1, 1

Enter

Initiate

End?

Yes

Exit

No

Sorting and Range Testing

Evaluating

Schematic Only
Details on Next Three Pages
Case 1, 1

\[ \tilde{\theta} = 0 \]
\[ \alpha = \delta \alpha / 2 \]
\[ \alpha_{lim} = 2\pi \]

Sort Configurations

Loop
From P-FC-12

(\(\alpha_{lim} - \alpha\)) 0

(\(\alpha_1 - \alpha_2\))

0 (\(\alpha_1 - \alpha_3\))

0 (\(\alpha_2 - \alpha_3\))

(\(\alpha_1 - \alpha_4\))

0 (\(\alpha_2 - \alpha_4\))

0 (\(\alpha_3 - \alpha_4\))

Exit

FC-10
Case 2, 1

\[ K = 1 \]
\[ \alpha_{\text{lim}} = \alpha_3 \]
\[ \alpha = \frac{\delta \alpha}{2} \]

\[ (\alpha - \alpha_{\text{lim}}) \geq 0 \] \(\Rightarrow\) \[ (\alpha_{1} - \alpha_{2}) \geq 0 \]

\[ (\alpha_{1} - \alpha_{2}) \geq 0 \] \(\Rightarrow\) \[ (\alpha - \alpha_{1})(\alpha - 2\pi) \geq 0 \]

\[ (\alpha - \alpha_{1})(\alpha - 2\pi) \geq 0 \] \(\Rightarrow\) \[ \hat{\beta} \times \gamma \]

Call Subroutine THET (\(\hat{\beta}\))

Call Subroutine BDI
APPENDIX VI
FORTRAN LISTINGS
SUBROUTINE ALBDO
SUBROUTINE BDI
SUBROUTINE XI
SUBROUTINE THET
SUBROUTINE ALBDO(SUM,BETAN,GAM,ALN,THETS, DALS,SGAM,CGAM,STHS,CTHS, SBN,CBN)

1    PY = 3.1415927
    PY2 = 1.5707963
    EP = .0000001
2    SUM = 0.

C    START THETA SEARCH
400   IF(THETS-GAM)400,401,401
    AL3 = PY
    AL4 = PY
    K = 1
    GO TO 408
401   IF(THETS-PY2)1402,402,405
    K = 2
    GO TO 407
402   IF(THETS-PY2)1406,575,575
    K = 3
405   IF(THETS-PY2-PY)1406,575,575
        K = 3
    GO TO 407
406   K = K+3
    GO TO 416
407   CAL3 = -SGAM * CTHS/(CGAM * STHS)
        AL3 = PY2 - ASINF(CAL3)
        AL4 = 2.*PY - AL3

C    START BETA SEARCH
408   IF(BETAN-PY2)409,409,410
409   AL1 = ALN + PY
        AL2 = ALN - PY
        GO TO 417
410   IF(BETAN-PY2)411,411,414
        K = K+3
        GO TO 416
411   IF(BETAN-PY2-PY)415,575,575
        K = K+6
414   CDAL = -CGAM * CBN/ (SGAM * SBN)
        DAL = PY2 - ASINF(CDAL)
        AL1 = ALN + DAL
        AL2 = ALN - DAL
416   IF(AL1-2.*PY)419,419,418
        AL1 = AL1 - 2.*PY
        AL2 = AL2 + 2.*PY
417   GO TO(422,423,424,425,426,427,428,429,430,431,432,433),K

C    START CASE 0-0
422   ALPHA = .5 * DALS
        BMIN = 0.
        BMAX = GAM
        ALIM = 2.*PY
423   IF(ALPHA-ALIM)424,575,575
424   CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
        GO TO 423

C    START CASE 0-1
425   ALPHA = .5 * DALS
        BMIN = 0.
        ALIM = 2.*PY
426   IF(ALPHA-ALIM)427,575,575
427 IF(AL1 - AL2) 428, 429
428 IF(ALPHA - AL1) 430, 431
429 IF(ALPHA * (ALPHA - AL2)) 432, 433

430 IF(ALPHA - AL1) 434, 435
431 BMAX = GAM
GO TO 432
432 CALL XI(BMAX, SBN, CBN, ALN, ALPHA)
433 CALL BD1(SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)
GO TO 426.

C START CASE 0-2

434 BMAX = GAM
ALPHA = AL2 + .5 * DALS
IF(AL1 - AL2) 435, 436
435 K = 1
ALIM = 2 * PY
GO TO 437

436 K = 2
ALIM = AL1
437 IF(ALPHA - ALIM) 440, 441, 442
438 IF(K-1) 439, 439

439 ALPHA = .5 * DALS
GO TO 436

440 CALL XI(BMIN, SBN, CBN, ALN, ALPHA)
CALL BD1(SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)
GO TO 426.

C START CASE 1-0

441 ALPHA = .5 * DALS

442 BMIN = 0
ALIM = 2 * PY
443 IF(ALPHA - ALIM) 444, 445, 446
444 CALL THET(BMIN, STHS, CTHS, SGAM, ALPHA)
GO TO 445

445 BMAX = GAM
446 CALL BD1(SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)
GO TO 442

C START CASE 2-0

447 K = 1
ALPHA = .5 * DALS
ALIM = AL3
448 IF(ALPHA - ALIM) 449, 450
449 CALL THET(BMIN, STHS, CTHS, SGAM, ALPHA)
CALL BD1(SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)
GO TO 445

450 IF(K-1) 451, 452, 453
451 K = 2
ALPHA = AL4 + .5 * DALS
ALIM = 2 * PY
GO TO 448

C START CASE 1-1

452 BMIN = 0
ALPHA = .5 * DALS
ALIM = 2 * PY
453 IF(ALPHA - ALIM) 454, 455

56
454 IF (AL1 - AL2) 465, 455
455 IF (AL1 - AL4) 462, 465
456 IF (AL2 - AL3) 460, 467
457 IF ((ALPHA - AL1) * (ALPHA - AL1)) 458, 602
458 IF ((ALPHA - AL2) * (ALPHA - AL4)) 461, 459
459 IF (ALPHA - AL2) 1604, 603
460 IF ((ALPHA - AL2) * (ALPHA - AL1)) 461, 602
461 IF ((ALPHA - AL3) * (ALPHA - AL4)) 461, 603
462 IF ((ALPHA - AL2) * (ALPHA - AL4)) 463, 602
463 IF (ALPHA - AL3) 601, 464
464 IF (ALPHA - AL3) 603, 503
465 IF (AL1 - AL3) 476, 466
466 IF (AL1 - AL4) 467, 473
467 IF (AL2 - AL4) 471, 468
468 IF ((ALPHA - AL1) * (ALPHA - AL2)) 469, 603
469 IF ((ALPHA - AL1) * (ALPHA - AL4)) 470, 470
470 IF ((ALPHA - AL1) * (ALPHA - AL4)) 472, 603
471 IF ((ALPHA - AL1) * (ALPHA - AL2)) 474, 603
472 IF ((ALPHA - AL3) * (ALPHA - AL4)) 475, 603
473 IF ((ALPHA - AL3) * (ALPHA - AL2)) 477, 603
474 IF ((ALPHA - AL4) * (ALPHA - AL1)) 478, 480
475 IF (ALPHA - AL4) 601, 602
476 IF (AL2 - AL3) 483, 477
477 IF (AL2 - AL4) 478, 481
478 IF ((ALPHA - AL1) * (ALPHA - AL4)) 479, 603
479 IF ((ALPHA - AL3) * (ALPHA - AL2)) 480, 480
480 IF ((ALPHA - AL3) * (ALPHA - AL2)) 481, 602
481 IF ((ALPHA - AL1) * (ALPHA - AL2)) 482, 603
482 IF ((ALPHA - AL3) * (ALPHA - AL4)) 483, 603
483 IF ((ALPHA - AL1) * (ALPHA - AL4)) 484, 603
484 IF ((ALPHA - AL2) * (ALPHA - AL3)) 485, 603
485 IF (ALPHA - AL2) 602, 601
486 CALL THET (BMAX, STHS, CTHS, SGAM, ALPHA)
487 IF (K = 4) 491, 487
488 CALL XI (BMAX, SBN, CBN, ALN, ALPHA)
489 IF (K = 4) 491, 490
490 BMAX = B1
491 CALL BOI (SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)
492 GO TO 453

500 ALIM = AL1
501 IF (AL1 - AL2) 501, 502
502 K = 1
503 ALPHA = .5 * DALS
GO TO 503
START CASE 2-1

502 K = 2
504 ALPHAM = AL2 + .5 * DALS
506 IF (ALPHA - ALIM) > 504, GO TO 503
507 ALIM = 2 * PY
509 IF (ALPHA - AL1) > 507, GO TO 508
510 CALL THET(BMAX, STHS, CTHS, SGAM, ALPHA)
511 CALL XI (BMIN, SBN, CBN, ALN, ALPHA)
512 CALL BDII(SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)
520 START CASE 2-2

C

520 K = 1
522 ALIM = AL3
524 IF (ALPHA - ALIM) > 524, GO TO 521
525 ALPHAM = .5 * DALS
527 IF (ALPHA - AL1) > 527, GO TO 528
528 BMAX = GAM
529 CALL XI (BMAX, SBN, CBN, ALN, ALPHA)
530 CALL THET(BMIN, STHS, CTHS, SGAM, ALPHA)
532 CALL BDII(SUM, ALPHA, DALS, ALN, BMAX, BMIN, SGAM, CGAM, STHS, CTHS, SBN, CBN)

C

540 BMAX = GAM
542 ALIM = AL1
544 ALIM = AL3
546 IF (AL1 - AL2) > 546, GO TO 545
548 ALPHAM = AL2 + .5 * DALS
549 IF (AL2 - AL3) > 549, GO TO 550
550 ALPHAM = AL4 + .5 * DALS
ALPHA = AL? + .5 * DALS
ALIM = AL1
IF(ALPHA - ALIM) 558, 554, 554
IF(K = 2) 555, 555
ALIM = 2 * PY
IF(AL2 - AL4) 556, 557, 557
ALPHA = AL4 + .5 * DALS
GO TO 553
ALPHA = AL2 + .5 * DALS
GO TO 553
CALL THET(BMIN, STHS, CTHS, SGAM, ALPHA)
IF(K = 3) 559, 561, 559
CALL XIIB1(SBN, CBN, ALN, ALPHA)
IF(B1 - BMIN) 561, 561, 561
SUM = SUM * DALS / (SGAM * 4. * PY)
IF(SUM < .00001) 576, 577, 577
SUM = c*
BETAN = GG
RETURN
END
SUBROUTINE BDI(SUM,ALPHA,DALS,ALPHN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,
   SBETN,CBETN)
   IF(BMIN =BMAX)1,2,2
   1 CALPD = COSRF(ALPHA -ALPHN)
   CALPH = COSRF(ALPHN)
   CBMAX = COSRF(BMAX)
   SBMAX = SINRF(BMAX)
   CBMIN = COSRF(BMIN)
   SBMIN = SINRF(BMIN)
   ARG1 = 1.-(SBMAX *SBMAX)/(SGAM * SGAM)
   ARG2 = 1.-(SBMIN *SBMIN)/(SGAM * SGAM)
   IF(ARG1)3,3,4
   3 CPMAX = 0.
   GO TO 5
   4 CPMAX = SQRTF(ARG1)
   5 IF(ARG2)6,6,7
   6 CPMIN = 0.
   GO TO 8
   7 CPMIN = SQRTF(ARG2)
   8 ARG3 =1. -CPMAX *CPMAX
   ARG4 =1. -CPMIN *CPMIN
   IF(ARG3)9,9,10
   9 PMAX = 0.
   GO TO 11
   10 PMAX = ASINF(SQRTF(ARG3))
   11 IF(ARG4)12,12,13
   12 PMIN = 0.
   GO TO 14
   13 PMIN = ASINF(SQRTF(ARG4))
   14 FE1 = SBETN* STHS * CALPH * CALPD
   FE2 = CBETN * CTHS
   FE3 = SBETN * CTHS * CALPD
   FE4 = CBETN * STHS * CALPH
   S1 =1*(FE1  5*( FE1 + FE2)*CGAM*CGAM) * (SGAM*(CBMAX*CPMAX
   1 - CBMIN *CPMIN) - CGAM*CGAM * LOGF(( CBMAX+ SGAM* CPMA)/( CBMIN
   2 + SGAM *CPMIN)) - (FE2 + FE1)*SGAM*SGAM*SGAM *(CBMAX* CPMA)*
   3 CPMA *CPMA - CBMIN * CPMA * CPMA * CPMA)
   S2 = (FE2 -FE3) * (SBMAX* CPMA *CPMA *CPMA - SBMIN * CPMA * CPMA
   PMIN)
   *CPMIN
   1 *CPMIN + 5 * ( SBMAX *CPMA - SBMIN *CPMIN +SGAM *(PMAX -PMIN))
   2 *SGAM*SGAM*SGAM
   S3 = (FE2 +FE1) * ( SBMAX**4 - SBMIN**4 )
   S4 = (5 *FE4 + 1.5 *FE3)*(BMAX -BMIN -SBMAX *CBMIN +SBMIN *CBMIN
   1 ) +(FE4 -FE3)*(SBMAX*SBMAX *SBMAX *CBMAX -SBMIN *SBMIN *CBMIN
   CBMIN)
   DSUM = S1 +S2 +S3 +S4
   SUM = SUM + DSUM
   2 ALPHA = ALPHA + DALS
   RETURN
END
SUBROUTINE XI(B, SBN, CBN, AN, AL)
   B = 0.
   TB = -CBN / (SBN * COSRF(AL - AN))
   B = ATANF(TB)
   RETURN
END

SUBROUTINE THET(B, STHS, CTHS, SGAM, AL)
   FF = -CTHS / (STHS * COSRF(AL))
   B = 0.
   B = ATANF(FF)
   FE = SINRF(B) / (COSRF(B) - SGAM)
   FE = ATANF(FF)
   B = FE - B
   RETURN
END
EVALUATION OF THE ALBEDO INTEGRAL FOR MARK I

This report is concerned with the development of a fast computer method for evaluating the albedo integral. This integral defines the illumination on an arbitrarily oriented surface element at any point in space about a diffusely reflecting sphere. It enters the calculation of simulation control parameters in the Arnold Engineering Development Center Aerospace Environmental Chamber (Mark I). The seminumerical method developed here is faster than ordinary numerical integration by a factor of about ten. A typical computer program, which formerly required about 30 minutes, now produces the same results in under four minutes.
mathematical analysis
albedo
integrals
computers
seminumerical integration
simulation control
planet radiance
illumination

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**Notes:**
- Mathematical analysis, albedo, integrals, computers, seminumerical integration, simulation control, planet radiance, illumination.