DISCRETE OPTIMIZATION VIA MARGINAL ANALYSIS

Bennett Fox

January 1966

P-3288

DISCRETE OPTIMIZATION VIA MARGINAL ANALYSIS

Bennett Fox

The RAND Corporation, Santa Monica, California

This paper was prepared for submission to Management Science.

ABSTRACT

Discrete optimization subject to one contraint is attacked by Lagrangian analysis. Incremental allocation schemes are given that generate undominated allocations. In an important special case, the complete family of undominated allocations is generated.

1. Introduction

In allocation problems, a marginal analysis of incremental return per additional dollar spent is intuitively appealing. We give conditions under which it is justified and applications. In some circles, some of our results are probably part of the folklore.

2. Problem

$$(*) \qquad \max \{\phi(x) : x \in S,\ C(x) \le M\}$$

where S is the set of n-tuples of nonnegative integers $(x_1, \ldots, x_n)$,

$$C(x) = \sum_{j=1}^{n} c_j x_j ,$$

$$M,\ c_j > 0,\ j = 1, \ldots, n,$$

and

$$\phi(x) = \sum_{j=1}^{n} \phi_j(x_j) .$$

3. Procedure

1. Start with the allocation $x^o = 0$.
2. $k = 1$.
3. $x^k = x^{k-1} + e_i$, where e_i is the ith unit vector and i is any index for which

$$[\phi_j(x_j^{k-1} + 1) - \phi_j(x_j^{k-1})]/c_j$$

is maximum.

4. If $C(x^k) > M$, terminate; otherwise $k \rightarrow k+1$ and go to step 3.

4. Variant 1

A slight variant of the foregoing procedure is to terminate when the objective function first exceeds a preassigned value instead of when the cost exceeds a preassigned value.

5. Variant 2

A second variant is to branch at step 3 whenever ties occur for the maximizing index. Let I^k be the set of maximizers. The procedure is successively initiated with the allocations $x^{k-1} + e_i$, $i \in I^k$.

6. Lagrangian Analysis

Allocations x satisfying

$$\phi(y) > \phi(x) \Rightarrow C(y) > C(x)$$

$$\phi(y) = \phi(x) > C(y) \geq C(x)$$

for all y are called undominated.

<u>Lemma 1.</u> If $\lambda \geq 0$ and $x^* \in \mathcal{S}$ maximizes the Lagrangian $\phi(x) - \lambda g(x)$ over all $x \in \mathcal{S}$, then x^* maximizes $\phi(x)$ over all those $x \in \mathcal{S}$ such that $g(x) \leq g(x^*)$.

<u>Proof.</u> This is a special case of a result of Everett [3].||

As λ varies over $(0,\infty)$ not all undominated allocations are necessarily generated. For a geometrical explanation of this fact, see Everett [3].

Let $m_i^*(\lambda)$ be the smallest nonnegative integer m (assumed to exist) satisfying $\phi_i(m + 1) - \phi_i(m) < \lambda c_i$, $E_i(\lambda)$ be the set of nonnegative integers m for which $\phi_i(m + 1) - \phi_i(m) = \lambda c_i$, $T_i(\lambda) = \{m_i^*(\lambda)\} \cup E_i(\lambda)$, and $T(\lambda) = X_{i=1}^n \, T_i(\lambda)$. In what follows, a function defined only on the integers is called concave if its first differences are decreasing.

<u>Theorem 1.</u> For any $\lambda > 0$, if $\phi_i(y)$ is concave, $i = 1, \ldots, n$, $x(\lambda) \in T(\lambda) \Rightarrow x(\lambda)$ is undominated.

<u>Proof.</u> Apply Lemma 1.||

If $\phi_i(y)$ is strictly concave, $m_i^*(\lambda)$ exists for all $\lambda > 0$. If $\phi_i(y)$ is differentiable, $T_i(\lambda)$ can be found by evaluating $\phi_i(y) - \lambda y c_i$ at the nonnegative integers neighboring the roots $\frac{d}{dy}\,\phi_i(y) = \lambda c_i$; of course, if $\phi_i(y)$ is strictly concave, there is a unique root.

<u>Theorem 2.</u> If $\phi_i(y)$ is concave and strictly increasing, $i = 1, \ldots, n$, the allocations generated by the incremental allocation procedure are undominated.

<u>Proof.</u> Set λ equal to the value of the maximum in step 3 of the procedure. From the definition of x^k, it is easily checked that $x^k \in T(\lambda)$. Now apply Theorem 1.||

From the proof of Theorem 2, we see that as λ varies over $(0,\infty)$ all the allocations generated by the incremental allocation procedure are found; with Variant 2, the converse is true as long as $C(x(\lambda)) \leq M$. An alternate proof of Theorem 2 can be found by adapting a proof of a theorem on redundancy optimization found in Barlow and Proschan [2], pp. 167-168.

In the incremental allocation procedure, we may replace the starting allocation $x^o = 0$ by any other undominated allocation obtained from Lagrangian analysis. By suitable iterations on trial values of λ, a better starting allocation can be generated. Conversely, having generated an allocation, we can immediately find a corresponding $\lambda = \max [\phi(x_i + 1) - \phi)x_i)]/c_i$, which can be interpreted as the approximate shadow price of the resource being allocated; it may be useful for sensitivity analysis.

<u>Theorem 3.</u> If $\phi_i(y)$ is concave and strictly increasing, $i = 1, \ldots, n$, $x^1, \ldots, x^m$ are the allocations generated by the procedure, and z is optimal for (*),

$$\phi(x^m) > \phi(z) \geq \phi(x^{m-1})$$

and

$$C(x^m) > C(z) \geq C(x^{m-1})$$

$$0 < C(x^m) - C(x^{m-1}) \leq \max_j c_j \, .$$

<u>Proof.</u> Use the definition of the procedure and apply Theorem 2.||

Let Z denote the s t of optimal allocations fo (*). Although it is not necessarily true that $x^{m-1} \in Z$, the inequalities in the theorem can be u: l to check if the procedure has generated an allocation sufficiently near optimal for practical purposes. If not, the exact solution can be found by dynamic programming, which requires much more computational effort.

In the important case of equal c_j's we can strengthen the preceding theorem.

Corollary 1. If, in addition to the conditions stated in Theorem 3, $c_1 = \ldots = c_n$, then the procedure generates an optimal allocation; i.e., $x^{m-1} \in Z$.

Proof. $C(x^m) = C(x^{m-1}) + c_1$. There is no allocation y such that $C(x^m) > C(y) > C(x^{m-1})$. From Theorem 3, it follows that $x^{m-1} \in Z$.|| Gross [4] proves this result directly.

Corollary 2. If, in addition to the conditions stated in Theorem 3, $c_1 = \cdots = c_n$, the solution to (*), say $\sigma(M)$, is concave and strictly increasing in M, when M is restricted to integral multiples of C_1.

Proof. Let u and v be optimal allocations corresponding to $\sigma(M - 1)$ and $\sigma(M)$, respectively. Assuming that incremental allocation was used, we have

$$\sigma(M + 1) - \sigma(M) = \phi_i(v_i + 1) - \phi_i(v_i)$$

$$\sigma(M) - \sigma(M - 1) = \phi_j(u_j + 1) - \phi_j(u_j),$$

say. By definition of the procedure, $\phi_j(u_j + 1) - \phi_j(u_j) \geq \phi_i(v_i + 1) - \phi_i(v_i)$; hence $\sigma(M) - \sigma(M - 1) \geq \sigma(M + 1) - \sigma(M)$.||

The concavity condition is essential. Examples can be easily constructed where the conclusion of Theorem 3 fails to hold when the concavity condition is dropped. If $\phi(x)$ is linear, say $\Sigma\, a_j\, x_j$, a direct proof of Theorem 3 is immediate, although the set of bounded allocations maximizing the Lagrangian in this case is either empty or consists only of the null vector, except when $\lambda = \max\, a_j/c_j$. With a linear objective function, we have a version of the knapsack problem, for which special methods are available to find the exact solution; see Gilmore and Gomory ([6], [7]) and the review paper of Balinski [1].

7. Nonconcave Objective Functions

We now drop the restriction that $\phi_i(y)$ be concave. Call $\tilde{\phi}_i(y)$ the least concave majorant of $\phi_i(y|N)$, the restriction of $\phi_i(y)$ to the set N of nonnegative integers, and let $Y_i = \{y \in N : \tilde{\phi}_i(y) = \phi_i(y)\}$. It is easily seen that the line $\lambda y c_i + \max\{\phi_i(x) - \lambda x c_i : x \in N\}$ never lies below $\phi_i(y|N)$; this motivates

<u>Lemma 2.</u> $y_i^* \in N$ maximizes $L_i(y) = \phi_i(y) - \lambda y c_i$ over $N \Rightarrow y_i^* \in Y_i$.

<u>Proof.</u> Suppose to the contrary that y_i^* is a maximizer but $y_i^* \notin Y_i$. Let x [z] be the largest [smallest] element of Y_i less [greater] than y_i^*. It follows that

$$\frac{\phi_i(z) - \phi_i(y_i^*)}{z - y_i^*} - \lambda c_i = \frac{L_i(z) - L_i(y_i^*)}{z - y_i^*} \le 0$$

and

$$\frac{\phi_i(y_i^*) - \phi_i(x)}{y_i^* - x} - \lambda c_i = \frac{L_i(y_i^*) - L_i(x)}{y_i^* - x} \ge 0 \,.$$

From these relations, it follows that

$$\frac{\phi_i(z) - \phi_i(y_i^*)}{z - y_i^*} \le \frac{\phi_i(y_i^*) - \phi_i(x)}{y_i^* - x} .$$

But since $y_i^* \notin Y_i$, we have

$$\frac{\phi_i(z) - \phi_i(y_i^*)}{z - y_i^*} > \frac{\phi_i(y_i^*) - \phi_i(x)}{y_i^* - x} ,$$

a contradiction.||

On the other hand, if $y' \in Y_i - \{0\}$, by setting $\lambda = \phi_i(y)/yc_i$ we see that there exists a λ such that y' maximizes $\phi_i(y) - \lambda c_i$.

Thus, we may easily modify the incremental allocation procedure to generate undominated allocations when each term of the objective function is strictly increasing but not necessarily concave. It is obvious that $x^o = 0$ is undominated. Inductively, suppose that $x^{k-1} = (x_1, \ldots x_n)$ is undominated and $x_i \in Y_i$, $i = 1, \ldots, n$. Let y_i be the smallest element of Y_i that is larger than x_i, $i = 1, \ldots, n$.

<u>Theorem 4.</u> If i is an index for which

$$\frac{\phi_j(y_j) - \phi_j(x_j)}{c_j(y_j - x_j)}$$

is maximum, the allocation $x^k = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ is undominated.

<u>Proof.</u> By Lemma 2 we may replace the set $\mathcal{S}$ in Lemma 1 by $X_{i=1}^n Y_i$. The rest of the proof is analogous to that of Theorem 2.||

A technique of Barlow and Proschan [2], pp. 167-168, can easily be extended, as in Feeney and Sherbrooke [4], to produce an alternate proof. However, our proof shows more clearly the relation to standard Lagrangian analysis; see the remarks following Theorem 2.

The suitability of incremental allocation for practical application depends on the spacings of the allocations generated. Feeney and Sherbrooke [4] describe an application where the successive undominated allocations generated are close together.

8. Nonlinear Constraint

Let $\phi_j(x_j)$ be concave and strictly increasing, $j = 1, \ldots, n$. If $C(x) = \Sigma\, c_j(x_j)$ with $c_j(x_j)$ convex and strictly increasing, $j = 1, \ldots, n$, and we replace the criterion of step 3 of the procedure by

$$\frac{\phi_j(x_j^{k-1} + 1) - \phi_j(x_j^{k-1})}{c_j(x_j^{k-1} + 1) - c_j(x_j^{k-1})} ,$$

all the allocations generated are undominated. The proof is analogous to that of Theorem 2.

A nonconvex constraint can be treated by an analog of the method of the preceding section.

9. Application 1

In the well-known flyaway-kit problem (see Karr and Geisler [9] and Geisler and Karr [5]), the expected stockout cost for item j when x_j units of item j are included in the kit is

$$\gamma_j(x_j) = b_j \sum_{k=x_j}^{\infty} (k - x_j)\, p_j(k)$$

where $b_j > 0$ and $p_j(k)$ is the probability that k units of item j are demanded. Setting $\phi_j(x_j) = -\gamma_j(x_j)$ [and replacing min $\Sigma\, \gamma_j(x_j)$ by max $\Sigma\, \phi_j(x_j)$], we have

$$\phi_j(x_j + 1) - \phi_j(x_j) = b_j\, P_j(x_j)$$

where

$$P_j(k) = \sum_{i=k}^{\infty} p_j(i) .$$

Since $P_j(x_j)$ is decreasing in x_j, $\phi_j(x_j)$ is concave; thus, the marginal analysis application in [5] and [9] is justified.

10. Application 2

Consider the problems

(A)

$$\min \sum_{j=1}^{n} p_j^{x_j}$$

$$\text{s.t.} \sum_{j=1}^{n} x_j = X$$

$$x_j = \text{nonnegative integer},\ j = 1, \ldots, n$$

(B)

$$\max \prod_{i=1}^{k} (1 - q_i^{y_i})$$

$$\text{s.t.} \sum_{i=1}^{k} y_i = Y$$

$$y_i = \text{nonnegative integer},\ i = 1, \ldots, k$$

The p_j's and q_i's are constants between 0 and 1. It is easily shown that the objective function in (A) is the sum of convex functions; hence, the procedure can be (legitimately) applied with $-\sum_{j=1}^{n} p_j^{x_j}$ as objective function. In (B), it is equivalent to maximize $\sum_{i=1}^{k} \log(1 - q_i^{y_i})$, each term of which is easily shown to be concave.

Landi [10] gives a war gaming problem, studied earlier but less generally by Piccarielo [11], in which the solutions of (A) and (B) are composed to obtain an optimal allocation of attacking missiles to an opposing missile launch complex. Denoting by F(X) and G(Y) the solutions of (A) and (B), respectively, let

$$H(T) = \min \{F(X)[1 - G(Y)] : X, Y \in S, X + Y = T\} .$$

In the missile allocation context, H(T) turns out to be the minimum expected number of missiles launched in retaliation after an attack by T missiles, if we interpret

$p_j \leftrightarrow$ probability of survival of launch pad j;

$q_i \leftrightarrow$ probability of survival of control center i;

$X \leftrightarrow$ the number of missiles allocated to attack launch pads;

$Y \leftrightarrow$ the number of missiles allocated to attack control centers;

$x_j \leftrightarrow$ the number of missiles allocated to attack launch pad j; and

$y_i \leftrightarrow$ the number of missiles allocated to attack control center j,

with the understanding that if at least one control center survives an enemy attack, all surviving missiles can be launched, and if all control centers are destroyed, none of the surviving missiles can be launched. It is assumed that of the N + C separate targets (N issiles, C redundant control centers) at most one can be destroyed by a single enemy missile.

From Corollary 2 it follows that F(X) and 1 - G(Y) are both convex and strictly decreasing. Unfortunately, this does not imply that - log F(X) and - log [1 - G(Y)] are concave. However, since we have effectively only two variables at this stage, the combinatorial problem can be solved by brute force without difficulty.

ACKNOWLEDGMENT

I thank Dale Landi for a careful review of an earlier draft.

REFERENCES

1. Balinski, M. L., "Integer Programming: Methods, Uses, Computation," Management Science, Vol. 12 (1965), pp. 253-313.

2. Barlow, R. E. and F. Proschan, Mathematical Theory of Reliability, John Wiley and Sons, New York 1965.

3. Everett, H., "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," Operations Research, Vol. 11 (1963), pp. 399-417.

4. Feeney, G. J. and C. C. Sherbrooke, "A System Approach to Base Stockage of Recoverable Items," RM-4720-PR, The RAND Corporation, Santa Monica, California, December 1965.

5. Geisler, M. and H. Karr, "The Design of Military Supply Tables for Spare Parts," Operations Research, Vol. 4 (1956), pp. 431-422.

6. Gilmore, P. C. and R. E. Gomory, "A Linear Programming Approach to the Cutting Stock Problem - Part II," Operations Research, Vol. 11 (1963), pp. 863-888.

7. ----- and -----, "Multistage Cutting Stock Problems of Two and More Dimensions," Operations Research, Vol. 13 (1965), pp. 94-120.

8. Gross, O. "A Class of Discrete-Type Minimization Problems," RM-1644-PR, The RAND Corporation, Santa Monica, California, February 1956.

9. Karr, H. and M. Geisler, "A Fruitful Application of Static Marginal Analysis," Management Science, Vol. 2 (1956), pp. 313-326.

10. Landi, D., unpublished notes, 1964.

11. Piccarielo, H., "A Missile Allocation Problem," Operations Research, Vol. 10 (1962), pp. 795-798.