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Thickness Shear Vibrations of an Ablating Rocket

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Thickness-Shear Vibrations of an Ablating Rocket*

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Abstract

The loss of momentum and stiffness due to ablation may significantly influence the vibrations of a solid propellant grain. This paper presents an analytical study of the axial shear vibrations of a long hollow cylinder that is subjected to time dependent body forces in the axial direction. The outer surface of the cylinder is bonded to a rigid case, and the inner radius increases monotonically with time. An expression is determined for the shear stress at the bond-interface. It is shown that the frequency of the shear-bond stress increases, and that its amplitude decreases towards burnout time. The shear stress is studied for various ablation rates. Conventional methods of analysis, such as separation of variables and Fourier-Bessel analysis, are not directly applicable in this problem, since the boundary conditions are prescribed on a time dependent surface. A modified Fourier-Bessel mode is defined that satisfies the boundary conditions. By substituting this mode into the equation of motion, a solution is obtained by asymptotic methods in the vicinity of the bond-interface. The analysis is extended to include the axial shear vibrations of an ablating viscoelastic cylinder. Viscoelasticity is introduced by means of the relaxation function in shear.

List of Major Symbols

a_o	= inner radius at $t = 0$
$a(t), b$	= inner and outer radius
m, n	= mode numbers
$q=(G/\rho)^{\frac{1}{2}}t_f/b$	= dimensionless burning time
r, θ, z	= polar coordinates
t	= time
t_f	= burning time
u_i	= displacement component
A_n, D_n	= constant Fourier-Bessel coefficients
C_n^1, C_n^2	= constants
$D(\tau)$	= dimensionless creep function
$F_i(x_j, t)$	= body force per unit mass
G	= shear modulus
$G(\tau)=Gg(\tau)$	= shear relaxation function
$H(\tau)$	= Heaviside unit function
$J_o(\ ), Y_o(\ )$	= Bessel functions of order zero
$N(\tau)$	= dimensionless body force
$S_n(\tau), T_n(\tau)$	= time dependent Fourier-Bessel coefficients
$U(\Omega_n, R)$	= mode of free vibration
$V_n[X_n(\tau), R]$	= ablation mode of free vibration
$W(R,\tau)$	[illegible] placement
$X_n(\tau)$	[illegible] frequencies"
$\gamma(\tau)$	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
λ	= [illegible]
ρ	= mass density
σ_{ij}	= stress tensor
τ	= dimensionless time
$\omega(\tau)$	= frequency function
Σ_{rz}	= dimensionless stress
Ω_n	= eigenfrequency
$\bar{\Omega}_n$	= $\Omega_n(1 - \beta)$
subscript $_v$	= viscoelastic solution
superscript *	= quasi-static solution
superscript $\bar{\ }$	= dynamic part of solution
superscript ^{A}	= ablation solution

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I. Introduction

The designing of solid propellant rocket motors requires a consideration of the dynamic response of a propellant-casing system to time-dependent body forces. The body forces may be associated with spinning motions, axial accelerations, etc. This paper focuses upon the axially-symmetric dynamic response due to axial acceleration, such as occurs in accelerated flight. Special attention is devoted to the shear bond stress at the propellant-casing interface.

The free and forced vibrations of encased elastic cylinders have been considered for various types of surface constraints.[1,2,3] The dynamics of encased viscoelastic cylinders have also been studied.[4,5] These papers present results for various values of the ratio of the internal and external radii of the propellant cylinder. In this way one obtains quasi-static information on the influence of an increasing inner radius. It was only recently that the influence of continuous ablation on the axially-symmetric plane strain vibration was studied for both an encased elastic[6] and an encased viscoelastic cylinder.[7] In the present paper an attempt is made to analyze the influence of continuous ablation on the axial response of a burning rocket. For simplicity we consider a case-bonded grain of infinite length.

To simplify the analysis the presence of any star points is neglected, and the propellant is represented as a thick-walled cylinder. Any star point material may be accounted for, however, by taking the inner radius of the propellant cylinder as the radius the cylinder would have if the star point material were uniformly distributed around the inner surface. It is assumed that the stiffness of the propellant is so small compared to the stiffness of the casing, that the latter may be assumed as infinite. In a study of a related free-vibration problem by Baltrukonis[8] it was shown that for axially symmetric shear deformations this assumption is acceptable for the stiffness ratios that are commonly encountered in solid propellant motors.

The paper is primarily concerned with the influence of grain ablation on the shear bond stress. It is shown that the loss of mass and stiffness of a burning grain significantly affects the magnitude and the frequency of the shear bond stress. Since most propellants are viscoelastic in shear the influence of viscoelastic damping is included in the third part of the paper.

The dynamic solutions that are presented in this paper are discussed in relation to the solutions of the analogous quasi-static problems. The quasi-static solutions for the elastic cylinder are simple. A quasi-static solution for the ablating viscoelastic cylinder is presented by Lindsey and Williams.[9]

The paper consists of three parts which are concerned with the axial shear vibrations of (1) an encased elastic cylinder of constant inner radius a_o, (2) an encased elastic cylinder of monotonically increasing radius $a(t)$, and (3) an encased viscoelastic cylinder of monotonically increasing radius $a(t)$. Conventional methods of analysis, such as separation of variables and Fourier-Bessel analysis, are not directly applicable in the last two problems, since the boundary conditions are prescribed on a time dependent surface. A modified Fourier-Bessel mode is defined that satisfies the boundary conditions. By substituting this mode into the equation of motion, a solution is obtained by asymptotic methods in the vicinity of the bond-interface. The method thus allows us to determine the shear-bond stress.

II. The Equations of Motion

In a solid body the stress tensor σ_{ij} satisfies the equation of motion

$$\sigma_{ij,j} + \rho F_i(x_j,t) = \rho\ddot{u}_i \tag{1}$$

where $F_i(x_j,t)$ is a body force per unit mass. A uniform body-force distribution is assumed, with a component in z-direction only. Since solutions of Eq. (1) are sought for a circular-cylindrical body subjected to axially symmetric loading and boundary conditions Eq. (1) reduces to

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = \rho\ddot{u}_r \tag{2a}$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho F_z(t) = \rho\ddot{u}_z \tag{2b}$$

The analysis is restricted to an infinitely long hollow cylinder that is rigidly encased at the outer surface and stress free at the time dependent inner surface. The boundary conditions for this configuration are

$$r = a(t) \qquad \sigma_r = \sigma_{rz} = 0 \tag{3a}$$

$$r = b \qquad u_r = u_z = 0 \tag{3b}$$

The cylinder is infinitely long and the system is anti-symmetric relative to any plane perpendicular to the z-axis. The stress component σ_{zz}, which is a symmetric quantity, must then vanish. The remaining stress and displacement components are independent of z, and equations (2) reduce to two uncoupled equations of motion. The equation of motion in radial direction, subject to the boundary conditions (3) and quiescent initial conditions, yields only the trivial solutions $\sigma_{rr} = \sigma_{\theta\theta} = 0$. The only remaining equation constitutes the governing equation of the present problem.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \rho F_z(t) = \rho\ddot{u}_z \tag{4}$$

The solution of Eq. (4) is subject to the boundary conditions

$$\text{at} \quad r = a(t) \qquad \sigma_{rz} = 0 \tag{5a}$$

$$\text{at} \quad r = b \qquad u_z = 0 \tag{5b}$$

Assuming an initially undisturbed cylinder the initial conditions are expressed as

$$u_z(r,t) = \dot{u}_z(r,t) = 0 \quad \text{for} \quad t < 0 \tag{6}$$

We shall determine solutions for an elastic cylinder as well as for a viscoelastic cylinder. By eliminating the stress from Eq. (4), the equation for the axial displacement $u_z(r,t)$ of the elastic cylinder is obtained as

$$\frac{G}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \rho F_z(t) = \rho\ddot{u}_z \tag{7}$$

If the cylinder is linearly viscoelastic the relation between shear stress and shear strain is expressed in the form

$$\sigma_{rz} = \int_{0^-}^{t} G(t-s)d(\partial u_z/\partial r) \tag{8}$$

where $G(t)$ is the relaxation function in shear. The equation of motion for a viscoelastic cylinder is obtained by substitution of Eq. (8) into Eq. (4).

For convenience the following dimensionless quantities are introduced

$$W = u_z/b \qquad \tau = t/t_f \tag{9a}$$

$$R = r/b \qquad N(\tau) = t_f^2 F_z(\tau)/b \tag{9b}$$

$$q = (G/\rho)^{\frac{1}{2}} t_f/b \qquad \beta = a_o/b \tag{9c}$$

In Eq. (9) t_f is the total burning time. We also define

$$a(t) = a_o\alpha(\tau), \text{ where } 1 \le \alpha(\tau) \le (b/a_o) \tag{9d}$$

The governing equation for the elastic cylinder, Eq. (7), can now be rewritten as

$$q^2 \left(\frac{\partial^2 W}{\partial R^2} + \frac{1}{R}\frac{\partial W}{\partial R} \right) + N(\tau) = \frac{\partial^2 W}{\partial \tau^2} \tag{10}$$

The governing equation for the viscoelastic cylinder is

$$\frac{q^2}{R}\frac{\partial}{\partial R}\left[R \int_{0^-}^{\tau} g(\tau - s)\, d(\partial W/\partial R) \right] + N(\tau) = \frac{\partial^2 W}{\partial \tau^2} \tag{11}$$

where the following substitution for the relaxation function was made

$$G(\tau) = G g(\tau) \tag{12}$$

The initial conditions on the dimensionless displacement are

$$W(R,0) = \dot{W}(R,0) = 0 \tag{13}$$

In terms of the dimensionless displacement the boundary conditions are expressed as

$$\text{at} \quad R = \beta\alpha(\tau) \qquad \partial W/\partial R = 0 \tag{14a}$$

$$\text{at} \quad R = 1 \qquad W = 0 \tag{14b}$$

III. Elastic Grain of Constant Inner Radius

For a cylinder of constant inner radius the boundary conditions simplify to

$$\text{at} \quad R = \beta \qquad \partial W/\partial R = 0 \tag{15a}$$

$$\text{at} \quad R = 1 \qquad W = 0 \tag{15b}$$

If a suddenly applied uniform body-force distribution is considered $N(\tau) = N H(\tau)$. The forced vibration problem is defined by the governing equation (10) and the initial (13) and boundary conditions (15). The solution to this problem is sought in the form of a Fourier-Bessel expansion in terms of the corresponding free vibration modes. The free vibration problem corresponding to Eqs. (10) and (15) was considered by Baltrukonis.[1] The modes of free vibration are

$$W = U(\Omega_n, R) e^{iq\Omega_n \tau} \tag{16}$$

where

$$U(\Omega_n, R) = J_o(\Omega_n R) Y_o(\Omega_n) - J_o(\Omega_n) Y_o(\Omega_n R) \tag{17}$$

The eigenfrequencies Ω_n are the positive roots of the frequency equation

$$J_1(\Omega_n \beta) Y_o(\Omega_n) - J_o(\Omega_n) Y_1(\Omega_n \beta) = 0 \tag{18}$$

It is not difficult to show that the functions $U(\Omega_n, R)$ form an orthogonal system on $[\beta, 1]$, with weight R.

$$\int_{\beta}^{1} R\, U(\Omega_n, R) U(\Omega_m, R)\, dR = B\, \delta_{nm} \tag{19}$$

where δ_{nm} is the Kronecker delta, and

$$B = -\frac{1}{2}\beta^2 \left[J_o(\Omega_n \beta) Y_o(\Omega_n) - J_o(\Omega_n) Y_o(\Omega_n \beta) \right]^2 + \frac{2}{\pi^2}\frac{1}{\Omega_n^2} \tag{20}$$

The solution of the forced vibration problem is written in the form of a Fourier-Bessel expansion as

$$W(R,\tau) = \sum_{n=0}^{\infty} T_n(\tau) U(\Omega_n, R) \tag{21}$$

where the sum is taken over the positive roots of the frequency equation (17). In Eq. (21) $T_n(\tau)$ are time dependent Fourier-Bessel coefficients which are, as yet, unknown functions of time. The forcing function $N(\tau)$ is also expanded in terms of $U(\Omega_n, R)$. For the suddenly applied uniform body-force distribution the Fourier-Bessel coefficients A_n of $N H(\tau)$ are obtained as

$$A_n = \pi N J_1^2(\Omega_n \beta) \Big/ \left[J_1^2(\Omega_n \beta) - J_o^2(\Omega_n) \right] \tag{22}$$

By substituting the Fourier-Bessel expansions of $W(R,\tau)$ and $N H(\tau)$ into the governing equation (10), and by assuming that term by term differentiation is allowable, it is found that $T_n(\tau)$ satisfies the ordinary differential equation

$$\ddot{T}_n + q^2 \Omega_n^2 T - A_n = 0 \tag{23}$$

The solution of Eq. (23) that satisfies the initial conditions (13) is

$$T_n(\tau) = -(A_n/\Omega_n^2 q^2)\cos(\Omega_n q\tau) + A_n/\Omega_n^2 q^2 \tag{24}$$

By substituting Eq. (24) and Eq. (17) into Eq. (21) the displacement solution of the problem at hand is obtained as

$$\frac{q^2}{\pi N} W(R,\tau) =$$

$$-\sum_{n=0}^{\infty} \frac{J_1^2(\Omega_n \beta)\cos(\Omega_n q\tau)\left[J_o(\Omega_n R) Y_o(\Omega_n) - J_o(\Omega_n) Y_o(\Omega_n R) \right]}{\Omega_n^2 \left[J_1^2(\Omega_n \beta) - J_o^2(\Omega_n) \right]}$$

$$+\sum_{n=0}^{\infty} \frac{J_1^2(\Omega_n \beta)\left[J_o(\Omega_n R) Y_o(\Omega_n) - J_o(\Omega_n) Y_o(\Omega_n R) \right]}{\Omega_n^2 \left[J_1^2(\Omega_n \beta) - J_o^2(\Omega_n) \right]} \tag{25}$$

We also consider the static solution of the present problem. The static solution $W^*(R)$ satisfies the equation

$$\frac{d^2 W^*}{dR^2} + \frac{1}{R}\frac{dW^*}{dR} = -\frac{N}{q^2} H(\tau) \tag{26}$$

The solution of Eq. (26) that satisfies the boundary conditions (15) is easily obtained as

$$W^*(R) = (N/2q^2)\left[\beta^2 \ln(R) - (1/2)(R^2 - 1) \right] H(\tau) \tag{27}$$

It can now be shown that the second expansion in

Eq. (25) is the Fourier-Bessel series with respect to the system (17) of $(q^2/\pi N)$ times the static solution $W^*(R)$. In the usual fashion the solution of the dynamic problem, Eq. (25), thus consists of a vibration about the static equilibrium position. We write

$$W(R,\tau) = W^*(R) + \bar{W}(R,\tau) \tag{28}$$

Of particular interest in the present problem is the dynamic overstress at the cylinder-casing interface. Let the dimensionless shear stress Σ_{rz} be defined as

$$\Sigma_{rz} = \sigma_{rz}/\rho F_z b \tag{29}$$

The dynamic overstress can then be written as

$$\bar{\Sigma}_{rz} = (q^2/N)\partial\bar{W}/\partial R \tag{30}$$

By employing Eq. (25) the shear-bond overstress at $R = 1$ is determined as

$$\bar{\Sigma}_{rz} = 2\sum_{n=0}^{\infty} \frac{J_1^2(\Omega_n\beta)\cos(\Omega_n q\tau)}{\Omega_n^2\left[J_1^2(\Omega_n\beta) - J_o^2(\Omega_n)\right]} \tag{31}$$

In Eq. (31) the following identity from the theory of Bessel functions was used[10]

$$J_1(z)Y_o(z) - J_o(z)Y_1(z) = 2/\pi z \tag{32}$$

IV. The Ablating Elastic Grain

In the previous section it is shown that for the non-ablating cylinder the displacement $W(R,\tau)$ can be expressed as the sum of the static solution $W^*(R)$ and a periodic function $\bar{W}(R,\tau)$. It is now assumed that for the ablating cylinder the displacement $W^A(R,\tau)$ can also be expressed in the form

$$W^A(R,\tau) = W^{A*}(R,\tau) + \bar{W}^A(R,\tau) \tag{33}$$

In Eq. (33) $W^{A*}(R,\tau)$ is the solution of the quasi-static problem, i.e. the solution of Eq. (26) that satisfies the boundary conditions (14). If the body-force system is suddenly applied, we easily obtain

$$W^{A*}(R,\tau) = (N/2q^2)[\beta^2\alpha^2(\tau)\ln(R) - (1/2)(R^2 - 1)]H(\tau) \tag{34}$$

By substitution of Eq. (33) for $W^A(R,\tau)$ in Eq. (10) it is found that for $\tau > 0$ the unknown function $\bar{W}^A(R,\tau)$ satisfies the equation

$$q^2\left[\frac{\partial^2\bar{W}^A}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{W}^A}{\partial R}\right] = \frac{\partial^2\bar{W}^A}{\partial\tau^2} + \frac{\partial^2 W^{A*}}{\partial\tau^2} \tag{35}$$

The initial conditions on $\bar{W}^A(R,\tau)$ at $\tau = 0^+$ are obtained from Eqs. (33) and (34) as

$$\bar{W}^A(R,0^+) = -(N/2q^2)[\beta^2\ln(R) - (1/2)(R^2-1)] \tag{36a}$$

$$\dot{\bar{W}}^A(R,0^+) = -(N/2q^2)[2\beta^2\dot{\alpha}(0^+)\ln(R)] \tag{36b}$$

The function $\bar{W}^A(R,\tau)$ must also satisfy the boundary conditions (14).

Since the boundary conditions on $\bar{W}^A(R,\tau)$, Eq. (14), are prescribed on a time dependent boundary, the conventional methods of separation of variables and Fourier analysis break down. In this paper it is proposed to seek a solution of Eq. (35) in terms of erosive free vibration modes that are defined as

$$V_n\left[X_n(\tau),R\right] = J_o\left[X_n(\tau)R\right]Y_o\left[X_n(\tau)\right] - J_o\left[X_n(\tau)\right]Y_o\left[X_n(\tau)R\right] \tag{37}$$

Eq. (37) represents the modes of free vibration of the constant-radius problem, Eq. (17), if the time dependent functions $X_n(\tau)$ are replaced by the constants Ω_n. It is noted that Eq. (37) automatically satisfies

$$V_n\left[X_n(\tau),R\right] = 0 \qquad \text{for} \quad R = 1 \tag{38}$$

We now require

$$\partial V_n\left[X_n(\tau),R\right]/\partial R = 0 \quad \text{for} \quad R = \beta\alpha(\tau) \tag{39}$$

The condition (39) yields an equation for $X_n(\tau)$

$$J_1\left[X_n(\tau)\beta\alpha(\tau)\right]Y_o\left[X_n(\tau)\right] - J_o\left[X_n(\tau)\right]Y_1\left[X_n(\tau)\beta\alpha(\tau)\right] = 0 \tag{40}$$

Equation (40) has the appearance of a frequency equation, but the "frequencies" are time dependent functions $X_n(\tau)$. On a constant boundary $R = \beta$ the frequency equation (40) reduces, of course, to Eq. (18). The frequency equation (18) of the constant inner radius problem was considered by Baltrukonis,[1] who tabulated the eigenfrequencies $\bar{\Omega}_n = \Omega_n(1-\beta)$ for various values of β. For the first three modes the tabulated $\bar{\Omega}_n$ are shown as functions of β in Fig. 1. It can be noted that Ω_n increases for increasing β, where Ω_n approaches infinity as β approaches unity. The curves of Fig. 1 are used to determine the solutions of Eq. (40). For a certain time $\tau = \tau_1$ the solution $X_n(\tau_1)$ is obtained as $[1-\beta\alpha(\tau_1)]^{-1}$ times the ordinate which corresponds [illegible] the abscissus $\beta\alpha(\tau_1)$. In fact, since the curves in Fig. 1 are almost straight lines we are justified in employing a linear approximation. More specifically, we can write for $\beta \geq 0.333$

$$X_1(\tau) = [2.0954 - .53\beta\alpha(\tau)]\,/\,[1-\beta\alpha(\tau)] \tag{41a}$$

$$X_2(\tau) = [4.9577 - .248\beta\alpha(\tau)]\,/\,[1-\beta\alpha(\tau)] \tag{41b}$$

$$X_3(\tau) = [8.0033 - .151\beta\alpha(\tau)]\,/\,[1-\beta\alpha(\tau)] \tag{41c}$$

The expressions (41) are valid for $\beta\alpha(\tau) < 0.99$, i.e. the present considerations are pertinent till just before burnout.

A solution of Eq. (35) is now attempted in the form

$$\vec{W}^A(R,\tau) = \sum_n S_n(\tau)V_n(R,\tau) \tag{42}$$

It is noticed that the terms of $\vec{W}^A(R,\tau)$ are not classical separation of variables solutions, since both $S_n(\tau)$ and $V_n(R,\tau)$ are functions of the dimensionless time τ. Nevertheless Eq. (42) is substituted in Eq. (35). By assuming that the nth term of the expansion satisfies Eq. (35), the substitution results in

$$- q^2X_n^2(\tau)S_n(\tau)V_n(R,\tau) =$$

$$= \frac{\partial^2}{\partial\tau^2}\left[S_n(\tau)V_n(R,\tau)\right] + Q(\tau)D_nU(\Omega_n,R) \tag{43}$$

The last term in Eq. (43) is yielded by the Fourier-Bessel expansion of $\partial^2 W^{A*}/\partial\tau^2$ in terms of the modes of free vibration, Eq. (17), of the cylinder of constant inner radius. The function $Q(\tau)$ and the Fourier-Bessel coefficient D_n are derived as

$$Q(\tau) = (N\beta^2/q^2)[\dot{\alpha}(\tau)^2 + \alpha(\tau)\ddot{\alpha}(\tau)] , \tag{44}$$

and

$$D_n = (\pi/\Omega_n\beta)J_o(\Omega_n)J_1(\Omega_n\beta) \Big/ \left[J_1^2(\Omega_n\beta)-J_o^2(\Omega_n)\right] \tag{45}$$

In an attempt to eliminate R from Eq. (43) we divide through by $V_n(R,\tau)$. As was expected $V_n(R,\tau)$ does not simply cancel out, and an exact solution of the type (42) is apparently not possible. We shall now, however, restrict the analysis to the vicinity of $R = 1$. Using L'Hospital's rule it can be shown that

$$\lim_{R\to1} [\dot{V}_n/V_n] = 0 \tag{46a}$$

$$\lim_{R\to1} [\ddot{V}_n/V_n] = 0 \tag{46b}$$

$$\lim_{R\to1} [U(\Omega_n,R)/V_n] = 1 \tag{46c}$$

In view of the above limits we may write in the vicinity of $R = 1$, that is, near the bond interface,

$$\ddot{S}_n(\tau) + q^2X_n^2(\tau)S_n(\tau) = - Q(\tau)D_n \tag{47}$$

The function $X_n(\tau)$ is defined by Eq. (41) for the first three modes. It is apparent that in general Eq. (47) cannot be solved exactly for arbitrary $\alpha(\tau)$. But an advantageous feature of Eq. (47) is that the real parameter q is large. An approximate method for solving Eq. (47) for large q is available.

We shall first consider the homogeneous equation. An asymptotic solution of the homogeneous equation can be determined by Horn's method.[11] The solution which is obtained by Horn's method is in the form of a series of descending powers of q, which are asymptotic to exact solutions of the differential equation. By using Horn's method the two independent complementary solutions of Eq. (47) are obtained as[11]

$$S_{n1}(\tau) = \left[X_n(\tau)\right]^{-\frac{1}{2}} \exp\left[iq\int_0^\tau X_n(s)ds\right] + O(1/q) \tag{48a}$$

$$S_{n2}(\tau) = \left[X_n(\tau)\right]^{-\frac{1}{2}} \exp\left[-iq\int_0^\tau X_n(s)ds\right] + O(1/q) \tag{48b}$$

It is easy to check that the Wronskian of S_{n1} and S_{n2} is $-2iq$. By using a well known theorem[11] the approximate solution of Eq. (47) is obtained as

$$S_n(\tau) = \left[X_n(\tau)\right]^{-\frac{1}{2}}\Big\{C_n^1 \sin\left[q\int_0^\tau X_n(s)ds\right] +$$

$$+ C_n^2 \cos\left[q\int_0^\tau X_n(s)ds\right]\Big\} + P(\tau) \tag{49a}$$

where

$$P(\tau) = \frac{-1}{2iq}\int_0^\tau\left[S_1(\tau)S_2(\xi) - S_2(\tau)S_1(\xi)\right]Q(\xi)d\xi \tag{49b}$$

A simplification of Eq. (49b) is achieved by introducing the new variable

$$v = \int_0^\xi X_n(s)ds \tag{50}$$

Eq. (49b) can then be rewritten as

$$P(\tau) = - \frac{1}{q}\left[X_n(\tau)\right]^{-\frac{1}{2}} \mathrm{Im}\left\{e^{iqv_1}\int_0^{v_1}\phi(v)e^{-iqv}dv\right\} \tag{51a}$$

in which

$$v_1 = \int_0^\tau X_n(s)ds \tag{51b}$$

and

$$\phi(v) = X_n\left[\xi(v)\right]^{-3/2} Q\left[\xi(v)\right] \tag{51c}$$

In evaluating the integral in Eq. (51a) we again take advantage of the fact that the exponential $\exp(-iqv)$ contains the very large real parameter q. Integrals containing such an exponential are suited for evaluation by the method of stationary phase.[11] The method of stationary phase entails the replacement of the real integral by a contour integral along the lines $v = -iw$ and $v = v_1 - iw$. Since q is a large real parameter the exponential $\exp(-qw)$ dies out very rapidly and the main contributions to the integral come from near $v = 0$ and $v = v_1$. The approximate evaluation of Eq. (51a) yields

$$P(\tau) = + q^{-2}X_n(\tau)^{-\frac{1}{2}}X_n(0^+)^{-3/2}Q(0^+)\cos\left[q\int_0^\tau X_n(s)ds\right] -$$

$$- q^{-2}X_n(\tau)^{-2}Q(\tau) + O(1/q) \tag{52}$$

The constants C_n^1 and C_n^2 in Eq. (49a) are determined from the initial conditions on $S_n(\tau)$. It is seen from Eq. (49b) that $P(0) = \dot{P}(0) = 0$. The initial value of $\overline{W}^A(R,\tau)$, Eq. (36), is then

$$\overline{W}^A(R,0) = \sum_n C_n^2 \left[X_n(0)\right]^{-\frac{1}{2}} V_n(R,0) \tag{53a}$$

Also, by neglecting terms of order $O(1/q)$, we derive

$$\dot{\overline{W}}^A(R,0) = \sum_n C_n^1 \, q \left[X_n(0)\right]^{+\frac{1}{2}} V_n(R,0) \tag{53b}$$

The initial conditions on $\overline{W}^A(R,\tau)$ are prescribed by Eq. (36). The expressions in Eqs. (36a) and (36b) are in terms of R, and they can be expanded in Fourier-Bessel series of $U(\Omega_n,R)$. It was observed before that $U(\Omega_n,R) = V_n(R,0)$, and the constants C_n^2 and C_n^1 can be obtained by equating Eqs. (53a) and (53b) to the Fourier-Bessel expansions of Eqs. (36a) and (36b) respectively. In this way the constant C_n^2 is obtained as

$$C_n^2 = -\frac{N\pi}{q^2} \frac{1}{\Omega_n^{3/2}} \frac{J_1^2(\Omega_n\beta)}{J_1^2(\Omega_n\beta) - J_o^2(\Omega_n)} \tag{54}$$

The constant C_n^1 is also easily obtained. It turns out that C_n^1 is of order $O(1/q)$ as compared to C_n^2, and C_n^1 is therefore neglected. Inspection of Eqs. (52), (44) and (54) shows that $P(\tau)$ is of order $O(1/q^2)$ as compared to C_n^2, thus $P(\tau)$ is also neglected. In the vicinity of $R = 1$ the function $\overline{W}^A(R,\tau)$ may then be written as

$$\overline{W}^A(R,\tau) = -\frac{N\pi}{q^2} \sum_{n=0}^{\infty} C(\tau) V_n(R,\tau) \cos\left[q \int_0^\tau X_n(s)ds\right] \tag{55}$$

where

$$C(\tau) = \frac{X_n(\tau)^{-\frac{1}{2}}}{\Omega_n^{3/2}} \frac{J_1^2(\Omega_n\beta)}{J_1^2(\Omega_n\beta) - J_o^2(\Omega_n)} \tag{56}$$

The dynamic overstress at the shear-bond interface $R = 1$ is finally obtained as

$$\overline{\Sigma}^A(R,\tau) = (q^2/N)\partial\overline{W}^A/\partial R$$

$$= 2\sum_{n=0}^{\infty} C(\tau)\cos\left[q\int_0^\tau X_n(s)ds\right] \tag{57}$$

Eq. (57) properly reduces to Eq. (31) for a non-ablating cylinder when $X_n(\tau) = X_n(0) = \Omega_n$.

V. The Ablating Viscoelastic Grain

The equation that governs the dynamic response of a viscoelastic core is derived in Section II, see Eq. (11). By removing the discontinuity at $\tau = 0$, and by integrating by parts Eq. (11) is rewritten as

$$\frac{q^2}{R}\frac{\partial}{\partial R}\left[R\frac{\partial W}{\partial R}\right] + \frac{q^2}{R}\frac{\partial}{\partial R}\left[R\int_{0^+}^{\tau} g'(\tau-s)\frac{\partial W}{\partial R}ds\right] +$$

$$+ N H(\tau) = \frac{\partial^2 W}{\partial\tau^2} \tag{58}$$

The dimensionless parameter q is defined by Eq. (9c), where G now denotes the glassy shear modulus of the viscoelastic material. It is noted that the body-force distribution is uniform and suddenly applied.

A linear viscoelastic material can in general be characterized by a discrete spectrum of relaxation times τ_i. The relaxation function $G(\tau)$ may then be expressed as

$$G(\tau) = G_R + \sum_{i=1}^{\ell} G_i \exp(-\tau/\tau_i) \tag{59}$$

In Eq. (59) G_R denotes the rubbery shear modulus. By comparison with Eq. (12) the function $g(\tau)$ is obtained as

$$g(\tau) = (G_R/G) + \sum_{i=1}^{\ell} (G_i/G)\exp(-\tau/\tau_i) \tag{60}$$

In this paper we shall consider viscoelastic materials whose relaxation functions in shear show a rapid decrease for very short times, and then a gradual decrease to the rubbery shear modulus. In terms of the relaxation spectrum this means that the discrete relaxation spectrum consists of a number of very small relaxation times $(1 < i \le \ell_1)$ and a number of larger relaxation times $(\ell_1 < i \le \ell)$. More specifically we assume that the viscoelastic material can be characterized by a discrete relaxation spectrum τ_i such that

$$(1/\tau_i) \gg q\Omega_1 \quad \text{for} \quad 1 \le i \le \ell_1 \tag{61a}$$

and

$$(1/\tau_i) \ll q\Omega_1 \quad \text{for} \quad \ell_1 < i \le \ell \tag{61b}$$

In Eqs. (61a) and (61b) $q\Omega_1$ is the first natural frequency of an elastic core with the glassy modulus as shear modulus. The function $g(\tau)$, Eq. (60), is now rewritten as

$$g(\tau) = (G_R/G) + g_1(\tau) + g_2(\tau) \tag{62}$$

In Eq. (62) $g_1(\tau)$ is the summation of exponentials over the very short relaxation times, Eq. (61a), and $g_2(\tau)$ covers the relaxation times defined by Eq. (61b). The expression (62) is differentiated and subsequently substituted for $g'(\tau-s)$ in Eq. (58). In view of the stipulation (61a) the integral containing $g_1'(\tau-s)$ can be simplified.

$$\int_{0^+}^{\tau} g_1'(\tau-s)\frac{\partial W}{\partial R}ds \cong -\frac{\partial W(R,\tau)}{\partial R}\sum_{i=1}^{\ell_1} G_i/G \tag{63}$$

Eq. (58) can then be rewritten as

$$\frac{p^2}{R}\frac{\partial}{\partial R}\left[R\frac{\partial W}{\partial R}\right]+\frac{q^2}{R}\frac{\partial}{\partial R}\left[R\int_{0^+}^{\tau}g'(\tau-s)\frac{\partial W}{\partial R}\,ds\right]+$$

$$+\ NH(\tau)=\frac{\partial^2 W}{\partial\tau^2} \tag{64}$$

where

$$p^2=q^2\left[1-\sum_{i=1}^{\ell_1}(G_i/G)\right] \tag{65}$$

We shall again seek a displacement solution in the form of an oscillation about the quasi-static displacement $W_V^{A*}(R,\tau)$.

$$W_V^A(R,\tau)=W_V^{A*}(R,\tau)+\bar{W}_V^A(R,\tau) \tag{66}$$

By substitution of Eq. (66) for $W(R,\tau)$ in Eq. (64) the governing equation for $\bar{W}_V^A(R,\tau)$ is obtained as

$$\frac{p^2}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\bar{W}_V^A}{\partial R}\right]+\frac{q^2}{R}\frac{\partial}{\partial R}\left[R\int_{0^+}^{\tau}g'(\tau-s)\frac{\partial\bar{W}_V^A}{\partial R}\,ds\right]=$$

$$=\frac{\partial^2\bar{W}_V^A}{\partial\tau^2}+\frac{\partial^2 W_V^{A*}}{\partial\tau^2} \tag{67}$$

The initial conditions on $\bar{W}_V^A(R,\tau)$ follow from the values of the quasi-static displacement and its time derivative at $\tau=0^+$. The quasi-static viscoelastic displacement is discussed next.

The stress σ_{rz} for the quasi-static problem is governed by Eq. (4) if the right-hand side of that equation is zero. The dimensionless quasi-static stress that also satisfies the boundary conditions (14) is easily obtained as

$$\Sigma_{rz}(R,\tau)=(1/2)\left[\beta^2\alpha^2(\tau)/R-R\right]H(\tau) \tag{68}$$

It is noted that Eq. (68) does not contain an elastic constant and the stress solution is thus independent of the material behavior. It follows that Eq. (68) is also the quasi-static stress for a viscoelastic grain. It was pointed out elsewhere that the quasi-static viscoelastic strain is then formally obtained by means of the creep function as

$$\partial W_V^{A*}/\partial R=\int_{0^-}^{\tau}D(\tau-s)\,d\Sigma(R,s) \tag{69}$$

In Eq. (69) $D(\tau)$ is the dimensionless creep function. Integration with respect to R yields the quasi-static displacement $W_V^{A*}(R,\tau)$.

$$W_V^{A*}=\frac{1}{2}\int_{0^-}^{\tau}D(\tau-s)\,d\left[\beta^2\alpha^2(s)\ \ln(R)-\frac{1}{2}(R^2-1)\right] \tag{70}$$

The viscoelastic material can also be characterized by a discrete spectrum of retardation times. This characterization implies that the creep function may be written as

$$D(\tau)=(N/q^2)\left[1+\sum_{j=1}^{k}d_j(1-e^{-\tau/\tau_j})\right] \tag{71}$$

In Eq. (71) d_i are dimensionless constants and τ_j are retardation times. The assumption that the relaxation spectrum consists of a number of very small relaxation times and a number of larger relaxation times implies a similar distribution of retardation times in the retardation spectrum. Let $D_2(\tau)$ cover the exponentials with retardation times much larger than the period associated with the first natural frequency of shear vibrations. Following the same procedure which yielded Eq. (67), the quasi-static viscoelastic solution can be rewritten as

$$W_V^{A*}=\frac{M}{2q^2}\left[\beta^2\alpha^2(\tau)\ \ln(R)-\frac{1}{2}(R^2-1)\right]H(\tau)+$$

$$+\int_{0^+}^{\tau}D_2'(\tau-s)\,d\Sigma(R,s) \tag{72}$$

where

$$M=N+\sum_{j=1}^{i}d_i \tag{73}$$

The initial conditions on $\bar{W}_V^A$ are now determined from Eqs. (66) and (72). It is noted that by this procedure the influence of the very short retardation times is expressed in the initial conditions on $\bar{W}_V^A$ rather than in the forcing term $\partial^2 W_V^{A*}/\partial\tau^2$ on the right-hand side of Eq. (67). As in the elastic problem (section IV) the particular solution due to the forcing function is of order $O(1/\tau_k^2 q^2)$ as compared to the solution due to the initial conditions. Since the remaining part of the creep function $D_2(\tau)$ contains only retardation times much larger than the period of vibration we may completely neglect the influence of the forcing function. Eq. (67) can thus be simplified to

$$\frac{p^2}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\bar{W}_V^A}{\partial R}\right]+\frac{q^2}{R}\frac{\partial}{\partial R}\left[R\int_{0^+}^{\tau}g'(\tau-s)\frac{\partial\bar{W}_V^A}{\partial R}\,ds\right]=$$

$$=\frac{\partial^2\bar{W}_V^A}{\partial\tau^2} \tag{74}$$

Guided by the solution of the analogous elastic problem, Eqs. (42) and (49a), Horn's method is extended and a solution of Eq. (74) is sought in the form

$$\left[\bar{W}_V^A\right]_n=\left[S_V(\tau)\right]_n V_n(R,\tau) \tag{75}$$

where $V_n(R,\tau)$ is defined by Eq. (37), and

$$\left[s_V(\tau)\right]_n = \{m(\tau)\cos[q\omega(\tau)] +$$

$$+ [p(\tau)/q]\sin[q\omega(\tau)\}f(\tau,q) \quad (76a)$$

In Eq. (76a)

$$f(\tau,q) = 1 + \sum_{j=1}^{\infty} f_j(\tau)/q^j \quad (76b)$$

As in the elastic problem all terms which contain q in the denominator are neglected in first approximation,

$$\left[\vec{W}_V^A\right]_n = m(\tau)\cos[q\omega(\tau)]\, V_n(R,\tau) \quad (77)$$

The postulated solution Eq. (77) is substituted into the integrodifferential equation Eq. (74). For the nth term this results in the equation

$$- p^2 X_n^2(\tau)\, m(\tau)\cos[q\omega(\tau)]\, V_n(R,\tau) -$$

$$- q^2 \int_{0^+}^{\tau} g_0'(\tau-s)\, X_n^2(s)\, m(s)\cos[q\omega(s)]\, V_n(R,s)ds =$$

$$= \frac{\partial^2 \vec{W}_V^A}{\partial\tau^2} \quad (78)$$

In evaluating the integral in Eq. (78) we take advantage of the stipulation that $g_0(\tau)$ covers the part of the relaxation spectrum that consists of larger relaxation times, Eq. (61b). By integration by parts, and by invoking the stipulation Eq. (61b) the integral is evaluated as

$$\int_{0^+}^{\tau} g_0'(\tau-s)\, X_n^2(s)\, m(s)\cos[q\omega(s)]\, V_n(R,s)ds \simeq$$

$$\simeq \frac{m(\tau)\, X_n^2(\tau)\, g_0'(0)}{q\dot{\omega}(\tau)}\sin[q\omega(\tau)]\, V_n(R,\tau) \quad (79)$$

The right-hand side of Eq. (78) is obtained by straightforward differentiation of Eq. (75). As in solving the elastic problem Eq. (78) is then divided through by $V_n(R,\tau)$, and the domain of interest is narrowed to the vicinity of $R = 1$. In view of the limits (46a) and (46b) the terms containing $\partial V_n(R,\tau)/\partial\tau$ and $\partial^2 V_n(R,\tau)/\partial\tau^2$ drop out.

In the vicinity of $R = 1$ Eq. (78) reduces to

$$- p^2 X_n^2(\tau)\, m(\tau)\cos[q\omega(\tau)] -$$

$$- q\, m(\tau)\, X_n^2(\tau) g_0'(0)\sin[q\omega(\tau)] / \dot{\omega}(\tau) =$$

$$= \ddot{m}(\tau)\cos[q\omega(\tau)] - 2q\,\dot{m}(\tau)\,\dot{\omega}(\tau)\sin[q\omega(\tau)] -$$

$$- q\, m(\tau)\,\ddot{\omega}(\tau)\sin[q\omega(\tau)] -$$

$$- q^2\, m(\tau)\,\dot{\omega}(\tau)^2\cos[q\omega(\tau)] \quad (80)$$

Following the general procedure of Horn's method[11] the coefficients of terms of order q^2 and q are taken to vanish separately. This results in

$$\dot{\omega}(\tau) = p\, X_n(\tau)/q\ , \quad (81a)$$

and in the ordinary differential equation

$$2\,\dot{m}\,\dot{\omega} + m\,\ddot{\omega} - g_0'(0)\, m\, X_n^2 / \dot{\omega} = 0 \quad (81b)$$

Eq. (81b) is satisfied by

$$m(\tau) = K_n \exp\left[q^2 g_0'(0)\tau/2p^2\right] \Big/ X_n^{\frac{1}{2}}(\tau) \quad (82)$$

where K_n is a constant. By substitution of Eqs. (81a) and (82) into Eq. (77) we obtain

$$\left[\vec{W}_V^A\right]_n = m(\tau)\cos\left[p\int_0^{\tau} X_n(s)ds\right] V_n(R,\tau) \quad (83)$$

It is noted that in first approximation the damping is primarily determined by the larger relaxation times. The very short relaxation times influence the frequency of the vibration. For $g_0(0) = 0$ and $p = q$ the solution reduces to the elastic solution.

The complete solution $W_V^A(R,\tau)$ consists of a summation over n modes of the type (83). The constants K_n are determined from the initial condition on the displacement. For the viscoelastic problem the displacement at $\tau = 0^+$ is obtained from Eq. (72). It is noticed that this initial value differs by a multiplicative constant N/M from the displacement at $\tau = 0^+$ of the previously considered elastic problem. It follows that the viscoelastic displacement may be written as

$$W_V^A(R,\tau) = -\frac{M\pi}{q^2}\sum_n K(\tau)\, V_n(R,\tau)\cos\left[p\int_0^{\tau} X_n(s)ds\right] \quad (84)$$

where

$$K(\tau) = C(\tau)\exp\left[q^2 g_0'(0)\tau/p^2\right] \quad (85)$$

and $C(\tau)$ is defined by Eq. (56). Similarly the bond-stress at $R = 1$ is obtained as

$$\Sigma_V^A(1,\tau) = 2(Mp^2/Nq^2)\sum_n K(\tau)\cos\left[p\int_0^{\tau} X_n(s)ds\right] \quad (86)$$

VI. Discussion of the Results

The axial shear vibrations of an encased elastic cylinder of constant inner radius a_0, and of elastic and viscoelastic cylinders of monotonically increasing inner radii $a(t)$ were studied in this paper. The solutions of these three problems involved the assumption that the uniform body-force distribution is suddenly applied. The Heaviside unit function $H(\tau)$ is used because the dynamic effects are illustrated most effectively by a suddenly applied load. For many practical problems the load may, however, have a finite rise time. For such problems the solutions in this paper present upper limits on the intensity of the dynamic effects that may be expected.

The influence of a finite rise time can also be studied directly by modifying the present solutions. For the dynamic response of the elastic cylinder of constant inner radius a_o this can be done very easily by solving Eq. (23) for arbitrary time dependence of $A_n(\tau)$. It is easily shown that the amplitudes of the vibrations decrease with increasing rise time of the load. Inclusion of a finite rise time of the load, for the elastic cylinder with ablating inner surface is somewhat more troublesome, since it is then less obvious what terms can be ignored in Eq. (49a). Dynamic effects are, of course, less pronounced for an increasing rise time. More care must also be exercised in considering a gradually applied load in the viscoelastic cylinder. If the load is suddenly applied the influence of the short relaxation times may be included in the initial conditions, Eq. (72). In this way the damping influence on the dynamic solution of the very small relaxation times is ignored. If the load is gradually applied the influence of the short relaxation times further diminishes. A complication arises, however, when, depending on the rise time, more terms are needed in Eq. (76a).

For the elastic grain of constant inner radius $R = \beta$ the shear-bond overstress at the propellant casing interface, Eq. (31), is computed for $\beta = 0.4$, $\beta = 0.6$ and $\beta = 0.8$. The eigenfrequencies Ω_n, Eq. (18), corresponding to the different values of β, can conceivably be obtained from Fig. 1. The difference of Bessel functions in Eq. (31) is, however, very sensitive to small deviations in Ω_n, and more accurate values of Ω_n[12] are therefore used to compute the Bessel functions in Eq. (31) and in Eq. (56). It is found that the amplitudes of the modes of Σ_{rz} decrease rapidly and it is sufficient to retain only the first three modes. In Fig. 2 the sum of the first three modes and the first mode of the shear-bond overstress are shown for $\beta = 0.4$. The sum of the first three modes of the shear-bond overstress is shown in Fig. 3 for $\beta = 0.6$ and $\beta = 0.8$. It is noted that for greater values of β the amplitudes are smaller and the frequencies are greater. The relatively soft material of the cylinder is defined by $q = 5.10^3$, Eq. (9c).

The shear-bond overstress at $R = 1$ for the ablating elastic grain is examined for various ablation rates. The influence of the ablation function $\alpha(\tau)$ is shown by considering the functions

$$\alpha(\tau) = 1 + \lambda\tau \quad , \text{ where } \lambda = (1/\beta) - 1 \tag{87a}$$

and

$$\alpha(\tau) = (1-\kappa\tau)^{-\frac{1}{2}} \quad , \text{ where } \kappa = 1-\beta^2 \tag{87b}$$

For $\beta = 0.4$ the ablation functions (87a) and (87b) are shown in Fig. 4.

From the expression for $\overline{\Sigma}^A(R,\tau)$, Eq. (57), it is noted that the amplitudes of the modes decrease and the frequencies increase towards burnout. We define the frequency function

$$\phi_{ni} = \int_0^\tau X_n(s)ds \tag{88}$$

The subscripts of ϕ_{ni} refer to the modes ($n=1,2,3$) and the ablation functions ($i=1,2,3$). More specifically, $i = 1$ indicates ablation according to Eq. (87a), $i = 2$ ablation according to Eq. (87b) and $i = 3$ indicates no ablation $\alpha(\tau) \equiv 1$. According to Eq. (41) the functions $X_n(\tau)$ may be expressed as

$$X_n(\tau) = [c_n - d_n\beta\alpha(\tau)]/[1-\beta\alpha(\tau)] \tag{89}$$

where the constants $c_{1,2,3}$ and $d_{1,2,3}$ are shown in Eqs. (41a,b,c). By substitution of Eqs. (87a), (87b) and (89) into Eq. (88) the frequency functions are evaluated as

$$\phi_{n1} = d_n\tau - [(c_n-d_n)/(1-\beta)]\,\ln\,(1-\tau) \tag{90a}$$

$$\begin{aligned}\phi_{n2} = &-[c_n/(1-\beta^2)][(1-\kappa\tau)^{\frac{1}{2}} + \beta]^2 + c_n(1+\beta)/(1-\beta) + \\ &+ 2d_n\beta[(1-\kappa\tau)^{\frac{1}{2}} - 1]/(1-\beta^2) - \\ &- [2\beta^2(c_n-d_n)/(1-\beta^2)]\,\ln\left\{[(1-\kappa\tau)^{\frac{1}{2}}-\beta]/(1-\beta)\right\}\end{aligned} \tag{90b}$$

$$\phi_{n3} = \Omega_n\tau \tag{90c}$$

The equations (90a) and (90b) indicate that the frequencies increase rapidly towards burnout as τ approaches unity. From Eq. (56) it is noted, however, that the amplitudes approach zero as τ approaches unity.

The frequency functions ϕ_{1i} for the first mode are shown in Fig. 5. The frequency functions are determined up to $\tau = 0.95$. It is noticed that linear ablation according to Eq. (87a) results in a much more rapid increase of the frequencies. The frequency functions ϕ_{ni} for higher modes are shown in Fig. 6.

The observations on amplitudes and frequencies are further illustrated by Figures 7, 8, 9 and 10. The figures 7 and 9 show the first mode of Eq. (57), respectively near $\tau = 0.5$ and $\tau = 0.9$. The sum of the first three modes of the shear-bond stress, Eq. (57), is shown for $\tau = 0.5$ and $\tau = 0.9$ in Fig. 8 and Fig. 10 respectively.

The results presented in this paper show that the frequencies of axial shear vibrations increase significantly during ablation. This effect may influence the structural integrity of the propellant-casing system.

References

1. Baltrukonis, J. H., Gottenberg, W. G., "Thickness-Shear Vibrations of Circular Bars," J.A.S.A. 31, 734 (1959).

2. Baltrukonis, J. H., Gottenberg, W. G. and Schreiner, R. N., "Dynamics of a Hollow, Elastic Cylinder Contained by an Infinitely Long Rigid Circular-Cylindrical Tank," J.A.S.A. 32, 1539 (1960).

3. Bird, J. F., Hart, R. W., and McClure, F. T., "Vibrations of Thick-Walled Hollow Cylinders -

Exact Numerical Solutions," J.A.S.A. 32, 1404-1412 (1960).

[4]Freudenthal, A. M., Henry, L. A., "Forced Vibrations of a Viscoelastic Cylinder Case-Bonded in a Thin Elastic Shell," AIAA Paper No. 65-173 (February 1965).

[5]Achenbach, J. D., "The Dynamic Response of an Encased Viscoelastic Cylinder," AIAA J. 3, 673-678 (1965).

[6]Achenbach, J. D., "The Dynamic Response of an Encased Elastic Cylinder with Ablating Inner Surface," AIAA J. 3, 1142-1144 (1965).

[7]Achenbach, J. D., "Forced Vibrations of a Burning Rocket," To be published as a Technical Note in AIAA J. (July 1965).

[8]Baltrukonis, J. H., Gottenberg, W. G., and Schreiner, R. N., "Axial Shear Vibrations of an Infinitely Long Composite Circular Cylinder," J.A.S.A. 33, 1447 (1961).

[9]Lindsey, G. H., Williams, M. L., "The Structural Integrity of an Ablating Rocket Subjected to Axial Acceleration," AIAA J. 3, 258-262 (1965).

[10]McLachlan, N. W., Bessel Functions for Engineers, Oxford University Press, London, 1961, p. 197.

[11]Jeffreys, H., Asymptotic Approximations, Oxford University Press, London, 1962, pp. 52, 46.

[12]Abramowitz, A., and Stegun, I. A. (ed.), Handbook of Mathematical Functions, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, 55, p. 415.

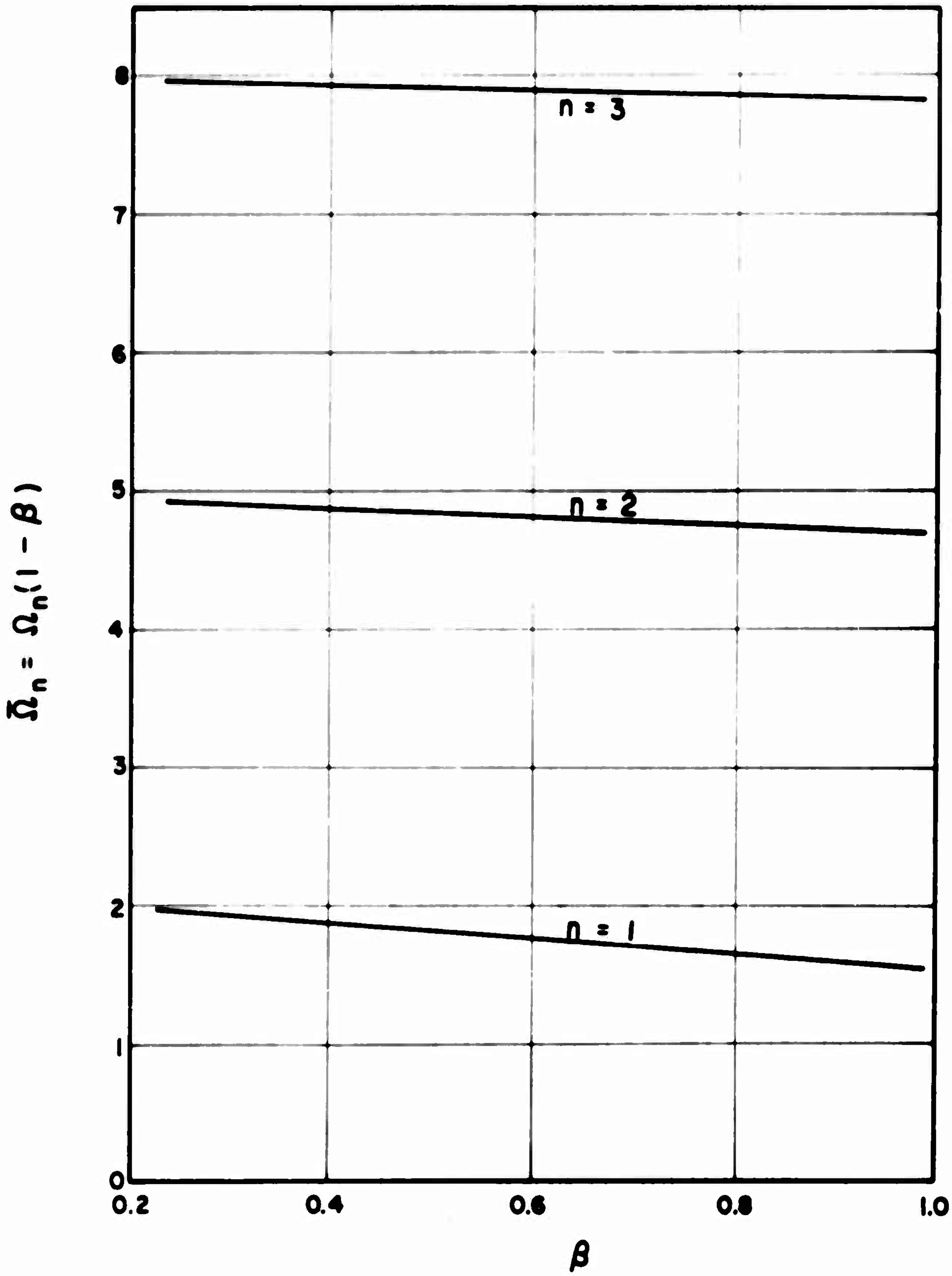

Fig. 1 Frequency coefficients $\bar{\Omega}_n$ versus β.

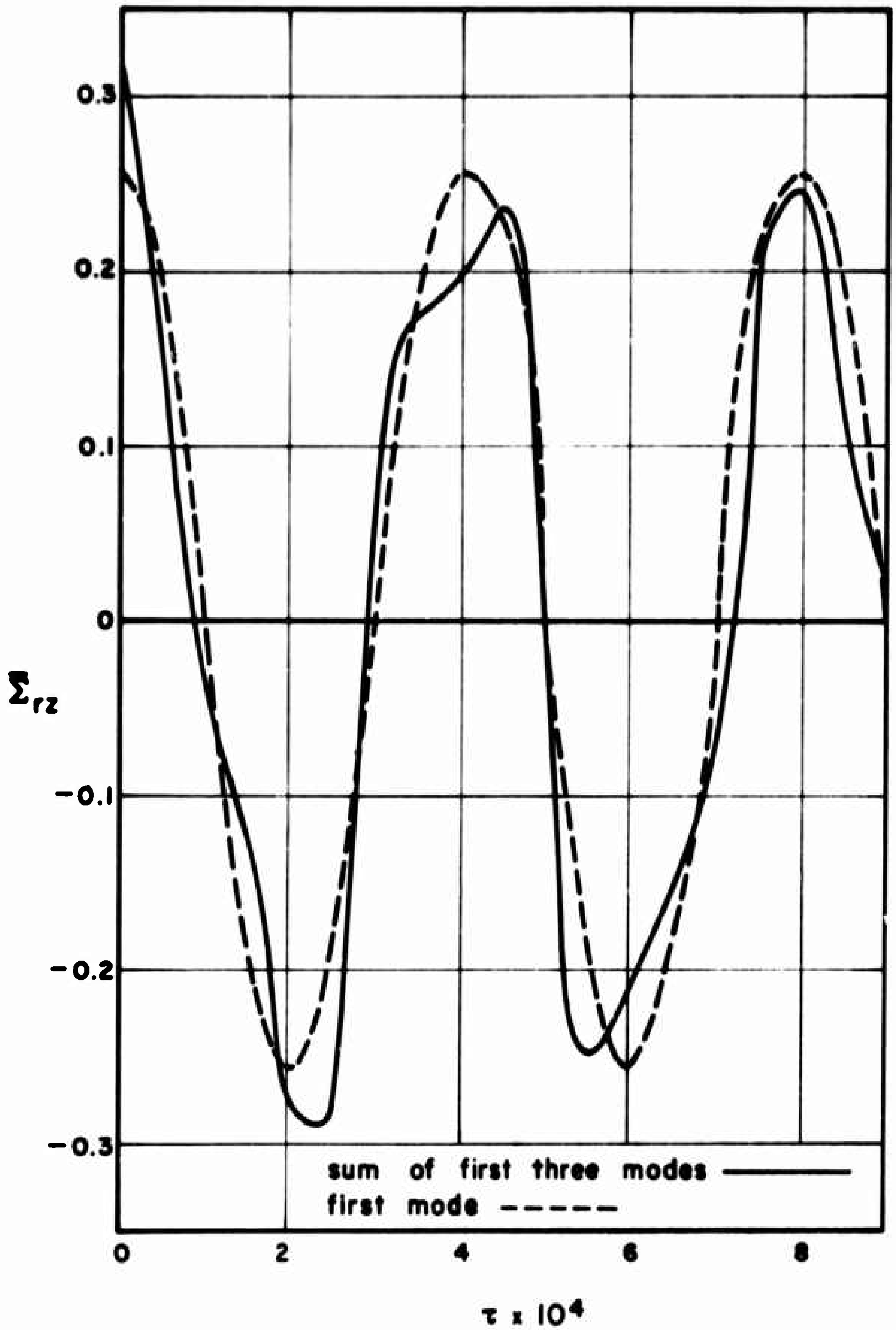

Fig. 2 Shear-bond overstress; no ablation $\beta = 0.4$

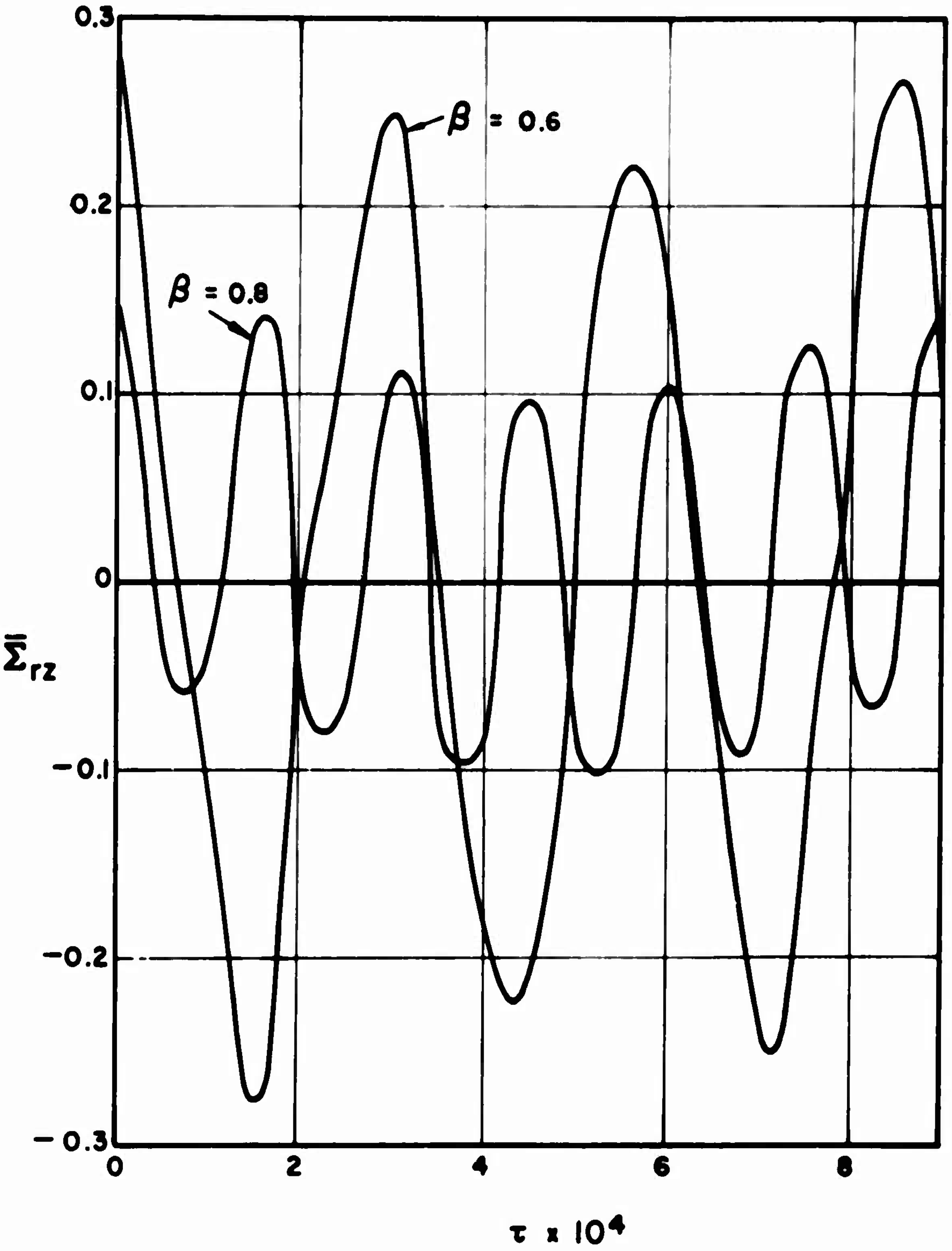

Fig. 3 Shear-bond overstress; no ablation $\beta = 0.6$ and $\beta = 0.8$

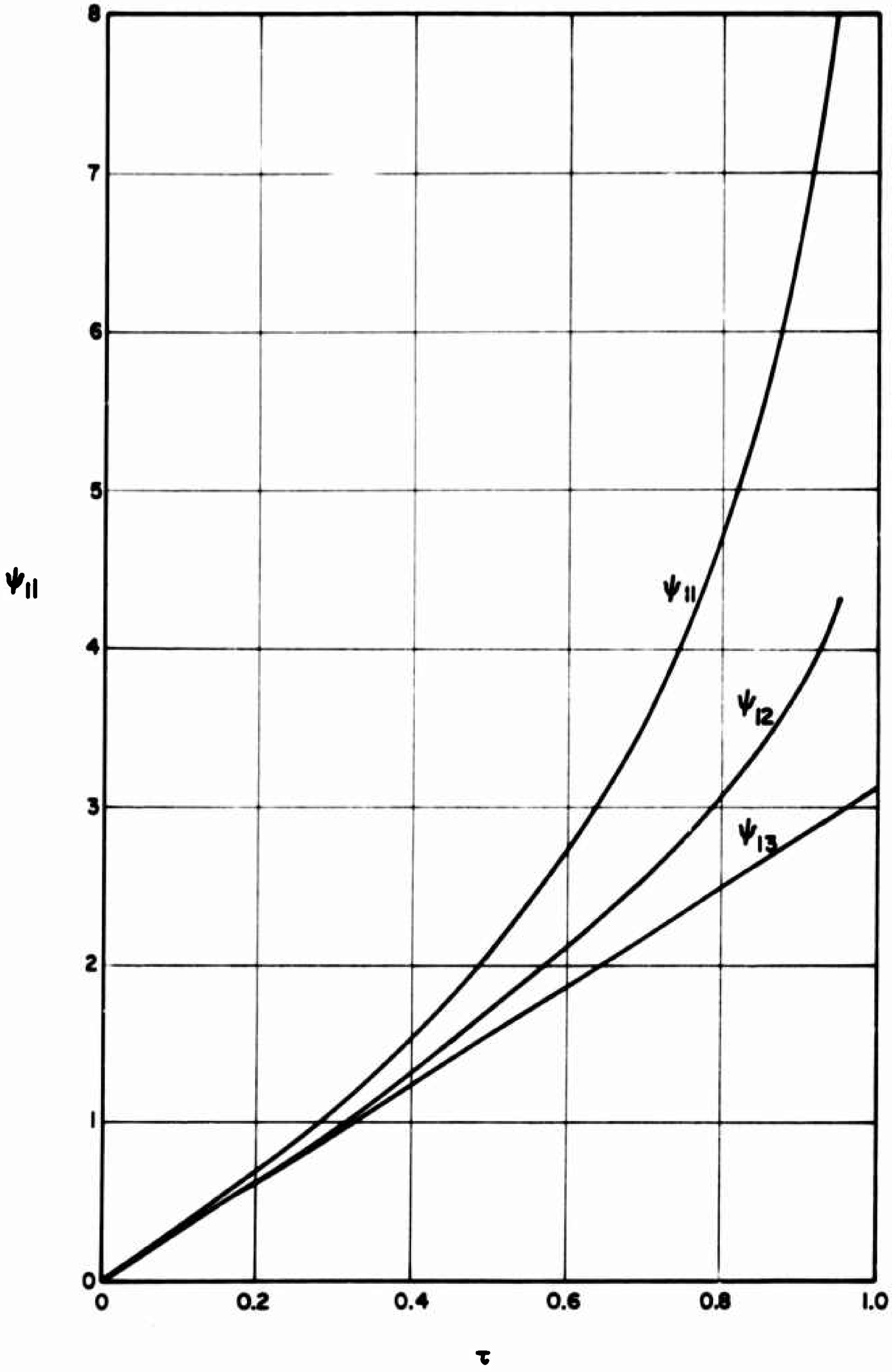

Fig. 5 Frequency function ψ_{11} versus τ

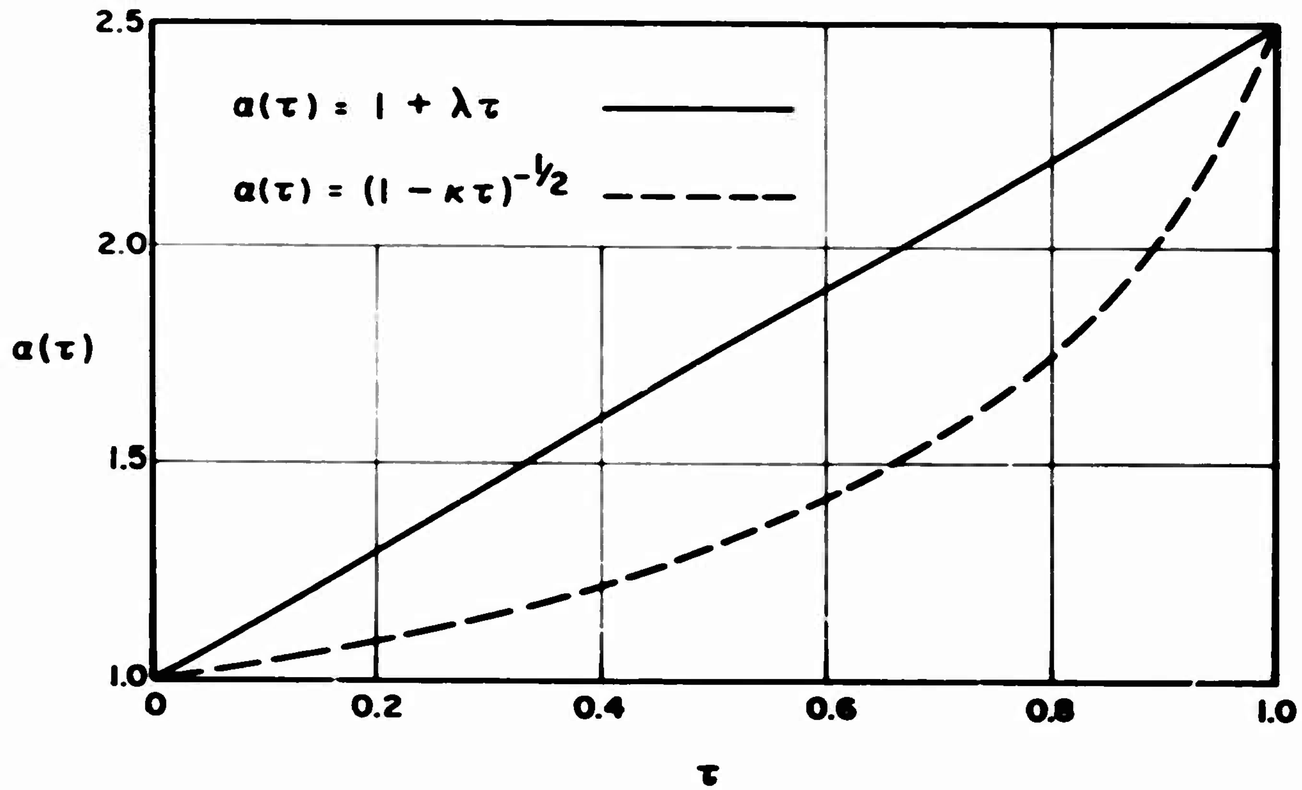

Fig. 4 Ablation function $a(\tau)$ versus τ for $\beta = 0.4$

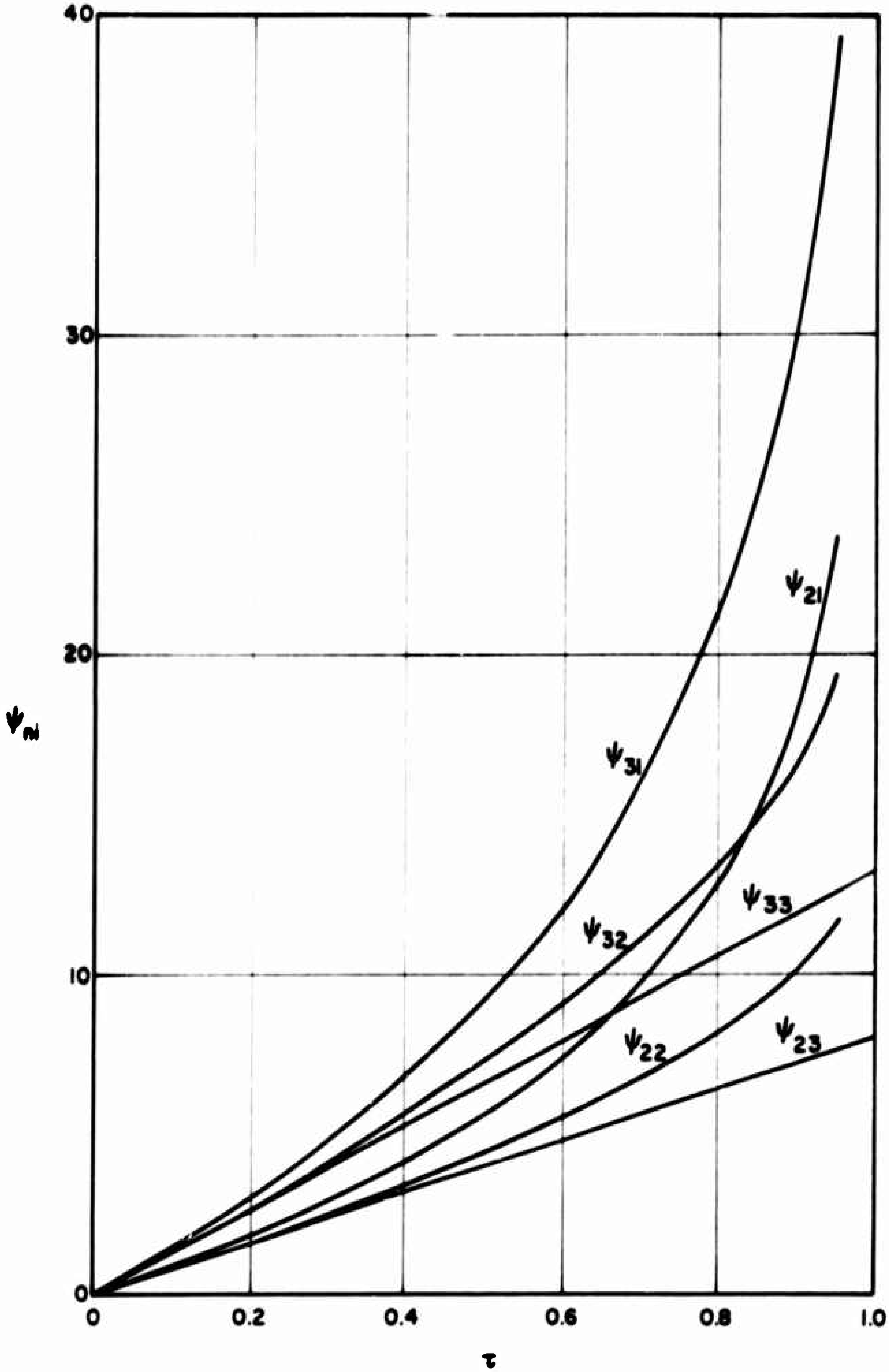

Fig. 6 Frequency functions ψ_{nl} versus τ for higher modes.

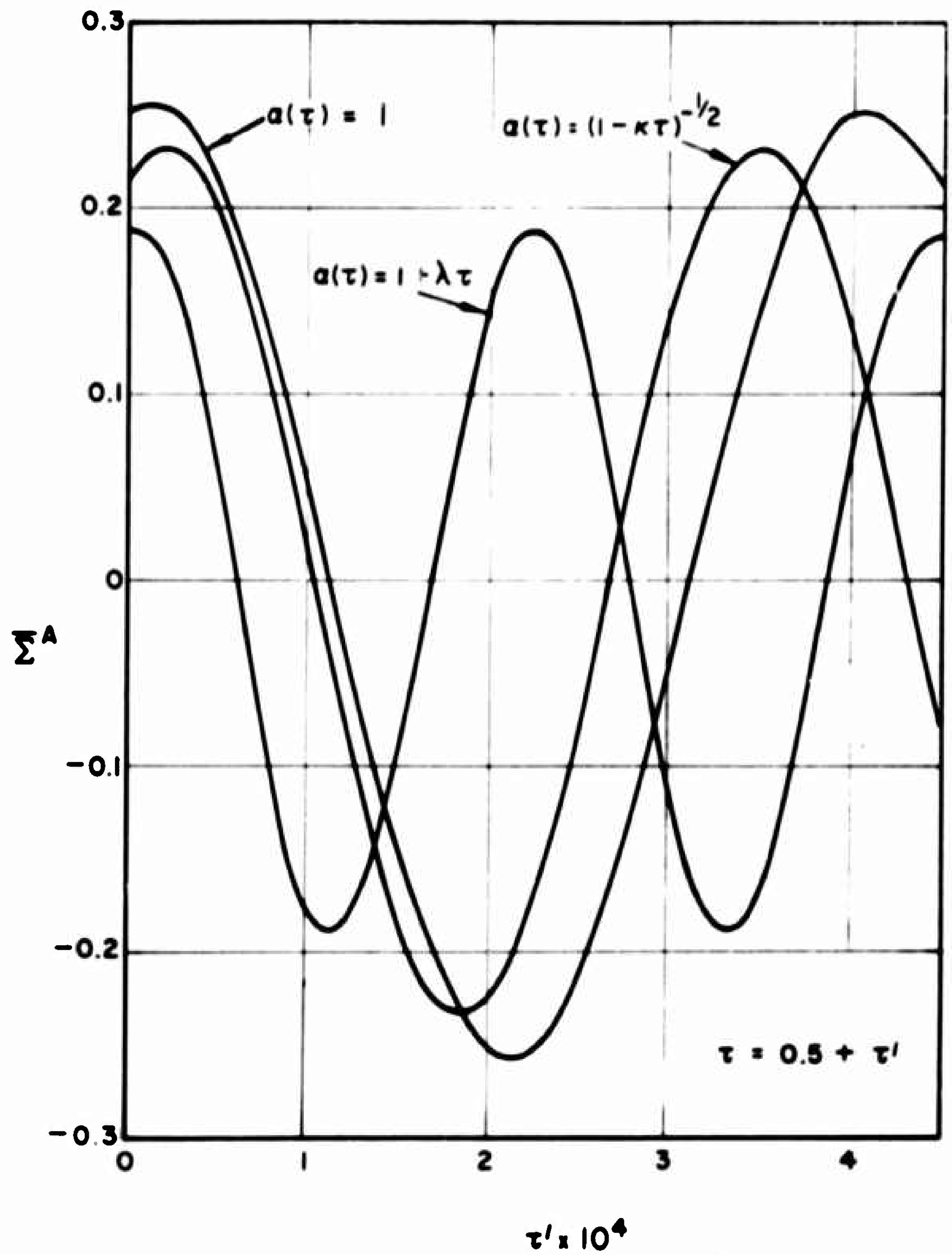

Fig. 7 The first mode of the shear-bond overstress for various ablation functions; $\tau = 0.5 + \tau'$.

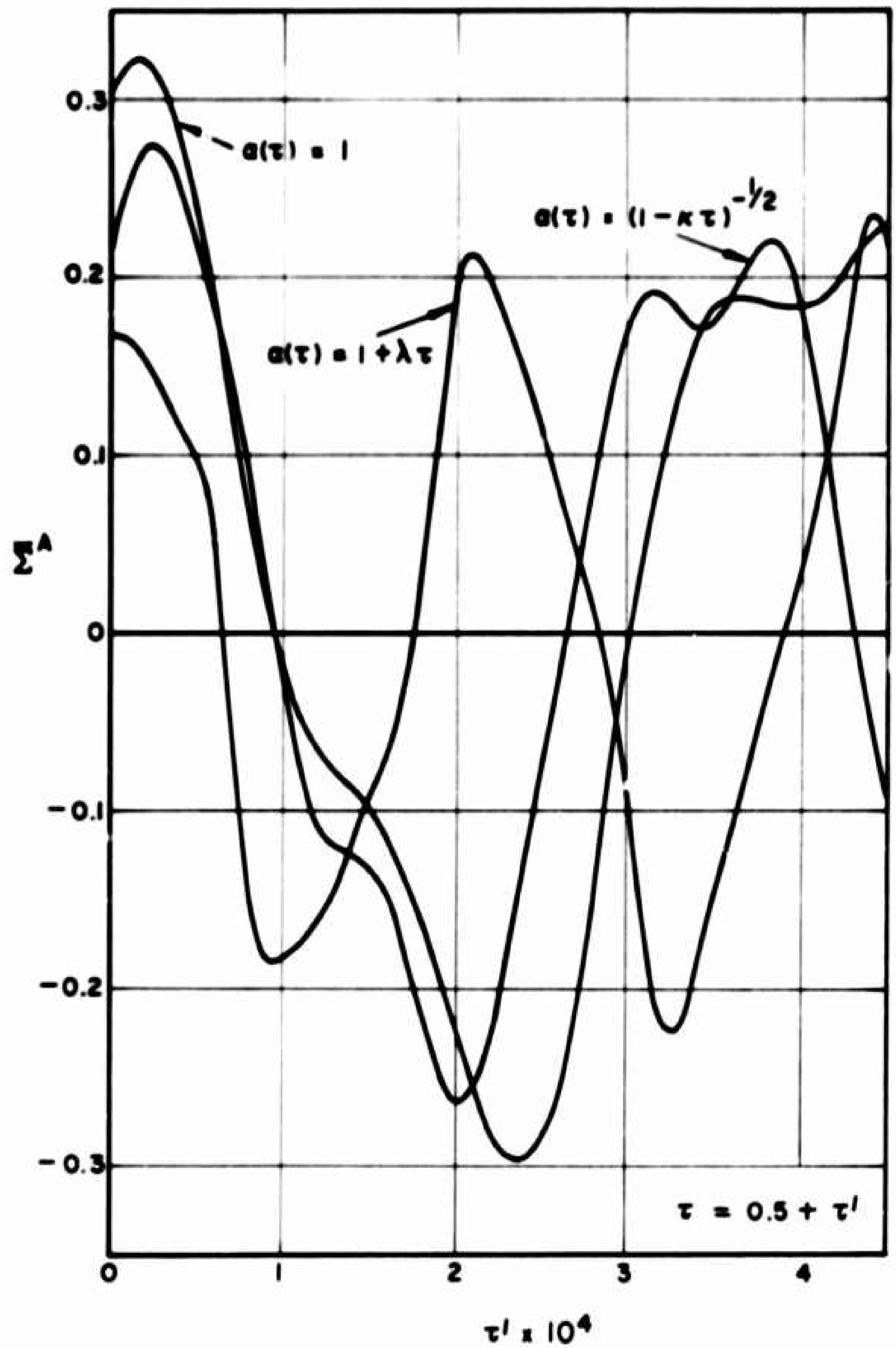

Fig.8 Shear-bond overstress for $\tau = 0.5 + \tau'$

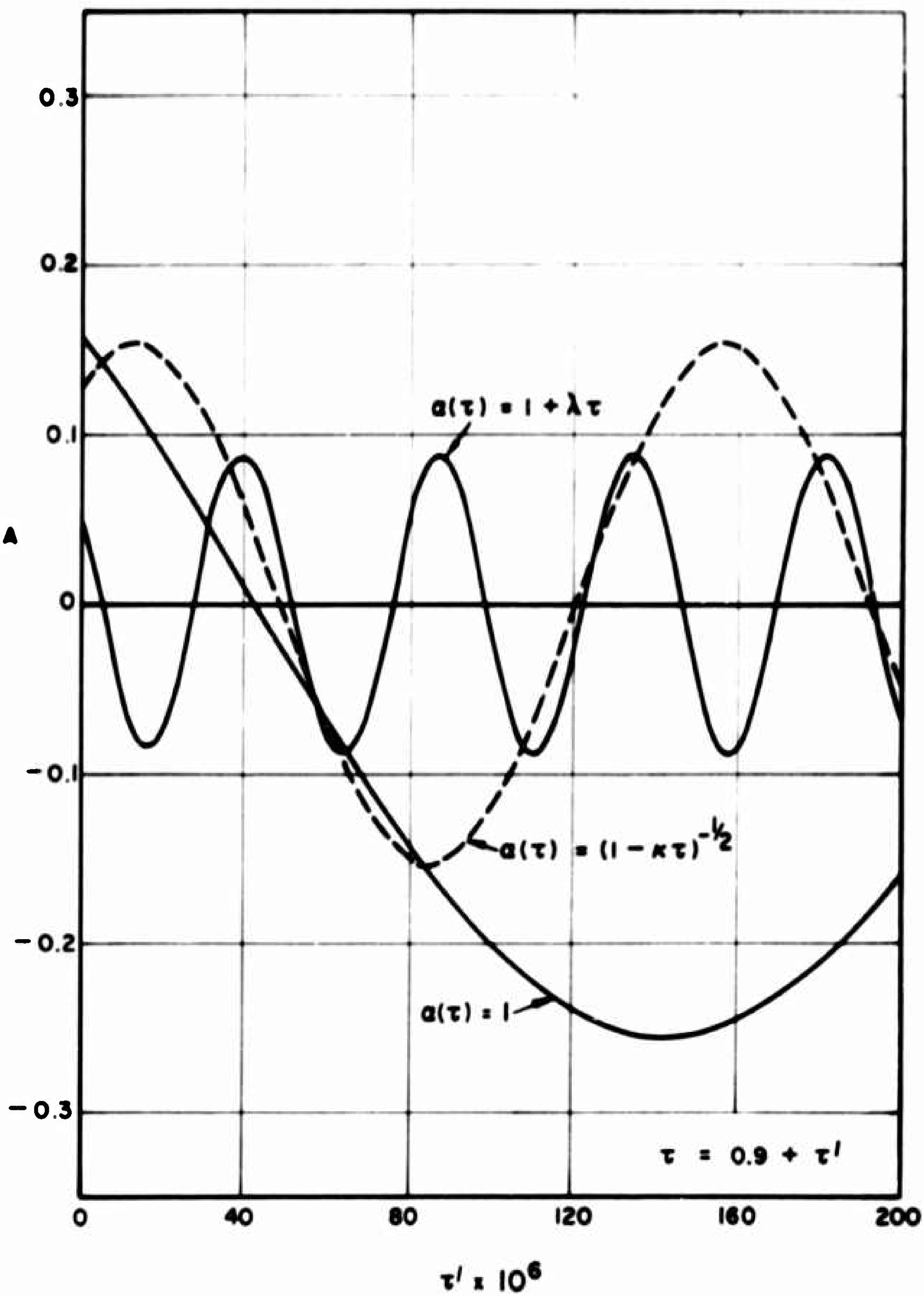

Fig. 9 First mode of the shear-bond overstress for $\tau = 0.9 + \tau'$

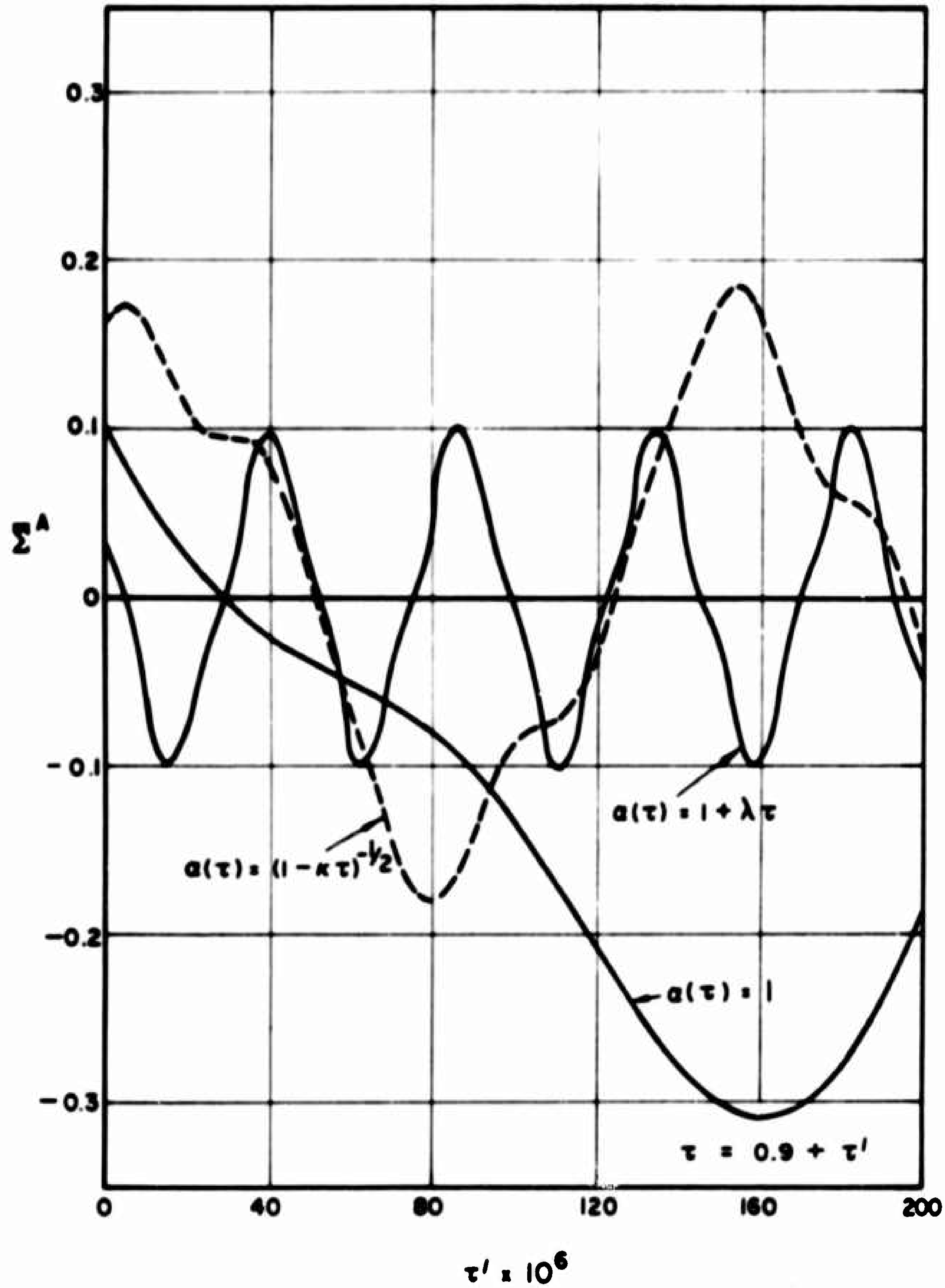

Fig. 10 Shear-bond overstress for $\tau = 0.9 + \tau'$