ESD-TDR 65-427

H. E. MOSES, *et al.*
1° Aprile 1965
Il Nuovo Cimento
Serie X, Vol. 36, pag. 788-804

The Kinematics of the Angular Momentum of a Particle.

I. – Translation Broadening of Angular Momentum.

H. E. MOSES and S. C. WANG

Lincoln Laboratory (*), *Massachusetts Institute of Technology - Lexington, Mass.*

(ricevuto il 24 Agosto 1964)

Summary. — The way a particle changes its angular momentum under inhomogeneous Lorentz transformations is well-known classically. The object of the present paper and a later one is to consider the problem quantum-mechanically for particles of any mass and any spin. In the present paper we shall consider in detail the case where a particle has a definite angular momentum in one frame of reference and we shall calculate the probability distribution of angular momentum in a frame translated with respect to the original frame. In a later paper we shall treat the case where the two frames of reference are moving with respect to one another. The basic mathematical tool is the form for the infinitesimal generators of the inhomogeneous Lorentz group devised by Lomont and Moses in which the Hamiltonian, square of the angular momentum, z component of angular momentum, and helicity are diagonal. The present paper and the projected one are important in multiple scattering problems, for it is possible using the results to take into account, to a certain degree at least, the effect of the selection rules. These rules are almost always ignored in multiple scattering problems. For example, it is shown that when the density of a gas is sufficiently low, radiative cooling goes on at a much faster rate when selection rules are taken into account than when they are ignored.

1. – Introduction.

In classical mechanics the problem of determining how the components of the angular momentum of a particle transform when the frame of reference is transformed under an inhomogeneous Galilean or Lorentz transformation has an almost trivial solution. In quantum mechanics, on the other hand, the

(*) Operated with support from the U. S. Advanced Research Projects Agency.

problem is very complicated because the components of the angular momentum are operators which do not commute and hence are not simultaneously measurable. Instead of asking for the values of components in different frames of reference, one either (*a*) asks for the transformation properties of the expectation value of the square of the angular momentum or (*b*) assigns to each frame of reference probabilities that the square of the angular momentum has a certain value. It is seen that the problem is further complicated if we wish to consider particles of any spin and any mass.

In the present paper and a subsequent paper we shall consider the quantum-mechanical problem where the frames of reference are related by an inhomogeneous Lorentz transformation. Since the Lorentz transformation includes the Galilean transformation together with corrections such as the Thomas factor as a limiting case and since the relativistic theory includes particle spin in a very natural way, it is seen that the case of the Lorentz transformation is far more interesting than the Galilean transformation.

The principal tool which we shall use will be the form of the infinitesi[illegible]l generators of the inhomogeneous Lorentz group in terms of a representatio[illegible] in which the Hamiltonian, square of the angular momentum, z-component of the angular momentum, and helicity are diagonal. This form was given in ref. (1).

Since the effect of rotations of frames of reference on the angular momentum has been studied before, we shall not discuss them further. Instead, we shall limit our attention in the present paper to the case in which a particle of given spin and mass has a given value of square of the angular momentum with respect to one frame of reference. We shall then consider the expectation value of the square of the angular momentum in a frame of reference displaced with respect to the original one. We shall also obtain analytically and numerically probabilities that the particles in the new frame of reference have certain values for the square of the angular momentum.

In the subsequent [illegible]r we shall consider the case where the two frames of reference are movi[illegible] espect to each other with a constant velocity.

The problem of th[illegible] [illegible]ties of the angular momentum as the frame of reference changes is importa[illegible] if one wishes to take into account selection rules in certain types of multiple scattering problems.

Let us consider, as an example, a gas in which photo-ionization and photo-recombination take place. One might consider the gas as being contained in a transparent container which is illuminated by an external source of photo-ionizing radiation for an instant and ask for the rate of decay of the number of free electrons due to photo-recombination. The decay rate will depend upon multiple absorption and emission processes and thus upon the probability that the electron is recaptured by an atom with the subsequent emission of a

---

(1) J. S. LOMONT and H. E. MOSES: *Journ. Math. Phys.*, 5, 294 (1964).

BLANK PAGE

photon. Because of the selection rules the photoelectron has only a finite number of possibilities for its angular momentum (usually $\frac{1}{2}$ or $\frac{3}{2}$ since the orbital angular momentum changes by 1) with respect to the emitting atom. To be captured in a permitted process by another atom, the electron must have the certain specific angular momenta with respect to the capturing atom, usually the same angular momenta as it has with respect to the emitting atom. Considering the emitting and capturing atom as constituting two frames of reference we are immediately led to the problem of determining the probability that the electron has a given angular momentum in a second frame when it is prescribed in the first frame. If the probability of having a suitable angular momentum is very small with respect to the capturing atom, the probability of multiple absorption and emission processes is very small. One may then neglect the multiple absorption and emission processes which go on in the gas and regard the atoms as being free in so far as photoionization processes are concerned (*).

As a second example of multiple scattering where the selection rules may play a decisive part, let us consider the radiative cooling of a gas. Consider a gas of one kind of atoms, which for simplicity of discussion, can have only one excited state. In the gas some of the atoms will be in the excited state and some will be in the ground state. If we restrict our attention to the way gas cools by emission of radiation only, some of the atoms will drop from the excited state to the ground state by emitting photons. Of these photons some will escape the gas completely. Some, however, will be captured by atoms in the ground state and excite these atoms. The excited atoms will then emit photons of the same frequency. The absorption and re-emission process slows down the rate at which gas cools by emission of radiation.

As in the previous example each of the photons which is emitted from an atom has a specific angular momentum whith respect to the atom (the angular momentum is 1) because of the selection rules. To be absorbed by an atom in the ground state the photon must have the correct angular momentum with respect to the absorbing atom (again 1). Hence, one must evaluate the probability that the photon has the correct angular momentum with respect to the absorbing atom when it has a given angular momentum with respect to the emitting atom.

One can think of many other examples. Generally speaking a complex particle will emit a given particle, the process being subject to selection rules. For absorption by another particle one must calculate the probability that the right angular momentum conditions are satisfied.

(*) We are indebted to Dr. ALI NAQVI of the Geophysics Corporation of America for his observation of the effect of selection rules on the multiple processes in this example. Indeed, his observation in this case led us to the discussion of the selection rules which follows and ultimately to our interest in the transformation properties of the angular momentum in quantum mechanics.

In all of these multiple scattering processes, the cross-section σ for absorption must be replaced by $W\sigma$ where W is the probability that the angular momentum has the correct value.

As far as the author knows, no one has taken the selection rules seriously into account in processes of the type described above, possibly because no one has heretofore calculated the probability W.

We may expect that when the gas has low densities so that the average distance between atoms is great, W will be small so that multiple processes can be ignored. Likewise at high (translational) temperatures such that the atoms have a high relative velocity on the average with respect to each other, W will be small and the effects of multiple processes are small. From such considerations it is clear that if selection rules are taken into account, the effect of multiple processes is less than if the selection rules are ignored.

Speaking more abstractly, most calculations of multiple scattering take into account conservation of energy but not conservation of angular momentum. Use of the selection rules has the effect of taking into account conservation of angular momentum.

In the present paper it is shown by numerical calculation that when the density is such that the average distance between the emitting particles is of the order of half a De Broglie wave length of the emitted particle, W will be of the order of 0.2 or less. (It should be noted that $W\sigma$ is still orders of magnitude much greater than cross-sections for forbidden transitions.) When the interatomic distance is greater than of the order $\frac{1}{6}$ the De Broglie wave length of the emitted particle W becomes significantly less than 1. Hence we are able to establish a critical density, such that if the density of the gas is less than the critical density it is extremely important to take the selection rules into account (*).

2. – Classical and quantum description of particles.

The present Section is intended to give the background to the analytical and numerical calculations which follow.

Classically, a relativistic particle of mass μ is specified by the set of dynamical variables which consists of the co-ordinates and momenta $\{x^i, p^i\}$ $(i=1, 2, 3)$, the relativistic Hamiltonian $H=c\left[\sum_i (p^i)^2+\mu^2c^2\right]^{\frac{1}{2}}$ and the com-

(*) It seems possible that one could have obtained such estimates as the above by a clever use of the uncertainly principle. These estimates are made for the case that the translational temperature is (strictly speaking) zero. However, they are still valid for any temperature, since if the atoms move with respect to each other, the « broadening » of the angular momentum is even more severe.

ponents of the angular momentum $J^1 = x^2p^3 - x^3p^2$, $J^2 = x^3p^1 - x^1p^3$, $J^3 = x^1p^2 - x^2p^1$. (We are anticipating relativistic notation in using superscripts rather than subscripts for components.)

Then, as is well-known, we can write a space-time 4-vector x^α and a momentum 4-vector p^α where $\alpha = 0, 1, 2, 3$, on defining $x^0 = ct$ and $p^0 = H/c$.

However, in order to write the components of the angular momentum in a relativistic notation, it is necessary to introduce a 4×4 dimensional angular momentum tensor

$$J^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha . \tag{2.1}$$

Then

$$J^1 = J^{23}, \qquad J^2 = J^{31}, \qquad J^3 = J^{12} . \tag{2.2}$$

The remaining components of the angular momentum tensor have vector transformation properties which can be exhibited by introducing new dynamical variables $\mathscr{J}^i$ defined by

$$\begin{cases} \mathscr{J}^i = -J^{0i} & (i = 1, 2, 3), \\ \quad = -ctp^i + x^i \dfrac{H}{c} = \dfrac{1}{c}[Hx^i - c^2tp^i] . & \end{cases} \tag{2.3}$$

We can now return to nonrelativistic notation and say that relativistically a particle is completely specified by the following dynamical variables: x^i, p^i, J^i, $\mathscr{J}^i$ and H.

Since we shall be interested in how the angular momentum changes under translation, let us consider a particularly simple translation, namely a translation along the x^3 axis. (It is clear that by a suitable rotation any translation can be converted to such a translation.)

Then in terms of the new frame of reference

$$\begin{cases} x^{1'} = x^1 \\ x^{2'} = x^2 \\ x^{3'} = x^3 - a \\ p^{i'} = p^i . \end{cases} \tag{2.4}$$

Hence, from (2.1) and (2.2)

$$\begin{cases} J^{1'} = J^1 + ap^2 \\ J^{2'} = J^2 - ap^1 \\ J^{3'} = J^3 . \end{cases} \tag{2.5}$$

Also denoting the square of the angular momentum by $\boldsymbol{J}^2$ where

$$\boldsymbol{J}^2 = \sum_i (J^i)^2 \tag{2.6}$$

we have in the new frame of reference

$$\boldsymbol{J}^{2\prime} = \boldsymbol{J}^2 + 2a(J^1p^2 - J^2p^1) + a^2(\boldsymbol{p}^2 - (p^3)^2)\,, \tag{2.7}$$

where $\boldsymbol{p}^2$ is the square of the momentum: $\boldsymbol{p}^2 = \sum_i (p^i)^2$.

To quantize the classical theory of the free relativistic particle we use the canonical approach. Namely, we calculate the Poisson brackets of all of the dynamical variables and replace the Poisson brackets by commutators of the operators which correspond to the dynamical variables.

As an example let us evaluate the Poisson bracket of $\mathcal{J}^1$ and p^1. Be definition of the Poisson bracket:

$$\{\mathcal{J}^1, p^1\} = \sum_i \left[\frac{\partial \mathcal{J}^1}{\partial x^i}\frac{\partial p^1}{\partial p^i} - \frac{\partial p^1}{\partial x^i}\frac{\partial \mathcal{J}^1}{\partial p^i}\right].$$

But

$$(\partial p^1/\partial p^i) = \delta_{1i}\,, \qquad (\partial p^1/\partial x^i) = 0\,,$$

and from (2.3)

$$(\partial \mathcal{J}^1/\partial x^i) = \frac{H}{c}\,\delta_{1i}\,.$$

Thus

$$\{\mathcal{J}^1, p^1\} = \frac{H}{c}\,.$$

On replacing the variables $\mathcal{J}^1$, p^1, and H by operators and on using the fact that the quantum Poisson bracket of any two operators A and B is related to the commutator $[A, B] = AB - BA$ by

$$\{A, B\} = (i\hbar)^{-1}[A, B]\,,$$

we obtain the commutation relation

$$[\mathcal{J}^1, p^1] = i\frac{\hbar}{c}H\,.$$

By replacing the classical dynamical variables by operators and Poisson brackets by commutators, we obtain quantum conditions on the dynamical variables in the form of a commutator algebra.

3078

In relativistic quantum mechanics, however, the co-ordinate operators x^i have serious difficulties associated with them. For example, from the commutation rules which are derived in the above manner, it can be shown that the co-ordinate operators do not commute with the sign of the energy and hence particles with positive energy cannot be localized. Because of this and similar difficulties, it is customary to ignore the co-ordinate operators x^i. One thus regards the momentum operators p^i, the Hamiltonian H, the angular momentum operators J^i, and the « space-time » operators $\mathcal{J}^i$ as constituting the entire set of dynamical variables which describe a free relativistic particle in quantum mechanics. The commutation relations which they satisfy are derived from the Poisson bracket approach described above. The commutators of interest also form a commutator algebra. We shall write the set of commutation rules. Since we shall have no further occasion to use relativistic notation, we shall replace superscripts by subscripts.

$$
(2.8)\qquad
\left\{
\begin{aligned}
&[p_i, p_j] = 0\,,\\
&[H, p_i] = 0\,,\\
&[J_1, J_2] = i\hbar J_3\,,\\
&[J_2, J_3] = i\hbar J_1\,,\\
&[J_3, J_1] = i\hbar J_2\,,\\
&[J_i, p_i] = 0\,,\\
&[J_1, p_2] = i\hbar p_3 = [p_1, J_2]\,,\\
&[J_2, p_3] = i\hbar p_1 = [p_2, J_3]\,,\\
&[J_3, p_1] = i\hbar p_2 = [p_3, J_1]\,,\\
&[J_i, H] = 0\,,\\
&[J_i, \mathcal{J}_i] = 0\,,\\
&[J_1, \mathcal{J}_2] = i\hbar \mathcal{J}_3 = [\mathcal{J}_1, J_2]\,,\\
&[J_2, \mathcal{J}_3] = i\hbar \mathcal{J}_1 = [\mathcal{J}_2, J_3]\,,\\
&[J_3, \mathcal{J}_1] = i\hbar \mathcal{J}_2 = [\mathcal{J}_3, J_1]\,,\\
&[\mathcal{J}_i, p_j] = i(\hbar/c)H\delta_{ij}\,,\\
&[\mathcal{J}_1 \mathcal{J}_2] = i\hbar J_3\,,\\
&[\mathcal{J}_i, H] = i\hbar c p_i\,,\\
&[\mathcal{J}_2, \mathcal{J}_3] = -i\hbar J_1\,,\\
&[\mathcal{J}_3, \mathcal{J}_1] = -i\hbar J_2\,.
\end{aligned}
\right.
$$

In most discussions of the commutation rules (2.8) $\hbar$ and c are set equal to unity to simplify the discussion. Since we are interested in obtaining numerical values, we shall instead introduce the « reduced » operators $\hat{p}_i = p_i/\hbar$, $\hat{H} = H/\hbar c$, $\hat{J}_i = J_i/\hbar$, $\hat{\mathcal{J}}_i = \mathcal{J}_i/\hbar$. These reduced operators satisfy the commutation rules (2.8) but with $\hbar = c = 1$.

It is a remarkable fact that the reduced operators can be interpreted as the infinitesimal generators of the ray representations of the proper orthochronous inhomogeneous Lorentz group (references (2) and (3)). They have been studied in this context and all irreducible representations have been found. Representations exist which can be interpreted as corresponding to particles of any mass and spin. Various forms for the infinitesimal generators or, equivalent, dynamical variables are given in references (3-7).

In references (3-7) the dynamical variables are given in representations where the momentum operators p_i are diagonal (*).

However, for the purpose of studying the transformation properties of the angular momentum, it is much more desirable to have an angular momentum basis. The angular momentum basis is described in detail in reference (1). In the present paper we shall describe only so much of the form of the operators in terms of this basis as is needed for the calculations of this paper.

In addition to the dynamical variables p_i, J_i, $\mathcal{J}_i$, and H introduced above, it will be convenient to introduce the helicity or circular polarization operator w defined by

$$w = \frac{\sum_i p_i J_i}{\sqrt{H^2 - \mu^2}} . \tag{2.9}$$

For the case of particles of nonvanishing mass μ and spin s we introduce a Hilbert space of functions $\varphi(E, j, m, \alpha)$ where the range of E is $\mu < E < \infty$, the values taken on by α are $\alpha = -s, -s+1, \ldots, s-1, s$, and for a fixed value of α the range of j is given by $j = |\alpha|, |\alpha|+1, |\alpha|+2, \ldots$ For a fixed value of j, m takes on the values $m = -j, -j+1, \ldots, j-1, j$.

The inner product of two states if given by

$$\sum_{\alpha, j, m} \int_\mu^\infty \frac{\mathrm{d}E}{p} \varphi^{(1)*}(E, j, m, \alpha) \varphi(E, j, m, \alpha) , \quad \left(p = \frac{1}{c}\sqrt{E^2 - \mu^2 c^4}\right) .$$

(2) E. P. Wigner: *Ann. Math.*, **40**, 149 (1939).
(3) V. Bargmann and E. P. Wigner: *Proc. Nat. Acad. Sci. U. S.*, **34**, 211 (1948).
(4) L. L. Foldy: *Phys. Rev.*, **102**, 568 (1956).
(5) Iu. M. Shirokov: *Sov. Phys. J.E.T.P.*, **6**, 919 (1958).
(6) V. I. Ritus: *Sov. Phys. J.E.T.P.*, **8**, 990 (1959).
(7) J. S. Lomont and H. E. Moses: *Journ. Math. Phys.*, **3**, 405 (1962).
(*) In ref. (4) the operators are given in terms of an x-representation which is essentially the Fourier transform of the momentum representation.

In terms of this basis we have

$$
(2.10)\quad
\begin{cases}
J^2\varphi(E, j, m, \alpha) = \hbar^2 j(j+1)\varphi(E, j, m, \alpha)\,, \\
J_3\varphi(E, j, m, \alpha) = \hbar m\varphi(E, j, m, \alpha)\,, \\
H\varphi(E, j, m, \alpha) = E\varphi(E, j, m, \alpha)\,, \\
w\varphi(E, j, m, \alpha) = \hbar\alpha\varphi(E, j, m, \alpha)\,, \\
(J_2 + iJ_1)\varphi(E, j, m, \alpha) = \hbar\sqrt{(j-m)(j+m+1)}\,\varphi(E, j, m+1, \alpha)\,, \\
(J_2 - iJ_1)\varphi(E, j, m, \alpha) = \hbar\sqrt{(j+m)(j-m+1)}\,\varphi(E, j, m-1, \alpha)\,.
\end{cases}
$$

$$
(2.11)\quad
\begin{cases}
p_3\varphi(E, j, m, \alpha) = p\left[\dfrac{m\alpha}{j(j+1)}\,\varphi(E, j, m, \alpha) + \dfrac{1}{(j+1)}\,\cdot\right. \\
\quad\cdot\sqrt{\dfrac{(j-m+1)(j+m+1)(j-\alpha+1)(j+\alpha+1)}{(2j+1)(2j+3)}}\,\varphi(E, j+1, m, \alpha)\Bigg] + \\
\qquad\left. + \dfrac{1}{j}\sqrt{\dfrac{(j-m)(j+m)(j-\alpha)(j+\alpha)}{(2j-1)(2j+1)}}\,\varphi(E, j-1, m, \alpha)\right], \\
(p_2 + ip_1)\varphi(E, j, m, \alpha) = \\
= p\left[\dfrac{\alpha}{j(j+1)}\sqrt{(j-m)(j+m+1)}\,\varphi(E, j, m+1, \alpha) - \dfrac{1}{(j+1)}\,\cdot\right. \\
\quad\cdot\sqrt{\dfrac{(j+m+1)(j+m+2)(j-\alpha+1)(j+\alpha+1)}{(2j+1)(2j+3)}}\,\varphi(E, j+1, m+1, \alpha) + \\
\qquad\left. + \dfrac{1}{j}\sqrt{\dfrac{(j-m-1)(j-m)(j-\alpha)(j+\alpha)}{(2j-1)(2j+1)}}\,\varphi(E, j-1, m+1, \alpha)\right], \\
(p_2 - ip_1)\varphi(E, j, m, \alpha) = \\
= p\left[\dfrac{\alpha}{j(j+1)}\sqrt{(j+m)(j-m+1)}\,\varphi(E, j, m-1, \alpha) + \dfrac{1}{(j+1)}\,\cdot\right. \\
\quad\cdot\sqrt{\dfrac{(j-m+1)(j-m+2)(j-\alpha+1)(j+\alpha+1)}{(2j+1)(2j+3)}}\,\varphi(E, j+1, m-1, \alpha) - \\
\qquad\left. - \dfrac{1}{j}\sqrt{\dfrac{(j+m-1)(j+m)(j-\alpha)(j+\alpha)}{(2j-1)(2j+1)}}\,\varphi(E, j-1, m-1, \alpha)\right].
\end{cases}
$$

If we take a state function which is normalized to unity, *i.e.*

$$
\sum_{j,m,\alpha}\int_{\mu}^{\infty}\frac{\mathrm{d}E}{p}\,|\varphi(E, j, m, \alpha)|^2 = 1\,, \qquad \text{then} \qquad \int_{\mu}^{\infty}\frac{\mathrm{d}E}{p}\,|\varphi(E, j, m, \alpha)|^2
$$

gives the probability that a measurement of $\boldsymbol{J}^2$ will give a value of $\hbar^2 j(j+1)$, J_3 will give a value of $\hbar m$, and w will give a value of $\hbar\alpha$. It will be useful to introduce states which are simultaneous eigenstates of $\boldsymbol{J}^2$, J_3 and w such that the particle will have its energy within a narrow range. Accordingly we introduce normalized states

$$\varphi_0(E, j, m, \alpha) = g(E)\,\delta_{j,j_0}\delta_{m,m_0}\delta_{\alpha,\alpha_0}\,, \tag{2.12}$$

where $g(E)$ is a function of E such that $|g(E)|$ has a sharp maximum for $E=E_0$ and

$$\int\limits_{\mu}^{\infty}\frac{\mathrm{d}E}{p}\,|g(E)|^2 = 1\,.$$

Thus these states are in a sense eigenstates of H with the eigenvalue E_0.

For the massless case we set $\mu=0$ in the above expressions. But, in addition, w is a scalar and α takes on one value only. The number α is either $+s$ or $-s$ where s is the spin as before. In the definition of inner product the summation over α is omitted. The eigenfunctions which correspond to those of (2.12) have the factor δ_{α,α_0} replaced by 1.

3. – Transformation properties of the angular momentum operator. The Heisenberg and Schrödinger pictures of invariance.

In the present Section we shall consider a translation of frames of reference along the z-axis, namely the translation which would lead classically to the results (2.4) and (2.5) for the new values of the co-ordinates, components of momentum, and components of angular momentum. We want to describe quantum-mechanically how the dynamical variables and states in the new frame of reference are related to those in the old.

It is customary to use one of two pictures for describing the change. These two pictures are generalizations of the Heisenberg and Schrödinger pictures for the equations of motion describing the time-development of a quantum-mechanical system. We shall call the corresponding generalizations also Heisenberg and Schrödinger pictures of invariance.

In the Heisenberg picture of invariance, the states in the two frames of reference are the same but the dynamical variables change. In fact, the relation of the dynamical variables in the two frames of reference are the same as in classical theory. Hence the momentum and angular momentum operators in the new frame of reference are related to the old by (2.4) and (2.5).

9077

From the general requirements of invariance it can be shown that the primed operators must be related to the unprimed variables by means of a unitary transformation. If A is any dynamical variable in the original frame of reference and A' is the dynamical variable in the new frame then

$$A' = U^{-1} A U \,. \tag{3.1}$$

For the translation along the z-axis it can be shown that

$$U = \exp\left(i \frac{a}{\hbar} p_3\right). \tag{3.2}$$

In the Schrödinger picture of invariance, on the other hand, the operators remain unchanged but the states change. Let us denote by ψ' a state in th new frame when the state in the original frame was given by ψ. The Schrödinger state ψ' is required to satisfy

$$(\psi', A\psi') = (\psi, A'\psi) \,. \tag{3.3}$$

One takes

$$\psi' = U\psi \,. \tag{3.4}$$

4. – The expectation value of the square of the angular momentum.

In the present Section we shall give *exact* expressions for the expectation value of the square of the angular momentum operator in the second frame of reference when in the original frame of reference the particle had a definite value for J^2, J_3, and w.

It will be convenient to use the Heisenberg picture. Hence the components of angular momentum transform according to (2.5). Thus

$$J'^2 = J^2 + a(J_1 p_2 + p_2 J_1 - J_2 p_1 - p_1 J_2) + a^2\left(\frac{H^2}{c^2} - \mu^2 c^2 - p_3^2\right). \tag{4.1}$$

Let us assume that we are in the state given by (2.12). Then on using (2.10) and (2.11) we find for the expectation values for J'^2

$$\langle J'^2 \rangle = \hbar^2 j_0(j_0+1) + \frac{2a^2 p_0^2}{(2j_0-1)(2j_0+3)}\left[j_0(j_0+1) - 1 + m_0^2 + \alpha_0^2 - \frac{3\alpha_0^2 m_0^2}{j_0(j_0+1)}\right]. \tag{4.2}$$

In (4.2) p_0 is given by $p_0 = [(E_0^2/c^2) - \mu^2 c^2]^{\frac{1}{2}}$ and is the absolute value of the momentum of the particle in both the original and translated frames. In de-

8078

riving the expression (4.2) we have used very heavily the peaking property of $g(E)$ in getting the dependence on p_0. For the massless case, one should replace α by the spin s.

If in the original frame of reference one did not know the value of m_0, it would be natural to average over all values of this quantum number. Denoting the result of this average by $\langle J'^2 \rangle$ we find an extraordinarilly simple formula

$$\langle J'^2 \rangle = \hbar^2 j_0(j_0+1) + \tfrac{2}{3} a^2 p_0^2 \,. \tag{4.3}$$

This formula is valid for particles of any mass, any spin, and any circular polarization.

It is interesting to compare this formula with the classical formula for J'^2. The classical counterpart of averaging over m_0 is to average over all directions of the vector $\boldsymbol{p}$. From (2.7) we find this process yields

$$J'^2 = J^2 + \tfrac{1}{3} a^2 p^2 \,. \tag{4.4}$$

Thus in quantum mechanics the square of the angular momentum changes faster than in classical mechanics under the translation of the frame.

In obtaining (4.3) we used the well-known expression for the average of m_0^2, namely,

$$\overline{m_0^2} = \frac{1}{2j_0+1} \sum_{-j_0}^{j_0} m_0^2 = \tfrac{1}{3} j_0(j_0+1) \,. \tag{4.5}$$

5. – Probabilities of various angular momenta. Analytic calculations.

In the present Section we shall calculate to the fourth order the probabilities that the square of the angular momentum has a certain value when in the original frame of reference it is in an eigenstate of the square of the angular momentum. That is, in the original frame of reference, the state is given by (2.12). In the new frame of reference, the state is given by

$$\varphi_a(E, j, m, \alpha) = U \varphi_0(E, j, m, \alpha) \tag{5.1}$$

in the Schrödinger picture of invariance, where U is the unitary operator of (3.3), which depends on a. Let us define $W_n(a)$ to be the probability that in the new frame of reference J^2 has the eigenvalue $\hbar^2(j_0+n)(j_0+n+1)$, J_3 has the eigenvalue $\hbar m_0$ and w has the eigenvalue $\hbar \alpha_0$. (The probability that J_3 and w have eigenvalues other than those given above is zero. Also the range of energies is unchanged in a translation of frames of reference so that the particle continues to have energies near E_0.)

Then

$$W_n(a) = \int_{\mu}^{\infty} \frac{\mathrm{d}E}{p} \, |\varphi_a(E, j_0 + n, m_0, \alpha_0)|^2 \tag{5.2}$$

where n is either a positive or negative integer or zero. We have acted as though we are discussing the nonzero mass case only, but a slight modification of notation includes the massless case also.

In order to obtain analytic expressions, we expand U in terms of a power series in a and evaluate $W_n(a)$ using (5.2). We have carried out this program to the fourth order in a. Since the state φ_0 has a sharp peak at $E = E_0$, $W_n(a)$ is actually expanded in a power series in $ap_0/\hbar$ in which only even powers appear.

Let us write

$$W_n(a) = W_n^{(0)} + (ap_0/\hbar)^2 W_n^{(1)} + (ap_0/\hbar) W_n^{(2)} + \dots \,. \tag{5.3}$$

Then

$$\left\{\begin{aligned}
& W_n^{(0)} = \delta_{n,0}\,, \\
& W_n^{(1)} = 0 \text{ for all } n \text{ except } n = 1 \text{ or } -1, \text{ or } n = 0 \\
& W_0^{(1)} = -\frac{(j_0^2 - m_0^2)(j_0^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 + 1)} \frac{1}{j_0^2} - \frac{((j_0 + 1)^2 - m_0^2)((j_0 + 1)^2 - \alpha_0^2)}{(2j_0 + 1)2j_0 + 3)} \frac{1}{(j_0 + 1)^2}\,, \\
& W_1^{(1)} = \frac{((j_0 + 1)^2 - m_0^2)((j_0 + 1)^2 - \alpha_0^2)}{(2j_0 + 1)(2j_0 + 3)} \frac{1}{(j_0 + 1)^2}\,, \\
& W_{-1}^{(1)} = \frac{(j_0^2 - m_0^2)(j_0^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 + 1)} \frac{1}{j_0^2}\,. \\
& W_n^{(2)} = 0 \text{ for all } n \text{ except for } n = 0,\, 1,\, -1,\, 2,\, -2. \\
& W_0^{(2)} = \frac{1}{12}\left\{\frac{4m_0^2\alpha_0^2}{j_0^2(j_0 + 1)^2(j_0 + 2)^2} \frac{((j_0 + 1)^2 - m_0^2)((j_0 + 1)^2 - \alpha_0^2)}{(2j_0 + 1)(2j_0 + 3)(j_0 + 1)^2} + \right. \\
& \quad + \frac{4m_0^2\alpha_0^2}{j_0^2(j_0 + 1)^2(j_0 - 1^2)} \frac{(j_0^2 - m_0^2)(j_0^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 + 1)j_0^2} + \\
& \quad + 8 \frac{(j_0^2 - m_0^2)(j_0^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 + 1)j_0^2} \frac{((j_0 + 1)^2 - m_0^2)((j_0 + 1)^2 - \alpha_0^2)}{(2j_0 + 1)(2j_0 + 3)(j_0 + 1)^2} + \\
& \quad + 4\left[\frac{((j_0 + 1)^2 - m_0^2)((j_0 + 1)^2 - \alpha_0^2)}{(2j_0 + 1)(2j_0 + 3)(j_0 + 1)^2}\right]^2 + 4\left[\frac{(j_0^2 - m_0^2)(j_0^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 + 1)j_0^2}\right]^2 + \\
& \quad + \frac{((j_0 + 1)^2 - m_0^2)((j_0 + 1)^2 - \alpha_0^2)}{(2j_0 + 1)(2j_0 + 3)(j_0 + 1)^2} \frac{((j_0 + 2)^2 - m_0^2)((j_0 + 2)^2 - \alpha_0^2)}{(2j_0 + 3)(2j_0 + 5)(j_0 + 2)^2} + \\
& \quad \left. + \frac{(j_0^2 - m_0^2)(j_0^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 + 1)j_0^2} \frac{((j_0 - 1)^2 - m_0^2)((j_0 - 1)^2 - \alpha_0^2)}{(2j_0 - 1)(2j_0 - 3)(j_0 - 1)^2}\right\},
\end{aligned}\right. \tag{5.4}$$

1080

$$
\left\{
\begin{aligned}
W_1^{(2)} = -\frac{1}{3}\Bigg\{ & \frac{m_0^2\alpha_0^2}{j_0^2(j_0+1)^2(j_0+2)^2}\,\frac{((j_0+1)^2-m_0^2)((j_0+1)^2-\alpha_0^2)}{(2j_0+1)(2j_0+3)(j_0+1)^2} + \\
& + \frac{(j_0^2-m_0^2)(j_0^2-\alpha_0^2)}{(2j_0-1)(2j_0+1)j_0^2}\,\frac{((j_0+1)^2-m_0^2)((j_0+1)^2-\alpha_0^2)}{(2j_0+1)(2j_0+3)(j_0+1)^2} + \\
& + \left[\frac{((j_0+1)^2-m_0^2)((j_0+1)^2-\alpha_0^2)}{(2j_0+1)(2j_0+3)(j_0+1)^2}\right]^2 + \\
& + \frac{((j_0+1)^2-m_0^2)((j_0+1)^2-\alpha_0^2)}{(2j_0+1)(2j_0+3)(j_0+1)^2}\,\frac{((j_0+2)^2-m_0^2)((j_0+2)^2-\alpha_0^2)}{(2j_0+3)(2j_0+5)(j_0+2)^2}\Bigg\}, \\
W_{-1}^{(2)} = -\frac{1}{3}\Bigg\{ & \frac{m_0^2\alpha_0^2}{(j_0-1)^2j_0^2(j_0+1)^2}\,\frac{(j_0^2-m_0^2)(j_0^2-\alpha_0^2)}{(2j_0-1)(2j_0+1)j_0^2} + \\
& + \frac{(j_0^2-m_0^2)(j_0^2-\alpha_0^2)}{(2j_0-1)(2j_0+1)j_0^2}\,\frac{((j_0+1)^2-m_0^2)((j_0+1)^2-\alpha_0^2)}{(2j_0+1)(2j_0+3)(j_0+1)^2} + \\
& + \left[\frac{(j_0^2-m_0^2)(j_0^2-\alpha_0^2)}{(2j_0-1)(2j_0+1)j_0^2}\right]^2 + \frac{(j_0^2-m_0^2)(j_0^2-\alpha_0^2)}{(2j_0-1)(2j_0+1)j_0^2}\cdot \\
& \cdot\frac{((j_0-1)^2-m_0^2)((j_0-1)^2-\alpha_0^2)}{(2j_0-3)(2j_0-1)(j_0-1)^2}\Bigg\}, \\
W_2^{(2)} = \frac{1}{4} & \frac{((j_0+1)^2-m_0^2)((j_0+1)^2-\alpha_0^2)}{(2j_0+1)(2j_0+3)(j_0+1)^2}\,\frac{((j_0+2)^2-m_0^2)((j_0+2)^2-\alpha_0^2)}{(2j_0+3)(2j_0+5)(j_0+2)^2}, \\
W_{-2}^{(2)} = \frac{1}{4} & \frac{((j_0-1)^2-m_0^2)((j_0-1)^2-\alpha_0^2)}{(2j_0-3)(2j_0-1)(j_0-1)^2}\,\frac{(j_0^2-m_0^2)(j_0^2-\alpha_0^2)}{(2j_0-1)(2j_0+1)j_0^2}.
\end{aligned}
\right.
\tag{5.4}
$$

For the massless case one should substitute the spin quantum number s for α_0 in (5.4). The expressions (5.3) and (5.4) give accuracy to at least two significant figures for $|ap_0/\hbar|\leqslant 1$ when one compares these results with the numerical results discussed in the next Section.

6. – Numerical results.

In order to obtain the probabilities $W_n(a)$ for larger values of a, we have resorted to numerical computation using a large computer. The objective of the computarion was to avoid expansions of the type indicated in the previous Section. This objective was accomplished by replacing the operator p_z which, according to (2.11), is an infinite-dimensional matrix in the quantum number j by a truncated finite-dimensional matrix. The finite-dimensional matrix can then be diagonalized using standard computer techniques. Then φ_a can be obtained in this representation for all a by simple exponentiation. Finally, one transforms back to the original representation. One can check the accuracy of a given computation by increasing the size of the finite-dimensional matrix which replaces p_3 and seeing whether the results are sensitive to this increase in size.

In Fig. 1 through 4 typical results are given graphically. The dimension of the matrices was 25×25 in Fig. 1 through 3 and 10×10 in Fig. 4. Since the amount of data which one can obtain by means of this technique is enormous, we have averaged over m_0 and, in the case of nonvanishing mass, over α_0.

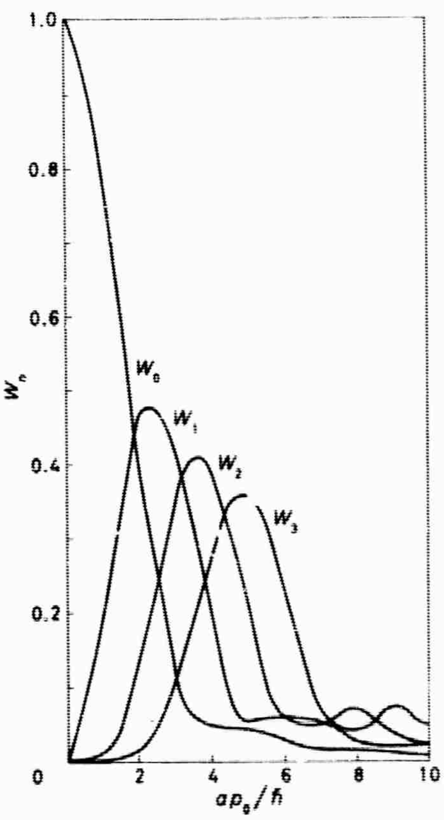

Fig. 1. – Electron or neutrino, $s=\frac{1}{2}$, $j_0=\frac{1}{2}$.

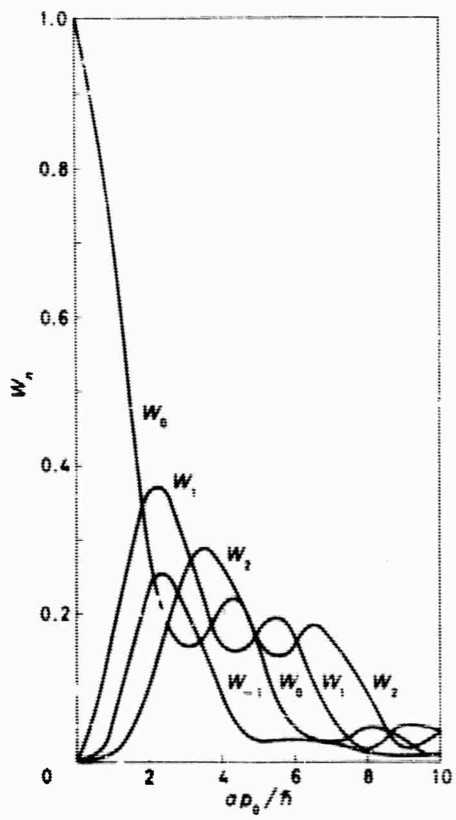

Fig. 2. – Electron or neutrino, $s=\frac{1}{2}$, $j_0=\frac{3}{2}$.

These averaged quantities are the interesting ones from a statistical point of view and in the examples of applications given in the Introduction.

In Fig. 1 we have given graphically W_n as a function of $ap_0/\hbar$ for various values of n for particles for spin $\frac{1}{2}$ and for $j_0=\frac{1}{2}$. In Fig. 2 results are given for particles of spin $\frac{1}{2}$ and for $j_0=\frac{3}{2}$. In both of these cases it does not matter whether the particle has zero or nonzero mass in taking the average over m_0. No average has to be taken over α_0 since these quantities are symmetric in α_0 and since $\alpha_0=s=\frac{1}{2}$. The particle being considered is an electron in the case that the mass is not zero and is a neutrino if the mass is zero.

In Fig. 3 we have given W_n for massless particles of spin 1 and $j_0 = 1$. The cases which we have given above are of interest in the applications discussed in the Introduction.

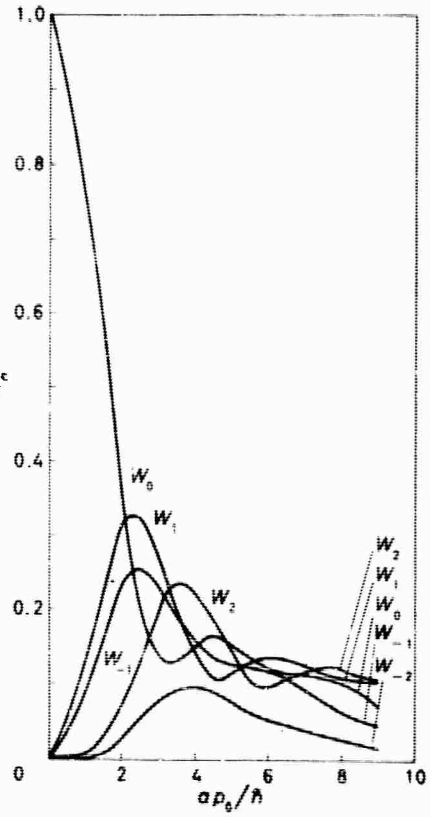

Fig. 3. – Photon, mass $= 0$, $s = 1$, $j_0 = 1$.

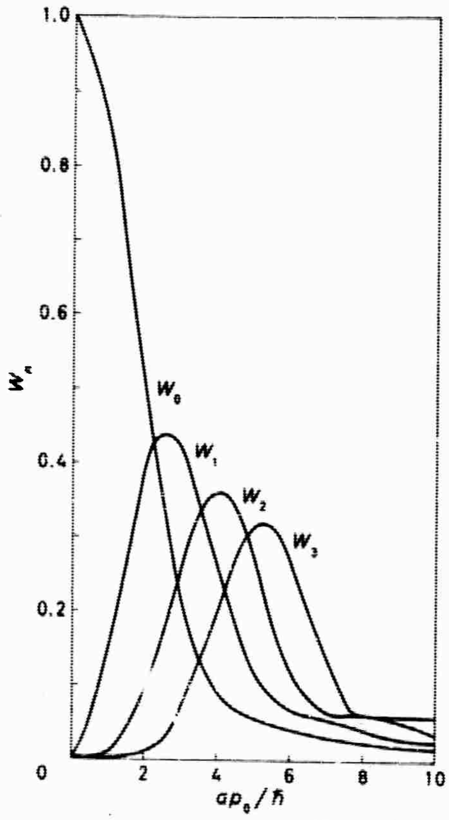

Fig. 4. – Mass $\neq 0$, $s = \frac{5}{2}$, $j_0 = \frac{7}{2}$.

In Fig. 4 we have considered the case for which the mass does not vanish and for which $s = \frac{5}{2}$ and $j_0 = \frac{7}{2}$. Such a particle might be a complex particle ejected from a nucleus. We have given this example to show what happens for particles of higher spin and angular momentum.

In all of these calculations there is quantum « noise » for values of $ap_0/\hbar$ such that W_n is small (less than 0.4). That is, small maxima and minima are superposed on the general curve. These effects are especially pronounced in the curves of W_n before averaging.

The calculations were taken for values of $(ap_0/\hbar) = 0, 1, 2, \ldots, 10$ and some of the « noise » may have been missed because small intervals in this variable were not taken (*).

(*) The program calculates W_n for particles of any spin, j_0 and $ap_0/\hbar$. It is thus possible to produce data for other cases in the form of tables or of graphs, if needed.

* * *

The authors are grateful to Prof. J. S. LOMONT of Polytechnic Institute of Brooklyn for his criticism.

RIASSUNTO (*)

Il modo in cui una particella cambia il suo momento angolare in seguito a trasformazioni di Lorentz inomogenee è classicamente ben noto. L'oggetto del presente lavoro, e di uno successivo, è di considerare il problema dal punto di vista della meccanica quantistica per particelle di massa e spin qualsiasi. Nel presente lavoro esamineremo in dettaglio il caso in cui una particella ha un momento angolare definito in un sistema di riferimento e calcoleremo la probabilità di distribuzione del momento angolare in un sistema di riferimento traslato rispetto al sistema originale. In un successivo lavoro tratteremo il caso in cui i due sistemi di riferimento si muovono l'uno rispetto all'altro. Lo strumento matematico fondamentale è la forma dei generatori infinitesimali del gruppo inomogeneo di Lorentz ideati da Lomont e Moses, in cui l'hamiltoniana, il quadrato del momento angolare, la componente z del momento angolare, e l'elicità sono diagonali. Il presente lavoro e quello progettato sono importanti nei problemi di scattering multiplo, in quanto è possibile, utilizzandone i risultati, di tener conto, almeno fino a un certo limite, dell'effetto delle regole di selezione. Nei problemi di scattering multiplo tali regole sono quasi sempre ignorate. Si dimostra, ad esempio, che quando la densità di un gas è sufficientemente bassa, il raffreddamento radiativo procede molto più rapidamente se si tien conto delle regole di selezione di quanto non avvenga se si trascurano.

(*) *Traduzione a cura della Redazione.*