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by

Stanley C. Gianzero, Jr.

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POLYTECHNIC INSTITUTE OF BROOKLYN

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PIBAL REPORT NO. 873

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Stanley C. Cianzero, Jr.

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RADIATION BY A UNIFORMLY ROTATING
LINE CHARGE IN A PLASMA†

by

Stanley C. Gianzero, Jr. ‡
Polytechnic Institute of Brooklyn

SUMMARY

A theoretical investigation is conducted for the radiation produced by a linear distribution of electric charge executing circular motion both inside and outside a cylindrical plasma column. The analysis includes the effects of compressibility and anisotropy of the plasma upon the radiation characteristics of the charge distribution.

In the incompressible isotropic case, a dipole resonance phenomenon is exhibited for the first harmonic of the angular frequency of rotation of the charge when the charge moves at non-relativistic velocities. If the charge moves at extremely small velocities, the resonance becomes a singularity. The influence of compressibility upon these radiation characteristics is discussed and is shown to be negligible. In the case of the presence of a magnetic field, i.e. for an anisotropic plasma, the dipole resonance is shifted. Moreover, a multipole resonance is possible for a sufficiently higher order harmonic.

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Furthermore, in a frequency range just above this multipole resonance, Cerenkov radiation contributes to the existent Bremsstahlung radiation for a single harmonic in the neighborhood of the singularity of the index of refraction. Thereafter, the radiation contributions of the remaining harmonics is negligible.

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LIST OF SYMBOLS

$\mathcal{E}_\rho$, $\mathcal{E}_\varphi$	components of the electric field
$\mathcal{U}_\rho$, $\mathcal{U}_\varphi$	components of the velocity field
$\mathcal{H}_z$	component of the magnetic field
$\mathcal{N}$	density fluctuation
j	electric current density
m	order of the harmonics
q	electric charge per unit length
t	time
W	radiated energy per unit time
P	mechanical energy per unit time
δ	Dirac-delta function
ϵ_o, μ_o	dielectric and magnetic permeabilities of free space
ϵ_1, ϵ_2, ϵ_3	components of tensor dielectric permeability of plasma
β_a	ratio of charge velocity to the velocity of sound in medium
β_c	ratio of charge velocity to the velocity of light in free space
κ	index of refraction
ρ_o	radius of the orbit
ρ_1	radius of the column
ρ, φ, z	cylindrical coordinates
ω_c	cyclotron frequency
ω_o	angular frequency of the charge
ω_p	plasma frequency
k	propagation constant

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SECTION I

INTRODUCTION

The current interest in radio communication with space vehicles has stimulated the study of wave propagation in a plasma. Such a study is of importance in providing a knowledge of both the possible wave types which the plasma admits and their means of excitation. Extensive research on these topics has produced papers which study the plasma both in the absence and presence of localized electromagnetic sources.

Studies of the plasma in the absence of localized electromagnetic sources are conducted for the sole purpose of obtaining the possible types of waves which propagate in a plasma. The usual method for obtaining these wave types is through a macroscopic hydrodynamic approach, which couples the linearized Euler equations of motion with the linearized Maxwell equations of electrodynamics. Allis and Papa[1,2] and Stix[3], in studying both compressible and incompressible plasmas with and without a static magnetic field, classified these wave types in terms of normal wave surfaces. Oster[4] examined the same cases but also from the point of view of a microscopic gas treatment, using the Boltzmann equation with the Maxwell equations. The importance of this paper is that it demonstrates the validity of both the macroscopic and microscopic approaches.

Having thus considered the methods of approach, it is now desirable to discuss more fully the results of these papers. Firstly, in the case of an incompressible isotropic plasma, the plasma behaves as a dielectric, with one exception: the phase velocity of the electromagnetic waves is greater than the velocity of light in free space. This wave type is known as the ordinary optic mode of propagation. If, on the other hand, the plasma is compressible, it retains its isotropy but admits an additional wave type, a longitudinal wave which propagates at the speed of sound.

A more complex situation occurs if a static magnetic field is applied to an incompressible plasma. In such a situation, the plasma may be viewed as an anisotropic medium possessing a tensor dielectric permeability. Its principal waves are separated into either of two classifications, depending upon whether they are propagated along or across the magnetic field direction. The transverse electromagnetic waves (T.E.M.), which propagate along the direction of the magnetic field, decompose naturally into right and left circularly polarized waves. This natural decomposition is the basis for Faraday rotation which has application in optics. The transverse electric (T.E.) and transverse magnetic (T.M.) waves, those waves which propagate perpendicular to the magnetic field, comprise the second of the two classifications. The T.E. wave characteristics are identical to the isotropic plasma waves mentioned previously. However, the T.M. waves, which are a consequence of the anisotropy of the plasma, are called the extraordinary waves of propagation because of their resemblance to the extraordinary waves in double refracting crystals in optics.

When a static magnetic field is applied to a compressible plasma, the T.E.M. waves are modified slightly, these and the longitudinal plasma wave remaining uncoupled. However, in the case of propagation perpendicular to the field, the extraordinary wave is coupled to the plasma wave. Seshadri[5], exciting this same wave type (the extraordinary wave) did not expound upon the coupling phenomenon. It is the intention of this paper to interpret the radiated spectrum of energy more readily, by precisely clarifying this wave coupling. It may be considered from this brief survey of the literature that the source free case has been otherwise sufficiently explored, and so it shall not be the intent of this paper to investigate this particular area further.

It is now fitting to elaborate upon some of the numerous applications of wave propagation properties in plasma diagnostics. For one, plasmas may be inserted in microwave cavities and wave guides. Here, the presence of the plasma affects the resonant frequency of the system and increases its loss. From the frequency shift, the electron density can be calculated; from the loss, the collision frequency can be calculated. A more refined technique, involving a dipole resonance phenomenon for measuring electron density, is used by Crawford[6, 7]. Specifically, if the plasma is a cylindrical column, thereby possessing a finite dimension in the direction of impinging electromagnetic waves, it experiences a resonance when the excitation frequency of the plasma is $\omega = \frac{\omega_p}{\sqrt{2}}$. The chief advantage of incorporating a dipole resonance phenomenon, as opposed to studying perturbations of the resonant frequency of a microwave cavity, is that the dipole resonance is a first-order effect. The author will also consider resonances of a dipole type in his configuration. It is speculated that such a configuration, applied to the task of detecting electron densities, will yield a greater range of operating frequencies by merely adjusting the intensity of the magnetic field external to the plasma. The dipole resonance phenomenon is retained regardless of the consequent variation of velocity of the charge distribution, brought about by altering the magnetic field intensity.

An understanding of the physical phenomenon occurring in the configuration considered by the author can only be obtained through a careful study of the radiation produced by localized electromagnetic sources. Indeed, studies of the plasma in the presence of localized electromagnetic sources possess, by far, a great number of interesting physical implications. The author is particularly concerned with the presence of moving sources. It is a well known fact that if a uniformly moving charge distribution moves with a velocity which exceeds the phase velocity of light in the medium, a radiative process called Čerenkov radiation, exists. The radiated spectrum of this process consists of a continuum of frequencies.[8] A possible application of this radiation process in a plasma is the generation or amplification of electromagnetic waves in the microwave range. The radiation characteristics of a uniformly moving charge distribution have been the subject of many current papers. Interest in this type of motion, particularly from the aspect of Cerenkov radiation, has been considered not only by Tuan and Seshadri but is also contained in the papers of Majumdar and Abele.

Tuan and Seshadri's[9] investigations in this area consisted of a determination of the radiation characteristics of a point charge moving uniformly along the direction of a static magnetic field of an unbounded incompressible plasma. Here, for the first time, an author explicitly evaluates the multiple Cerenkov rays (which correspond to different frequency components) propagating in the same direction.

Majumdar[10], still considering the uniformly moving point charge, extended their results to include the case of a compressible magnetoplasma. Tuan and Seshadri[5] have also analyzed the radiation produced by the rectilinear motion of a line charge in a compressible magnetoplasma where the charge moves both along and perpendicular to the direction of an impressed magnetic field. It is in this investigation that they experienced the excitation of the coupled modes which the author will discuss more fully later.

Abele[11] examined the spectral distribution of energy produced by a uniformly moving line charge, not only in the previously studied unbounded case, but also in a bounded compressible plasma. His conclusion that radiation is also possible for a charge distribution moving outside a plasma finds application in the cases to be discussed by the author.

The subject of sources executing circular motion has also received considerable attention in current literature. In the case of circular motion of a charge distribution, the radiation may be confined to certain discrete frequencies within a specific range. In free space, a rotating point charge radiates a spectrum of lines which corresponds to the harmonics of the angular frequency of motion[12]. For non-relativistic velocities of the charge, the dominant part of the radiation is confined to the first few lines of the spectrum. For relativistic velocities, the spectral distribution of the radiated energy at first increases with the order of harmonics, reaches a maximum, and decreases thereafter.

A different behavior is expected if the charge rotates in a magneto-plasma[13]. In this case, the medium is highly dispersive and a resonant condition is expected. Moreover, the significant part of the radiation is confined precisely to the particular harmonic where the resonant condition exi ts. Here, the process of Cerenkov radiation contributes to the ordinary Bremsstahlung radiation[14].

Canobbio[15] investigated the radiation produced by a density modulated beam of ions in an infinite plasma for the case where the beam is an infinite plane parallel to the static mag ietic field, and for the case where the beam is an infinite cylindrical surface parallel to the magnetic field. In both situations, he studies resonances in the radiated energy, as will the author of this paper.

Twiss and Roberts[13], investigating the radiation produced by an electron moving in a circle in an incompressible anisotropic unbounded plasma, showed that of the two modes that are excited (ordinary and extraordinary) the radiation is emitted predominantly in the extraordinary mode. Although the corresponding problem of a line charge excites only the extraordinary mode, it is now clear that this is the only mode of importance in this type of investigation. The author, therefore, finds justification in considering the two-dimensional problem in preference to the three-dimensional one.

Finally, extensive research is found in the Russian literature. Here, considerable attention has been paid to the case where the radiation is produced by a point charge executing circular motion, but now in a compressible unbounded plasma. It has been found that an appreciable amount of the radiated energy can be associated with the longitudinal plasma waves[16,17,18].

Having thus considered the contribution of Cerenkov radiation to the ordinary Bremsstahlung radiation in the unbounded cases, it is prudent to once again mention the fact that if the charge is moving in the vicinity of a plasma column, a dipole resonance is anticipated.

It is clear, from the above synopsis, that a systematic study of the problem of radiation from a rotating line charge in both an unbounded compressible plasma and a bounded compressible plasma remains to be determined. This paper, therefore, proposes to fully investigate the radiation characteristics of a line charge in the presence of a plasma column.

The author is indebted to Dr. Manlio Abele for his innumerable suggestions, his erudite opinions, and his constant guidance.

SECTION II

BASIC EQUATIONS

The present investigation is conducted for the case of a uniform linear distribution of electric charge, oriented parallel to the z axis of a cylindrical coordinate system. The charge distribution rotates about the z axis (which is parallel to an impressed static magnetic field) with a constant angular velocity ω_o. q and ρ_o denote the charge per unit length and the radius of the orbit respectively.* A cylindrical plasma column of radius ρ_1 is located within the orbit of the charge distribution, oriented parallel to its axis of rotation. The boundary of the plasma is assumed to be perfectly rigid; the plasma medium is assumed to be compressible and lossless. It is further assumed that the angular frequency of motion of the charge distribution is sufficiently large so that the ion motion may be neglected. Finally, the intensity of the electromagnetic field is assumed small enough so that the equations of motion may be linearized. The charge distribution produces a current density which may be described as a continuous current distribution in the following manner:

$$j_\rho = j_z = 0 \quad , \quad j_\varphi = \omega_o q \delta(\rho - \rho_o) \delta(\omega_o t - \varphi) \tag{2-1}$$

where $\delta(\rho-\rho_o)$ and $\delta(\omega_o t-\varphi)$ are Dirac-delta functions. The φ component

---

*The fact that the angular frequency of rotation of the charge distribution is not synchronized with the cyclotron frequency is not inconsistent, since a physical situation can always be realized wherein the intensity of the magnetic field outside the plasma differs from the intensity inside.

can be written as,

$$j_{\varphi} = \frac{\omega_o q}{2\pi} \delta(\rho - \rho_o) \sum_{m=-\infty}^{m=\infty} e^{-im\omega_o \tau} \tag{2-2}$$

where τ is related to the time t and the angular position φ, as

$$\tau = t - \varphi/\omega_o \tag{2-3}$$

The current density induces in the plasma a density fluctuation, $\mathcal{N}$; an electron motion with velocity components $\mathcal{U}_\rho, \mathcal{U}_\varphi$; and an electromagnetic field with components $\mathcal{E}_\rho, \mathcal{E}_\varphi$, of the electric field, and the component $\mathcal{H}_z$ of the magnetic field.

The chosen method of solution for the above field quantities requires first, a separation of the solution into cylindrical regions about ρ_o and ρ_1. Then, a particular solution of the linearized Maxwell equations and the linearized hydrodynamic equations for $\mathcal{H}_z$ and $\mathcal{N}$ (which is consistent with the current density j_φ) is used to find the remaining field components in all regions. Finally, the field components are matched at ρ_o and ρ_1 with the aid of specified boundary conditions.

Thus, the governing equations inside the plasma medium are,

$$\frac{\partial \mathcal{N}}{\partial t} + n_o \nabla \cdot \underset{\sim}{\mathcal{U}} = 0 \tag{2-4}$$

$$\frac{\partial \underset{\sim}{\mathcal{U}}}{\partial t} + \frac{q_o}{m_e} \underset{\sim}{\mathcal{E}} + \underset{\sim}{\mathcal{U}} \times \underset{\sim}{\omega}_c + \frac{a^2}{n_o} \nabla \mathcal{N} = 0 \tag{2-5}$$

$$\nabla \times \underset{\sim}{\mathcal{E}} + \mu_o \frac{\partial \underset{\sim}{\mathcal{H}}}{\partial t} = 0 \tag{2-6}$$

$$\nabla \times \underset{\sim}{\mathcal{H}} - \epsilon_o \frac{\partial \underset{\sim}{\mathcal{E}}}{\partial t} + n_o q_o \underset{\sim}{\mathcal{U}} = 0 \tag{2-7}$$

where m_e and q_o are the electron mass and electric charge, respectively;

n_o is the equilibrium value of the electron density $\mathcal{N}$; a is the speed of sound of the electron gas of equilibrium temperature, T_o, given by,

$$a = (\gamma kT_o/m_e)^{\frac{1}{2}} \tag{2-8}$$

Also, k is the Boltzmann constant; and γ is related to the number of degrees of freedom, ℓ, of the electron adiabatic motion through the equation,

$$\gamma = (\ell+2)/\ell \tag{2-9}$$

ϵ_o, μ_o are the dielectric and magnetic permeabilities of free space, respectively; $\underset{\sim}{\mathcal{E}}, \underset{\sim}{\mathcal{H}}$ are the intensities of the electric and magnetic fields; and $\underset{\sim}{\mathcal{U}}$ is the electron macroscopic velocity. Finally, the magnitude of the cyclotron frequency ω_c is given by,

$$\omega_c = \frac{\mu_o q_o \mathcal{H}_o}{m_e} \tag{2-10}$$

A particular solution of the governing equations for the field components, which is consistent with the current density, is,

$$\mathcal{A}(\rho, \varphi, t) = \sum_{m=-\infty}^{m=\infty} A_m(\rho) e^{-im\omega_o \tau} \tag{2-11}$$

By virtue of Eq. (2-11), the governing equations (2-4), (2-5), (2-6), and (2-7) may be written in the following component form,

$$-im\omega_o N_m + n_o \left[\frac{1}{\rho} \frac{d(\rho U_{\rho m})}{d\rho} + \frac{im}{\rho} U_{\varphi m} \right] = 0 \tag{2-12}$$

$$-im\omega_o U_{\rho m} + \frac{q_o}{m_e} E_{\rho m} + \omega_c U_{\varphi m} + \frac{a^2}{n_o} \frac{dN_m}{d\rho} = 0 \tag{2-13}$$

$$-im\omega_o U_{\varphi m} + \frac{q_o}{m_e} E_{\varphi m} - \omega_c U_{\rho m} + \frac{ima^2}{n_o \rho} N_m = 0 \tag{2-14}$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho E_{\varphi m}) - \frac{im}{\rho} E_{\rho m} - im\omega_o \mu_o H_{zm} = 0 \tag{2-15}$$

$$\frac{im}{\rho} H_{zm} + im\omega_o \epsilon_o E_{\rho m} + n_o q_o U_{\rho m} = 0 \tag{2-16}$$

$$-\frac{dH_{zm}}{d\rho} + im\omega_o \epsilon_o E_{\varphi m} + n_o q_o U_{\varphi m} = 0 \tag{2-17}$$

Solving Eqs. (2-13) and (2-14) for the velocity components, gives

$$n_o q_o U_{\rho m} = \frac{1}{m^2\omega_o^2 - \omega_c^2}\left\{-\epsilon_o \omega_p^2 [im\omega_o E_{\rho m} + \omega_c E_{\varphi m}] - q_o a^2 [im\omega_o \frac{dN_m}{d\rho} + \frac{im\omega_c}{\rho} N_m]\right\} \tag{2-18}$$

$$n_o q_o U_{\varphi m} = \frac{1}{m^2\omega_o^2 - \omega_c^2}\left\{\epsilon_o \omega_p^2 [-im\omega_o E_{\varphi m} + \omega_c E_{\rho m}] + q_o a^2 [\frac{m^2\omega_o}{\rho} N_m + \omega_c \frac{dN_m}{d\rho}]\right\} \tag{2-19}$$

where the plasma frequency is defined as,

$$\omega_p^2 = \frac{n_o q_o^2}{\epsilon_o m_e} \tag{2-20}$$

Substituting Eqs. (2-18) and (2-19) into (2-16) and (2-17), and solving for the transverse field components $E_{\rho m}$ and $E_{\varphi m}$ in terms of the longitudinal field components H_{zm} and the electron density term N_m, yields,

$$E_{\rho m} = \frac{1}{\epsilon_o k_{sm}^2 k_{\ell m}^2}\left[\frac{q_o m^2 \omega_o \omega_c}{\rho c^2} N_m + q_o k_{om}^2 \frac{dN_m}{d\rho} - \frac{m^2 \omega_o k_{am}^2}{\rho c^2} H_{zm} - \frac{\omega_c \omega_p^2}{a^2 c^2}\frac{dH_{zm}}{d\rho}\right] \tag{2-21}$$

$$E_{\varphi m} = \frac{i}{\epsilon_o k_{sm}^2 k_{\ell m}^2}\left[\frac{m q_o k_{om}^2}{\rho} N_m + \frac{q_o m \omega_o \omega_c}{c^2}\frac{dN_m}{d\rho} - \frac{m \omega_c \omega_p^2}{\rho a^2 c^2} H_{zm} - \frac{m \omega_o k_{am}^2}{c^2}\frac{dH_{zm}}{d\rho}\right] \tag{2-22}$$

where $c = 1/\sqrt{\mu_o \epsilon_o}$ is the speed of light in free space. The propagation constants k_{sm}, $k_{\ell m}$ are given in terms of k_{om}, k_{am} as follows:

$$k^2_{sm} = \frac{k^2_{om} + k^2_{am}}{2} + \frac{1}{2}\sqrt{(k^2_{om} - k^2_{am})^2 + \frac{4\omega_p^2 \omega_c^2}{a^2 c^2}} \tag{2-23a}$$

$$k^2_{\ell m} = \frac{k^2_{om} + k^2_{am}}{2} - \frac{1}{2}\sqrt{(k^2_{om} - k^2_{am})^2 + \frac{4\omega_p^2 \omega_c^2}{a^2 c^2}} \tag{2-23b}$$

where k_{am}, k_{om} are given by,

$$k^2_{om} = \frac{m^2 \omega_o^2 - \omega_p^2}{c^2} \tag{2-24}$$

$$k^2_{am} = \frac{m^2 \omega_o^2 - \omega_p^2 - \omega_c^2}{a^2} \tag{2-25}$$

The propagation constant k_{om} is the ordinary optic mode of an isotropic plasma, and k_{am} is the corresponding acoustic mode which is modified somewhat due to the presence of the magnetic field.

Because of their complexity, the propagation constants defined by Eqs. (2-23) warrant explanation and, therefore, a slight degression from the present discussion is desirable. It must be recalled from Section I that in a compressible, anisotropic plasma, the extraordinary electromagnetic mode of propagation is coupled to the acoustic mode of propagation. Equation (2-23) is a statement of this physical phenomenon. A clear understanding of this mode coupling can be obtained from a study of the corresponding indices of refraction. In such a consideration, the excitation frequency $m\omega_o$ is assumed to be equal to ω, a continuous variable, and therefore the subscript m will be omitted in all the defining relations. If it is further assumed that the speed of sound in the electron gas is much smaller than the speed of light, the propagation constants k_s and k_ℓ, corresponding to Eq. (2-23), may be approximated as follows:

$$
\begin{aligned}
k_s^2 &= k_a^2 + \frac{\omega_p^2 \omega_c^2}{a^2 c^2 k_a^2} + \frac{k_o^4}{4k_a^2} \\
k_\ell^2 &= k_e^2 - \frac{k_o^4}{4k_a^2}
\end{aligned}
\tag{2-26}
$$

where $k_e^2 = \frac{\omega^2}{c^2} \frac{(\epsilon_1^2 - \epsilon_2^2)}{\epsilon_1}$ is the extraordinary mode of propagation for an incompressible, anisotropic plasma described by a tensor dielectric permeability. The tensor dielectric permeability being defined as,

$$
\underset{\approx}{\epsilon} = \begin{pmatrix} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}
\tag{2-27}
$$

where

$$
\begin{aligned}
\epsilon_1 &= 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \\
\epsilon_2 &= \frac{\omega_c \omega_p^2}{(\omega^2 - \omega_c^2)\omega} \\
\epsilon_3 &= 1 - \frac{\omega_p^2}{\omega^2}
\end{aligned}
\tag{2-28}
$$

It is important to note that the previous approximations in Eq. (2-26) are valid for all frequencies except those frequencies which satisfy the following relation:

$$
\omega^2 - \omega_p^2 - \omega_c^2 \cong 0 \tag{2-29}
$$

Figure 1 is a plot of the indices of refraction, corresponding to Eqs. (2-23) and their approximate forms [Eqs. (2-26)] for arbitrary values of non-dimensionalized frequency ω/ω_c. These results show that a reasonably good approximation to the coupled propagation constants may be obtained by merely retaining the first terms in the expansions in Eq. (2-26).

Specifically,

$$\begin{aligned} k_s &\sim k_e \\ k_\ell &\sim k_a \end{aligned} \quad ; \quad 0 \le \frac{\omega}{\omega_c} < \sqrt{1 + \left(\frac{\omega_p}{\omega_c}\right)^2}$$

and (2-30)

$$\begin{aligned} k_s &\sim k_a \\ k_\ell &\sim k_o \end{aligned} \quad ; \quad \sqrt{1 + \left(\frac{\omega_p}{\omega_c}\right)^2} < \frac{\omega}{\omega_c} \le \infty$$

It is evident that in this approximation the region defined by strong coupling actually comprises only a narrow range of frequencies which separate the entire frequency spectrum into two ranges, wherein the modes are effectively uncoupled. Moreover, the coupled propagation constants assume alternate roles in these ranges of frequency. Specifically, for low frequencies, k_s behaves as the extraordinary mode of propagation of an incompressible, anisotropic plasma, where as k_ℓ behaves as a modified acoustic mode of propagation of a compressible plasma. For high frequencies, a reversal is evidenced. k_s now behaves as the modified acoustic mode and k_ℓ behaves as the ordinary optic mode of propagation. It is interesting to note that this switching of modes occurs precisely in the frequency range where a singularity of the index of refraction would exist if the plasma were incompressible. The results of this interesting phenomenon will be applied to one of the cases to be discussed later.

Returning to the previous discussion, the velocity components

$U_{\rho m}$ and $U_{\varphi m}$, corresponding to Eqs. (2-21) and (2-22), are found by substituting Eqs. (2-21) and (2-22) into Eqs. (2-18) and (2-19). Then, simplifying them yields,

$$U_{\rho m} = \frac{-i}{k^2_{sm} k^2_{\ell m}} \left[\frac{m^3 \omega_o^2 \omega_c}{n_o c^2 \rho} N_m + \frac{m \omega_o k^2_{om}}{n_o} \frac{dN_m}{d\rho} - \frac{m \omega_p^2 k^2_{om}}{n_o q_o a^2 \rho} H_{zm} - \frac{m \omega_o \omega_c \omega_p^2}{n_o q_o a^2 c^2} \frac{dH_{zm}}{d\rho} \right] \tag{2-31}$$

$$U_{\varphi m} = \frac{1}{k^2_{sm} k^2_{\ell m}} \left[\frac{m^2 \omega_o k^2_{om}}{n_o \rho} N_m + \frac{m^2 \omega_o^2 \omega_c}{n_o c^2} \frac{dN_m}{d\rho} - \frac{m^2 \omega_o \omega_c \omega_p^2}{n_o q_o a^2 c^2 \rho} H_{zm} - \frac{\omega_p^2 k^2_{om}}{n_o q_o a^2} \frac{dH_{zm}}{d\rho} \right] \tag{2-32}$$

The equations which the longitudinal field components satisfy are found by substituting Eqs. (2-21), (2-22), (2-31), and (2-32) into Eqs. (2-12) and (2-15). Then, after some simplification,

$$\frac{k^2_{am}}{q_o \omega_c} \left\{ \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dH_{zm}}{d\rho} \right) + \left[\frac{m^2 \omega_o^2}{c^2} - \frac{\omega_p^2 k^2_{om}}{a^2 k^2_{am}} - \frac{m^2}{\rho^2} \right] H_{zm} \right\} - \left\{ \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dN_m}{d\rho} \right) - \frac{m^2}{\rho^2} N_m \right\} = 0 \tag{2-33}$$

$$\frac{q_o a^2 c^2 k^2_{om}}{\omega_c \omega_p^2} \left\{ \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dN_m}{d\rho} \right) + \left[\frac{m^2 \omega_o^2 k^2_{am}}{c^2 k^2_{om}} - \frac{\omega_p^2}{a^2} - \frac{m^2}{\rho^2} \right] N_m \right\} - \left\{ \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dH_{zm}}{d\rho} \right) - \frac{m^2}{\rho^2} H_{zm} \right\} = 0 \tag{2-34}$$

The above equations can be uncoupled by first assuming the solutions to be linear combinations of N_m and H_{zm} with arbitrary coefficients.

These coefficients are then adjusted to yield two separate Bessel equations in the assumed solutions.* The result is

$$
N_m = \begin{cases} \dfrac{1}{(k^2_{sm} - k^2_{\ell m})} \Big[(k^2_{om} - k^2_{\ell m}) b_{im} J_m(k_{\ell m}\rho) - \\ \qquad (k^2_{om} - k^2_{sm}) a_{im} J_m(k_{sm}\rho) \Big] \; ; \quad \rho < \rho_1 & \text{(2-35a)} \\ 0 \; ; \quad \rho_1 < \rho < \infty & \text{(2-35b)} \end{cases}
$$

$$
H_{zm} = \begin{cases} \dfrac{q_o \omega_c}{(k^2_{sm} - k^2_{\ell m})} \Big[a_{im} J_m(k_{sm}\rho) - b_{im} J_m(k_{\ell m}\rho) \Big] \; ; \; \rho < \rho_1 & \text{(2-36a)} \\ a'_{im} J_m(k_m \rho) + a''_{im} Y_m(k_m \rho) \; ; \; \rho_1 < \rho < \rho_0 & \text{(2-36b)} \\ a_{em} H^{(1)}_m(k_m \rho) \; ; \; \rho_0 < \rho < \infty & \text{(2-36c)} \end{cases}
$$

where J_m, Y_m, and $H^{(1)}_m$ are the Bessel, Neumann, and Hankel functions respectively; and a_{im}, b_{im}, a'_{im}, a''_{im}, and a_{em} are the constants of integration which are to be determined from the boundary conditions at ρ_o and ρ_1. Note that the magnetic field components in the regions outside the plasma column have been obtained from the solutions of Eq. (2-33) in the limit of $\omega_p = 0$. Also, the propagation constant k_m in these regions is defined as

$$
k_m = \frac{m\omega_o}{c} \qquad \text{(2-37)}
$$

It is important to mention that the Hankel function of the first kind has been chosen in order to insure outgoing waves for positive and

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*See Appendix for explicit calculation.

negative m for the specified time dependence. The transverse field components in all regions can now be obtained by substituting Eq. (2-35a) and Eq. (2-36a) into Eqs. (2-21) and (2-22), and then by substituting the remaining equations in (2-35) and (2-36) into Eqs. (2-21) and (2-22) in the limit of $\omega_p = 0$.

$$E_{\rho m} = \begin{cases} \dfrac{-q_o}{\epsilon_o(k_{sm}^2 - k_{\ell m}^2)} \left\{ a_{im}\left[\dfrac{m^2\omega_o\omega_c}{\rho c^2 k_{sm}^2} J_m(k_{sm}\rho) + \dfrac{(k_{om}^2 - k_{sm}^2)}{k_{sm}} J'_m(k_{sm}\rho)\right] \right. \\ \qquad \left. -b_{im}\left[\dfrac{m^2\omega_o\omega_c}{\rho c^2 k_{\ell m}^2} J_m(k_{\ell m}\rho) + \dfrac{(k_{om}^2 - k_{\ell m}^2)}{k_{\ell m}} J'_m(k_{\ell m}\rho)\right]\right\} ; \rho < \rho_1 & \text{(2-38a)} \\ \dfrac{-m^2\omega_o\mu_o}{\rho k_m^2}\left[a'_{im} J_m(k_m\rho) + a''_{im} Y_m(k_m\rho)\right] ; \quad \rho_1 < \rho < \rho_o & \text{(2-38b)} \\ \dfrac{-m^2\omega_o\mu_o}{\rho k_m^2} a_{em} H_m^{(1)}(k_m\rho) ; \quad \rho_o < \rho < \infty & \text{(2-38c)} \end{cases}$$

$$E_{\varphi m} = \begin{cases} -\dfrac{imq_o}{\epsilon_o(k_{sm}^2 - k_{\ell m}^2)} \left\{ a_{im}\left[\dfrac{(k_{om}^2 - k_{sm}^2)}{\rho k_{sm}^2} J_m(k_{sm}\rho) + \dfrac{\omega_o\omega_c}{c^2 k_{sm}} J'_m(k_{sm}\rho)\right] \right. \\ \qquad \left. -b_{im}\left[\dfrac{(k_{om}^2 - k_{\ell m}^2)}{\rho k_{\ell m}^2} J_m(k_{\ell m}\rho) + \dfrac{\omega_o\omega_c}{c^2 k_{\ell m}} J'_m(k_{\ell m}\rho)\right]\right\} ; \rho < \rho_1 & \text{(2-39a)} \\ -\dfrac{im\omega_o\mu_o}{k_m}\left[a'_{im} J'_m(k_m\rho) + a''_{im} Y'_m(k_m\rho)\right] ; \quad \rho_1 < \rho < \rho_o & \text{(2-39b)} \\ -\dfrac{im\omega_o\mu_o}{k_m} a_{em} H_m^{(1)\prime}(k_m\rho) ; \quad \rho_o < \rho < \infty & \text{(2-39c)*} \end{cases}$$

---

*Note that all derivatives of the Bessel functions are taken with respect to the entire argument.

The necessary boundary conditions which are to be used for the evaluation of the constants of integration when expressed in terms of the total field components are

$$\left.\begin{aligned}
&\left[\mathcal{H}_z\right]_{i'} - \left[\mathcal{H}_z\right]_i = 0 \\
&\left[\mathcal{E}_\varphi\right]_{i'} - \left[\mathcal{E}_\varphi\right]_i = 0 \\
&\mathcal{U}_\rho = 0^{*}
\end{aligned}\right\} ; \ \rho = \rho_1 \tag{2-40}$$

$$\left.\begin{aligned}
&\left[\mathcal{H}_z\right]_e - \left[\mathcal{H}_z\right]_{i'} = -\lim_{\delta \to 0} \int_{\rho_o - \delta}^{\rho_o + \delta} j_\varphi \, d\rho \\
&\left[\mathcal{E}_\varphi\right]_e - \left[\mathcal{E}_\varphi\right]_{i'} = 0
\end{aligned}\right\} \rho = \rho_o \tag{2-41}$$

where the subscript i represents the region defined by $\rho < \rho_1$; the primed subscript i' represents the region defined by $\rho_1 < \rho < \rho_o$; and the subscript e represents the region defined by $\rho_o < \rho < \infty$.

The previous conditions can also be expressed in terms of the harmonic field components as

$$\left.\begin{aligned}
&\left[H_{zm}\right]_{i'} - \left[H_{zm}\right]_i = 0 \\
&\left[E_{\varphi m}\right]_{i'} - \left[E_{\varphi m}\right]_i = 0 \\
&U_{\rho m} = 0
\end{aligned}\right\} ; \ \rho = \rho_1 \tag{2-42}$$

---

*This dynamical boundary condition, which is a consequence of a rigid wall, is used only in the compressible plasma cases.

$$\left.\begin{aligned} \left[H_{zm}\right]_e - \left[H_{zm}\right]_{i'} &= -\frac{\omega_o q}{2\pi} \\ \left[E_{\varphi m}\right]_e - \left[E_{\varphi m}\right]_{i'} &= 0 \end{aligned}\right\} ; \rho = \rho_o \tag{2-43}$$

Conditions (2-42) and (2-43) lead to the following system of equations:

$$[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_m\rho_1)] - \frac{q_o \omega_c}{(k^2_{sm} - k^2_{\ell m})}$$

$$[a_{im} J_m(k_{sm}\rho_1) - b_{im} J_m(k_{\ell m}\rho_1)] = 0 \tag{2-44}$$

$$[a'_{im} J'_m(k_m\rho_1) + a''_{im} Y'_m(k_m\rho_1)] - \frac{q_o k_m}{(k^2_{sm} - k^2_{\ell m})}$$

$$\left\{ a_{im}\left[\frac{c^2(k^2_{om} - k^2_{sm})}{\omega_o \rho_1 k^2_{sm}} J_m(k_{sm}\rho_1) + \frac{\omega_c}{k_{sm}} J'_m(k_{sm}\rho_1)\right]\right.$$

$$\left. + b_{im}\left[\frac{c^2(k^2_{om} - k^2_{\ell m})}{\omega_o \rho_1 k^2_{\ell m}} J_m(k_{\ell m}\rho_1) + \frac{\omega_c}{k_{\ell m}} J'_m(k_{\ell m}\rho_1)\right]\right\} = 0$$

(2-45)

$$[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_m\rho_1)] - \frac{q_o k^2_m}{(k^2_{sm} - k^2_{\ell m})}$$

$$\left\{ a_{im}\left[\frac{\omega_c}{k^2_{sm}} J_m(k_{sm}\rho_1) + \frac{\rho_1 c^2(k^2_{om} - k^2_{sm})}{m^2 \omega_o k_{sm}} J'_m(k_{sm}\rho_1)\right]\right.$$

$$\left. - b_{im}\left[\frac{\omega_c}{k^2_{\ell m}} J_m(k_{\ell m}\rho_1) + \frac{\rho_1 c^2(k^2_{om} - k^2_{\ell m})}{m^2 \omega_o k_{\ell m}} J'_m(k_{\ell m}\rho_1)\right]\right\} = 0$$

(2-46)

$$a_{em} H_m^{(1)}(k_m \rho_o) - [a'_{im} J_m(k_m \rho_o) + a''_{im} Y_m(k_m \rho_o)] = -\frac{\omega_o q}{2\pi} \tag{2-47}$$

$$a_{em} H_m^{(1)'}(k_m \rho_o) - [a'_{im} J'_m(k_m \rho_o) + a''_{im} Y'_m(k_m \rho_o)] = 0 \tag{2-48}$$

Normally, the above equations would be solved for the constant of integration, a_{em}, since the field quantities [see Eqs. (2-36c) and (2-39c)], where the radiation is to be evaluated, involve only this constant. However, an explicit evaluation of a_{em} will be temporarily postponed.

In order to obtain W, the power radiated by the charge distribution per unit length, the flux of the Poynting vector upon a cylinder of unit height at an arbitrary radius $\rho > \rho_o$, coaxial with the z axis, will be evaluated from Eqs. (2-36c) and (2-39c). Rewriting the necessary field components

$$\left[\mathcal{H}_z\right]_e = \sum_{m=-\infty}^{m=\infty} a_{em} H_m^{(1)}(k_m \rho) e^{-im\omega_o \tau} \tag{2-49}$$

$$\left[\mathcal{E}_\varphi\right]_e = -i\omega_o \mu_o \sum_{n=-\infty}^{n=\infty} \frac{n}{k_n} a_{en} H_n^{(1)'}(k_n \rho) e^{-in\omega_o \tau} \tag{2-50}$$

Then,

$$W = \int_0^{2\pi} \rho \left[\mathcal{E}_\varphi\right]_e \left[\mathcal{H}_z\right]_e d\varphi =$$

$$-i\omega_o \mu_o \rho \int_0^{2\pi} \sum_{m=-\infty}^{m=\infty} \sum_{n=-\infty}^{n=\infty} \frac{n}{k_n} a_{en} a_{em} H_n^{(1)'}(k_n \rho) H^{(1)}(k_m \rho) e^{-i(n+m)\omega_o \tau} d\varphi \tag{2-51}$$

It is apparent that the integration over the angle φ is non-vanishing only for $m = -n$. Thus, the double sum reduces to the following single sum:

$$W = -2\pi i\omega_o\mu_o\rho \sum_{m=-\infty}^{m=\infty} \frac{m}{k_m} a_{e-m} a_{em} H_{-m}^{(1)'}(-k_m\rho) H_m^{(1)}(k_m\rho) \tag{2-52}$$

where use was made of the fact that $k_{-m} = -k_m$.

If the sum is now divided into two sums for positive and negative m respectively, and the negative sum is converted into a positive sum by replacing +m by -m wherever it occurs, the result is,

$$W = -2\pi i\omega_o\mu_o\rho \sum_{m=1}^{m=\infty} \frac{m}{k_m} a_{e-m} a_{em} [H_{-m}^{(1)'}(-k_m\rho) H_m^{(1)}(k_m\rho) + H_m^{(1)'}(k_m\rho) H_{-m}^{(1)}(-k_m\rho)] \tag{2-53}$$

Finally,

$$W = -2\pi i\omega_o\mu_o\rho \sum_{m=1}^{m=\infty} \frac{m}{k_m} a_{e-m} a_{em} [H_m^{(1)}(k_m\rho) H_m^{(2)'}(k_m\rho) - H_m^{(2)}(k_m\rho) H_m^{(1)'}(k_m\rho)] =$$

$$-8\omega_o\mu_o \sum_{m=1}^{m=\infty} \frac{m}{k_m^2} a_{e-m} a_{em} \tag{2-54}$$

since,

$$H_m^{(1)}(k_m\rho) H_m^{(2)'}(k_m\rho) - H_m^{(2)}(k_m\rho) H_m^{(1)'}(k_m\rho) = \frac{-4i}{\pi k_m \rho} \;*$$

*See reference 19.

SECTION III

THE CHARGE DISTRIBUTION OUTSIDE THE PLASMA COLUMN

The first case to be examined is that of an incompressible, isotropic plasma column. Here, it is assumed that the thermal velocity of the electrons and the intensity of the magnetic field in the plasma are vanishingly small. Thus, Eqs. (2-44) through (2-48), which are a result of the boundary conditions, reduce to the following.

$$[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_m\rho_1)] - a_{im} J_m(k_{om}\rho_1) = 0 \tag{3-1}$$

$$[a'_{im} J'_m(k_m\rho_1) + a''_{im} Y'_m(k_m\rho_1)] - \frac{k_m}{k_{om}} a_{im} J'_m(k_{om}\rho_1) = 0 \tag{3-2}$$

$$a_{em} H^{(1)}_m(k_m\rho_o) - [a'_{im} J_m(k_m\rho_o) + a''_{im} Y_m(k_m\rho_o)] = \frac{-\omega_o q}{2\pi} \tag{3-3*}$$

$$a_{em} H^{(1)\prime}_m(k_m\rho_o) - [a'_{im} J'_m(k_m\rho_o) + a''_{im} Y'_m(k_m\rho_o)] = 0 \tag{3-4}$$

A simultaneous solution of the above equations for the constant of integration a_{em}, the only constant of importance, results in,

$$a_{em} = \frac{-\omega_o \rho_o k_m q}{4}\left[\frac{\mathcal{R}_e \Delta_{om} Y'_m(k_m\rho_o) - \mathcal{I}_m \Delta_{om} J'_m(k_m\rho_o)}{\Delta_{om}}\right] \tag{3-5}$$

where

$$\Delta_{om} = \frac{1}{k_m} J_m(k_{om}\rho_1) H^{(1)\prime}_m(k_m\rho_1) - \frac{1}{k_{om}} J'_m(k_{om}\rho_1) H^{(1)}_m(k_m\rho_1) \tag{3-6}$$

---

*Equations (3-3) and (3-4) are repeated for sake of completeness.

Direct substitution of these results into Eq. (2-54) gives the expression for the electromagnetic radiation outside the plasma column.

$$W = + \frac{\omega_o q^2 \beta_c^2}{2\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m\left[\mathcal{R}_e(\Delta_{om}) Y'_m(m\beta_c) - \mathcal{J}_m(\Delta_{om}) J'_m(m\beta_c)\right]^2}{\Delta_{om}\Delta^*_{om}} \qquad (3\text{-}7)$$

where $\beta_c = \frac{\omega_o \rho_o}{c}$ is a measure of the ratio of the charge velocity to the velocity of light in free space, and the asterisk denotes the complex conjugate of a quantity.

If now it is assumed that the charge moves with extremely non-relativistic velocities, $\beta_c << 1$, the plasma column sees effectively a uniform static electric field when the dimensions of the orbit of the charge are much greater than the plasma column. Therefore, the physical situation approaches the conditions necessary for the plasma to exhibit a dipole resonance.

Now for $\beta_c << 1$, the Bessel functions may be approximated for small arguments.

In particular,

$$J_m(m\beta_c) \simeq \frac{(m\beta_c)^m}{2^m(m!)} \quad ; \quad Y_m(m\beta_c) \simeq \frac{-(m-1)!\, 2^m}{\pi(m\beta_c)^m} \qquad (3\text{-}8)^*$$

Applying the approximations of Eqs. (3-8) to Eq. (3-7) gives,

$$W = \frac{\omega_o q^2}{2\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^2 m(m!)^2} \left\{ 1 + \frac{(\rho_1/\rho_o)^2 m\left(\frac{\omega_p}{m\omega_o}\right)^2}{\left[2 - \left(\frac{\omega_p}{m\omega_c}\right)^2\right]} \right\}^2 \qquad (3\text{-}9)^{**}$$

---

*See reference (19).

**The appearance of the factor ρ_1/ρ_o in Eq. (3-9) is explicable in terms of a scattering cross section.

It is evident from an inspection of the above equation that a resonant condition in the electromagnetic radiation exists for those frequencies which satisfy the following relation,

$$\frac{\omega_p}{m\omega_o} = \sqrt{2} \tag{3-10}$$

The significance of this result is realized when comparing it to the corresponding free space problem for non-relativistic velocities, the solution of which can be deduced directly from Eq. (3-9) by setting ω_p equal to zero.

The result is,

$$W_o = \frac{\omega_o q^2}{2\varepsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^{2m}(m!)^2} \tag{3-11}$$

A comparison of Eq. (3-9) with (3-11) shows that the energy W_m radiated at the mth harmonic compared to the energy W_{mo} radiated in free space at the same harmonic is

$$W_m/W_{mo} = \left\{1 + \frac{(\rho_1/\rho_o)^2 m(\frac{\omega_p}{m\omega_o})^2}{\left[2 - (\frac{\omega_p}{m\omega_o})^2\right]}\right\}^2 \tag{3-12}$$

Result (3-12) has been obtained for $\beta_c \to 0$. The singularity which is found at $\omega_p = \sqrt{2}\, m\,\omega_o$ disappears when the effect of a small but finite value of β_c is included in the analysis of (3.7). For the sake of simplicity, the subsequent discussion is confined to the fundamental frequency (m=1).

If then, Eq. (3-7) is maximized with respect to the independent variable, $\frac{\omega_p}{\omega_o}$, the condition for maximum reduces to the following transcendental equation exactly,

$$J_1'(\beta_c)\mathcal{J}_e\Delta_{01} + Y_1'(\beta_c)\mathcal{J}_m\Delta_{01} = 0 \qquad (3\text{-}13)$$

It is important to note that the above result is also maintained in a compressible, isotropic plasma as well as in an incompressible, anisotropic plasma.

An insertion of higher order corrections to the Bessel functions into Eq. (3-13) yields the following frequency condition,

$$\frac{\omega_p}{\omega_o} = \sqrt{2}\left[1 - \frac{\beta_c^2(\rho_1/\rho_o)^2}{4}\,\frac{\ln \gamma\beta_c(\rho_1/\rho_o)}{2}\right] \qquad (3\text{-}14)$$

Thus, the maximum is seen to occur at a frequency shifted from the dipole singularity in the direction of increasing plasma frequency.

Substituting Eq. (3-13), in total, into Eq. (3-7) yields,

$$[W_1]_{max} = \frac{\omega_o q^2\beta_c^2}{2\epsilon_o}\left[J_1'^2(\beta_c) + Y_1'^2(\beta_c)\right] \qquad (3\text{-}15)$$

which reduces to

$$[W_1]_{max} \cong \frac{2\omega_o q^2}{\epsilon_o \pi^2 \beta_c^2} \qquad (3\text{-}16)$$

The results of a numerical calculation which extends the previous case to include relativistic velocities of the charge distribution, are found in Fig. 4. An inspection of the figure reveals that, for extremely low frequencies, there are fluctuations in the electromagnetic radiation. These fluctuations are attributed to electromagnetic interference phenomena. Thus, when the plasma is overdense, $\frac{\omega_p}{\omega_o} > 1$, the plasma acts as a reflector of electromagnetic waves and resonant conditions are found for certain values of the ratio ρ_1/ρ_o.

More significant is the fact that even for almost relativistic velocities of the charge, the radiation experiences a large maximum. In particular, for $\beta_c = .9$ in Fig. 4, the maximum radiation is of the order of ten times that of free space. Furthermore, the maximum occurs for $\frac{\omega_p}{\omega_o}$ shifted slightly from the value $\sqrt{2}$. [This result is consistent with

Eq. (3-14)]. Thus, it is apparent that the dipole resonance has to be expected even at large velocities of the charge distribution.

The effect of compressibility upon the radiation characteristics of the charge distribution will now be examined. In this case, Eqs. (2-44) through (2-48) reduce to,

$$[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_m\rho_1)] - a_{im} J_m(k_{om}\rho_1) = 0 \tag{3-17}$$

$$[a'_{im} J'_m(k_m\rho_1) + a''_{im} Y'_m(k_m\rho_1)] + \left[\frac{q_o c^2 k_m}{\omega_o \rho_1 k^2_{am}} b_{im} J_m(k_{am}\rho_1) - \frac{k_m}{k_{om}} a_{im} J'_m(k_{om}\rho_1)\right] = 0 \qquad (3\text{-}18)^*$$

$$[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_m\rho_1)] + \left[\frac{q_o \rho_1 c^2 k^2_m}{m^2 \omega_o k_{am}} b_{im} J'_m(k_{am}\rho_1) - \frac{k^2_m}{k^2_{om}} a_{im} J_m(k_{om}\rho_1)\right] = 0 \tag{3-19}$$

$$a_{em} H^{(1)}_m(k_m\rho_o) - \left[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_m\rho_o)\right] = -\frac{\omega_o q}{2\pi} \tag{3-20}$$

$$a_{em} H^{(1)\prime}_m(k_m\rho_o) - \left[a'_{im} J'_m(k_m\rho_o) + a''_{im} Y'_m(k_m\rho_o)\right] = 0 \tag{3-21}$$

Simultaneous solution of the above, for a_{em}, leads to,

$$a_{em} = \frac{-\omega_o \rho_o k_m q}{4} \; \frac{[\mathcal{R}_e(\Delta_{am}) Y'_m(k_m\rho_o) - \mathcal{I}_m(\Delta_{am}) J'_m(k_m\rho_o)]}{\Delta_{am}} \tag{3-22}$$

---

*In a compressible, isotropic plasma, $k^2_{am} = \dfrac{m^2\omega_o^2 - \omega_p^2}{a^2}$.

Although the same symbol k_{am} has been used in Eq. (2-25), no ambiguity arises since they never appear together.

where

$$\Delta_{am} = \frac{\omega_p^2}{k_{am}\rho_1 a^2} J_m(k_{om}\rho_1) J_m(k_{am}\rho_1) H_m^{(1)}(k_m\rho_1) + \frac{\omega_o^2 \rho_1 k_{am}^2}{c^2} \Delta_{om} J_m'(k_{am}\rho_1)$$

(3-23)

If Eq. (3-22) is substituted into equation (2-54), the result is,

$$W = + \frac{\omega_o q^2 \beta_c^2}{2\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m[\mathcal{R}_e(\Delta_{am}) Y_m'(m\beta_c) - \mathcal{I}_m(\Delta_{am}) J_m'(m\beta_c)]^2}{\Delta_{am}\Delta_{am}^*} \qquad (3\text{-}24)$$

Although it is not explicitly shown in Eq. (3-24), there are terms which are written in terms of the quantity $\beta_a = \frac{\omega_o \rho_o}{a}$, which is a measure of the ratio of the speed of the charge to the speed of sound in the plasma medium. A physically plausible situation exists when the charge distribution moves with highly supersonic and non-relativistic velocities (i. e. $\beta_a >> 1$, $\beta_c << 1$). In this case, the terms with arguments proportional to β_a may be replaced by their asymptotic forms, and the terms with arguments proportional to β_c may be approximated according to Eq. (3-8).

The asymptotic form is,

$$J_m(m\beta_a) \cong \sqrt{\frac{2}{\pi m\beta_a}} \cos(m\beta_a - \pi/4 - m\pi/2) \qquad (3\text{-}25)$$

If the approximations of Eqs. (3-8) and (3-25) are now applied to Eq. (3-24), the results reduce to Eq. (3-9) exactly. Thus, it is concluded, that the influence of compressibility upon the plasma is to leave the radiation characteristics effectively unchanged.

Having found that the effect of compressibility is negligible, it is now desirable to study the incompressible plasma under the influence of a magnetic field. That is, the intensity of the magnetic field in the plasma is now appreciable. The description of the plasma, in terms of a tensor dielectric permeability, is now applicable. Conditions (2-44) through (2-48) simplify to the following system of equations.

$$[a'_{im} J_m(k_m \rho_1) + a''_{im} Y_m(k_m \rho_1)] - a_{im} J_m(k_{em} \rho_1) = 0 \tag{3-26}$$

$$[a'_{im} J'_m(k_m \rho_1) + a''_{im} Y'_m(k_m \rho_1)] - \left[\frac{m k_m \epsilon_2}{k^2_{em} \rho_1 \epsilon_1} a_{im} J_m(k_{em} \rho_1) + \frac{k_m}{k_{em}} a_{im} J'_m(k_{em} \rho_1)\right] = 0 \tag{3-27}$$

$$a_{em} H^{(1)}_m(k_m \rho_o) - [a'_{im} J_m(k_m \rho_o) + a''_{im} Y_m(k_m \rho_o)] = -\frac{\omega_o q}{2\pi} \tag{3-28}$$

$$a_{em} H^{(1)\prime}_m(k_m \rho_o) - [a'_{im} J'_m(k_m \rho_o) + a''_{im} Y'_m(k_m \rho_o)] = 0 \tag{3-29}$$

where k_{em}, ϵ_1, and ϵ_2 are the definitions applied to the case of a rotating charge distribution. (See page 12). Solving the system of Eqs. (3-26) through (3-29) for the constant of integration a_{em}, gives,

$$a_{em} = -\frac{\omega_o \rho_o q k_m}{4} \; \frac{[\mathcal{R}_e(\Delta_{em}) Y'_m(k_m \rho_o) - \mathcal{I}_m(\Delta_{em}) J'_m(k_m \rho_o)]}{\Delta_{em}} \tag{3-30}$$

where

$$\Delta_{em} = \frac{1}{k_m} J_m(k_{em} \rho_1) H^{(1)\prime}_m(k_m \rho_1) - \frac{1}{k_{em}} J'_m(k_{em} \rho_1) H^{(1)}_m(k_m \rho_1) - \frac{m \epsilon_2}{k^2_{em} \rho_1 \epsilon_1} J_m(k_{em} \rho_1) H^{(1)}_m(k_m \rho_1) \tag{3-31}$$

In the same manner as before, the radiation in this case is,

$$W = + \frac{\omega_o q^2 \beta_c^2}{2\epsilon_o} \sum_{m=1}^{m=\infty} m \frac{[\mathcal{R}_e(\Delta_{em}) Y'_m(m\beta_c) - \mathcal{J}_m(\Delta_{em}) J'_m(m\beta_c)]^2}{\Delta_{em}\Delta^*_{em}} \qquad (3\text{-}32)$$

Because of the complexity of an anisotropic medium as opposed to the previously studied case of an isotropic medium, a closer study of the dependence of the radiated energy upon the dispersive properties of the plasma is necessary. The dispersive properties of the plasma are determined by $\kappa_e = \sqrt{\frac{\epsilon_1^2 - \epsilon_2^2}{\epsilon_1}}$, the index of refraction for the propagation of a plane electromagnetic wave perpendicular to the direction of the magnetic field. The index of refraction for the medium can be written more explicitly as follows:

$$\kappa_e^2 = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_p^2 - \omega_c^2} \qquad (3\text{-}33)$$

The accompanying Fig. 2, is a plot of κ_e^2 versus ω/ω_c (where ω is now assumed) for a particular value of ω_p/ω_c. A study of this figure reveals that the singularities occur at,

$$\frac{\omega}{\omega_c} = 0 \,, \quad \frac{\omega}{\omega_c} = \sqrt{1 + \left(\frac{\omega_p}{\omega_c}\right)^2} \qquad (3\text{-}34)$$

and κ_e vanishes for,

$$\frac{\omega}{\omega_c} = \sqrt{\frac{1}{4} + \left(\frac{\omega_p}{\omega_c}\right)^2} \pm \frac{1}{2} \qquad (3\text{-}35)$$

Furthermore, κ_e is larger than unity for,

$$\frac{\omega_p}{\omega_c} < \frac{\omega}{\omega_c} < \sqrt{1 + \left(\frac{\omega_p}{\omega_c}\right)^2} \qquad (3\text{-}36)$$

and is imaginary in the regions,

$$0 < \frac{\omega}{\omega_c} < \sqrt{\frac{1}{4} + \left(\frac{\omega_p}{\omega_c}\right)^2} - \frac{1}{2}; \quad \sqrt{1 + \left(\frac{\omega_p}{\omega_c}\right)^2} < \frac{\omega}{\omega_c} < \sqrt{\frac{1}{4} + \left(\frac{\omega_p}{\omega_c}\right)^2} + \frac{1}{2} \tag{3-37}$$

Again, it may be said that Eq. (3-33) and the above mentioned properties of the dispersion relation may be applied to the problem of a rotating charge distribution for $\omega = m\omega_o$. Further, assume that the angular frequency of the charge is equal to the cyclotron frequency, $\omega_o = \omega_c$, (i.e. the intensity of the magnetic field is the same both inside and outside the plasma). In this case, the range of frequency given by Eq. (3-36) becomes,

$$\frac{\omega_p}{\omega_o} < m < \sqrt{1 + \left(\frac{\omega_p}{\omega_o}\right)^2} \tag{3-38}$$

Due to the fact that

$$\sqrt{1 + \left(\frac{\omega_p}{\omega_o}\right)^2} - \frac{\omega_p}{\omega_o} \leq 1 \tag{3-39}$$

it is possible to find only one value of m where $\kappa_{em}\beta_c$ is greater than unity, that is, when the velocity of the charge distribution is greater than the phase velocity of the electromagnetic field in the medium. When this condition is fulfilled, a process of Cerenkov radiation contributes to the already existent Bremsstahlung radiation for this harmonic. It is to be noted that in the previously considered cases dealing with an isotropic plasma, Cerenkov radiation was not possible since the index of refraction κ_o, corresponding to the ordinary optic mode of propagation, was always less than unity.

The discussion will first be confined to those frequencies which do not lie in the neighborhood of the particular frequency for which Cerenkov radiation is possible. Furthermore, the charge distribution will again be considered to be moving with non-relativistic velocities. Therefore, $\kappa_{em}\beta_c << 1$ and Eq. (3-32) becomes in this approximation.

$$W = \frac{\omega_o q^2}{2\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^{2m}(m!)^2} \left\{ 1 + (\rho_1/\rho_o)^{2m} \frac{[1-(\epsilon_1-\epsilon_2)]}{[1+(\epsilon_1-\epsilon_2)]} \right\}^2 \qquad (3\text{-}40)$$

In order to see how the dipole resonance in the isotropic plasma case is generated from the anisotropic plasma, it is necessary to assume in Eq. (3-40) that the intensity of the magnetic field in the plasma is small. Therefore, the assumption that the charge distribution moves in synchronization with the cyclotron frequency will be dropped for the present. Under this new assumption, Equation (3-40) becomes,

$$W = \frac{\omega_o q^2}{2\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^{2m}(m!)^2} \left\{ 1 + \frac{(\rho_1/\rho_o)^{2m}\left(\frac{\omega_p}{m\omega_o}\right)^2\left[1+\left(\frac{\omega_c}{m\omega_o}\right)\right]}{2-\left(\frac{\omega_p}{m\omega_o}\right)^2\left[1+\left(\frac{\omega_c}{m\omega_o}\right)\right]} \right\}^2 \qquad (3\text{-}41)$$

where $\frac{\omega_c}{\omega_o}$ is assumed much smaller than unity in the above equation.

Equation (3-41) indicates that a resonance may occur for the m^{th} harmonic when the following condition is satisfied

$$\frac{\omega_p}{m\omega_o} \simeq \sqrt{2}\left(1-\frac{\omega_c}{m\omega_o}\right)^{\frac{1}{2}} \qquad (3\text{-}42)$$

Thus, the effect of the magnetic field, as far as the resonant frequency shift is concerned, decreases with the order of the harmonics.

The assumption of synchronized motion of the charge distribution with the cyclotron frequency will now be resumed in the subsequent discussion. The radiation will be analyzed for two specific ranges of the ratio of the plasma frequency to the cyclotron frequency.

First, it will be assumed that $\omega_p/\omega_o << 1$(underdense plasma). Then, the index of refraction may be approximated as follows:

$$\kappa^2_{em} \sim 1 - \frac{\left(\frac{\omega_p}{\omega_o}\right)^2}{1-m^2} \qquad (3\text{-}43)$$

which is effectively unity for all orders of the harmonics except the fundamental. Thus, for extremely non-relativistic velocities of the charge distribution, the radiation is confined entirely to the fundamental frequency and Eq. (3-40) reduces to:

$$W \sim \frac{\omega_o q^2 \beta_c^6}{32\epsilon_o} \qquad (3\text{-}44)$$

which is identical to free space.

In physical terms, if the index of refraction is effectively unity, then the electromagnetic radiation passes through the plasma undisturbed.

The region of most interest is the frequency range where $\omega_p/\omega_o >> 1$. In this region, the index of refraction may be approximated as,

$$\kappa^2_{em} \sim 1 - \frac{1}{m^2}\left(\frac{\omega_p}{\omega_o}\right)^2 \frac{1}{m^2 - \left(\frac{\omega_p}{\omega_o}\right)^2} \qquad (3\text{-}45)$$

It is evident from Equation (3-45), or equivalently through an inspection of Fig. (2), that the index of refraction possesses a singularity. Obviously, the conditions necessary for Cerenkov radiation to occur are satisfied for the particular harmonic in the neighborhood of the singularity. However, it must be recalled from the previous discussion, that a multipole resonance may also occur for a harmonic in the same frequency range. It is therefore essential to demonstrate that the Cerenkov effect does not mask the multipole resonance. Clearly, if the multipole resonance occurs, it must occur for the harmonic satisfying the relationship,

$$m = \frac{1}{\sqrt{2}} \left(\frac{\omega_p}{\omega_o} \right) \tag{3-46}$$

Equation (3-36) indicates that $\varkappa^2_{em} = 1$ when $m = \frac{\omega_p}{\omega_o}$. Furthermore, if Equation (3-33) is solved for the harmonic which makes $\varkappa^2_{em} = -1$, the result is approximately equal to Equation (3-46). Hence, the harmonic satisfying Equation (3-46) never occurs in the frequency range where Cerenkov radiation is possible.

Resuming the investigation of the radiation in terms of the index of refraction, Equation (3-45), it is apparent that Equation (3-32) permits a large number of harmonics for which $\varkappa_{em}$ is large in magnitude but imaginary. Hence, the necessary approximations upon the

Bessel functions here are,

$$i^{-m} J_m(m\beta_c \varkappa_{em}) = I_m(-im\beta_c \varkappa_{em}) \sim \sqrt{\frac{2}{-im\beta_c \varkappa_{em}}} \, e^{-im\beta_c \varkappa_{em}} \qquad (3\text{-}47)$$

where it is understood that $\varkappa_{em}$ is now purely imaginary and I_m is the modified Bessel function of the first kind.

An expression for W_m, a typical harmonic in this frequency range, is obtained by substituting the approximation of Equation (3-47) into Equation (3-32), which gives,

$$W_m = \frac{m\omega_o q^2}{2\epsilon_o} \frac{(m\beta_c)^{2m}}{2m(m!)^2} \left[1 - \left(\frac{\rho_1}{\rho_o} \right)^{2m} \right]^2 \qquad (3\text{-}48)$$

A comparison of Equation (3-48) with the corresponding free space case, Equation (3-11) is relevant. It is seen from this comparison that the radiation pattern of free space is modified by the factor $[1 - (\rho_1/\rho_o)^{2m}]^2$, for the case at hand. This result complies with the physical situation. Thus, in the low frequency range, the plasma is overdense and acts as a reflector of the electromagnetic radiation emitted by the charge distribution. The modification factor is proportional to the scattering cross-section. As the order of the harmonics increases, the plasma appears less dense; the modification factor goes to unity; and the radiation pattern approaches that of free space.

Finally, attention must be directed to the radiation produced by the single higher order harmonic in the vicinity of the singularity of $\varkappa_{em}$. Because this frequency may lie anywhere in the neighborhood of the singularity, the plasma density and the magnetic field intensity may be adjusted to yield any particular frequency within this range. It must be noted, that this is an extremely critical solution since Equation (3-39) indicates that the index of refraction has a very sharp resonant condition. Also, it is seen from Equation (3-45) that $\varkappa_{em}$ is extremely small for the subsequent harmonics beyond the singularity and, hence, the radiation contribution of these harmonics is negligible. Furthermore, for large values of m, it is apparent that a small change in $\varkappa_{em}$ leads to a large change in the argument $m\beta_c\varkappa_{em}$ of the Bessel functions. Thus, the radiated energy at the m^{th} harmonic becomes an extremely sensitive function of $\varkappa_{em}$. If the average value of W_m for the range $\varkappa_{em} >> 1$ is taken for Equation (3-32), the result is identical in form to Equation (3-48), and will therefore not be repeated. This result is not surprising since the only difference between the asymptotic forms of the Bessel functions, for real or imaginary arguments, is an oscillating part in time. This distinction no longer exists after an averaging process is carried out.

In summary, let it be recalled that in the case of a simple plasma (isotropic and incompressible), a multipole resonance is exhibited for the harmonic satisfying $m \cong \omega_p / \sqrt{2}\omega_o$ when the charge moves at non-relativistic velocities. If the charge velocity is greatly reduced, the resonance becomes a singularity and the harmonic satisfies exactly the relation $m = 1/\sqrt{2}\ \omega_p/\omega_o$. The influence of compressibility upon these radiation characteristics is negligible. However, even a weakly anisotropic plasma is sufficient to shift the multipole resonance. As the order of the harmonics increases, this shift decreases. Thus, the multipole resonance is effectively unchanged for a sufficiently higher order harmonic. Moreover, this resonance is not masked by the Cerenkov effect which occurs for a singular harmonic of an even higher order in a neighboring frequency range. Thereafter, the radiation contribution of the remaining harmonics is negligible.

SECTION IV

THE CHARGE DISTRIBUTION INSIDE THE PLASMA COLUMN

It is now desirable to consider the effect of moving the charge distribution onto the plasma column. That is, to set the radius of the orbit ρ_o of the charge distribution equal to the radius ρ_1 of the plasma column. It is seen in Eq. (3-32) that, in this limit, the form of the radiation expression remains unchanged. Consequently, it is concluded that a dipole resonance also occurs for the charge distribution when it is located on the periphery of the column. Moreover, this resonance may still be possible if the charge distribution is moved inside the plasma column. The physical implications of this speculation can be better understood through a complete solution for the radiation field of a rotating line charge immersed in a plasma column. Thus the method as adopted in the previous chapter, of investigating independently the influence of both compressibility and anisotropy upon the radiation characteristics of the rotating charge distribution will be repeated. The question of the physical mechanism of the rotation of the charge distribution then arises in the isotropic plasma case. A new physical configuration must be introduced to answer this question. Consider the charge distribution to be located within a thin vacuum gap, bounded by two concentric cylindrical regions of plasma. A magnetic field intensity, different from that in the plasma, can be maintained in this gap. The radiation field, computed in the ideal situation (that is, the charge distribution immersed in a uniform isotropic plasma column) is then valid for wavelengths of the electromagnetic radiation much greater than the thickness of the gap.

Since the effects of compressibility and anisotropy will be considered, it is expedient to formulate the problem in the most general case of a compressible anisotropic column. This necessitates an inspection of

Eqs. (2-33) and (2-34), which may be uncoupled in the same manner as that technique employed in the previous problem of the charge distribution located outside the plasma. In this case, the solution for the longitudinal field components is,

$$N_m = \begin{cases} \dfrac{1}{(k_{sm}^2 - k_{\ell m}^2)} \left\{ (k_{om}^2 - k_{\ell m}^2) b_{im} J_m(k_{\ell m}\rho) - (k_{om}^2 - k_{sm}^2) a_{im} J_m(k_{sm}\rho) \right\} & ; \rho < \rho_o \\ \dfrac{1}{(k_{sm}^2 - k_{\ell m}^2)} \left\{ (k_{om}^2 - k_{\ell m}^2)[b'_{im} J_m(k_{\ell m}\rho) + b''_{im} Y_m(k_{\ell m}\rho)] - \right. & \\ \left. (k_{om}^2 - k_{sm}^2)[a'_{im} J_m(k_{sm}\rho) + a''_{im} Y_m(k_{sm}\rho)] \right\} & ; \rho_o < \rho < \rho_1 \\ 0 & ; \rho_1 < \rho < \infty \end{cases} \tag{4-1}$$

$$H_{zm} = \begin{cases} \dfrac{q_o \omega_c}{(k_{sm}^2 - k_{\ell m}^2)} \left\{ a_{im} J_m(k_{sm}\rho) - b_{im} J_m(k_{\ell m}\rho) \right\} & ; \rho < \rho_o \\ \dfrac{q_o \omega_c}{(k_{sm}^2 - k_{\ell m}^2)} \left\{ [a'_{im} J_m(k_{sm}\rho) + a''_{im} Y_m(k_{sm}\rho)] - \right. & \\ \left. [b'_{im} J_m(k_{\ell m}\rho) + b''_{im} Y_m(k_{\ell m}\rho)] \right\} & ; \rho_o < \rho < \rho_1 \\ a_{em} H_m^{(1)}(k_m\rho) & ; \rho_1 < \rho < \infty \end{cases} \tag{4-2}$$

The transverse field components, obtained in the same manner as in Section Two, are

$$E_{\rho m} = \begin{cases} \left\{\dfrac{-q_o}{\epsilon_o(k_{sm}^2-k_{\ell m}^2)}\right\} a_{im}\left[\dfrac{m^2\omega_o\omega_c}{\rho k_{sm}^2 c^2} J_m(k_{sm}\rho) + \dfrac{(k_{om}^2-k_{sm}^2)}{k_{sm}} J_m'(k_{sm}\rho)\right] \\ \quad -b_{im}\left[\dfrac{m^2\omega_o\omega_c}{\rho k_{\ell m}^2 c^2} J_m(k_{\ell m}\rho) + \dfrac{(k_{om}^2-k_{\ell m}^2)}{k_{\ell m}} J_m'(k_{\ell m}\rho)\right]\Bigg\} & ; \ \rho<\rho_o \\ \left\{\dfrac{-q_o}{\epsilon_o(k_{sm}^2-k_{\ell m}^2)}\right\} a_{im}'\left[\dfrac{m^2\omega_o\omega_c}{\rho k_{sm}^2 c^2} J_m(k_{sm}\rho) + \dfrac{(k_{om}^2-k_{sm}^2)}{k_{sm}} J_m'(k_{sm}\rho)\right] \\ \quad +a_{im}''\left[\dfrac{m^2\omega_o\omega_c}{\rho k_{sm}^2 c^2} Y_m(k_{sm}\rho) + \dfrac{(k_{om}^2-k_{sm}^2)}{k_{sm}} Y_m'(k_{sm}\rho)\right] \\ \quad -b_{im}'\left[\dfrac{m^2\omega_o\omega_c}{\rho k_{\ell m}^2 c^2} J_m(k_{\ell m}\rho) + \dfrac{(k_{om}^2-k_{\ell m}^2)}{k_{\ell m}} J_m'(k_{\ell m}\rho)\right] \\ \quad -b_{im}''\left[\dfrac{m^2\omega_o\omega_c}{\rho k_{\ell m}^2 c^2} Y_m(k_{\ell m}\rho) + \dfrac{(k_{om}^2-k_{\ell m}^2)}{k_{\ell m}} Y_m'(k_{\ell m}\rho)\right]\Bigg\} & ; \ \rho_o<\rho<\rho_1 \\ -\dfrac{m^2\omega_o\mu_o}{\rho k_m^2} a_{em} H_m^{(1)}(k_m\rho) & ; \ \rho_1<\rho<\infty \end{cases} \qquad (4\text{-}3)$$

$$
E_{\varphi m} = \begin{cases}
\dfrac{-imq_o}{\epsilon_o(k^2_{sm}-k^2_{\ell m})}\left\{ a_{im}\left[\dfrac{(k^2_{om}-k^2_{sm})}{\rho k^2_{sm}} J_m(k_{sm}\rho) + \dfrac{\omega_o\omega_c}{k_{sm}c^2} J'_m(k_{sm}\rho)\right]\right. \\
\qquad \left. - b_{im}\left[\dfrac{(k^2_{om}-k_{\ell m})}{\rho k^2_{\ell m}} J_m(k_{\ell m}\rho) + \dfrac{\omega_o\omega_c}{k_{\ell m}c^2} J'_m(k_{\ell m}\rho)\right]\right\} & ; \ \rho < \rho_o \\
\dfrac{-imq_o}{\epsilon_o(k^2_{sm}-k^2_{\ell m})}\left\{ a'_{im}\left[\dfrac{(k^2_{om}-k^2_{sm})}{\rho k^2_{sm}} J_m(k_{sm}\rho) + \dfrac{\omega_o\omega_c}{k_{sm}c^2} J'_m(k_{sm}\rho)\right]\right. \\
\qquad + a''_{im}\left[\dfrac{(k^2_{om}-k^2_{sm})}{\rho k^2_{sm}} Y_m(k_{sm}\rho) + \dfrac{\omega_o\omega_c}{k_{sm}c^2} Y'_m(k_{sm}\rho)\right] \\
\qquad - b'_{im}\left[\dfrac{(k^2_{om}-k^2_{\ell m})}{\rho k^2_{\ell m}} J_m(k_{\ell m}\rho) + \dfrac{\omega_o\omega_c}{k_{\ell m}c^2} J'_m(k_{\ell m}\rho)\right] \\
\qquad \left. - b''_{im}\left[\dfrac{(k^2_{om}-k^2_{\ell m})}{\rho k^2_{\ell m}} Y_m(k_{\ell m}\rho) + \dfrac{\omega_o\omega_c}{k_{\ell m}c^2} Y'_m(k_{\ell m}\rho)\right]\right\} & ; \ \rho_o < \rho < \rho_1 \\
\qquad - \dfrac{im\omega_o\mu_o}{k_m} a_{em} H^{(1)\prime}_m(k_m\rho) & ; \ \rho_1 < \rho < \infty
\end{cases} \tag{4-4}
$$

The appropriate boundary conditions in terms of the total field components are,

$$
\left.\begin{aligned}
\left[\mathcal{H}_z\right]_{i'} - \left[\mathcal{H}_z\right]_i &= -\lim_{\delta\to 0} \int_{\rho_o-\delta}^{\rho_o+\delta} j_\varphi d_\rho \\
\left[\mathcal{N}\right]_{i'} - \left[\mathcal{N}\right]_i &= 0 \\
\left[\mathcal{E}_\varphi\right]_{i'} - \left[\mathcal{E}_\varphi\right]_i &= 0 \\
\left[\mathcal{U}_\rho\right]_{i'} - \left[\mathcal{U}_\rho\right]_i &= 0
\end{aligned}\right\} \quad ; \ \rho = \rho_o \tag{4-5}
$$

$$
\begin{aligned}
\left[\mathcal{H}_z\right]_e &- \left[\mathcal{H}_z\right]_{i'} = 0 \\
\left[\mathcal{E}_\varphi\right]_e &- \left[\mathcal{E}_\varphi\right]_{i'} = 0 \qquad ; \quad \rho = \rho_1 \qquad (4\text{-}6) \\
\mathcal{U}_\rho & \qquad 0
\end{aligned}
$$

The above conditions, when written in terms of the harmonic components, become,

$$
\begin{aligned}
\left[H_{zm}\right]_{i'} &- \left[H_{zm}\right]_i = -\frac{\omega_o q}{2\pi} \\
\left[N_m\right]_{i'} &- \left[N_m\right]_i = 0 \\
\left[E_{\varphi m}\right]_{i'} &- \left[E_{\varphi m}\right]_i = 0 \qquad ; \quad \rho = \rho_o \qquad (4\text{-}7) \\
\left[U_{\rho m}\right]_{i'} &- \left[U_{\rho m}\right]_i = 0
\end{aligned}
$$

$$
\begin{aligned}
\left[H_{zm}\right]_e &- \left[H_{zm}\right]_i = 0 \\
\left[E_{\varphi m}\right]_e &- \left[E_{\varphi m}\right]_{i'} = 0 \qquad ; \quad \rho = \rho_1 \qquad (4\text{-}8) \\
& U_{\rho m} = 0
\end{aligned}
$$

The above equations result in the following system of equations,

$$
\left\{ \left[a'_{im} J_m(k_{sm}\rho_o) + a''_{im} Y_m(k_{sm}\rho_o)\right] - \left[b'_{im} J_m(k_{\ell m}\rho_o) + b''_{im} Y_m(k_{\ell m}\rho_o)\right] - \left[a_{im} J_m(k_{sm}\rho_o) - b_{im} J_m(k_{\ell m}\rho_o)\right] \right\} = -\frac{\omega_o q (k^2_{sm} - k^2_{\ell m})}{2\pi \omega_c q_o} \qquad (4\text{-}9)
$$

$$
\begin{aligned}
&(k_{om}^2-k_{\ell m}^2)\left[b'_{im}J_m(k_{\ell m}\rho_o)+b''_{im}Y_m(k_{\ell m}\rho_o)\right]-(k_{om}^2-k_{sm}^2)\Big[a'_{im}J_m(k_{sm}\rho_o)\\
&\quad+a''_{im}Y_m(k_{sm}\rho_o)\Big]-\left[(k_{om}^2-k_{\ell m}^2)b_{im}J_m(k_{\ell m}\rho_o)-(k_{om}^2-k_{sm}^2)a_{im}J_m(k_{sm}\rho_o)\right]=0
\end{aligned}
\tag{4-10}
$$

$$
\begin{aligned}
&b'_{im}\left[\frac{(k_{om}^2-k_{\ell m}^2)}{\rho_o k_{sm}^2}J_m(k_{\ell m}\rho_o)+\frac{\omega_o\omega_c}{k_{\ell m}c^2}J'_m(k_{\ell m}\rho_o)\right]+b''_{im}\Bigg[\frac{(k_{om}^2-k_{\ell m}^2)}{\rho_o k_{\ell m}^2}Y_m(k_{\ell m}\rho_o)\\
&\quad+\frac{\omega_o\omega_c}{k_{sm}c^2}Y'_m(k_{\ell m}\rho_o)\Bigg]-a'_{im}\left[\frac{(k_{om}^2-k_{sm}^2)}{\rho_o k_{sm}^2}J_m(k_{sm}\rho_o)+\frac{\omega_o\omega_c}{k_{sm}c^2}J'_m(k_{sm}\rho_o)\right]\\
&\quad-a''_{im}\left[\frac{(k_{om}^2-k_{sm}^2)}{\rho_o k_{sm}^2}Y_m(k_{sm}\rho_o)+\frac{\omega_o\omega_c}{k_{sm}c^2}Y'_m(k_{sm}\rho_o)\right]-b_{im}\Bigg[\frac{(k_{om}^2-k_{\ell m}^2)}{\rho_o k_{\ell m}^2}J_m(k_{\ell m}\rho_o)\\
&\quad+\frac{\omega_o\omega_c}{k_{\ell m}c^2}J'_m(k_{\ell m}\rho_o)\Bigg]+a_{im}\left[\frac{(k_{om}^2-k_{sm}^2)}{\rho_o k_{sm}^2}J_m(k_{sm}\rho_o)+\frac{\omega_o\omega_c}{k_{sm}c^2}J'_m(k_{sm}\rho_o)\right]=0
\end{aligned}
\tag{4-11}
$$

$$
\begin{aligned}
&b'_{im}\left[\frac{m^2\omega_o\omega_c}{\rho_o k_{\ell m}^2c^2}J_m(k_{\ell m}\rho_o)+\frac{(k_{om}^2-k_{\ell m}^2)}{k_{\ell m}}J'_m(k_{\ell m}\rho_o)\right]+b''_{im}\Bigg[\frac{m^2\omega_o\omega_c}{\rho_o k_{\ell m}^2c^2}Y_m(k_{\ell m}\rho_o)\\
&\quad+\frac{(k_{om}^2-k_{\ell m}^2)}{k_{\ell m}}Y'_m(k_{\ell m}\rho_o)\Bigg]-a'_{im}\left[\frac{m^2\omega_o\omega_c}{\rho_o k_{sm}^2c^2}J_m(k_{sm}\rho_o)+\frac{(k_{om}^2-k_{sm}^2)}{k_{sm}}J'_m(k_{sm}\rho_o)\right]\\
&\quad-a''_{im}\left[\frac{m^2\omega_o\omega_c}{\rho_o k_{sm}^2c^2}Y_m(k_{sm}\rho_o)+\frac{(k_{om}^2-k_{sm}^2)}{k_{sm}}Y'_m(k_{sm}\rho_o)\right]-b_{im}\Bigg[\frac{m^2\omega_o\omega_c}{\rho_o k_{sm}^2c^2}J_m(k_{\ell m}\rho_o)\\
&\quad+\frac{(k_{om}^2-k_{\ell m}^2)}{k_{\ell m}}J'_m(k_{\ell m}\rho_o)\Bigg]+a''_{im}\left[\frac{m^2\omega_o\omega_c}{\rho_o k_{sm}^2c^2}J_m(k_{sm}\rho_o)+\frac{(k_{om}^2-k_{sm}^2)}{k_{sm}}J'_m(k_{sm}\rho_o)\right]\\
&\quad=\frac{q(k_{sm}^2-k_{\ell m}^2)}{q_o 2\pi\rho_o}
\end{aligned}
\tag{4-12}
$$

$$a_{em} H_m^{(1)}(k_m \rho_1) - \frac{q_o \omega_o}{(k_{sm}^2 - k_{\ell m}^2)} \left\{ \left[a'_{im} J_m(k_{sm}\rho_1) + a''_{im} Y_m(k_{sm}\rho_1) \right] - \left[b'_{im} J_m(k_{\ell m}\rho_1) + b''_{im} Y_m(k_{\ell m}\rho_1) \right] \right\} = 0 \tag{4-13}$$

$$\begin{aligned} a_{em} H_m^{(1)\prime}(k_m \rho_1) + \frac{q_o k_m}{(k_{sm}^2 - k_{\ell m}^2)} \Bigg\{ & b'_{im} \left[\frac{c^2 (k_{om}^2 - k_{\ell m}^2)}{\omega_o \rho_1 k_{\ell m}^2} J_m(k_{\ell m}\rho_1) \right. \\ & \left. + \frac{\omega_c}{k_{\ell m}} J'_m(k_{\ell m}\rho_1) \right] + b''_{im} \left[\frac{c^2 (k_{om}^2 - k_{\ell m}^2)}{\omega_o \rho_1 k_{\ell m}^2} Y_m(k_{\ell m}\rho_1) \right. \\ & \left. + \frac{\omega_c}{k_{\ell m}} Y'_m(k_{\ell m}\rho_1) \right] - a'_{im} \left[\frac{c^2 (k_{om}^2 - k_{sm}^2)}{\omega_o \rho_1 k_{sm}^2} J_m(k_{sm}\rho_1) \right. \\ & \left. + \frac{\omega_c}{k_{sm}} J'_m(k_{sm}\rho_1) \right] - a''_{im} \left[\frac{c^2 (k_{om}^2 - k_{sm}^2)}{\omega_o \rho_1 k_{sm}^2} Y_m(k_{sm}\rho_1) + \frac{\omega_c}{k_{sm}} Y'_m(k_{sm}\rho_1) \right] \Bigg\} = 0 \end{aligned} \tag{4-14}$$

$$a_{em} H_m^{(1)}(k_m \rho_1) + \frac{q_o k_m^2}{(k_{sm}^2 - k_{\ell m}^2)} \Bigg\{ b'_{im} \left[\frac{\omega_c}{k_{\ell m}^2} J_m(k_{\ell m}\rho_1) \right.$$

$$\left. + \frac{c^2 \rho_1 (k_{om}^2 - k_{\ell m}^2)}{m^2 \omega_o k_{\ell m}} J'_m(k_{\ell m}\rho_1) \right] + b''_{im} \left[\frac{\omega_c}{k_{\ell m}^2} Y_m(k_{\ell m}\rho_1) \right.$$

$$\left. + \frac{c^2 \rho_1 (k_{om}^2 - k_{\ell m}^2)}{m^2 \omega_o k_{\ell m}} Y'_m(k_{\ell m}\rho_1) \right] - a'_{im} \left[\frac{\omega_c}{k_{sm}^2} J_m(k_{sm}\rho_1) \right.$$

$$\left. + \frac{c^2 \rho_1 (k_{om}^2 - k_{sm}^2)}{m^2 \omega_o k_{sm}} J'_m(k_{sm}\rho_1) \right] - a''_{im} \left[\frac{\omega_c}{k_{sm}^2} Y_m(k_{sm}\rho_1) \right.$$

$$\left. + \frac{c^2 \rho_1 (k_{om}^2 - k_{sm}^2)}{m^2 \omega_o k_{sm}} Y'_m(k_{sm}\rho_1) \right] \Bigg\} = 0 \qquad (4\text{-}15)$$

With this in mind, the analysis will begin with a study of the radiation pattern for the case of an isotropic, incompressible plasma, Eqs. (4-9) through (4-15) reduce to,

$$\left[a'_{im} J_m(k_{om}\rho_o) + a''_{im} Y_m(k_{om}\rho_o) \right] - a_{im} J_m(k_{om}\rho_o) = -\frac{\omega_o q}{2\pi} \qquad (4\text{-}16)$$

$$\left[a'_{im} J'_m(k_{om}\rho_o) + a''_{im} Y'_m(k_{om}\rho_o) \right] - a_{im} J'_m(k_{om}\rho_o) = 0 \qquad (4\text{-}17)$$

$$a_{em} H_m^{(1)}(k_m\rho_o) - \left[a'_{im} J_m(k_{om}\rho_1) + a''_{im} Y_m(k_{om}\rho_1) \right] = 0 \qquad (4\text{-}18)$$

$$a_{em} H_m^{(1)'}(k_m\rho_1) \quad \frac{k_m}{k_{om}} \left[a'_{im} J'_m(k_{om}\rho_1) + a''_{im} Y'_m(k_{om}\rho_1) \right] = 0 \qquad (4\text{-}19)$$

If Eqs. (4-16) through (4-19) are solved for the constant of integration applicable outside the plasma, the result is,

$$a_{em} = \frac{\omega_o \rho_o q}{2\pi k_{om} \rho_1} \; \frac{J_m'(k_{om}\rho_o)}{\Delta_{om}} \tag{4-20}$$

Direct substitution of this result into Eq. (2-54), gives,

$$W = + \frac{2\omega_o q^2 \rho_o^4}{\epsilon_o \pi^4 \beta^4 \rho_1^4} \sum_{m=1}^{m=\infty} \frac{1}{m^2 \kappa_{om}^2} \; \frac{J_m'^2(m\beta_c \kappa_{om})}{\Delta_{om}\Delta_{om}^*} \tag{4-21}$$

where $\kappa_{om}^2 = 1 - \left(\frac{\omega_p}{m\omega_o}\right)^2$ is the index of refraction for the ordinary electromagnetic mode of propagation in an isotropic plasma.

Again, interest is confined to the case of extremely non-relativistic velocities of the charge distribution since a multipole resonance is possible here. Using the approximations of Eq. (3-8) in Eq. (4-21), gives,

$$W = \frac{2\omega_o q^2}{\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^{2m}(m!)^2} \; \frac{1}{\left[2 - \left(\frac{\omega_p}{m\omega_o}\right)^2\right]^2} \tag{4-22}$$

Thus, the multipole resonance is clearly evidenced in Eq. (4-22). A calculation analagous to the one conducted in Section Three, for higher velocities of the charge distribution, results in the same conclusions. Precisely, the dipole singularity is replaced by a maximum in the radiation for the same frequency given by Eq. (3-14). The radiation expression for the fundamental frequency now is,

$$\left[W_1\right]_{max} = \frac{2\omega_o q^2}{\epsilon_o \pi^2 \beta_c^2 (\rho_1/\rho_o)^4} \left[1 + \beta_c^2 (\rho_1/\rho_o)^2 \; \frac{\ln \gamma\beta_c(\rho_1/\rho_o)}{2}\right] \tag{4-23}$$

Furthermore, a numerical calculation extending these results to include relativistic velocities of the charge distribution is depicted in Fig. 3. A huge maximum is evidenced for frequencies shifted slightly from the dipole resonance condition. Hence, the conclusion that the dipole resonance phenomenon is maintained for all velocities of the charge distribution is applicable also to the case of the charge moving inside a plasma column.

For completeness, the effects of compressibility, if any, will be studied. Here, the conditions [Eqs. (4-9) through (4-15)] resulting from the boundary conditions simplify to,

$$\left[a'_{im} J_m(k_{om}\rho_o) + a''_{im} Y_m(k_{om}\rho_o)\right] - a_{im} J_m(k_{om}\rho_o) = -\frac{\omega_o q}{2\pi} \tag{4-24}$$

$$\left[b'_{im} J_m(k_{am}\rho_o) + b''_{im} Y_m(k_{am}\rho_o)\right] - b_{im} J_m(k_{am}\rho_o) = 0 \tag{4-25}$$

$$\frac{q_o c^2}{\omega_o \rho_o k^2_{am}}\left\{\left[b'_{im} J_m(k_{am}\rho_o) + b''_{im} Y_m(k_{am}\rho_o)\right] - \frac{1}{k_{om}}\left[a'_{im} J'_m(k_{om}\rho_o) + a''_{im} Y'_m(k_{om}\rho_o)\right]\right\}$$

$$-\left\{\frac{q_o c^2}{\omega_o \rho_o k^2_{am}} b_{im} J_m(k_{am}\rho_o) - \frac{a_{im}}{k_{om}} J'_m(k_{om}\rho_o)\right\} = 0 \tag{4-26}$$

$$\left\{\frac{q_o c^2}{k_{am}}\left[b'_{im} J'_m(k_{am}\rho_o) + b''_{im} Y'_m(k_{am}\rho_o)\right] - \frac{m^2 \omega_o}{\rho_o k^2_{om}}\left[a'_{im} J_m(k_{om}\rho_o)\right.\right.$$

$$\left.\left.+ a''_{im} Y_m(k_{om}\rho_o)\right]\right\} - \left\{\frac{q_o c^2}{k_{am}} b_{im} J'_m(k_{am}\rho_o)\right.$$

$$\left.- \frac{m^2 \omega_o}{\rho_o k^2_{om}} a_{im} J_m(k_{om}\rho_o)\right\} = \frac{qc^2}{2\pi\rho_o}$$

$$a_{em} H^{(1)}_m(k_m \rho_1) - \left[a'_{im} J_m(k_{om}\rho_1) + a''_{im} Y_m(k_{om}\rho_1)\right] = 0 \tag{4-28}$$

$$a_{em} H_m^{(1)\prime}(k_m \rho_1) + \left\{ \frac{q_o c^2 k_m}{\omega_o \rho_1 k_{am}^2} \left[b'_{im} J_m(k_{am}\rho_1) + b''_{im} Y_m(k_{am}\rho_1) \right] \right.$$

$$\left. - \frac{k_m}{k_{om}} \left[a'_{im} J'_m(k_{om}\rho_1) + a''_{im} Y'_m(k_{om}\rho_1) \right] \right\} = 0 \tag{4-29}$$

$$a_{em} H_m^{(1)}(k_m \rho_1) + \left\{ \frac{q_o \rho_1 c^2 k_m^2}{m^2 \omega_o k_{am}} \left[b'_{im} J'_m(k_{am}\rho_1) + b''_{im} Y'_m(k_{am}\rho_1) \right] \right.$$

$$\left. - \frac{k_m^2}{k_{om}^2} \left[a'_{im} J_m(k_{om}\rho_1) + a''_{im} Y_m(k_{om}\rho_1) \right] \right\} = 0 \tag{4-30}$$

Simultaneous solution of Eqs. (4-24) through (4-30), for a_{em}, gives,

$$a_{em} = \frac{\omega_o q}{2\pi} \left\{ \frac{\frac{\omega_o^2 k_{om}\rho_o}{a^2} J'_m(k_{om}\rho_o) J'_m(k_{am}\rho_1) - \frac{\omega_p^2}{k_{am}\rho_1 a^2} J_m(k_{am}\rho_o) J_m(k_{om}\rho_1)}{\Delta_{am}} \right\} \tag{4-31}$$

By virtue of Eqs. (2-54) and (4-31), the radiation is,

$$W = + \frac{2\omega_o q^2}{\epsilon_o \pi^2} \sum_{m=1}^{m=\infty} \frac{1}{m} \left\{ \frac{\frac{\omega_o^2}{a^2} m \varkappa_{om} \beta_c J'_m(m\beta_c \varkappa_{om}) J'_m(m\beta_a \varkappa_{om}\rho_1/\rho_o)}{\Delta_{am} \Delta_{am}^*} \right.$$

$$\left. - \frac{\frac{\omega_p^2}{a^2 m \beta_a \varkappa_{om} \rho_1/\rho_o} J_m(m\beta_a \varkappa_{om}) J_m(m\beta_c \varkappa_{om}\rho_1/\rho_o)}{\Delta_{am} \Delta_{am}^*} \right\}^2$$

It sufficies to say: if it is assumed that the charge distribution moves with supersonic, non-relativistic velocities (that is, $\beta_a >> 1$, and

and $\beta_c \ll 1$) and the approximations of Eqs. (3-8) and (3-25) are employed, Eq. (4-32) reduces to Eq. (4-22) identically. Thus, the influence of compressibility upon the radiation characteristics is to have them effectively unchanged.

It is now desirable to consider the influence of an impressed static magnetic field upon the radiation produced by the charge distribution moving in an incompressible plasma. This necessitates that Eqs. (4-9) through (4-15) be rewritten as,

$$\left[a'_{im} J_m(k_{em}\rho_o) + a''_{im} Y_m(k_{em}\rho_o)\right] - a_{im} J_m(k_{em}\rho_o) = \frac{-\omega_o q}{2\pi} \qquad (4\text{-}33)$$

$$\left\{\frac{m\,\epsilon_2}{\rho_o \epsilon_1}\left[a'_{im} J_m(k_{em}\rho_o) + a''_{im} Y_m(k_{em}\rho_o)\right] + k_{em}\left[a'_{im} J'_m(k_{em}\rho_o)\right.\right.$$

$$\left.\left. + a''_{im} Y'_m(k_{em}\rho_o)\right]\right\} - \left\{\frac{m\,\epsilon_2}{\rho_o \epsilon_1} a_{im} J_m(k_{em}\rho_o) + k_{em} a_{im} J'_m(k_{em}\rho_o)\right\} = 0$$

$$(4\text{-}34)$$

$$a_{em} H^{(1)}_m(k_m\rho_1) - \left[a'_{im} J_m(k_{em}\rho_1) + a''_{im} Y_m(k_{em}\rho_1)\right] = 0 \qquad (4\text{-}35)$$

$$a_{em} H^{(1)\prime}_m(k_m\rho_1) - \frac{k_m}{k^2_{em}}\left\{\frac{m\,\epsilon_2}{\rho_1 \epsilon_1}\left[a'_{im} J_m(k_m\rho_1) + a''_{im} Y_m(k_{em}\rho_1)\right]\right.$$

$$\left. + k_{em}\left[a'_{im} J'_m(k_{em}\rho_1) + a''_{im} Y'_m(k_{em}\rho_1)\right]\right\} = 0 \qquad (4\text{-}36)$$

Solving this set of equations for a_{em}, gives

$$a_{em} = \frac{\omega_o \rho_o q}{2\pi\rho_1 k_{em}}\left\{\frac{\frac{m}{k_{em}\rho_o}\frac{\epsilon_2}{\epsilon_1} J_m(k_{em}\rho_o) + J'_m(k_{em}\rho_o)}{\Delta_{em}}\right\} \qquad (4\text{-}37)$$

The radiation for this case is then,

$$W = \frac{2\omega_o q^2 \rho_o^4}{\epsilon_o \pi^2 \rho_1^2 \beta_c^2} \sum_{m=1}^{m=\infty} \frac{1}{m^3 \varkappa_{em}^2} \left\{ \frac{\left[\frac{1}{\beta_c \varkappa_{em}} \frac{\epsilon_2}{\epsilon_1} J_m(m\beta_c \varkappa_{em}) + J_m'(m\beta_c \varkappa_{em}) \right]^2}{\Delta_{em} \Delta_{em}^*} \right\} \tag{4-38}$$

For non-relativistic velocities of the charge, this equation reduces to,

$$W = \frac{2\omega_o q^2}{\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^{2m}(m!)^2} \frac{1}{[1 + (\epsilon_1 - \epsilon_2)]^2} \tag{4-39}$$

It may be assumed that even a weak magnetic field is sufficient to shift the multipole resonance for the lower order harmonics, since this is what occurs when the charge distribution moves outside the plasma column. A necessary approximation, that, $\frac{\omega_c}{\omega_o} << 1$, in Eq. (4-39) results in the following simplification.

$$W = \frac{2\omega_o q^2}{\epsilon_o} \sum_{m=1}^{m=\infty} \frac{m(m\beta_c)^{2m}}{2^{2m}(m!)^2} \frac{1}{\left\{ 2 - \left(\frac{\omega_p}{m\omega_o}\right)^2 \left[1 + \left(\frac{\omega_c}{m\omega_o}\right)^2 \right] \right\}^2} \tag{4-40}$$

Inspection of Eq. (4-40) affirms the above mentioned assumption.

Adjusting the intensity of the magnetic field to be uniform throughout all space implies that the charge moves in synchronization with the cyclotron frequency $(\omega_o = \omega_c)$. If it is then assumed that $\frac{\omega_p}{\omega_o} \ll 1$, the index of refraction may be approximated according to Eq. (3-43), and the resulting radiation is equivalent to free space, Eq. (3-11), as in the case of the charge distribution moving outside the plasma. Thus, when the influence of the plasma upon the radiation characteristics is small, it matters little whether the charge moves inside or outside the plasma because the radiation characteristics are identical to free space.

Again, the region of most interest occurs when the index of refraction of the medium may be approximated according to Eq. (3-45). Of course, the Cerenkov effect, here too, does not mask the multipole resonance that exists for the higher order harmonic. Moreover, the radiation for a typical lower order harmonic, W_m, may be obtained using the approximation of Eq. (3-47). The result is,

$$W_m = -\frac{2m\,\omega_o q^2 \beta_c^2 (m\beta_c \rho_1/\rho_o)^{2m}}{\epsilon_o 2^{2m}(m!)^2 \varkappa_{em}^2} \tag{4-41}$$

Finally, the average radiation of the single harmonic in the neighborhood of the singularity of the index of refraction is identical in form to this equation.

An interpretation of these results can be obtained through a study of the radiation characteristics for the charge distribution moving in an infinite plasma medium.

It is now expedient in this section to consider all the previous cases in the limit of $\rho_1 \to \infty$. That is, the plasma medium permeates throughout all space. Mathematically, then, the boundary conditions at ρ_1 are eliminated; the boundary conditions at ρ_o are retained intact. Since the present discussion of an unbounded plasma will be used merely as a means of comparison with the bounded plasma cases, only the more pertinent equations will be stated.

In the case of a simple plasma (incompressible and isotropic), the radiation expression reduces to,

$$W = \frac{\omega_o q^2 \beta_c^2}{2\epsilon_o} \sum_m m J_m'^2 (m\beta_c \varkappa_{om}) \tag{4-42}$$

where the sum is extended over those values of m for which $\varkappa_{om}$ is real.

If the charge distribution is assumed to move at very small velocities; then Eq. (4-42) converges to Eq. (3-11). Thus, the radiated energy converges geometrically from the fundamental frequency.

When it is assumed that the charge distribution moves at relativistic velocities, $\beta_c \sim 1$, the energy does not converge rapidly around a single harmonic but is distributed over a wide range of frequencies.

In order to analyze the radiated spectrum now, those Bessel functions whose arguments and orders are both large and comparable must be retained in the energy expression. For this purpose, the following relation is employed,

$$J_m(m\beta_c) \sim \frac{1}{\pi}\left[\frac{2^{\frac{1}{2}}(1-\beta_c)^{\frac{1}{2}}}{3^{\frac{1}{2}}\beta_c^{\frac{1}{2}}}\right] K_{\frac{1}{3}}\left[\frac{2^{\frac{3}{2}} m(1-\beta_c)^{\frac{3}{2}}}{3\beta_c^{\frac{1}{2}}}\right] \tag{4-43}$$

Equation (4-43) indicates that the argument of $K_{\frac{1}{3}}$ may be large or small depending upon the relative size of m. Therefore, Eq. (4-43) must be approximated for two ranges of m.

$$1 \ll m \ll m_o \quad , \quad m \gg m_o \tag{4-44}$$

where

$$m_o = 3/2\,(1-\beta_c^2)^{\frac{3}{2}}$$

In the first range of approximation, the energy expression reduces to,

$$W_m \sim \left\{\frac{\omega_o q^2}{2\epsilon_o}\;\frac{3^{\frac{1}{3}}\,\Gamma^2(2/3)}{2^{\frac{2}{3}}\,\pi^2}\right\} m^{-\frac{1}{3}} \tag{4-45}$$

for the m^{th} harmonic. The energy is seen to decrease very slowly with respect to increasing harmonics.

In the second range of m the m^{th} harmonic of the radiation is given by,

$$W_m \sim \left\{\frac{e^{-m/m_o}}{\sqrt{m}}\right\} \tag{4-46}$$

The radiation for extremely high order of harmonics decays rapidly.

Consider now the radiation from the line charge in a compressible isotropic plasma.

$$W = \frac{\omega_o q^2 \beta_c^2}{2\epsilon_o} \sum_m m J_m'^2 (m\beta_c \varkappa_{om})$$

$$- \frac{\pi}{4}\sqrt{\frac{\mu_o}{\epsilon_o}} \rho_o \omega_p^2 q^2 \sum_{m=1}^{m=\infty} \frac{1}{\varkappa_{om}} J_m(m\beta_a \varkappa_{om}) J_m'(m\beta_c \varkappa_{om}) \Big[J_m(m\beta_a \varkappa_{om} \rho/\rho_o) Y_m(m\beta_c \varkappa_{om} \rho/\rho_o)$$

$$- J_m(m\beta_c \varkappa_{om} \rho/\rho_o) Y_m(m\beta_a \varkappa_{om} \rho/\rho_o) \Big] \qquad (4\text{-}47)$$

and again it is understood that the first sum is for real values of $\varkappa_{om}$ only.

The appearance of a radially dependent term in this result is not surprising since part of the radiated energy is converted into mechanical energy of the medium. In order to determine the total power passing through a cylinder of unit height at an arbitrary radius $\rho > \rho_o$, the mechanical work done by the pressure must be included along with the radiated energy.

Adapting the procedure for calculating the radiated energy to calculating P, the mechanical energy results in,

$$P = \frac{\mu_o \omega_o q^2 \omega_p^2 \rho_o^2}{2\beta_c^2} \sum_m \frac{J_m^2(m\beta_a \varkappa_{om})}{m \varkappa_{om}^2}$$

$$+ \frac{\pi}{4}\sqrt{\frac{\mu_o}{\epsilon_o}} \rho_o \omega_p^2 q^2 \sum_{m=1}^{m=\infty} \frac{1}{\varkappa_{om}} J_m(m\beta_a \varkappa_{om}) J_m'(m\beta_c \varkappa_{om}) \Big[J_m(m\beta_a \varkappa_{om} \rho/\rho_o) Y_m(m\beta_c \varkappa_{om} \rho/\rho_o)$$

$$- J_m(m\beta_c \varkappa_{om} \rho/\rho_o) Y_m(m\beta_a \varkappa_{om} \rho/\rho_o) \Big] \qquad (4\text{-}48)$$

A superposition of Eqs. (4-47) and (4-48) results in an expression for W_t, which is just the work done by the electric field on the charge.

$$W_t = \frac{\omega_o q^2 \beta_c^2}{2\epsilon_o} \sum_m m J_m'^2(m\beta_c \varkappa_{om}) + \frac{\mu_o \omega_o q^2 \omega_p^2 \rho_o^2}{2\beta_c^2} \sum_m \frac{J_m^2(m\beta_a \varkappa_{om})}{m \varkappa_{om}^2}, \qquad (4\text{-}49)$$

the unspecified sums having the same meaning as before.

This result is a specific example of a more generalized Poynting theorem (see reference 20) which is,

$$\int_V \nabla \cdot [(\underset{\sim}{\mathcal{E}} \times \underset{\sim}{\mathcal{H}}) + p\underset{\sim}{\mathcal{U}}]\, dv = -\int_V \frac{\partial}{\partial t}\left[\frac{\mu_o}{2}\underset{\sim}{\mathcal{E}}^2 + \frac{\epsilon_o}{2}\underset{\sim}{\mathcal{H}}^2 + m\, n_o \underset{\sim}{\mathcal{U}}^2 + p\right] dv + \int \underset{\sim}{j} \cdot \underset{\sim}{\mathcal{E}}\, dv \qquad (4\text{-}50)$$

That the stored energy is time invarient for the case at hand is due to the fact that the fields created by the moving charge distribution are carried along with the charge. Therefore, the total flux of energy passing per second through a cylinder of unit height enclosing the charge cannot change with time.

Returning to the expression for the radiated energy, Eq. (4-47), a detailed analysis for supersonic, non-relativistic velocities (that is $\beta_a >> 1$, $\beta_c << 1$) reveals that the energy is primarily electromagnetic in nature. Therefore, the previous analysis for the incompressible plasma may be carried over completely.

It is desirable to calculate the total electric charge induced in the plasma per unit length in the z direction. The density profiles in the two cylindrical regions located about ρ_o are;

$$n = \begin{cases} n_o + i\dfrac{q\,\omega_p^2}{4q_o a^2}\displaystyle\sum_{m=-\infty}^{m=\infty} H_m^{(1)}(k_{am}\rho_o) J_m(k_{am}\rho) e^{-im\omega_o \tau} & ; \rho < \rho_o \\[2ex] n_o + i\dfrac{q\,\omega_p^2}{4q_o a^2}\displaystyle\sum_{m=-\infty}^{m=\infty} J_m(k_{am}\rho_o) H_m^{(1)}(k_{am}\rho) e^{-im\omega_o \tau} & ; \rho > \rho_o \end{cases} \qquad (4\text{-}51)$$

The electric charge density is given by,

$$\bar{\rho} = q_o(n_o - n) \qquad (4\text{-}52)$$

Therefore, the total electric charge induced in the region $\rho < \rho_o$ is,

$$\overline{\rho}_< = i\,\frac{q\,\omega_p^2}{4a^2}\sum_{m=-\infty}^{m=+\infty} H_m^{(1)}(k_{am}\rho_o)\int_0^{\rho_o}\rho J_m(k_{am}\rho)d\rho\int_0^{2\pi} e^{-i m\omega_o\tau}\,d\varphi \qquad (4\text{-}53)$$

It is seen from Eq. (4-53) that the only non-vanishing term in the sum is $m = 0$. Then,

$$\overline{\rho}_< = -i\,\frac{\pi q\,\omega_p^2}{2a^2}H_o^{(1)}(i\frac{\omega_p}{a}\rho_o)\int_0^{\rho_o}\rho J_o(i\frac{\omega_p}{a}\rho)d\rho = -\frac{\pi q\,\omega_p\rho_o}{2a}J_1(i\frac{\omega_p}{a}\rho_o)H_o^{(1)}(i\frac{\omega_p}{a}\rho_o) \qquad (4\text{-}54)$$

Similarly, the total electric charge induced in the region $\rho > \rho_o$ is,

$$\overline{\rho}_> = \frac{\pi q\,\omega_p\rho_o}{2a}J_o(i\frac{\omega_p}{a}\rho_o)H_1^{(1)}(i\frac{\omega_p}{a}\rho_o) \qquad (4\text{-}55)$$

The total electric charge induced in the plasma is,

$$\overline{\rho} = \frac{\pi q\,\omega_p\rho_o}{2a}\left[J_o(i\frac{\omega_p}{a}\rho_o)H_1^{(1)}(i\frac{\omega_p}{a}\rho_o) - J_1(i\frac{\omega_p}{a}\rho_o)H_o^{(1)}(i\frac{\omega_p}{a}\rho_o)\right]$$

$$= -\frac{\pi q\,\omega_p\rho_o}{2}\left[J_o(i\frac{\omega_p}{a}\rho_o)H_o^{(1)\prime}(i\frac{\omega_p}{a}\rho_o) - J_o'(i\frac{\omega_p}{a}\rho_o)H_o^{(1)}(i\frac{\omega_p}{a}\rho_o)\right] = -q \qquad (4\text{-}56)$$

Consequently, the total system of plasma and moving electric charge remains electrically neutral.

Consider now the radiated energy from the line charge in an incompressibie, anisotropic plasma.

$$W = \frac{\omega_o q^2\beta_c^2}{2\epsilon_o}\sum_m m\left[\frac{\epsilon_2}{\beta_c\varkappa_{em}\epsilon_1}J_m(m\beta_c\varkappa_{em}) + J_m'(m\beta_c\varkappa_{em})\right]^2 \qquad (4\text{-}57)$$

and the sum is extended to only real values of $\varkappa_{em}$.

For simplicity, it will again be assumed that $\omega_o = \omega_c$. Then, if it is further stipulated that $\omega_p/\omega_o << 1$, the index of refraction may be approximated by Eq. (3-43). The equivalent $\varkappa_{em}\beta_c$ for the plasma medium can be greater than unity for the fundamental frequency only. The energy radiated from the first line of the spectrum is,

$$W_1 = \frac{\omega_o q^2}{2\epsilon_o \kappa_{e1}^2}\left[(\beta_c \kappa_{e1})\, J_1(\beta_c \kappa_{e1}) + J_1'(\beta_c \kappa_{e1})\right]^2 = \frac{\omega_o q^2}{2\epsilon_o}\beta_c^2 J_2^2(\beta_c \kappa_{e1}) \tag{4-58}$$

The index of refraction is,

$$\kappa_{e1} = \sqrt{2 - \left(\frac{\omega_p}{\omega_o}\right)^2} \tag{4-59}$$

and converges to $\sqrt{2}$ in this approximation. Obviously, the condition for Cerenkov radiation $\kappa_{e1}\beta_c > 1$ exists only if the charge distribution moves at relativistic velocities. Therefore, if it assumed that the charge distribution moves with an extremely low velocity $\beta_c << 1$, the plasma behaves essentially as does free space. The energy radiated is then confined entirely to the fundamental harmonic and agrees with Eq. (3-44).

It is now desirable to consider the frequency range where $\omega_p/\omega_o >> 1$. In this region, the index of refraction may be approximated as Eq. (3-45). It is apparent from this equation that no radiation is possible for small orders of harmonics, and radiation only begins to become possible for those higher order harmonics in the neighborhood of the singularity of the index of refraction. Note that, in the finite plasma case, the radiation was not restricted to frequencies near the singularity. The first frequency where radiation is possible may lie anywhere in the neighborhood of the singularity, and both the plasma density and magnetic field intensity may be adjusted to yield any particular frequency within this range. It must be recalled that this is an extremely critical solution since, as stated before, Eq. (3-39) indicates that the index of refraction has a very sharp resonant condition. Also, once again, Eq. (3-45) states that κ_{em} is extremely small for the subsequent harmonics beyond the singularity. The radiation contribution of these harmonics is then vanishingly small. Directing attention to the radiation produced by the single harmonic in the vicinity of the singularity of κ_{em}, and using the asymptotic approximations for the Bessel functions in the expression for the radiated energy, gives,

$$< W_m > \sim \frac{\omega_o q^2 \beta_c}{2\pi\epsilon_o \varkappa_{em}} \tag{4-60}$$

where it is understood that the above equation is the average value of the energy for the m^{th} harmonic. A resemblance of this equation with Eq. (4-41) is attributed to the fact that, in this approximation when the plasma is extremely overdense, it is immaterial whether the plasma is finite or infinite in extent.

In summary, it may be said that, if the plasma frequency is much larger than the cyclotron frequency, the radiation field is confined entirely to a single harmonic which resides in the vicinity of the singularity of the index of refraction. The magnitude of the velocity of the charge distribution is immaterial since it is always possible to select the order of the harmonic such that the index of refraction is much greater than unity.

The radiation produced by the rotating charge for the more general case of a compressible, anisotropic plasma will now be studied. In this case, the coupled Eqs. (2-33) and (2-34) are uncoupled as in Sections Two and Three. In this case, the solution for the longitudinal field components becomes,

$$H_{zm} = \begin{cases} \dfrac{q_o \omega_c}{(k^2_{sm} - k^2_{\ell m})} \left[a_{im} J_m(k_{sm}\rho) - b_{im} J_m(k_{\ell m}\rho) \right] & ; \ \rho < \rho_o \\[2em] \dfrac{q_o \omega_c}{(k^2_{sm} - k^2_{\ell m})} \left[a_{em} H^{(1)}_m(k_{sm}\rho) - b_{em} H^{(1)}_m(k_{\ell m}\rho) \right] & ; \ \rho > \rho_o \end{cases} \tag{4-61}$$

$$N_{in} = \begin{cases} \dfrac{1}{(k_{sm}^2 - k_{\ell m}^2)}\left[(k_{om}^2 - k_{\ell m}^2)\, b_{im} J_m(k_{\ell m}\rho) - (k_{om}^2 - k_{sm}^2)\, a_{im} J_m(k_{sm}\rho)\right]; \rho < \rho_o \\ \\ \dfrac{1}{(k_{sm}^2 - k_{\ell m}^2)}\left[(k_{om}^2 - k_{\ell m}^2)\, b_{em} H_m^{(1)}(k_{\ell m}\rho) - (k_{om}^2 - k_{sm}^2)\, a_{em} H_m^{(1)}(k_{sm}\rho)\right]; \rho > \rho_o \end{cases} \tag{4-62}$$

Application of the boundary conditions results in,

$$a_{em} = -\frac{iq}{4q_o}\left[\frac{\omega_o \rho_o k_{sm}}{\omega_c}\,(k_{sm}^2 - k_{am}^2)\, J_m'(k_{sm}\rho_o) - \frac{\omega_p^2}{a^2} J_m(k_{sm}\rho_o)\right] \tag{4-63}$$

$$b_{em} = -\frac{iq}{4q_o}\left[\frac{\omega_o \rho_o k_{\ell m}}{\omega_c}\,(k_{\ell m}^2 - k_{am}^2)\, J_m'(k_{\ell m}\rho_o) - \frac{\omega_p^2}{a^2} J_m(k_{\ell m}\rho_o)\right]$$

For the sake of simplicity, an investigation of only the expression for the total power will be conducted. This has been shown to be equal to the work done on the charge by the electric field. See Eq. (4-50).

$$W_t = \frac{8\omega_o \rho_o^2 a^2}{\epsilon_o \omega_p^2}\left\{\sum_m \frac{m\, a_{em} a_{e-m}(k_{sm}^2 - k_{om}^2)}{k_{sm}^2 (k_{sm}^2 - k_{\ell m}^2)} + \sum_m \frac{m\, b_{em} b_{e-m}(k_{om}^2 - k_{\ell m}^2)}{k_{\ell m}^2 (k_{sm}^2 - k_{\ell m}^2)}\right\}, \tag{4-64}$$

where the first sum is extended to values of m for which k_{sm} is real and the second sum is extended to values of m for which $k_{\ell m}$ is real. The virtue of the technique indicated in Section Two can now be realized, since a direct substitution of Eq. (2-30) into (4-64) results in,

$$W_t = \frac{\omega_o q^2 \omega_p^2}{2\epsilon_o a^2} \sum_{m > \sqrt{1 + \left(\frac{\omega_p}{\omega_o}\right)^2}}^{\infty} \frac{m J_m^2(k_{am}\rho_o)}{k_{am}^2} \tag{4-65}$$

Now, Eq. (4-65) may be compared with Eq. (4-49). It is then evident that the effect of the coupling of the modes of propagation is to prohibit propagation of the electromagnetic mode, and to introduce a lower limit in frequency in the propagating acoustic mode. These results are entirely consistent with Tuan and Seshadri's paper [5] where the same mode of propagation is excited.

It suffices to say, that for supersonic velocities ($\beta_a >> 1$) of the charge distribution, the radiated spectrum decreases with increasing order of the harmonics.

SECTION V

CONCLUSIONS

The spectral distribution of the radiated energy of a line charge rotating both inside and outside a plasma column has been analyzed. The analysis was conducted to include the effects of compressibility and anisotropy of the plasma upon the radiation characteristics of the charge distribution.

If, in the incompressible isotropic plasma case, the charge moves at non-relativistic velocities, the plasma exhibits a resonance of a multipole type for harmonics of the angular frequency of rotation of the charge which satisfy the condition $m \cong \frac{1}{\sqrt{2}} \cdot \frac{\omega_p}{\omega_o}$. As the charge velocity approaches zero, the resonance becomes a singularity for those harmonics which satisfy exactly the relation $m = \frac{1}{\sqrt{2}} \frac{\omega_p}{\omega_o}$. The effect of compressibility is to leave the radiation field essentially unchanged.

A plasma which is slightly anisotropic is sufficient to shift this multipole resonance for the lower order harmonics. These harmonics correspond to reflected radiation when the charge moves outside the plasma; the charge moving inside the plasma sees an impervious infinite plasma medium.

In contrast, the plasma can still experience this multipole resonance for a sufficiently higher order harmonic. Furthermore, in the frequency range just above this multipole resonance, Cerenkov radiation contributes to the Bremsstahlung radiation for a single harmonic in the neighborhood of the singularity of the index of refraction.

If the plasma becomes highly anisotropic, the radiated energy of the charge is identical to free space for non-relativistic velocities of the charge.

Finally, the radiation characteristics are examined in the limit of an unbounded plasma. In the incompressible isotropic plasma, the radiation characteristics are similar in nature to the free space characteristics.

Moreover, compressibility is of no influence on these characteristics which are not unlike the radiation characteristics of the uniaxial plasma case. In the weakly anisotropic plasma, the radiation was confined to the singular harmonic near the resonance of the index of refraction.

In the more complex situation of a compressible anisotropic plasma, the affect of coupling is to prohibit the electromagnetic mode from propagating, and to introduce a lower limit in frequency of the propagating acoustic mode.

SECTION VI

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APPENDIX

An explicit evaluation of the uncoupling of equations (2-33) and (2-34) will now be conducted. First, the following abbreviated notation will be introduced. Thus,

$$A_1 = \frac{k_{am}^2}{q_o \omega_c} \quad , \quad B_1 = \frac{k_{am}^2}{q_o \omega_c}\left(\frac{m^2 \omega_o^2}{c^2} - \frac{\omega_p^2 k_{om}^2}{a^2 k_{am}^2}\right)$$

$$A_2 = \frac{q_o a^2 c^2 k_{om}^2}{\omega_c \omega_p^2} \quad , \quad B_2 = \frac{q_o a^2 c^2 k_{om}^2}{\omega_c \omega_p^2}\left(\frac{m^2 \omega_o^2}{c^2}\,\frac{k_{am}^2}{k_{om}^2} - \frac{\omega_p^2}{a^2}\right) \tag{A-1}$$

and the differential operator, $\mathcal{D}$, is defined as,

$$\mathcal{D} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) - \frac{m^2}{\rho^2} \tag{A-2}$$

Consequently, equations (2-33) and (2-34) can be written as,

$$(A_1 \mathcal{D} + B_1) H_{zm} - \mathcal{D} N_m = 0 \tag{A-3}$$

$$(A_2 \mathcal{D} + B_2) N_m - \mathcal{D} H_{zm} = 0 \tag{A-4}$$

Since the above equations are linear, a solution which is a linear combination of H_{zm} and N_m may be assumed. Specifically,

$$\begin{aligned} \psi_1 &= N_m + a H_{zm} \\ \psi_2 &= N_m + b H_{zm} \end{aligned} \tag{A-5}$$

or equivalently,

$$\begin{aligned} N_m &= \frac{b\psi_1 - a\psi_2}{b-a} \\ H_{zm} &= \frac{\psi_2 - \psi_1}{b-a} \end{aligned} \tag{A-6}$$

Direct substitution of equation (A-5) into equations (A-3) and (A-4) yields,

$$\left[(b+A_1)\mathcal{D}+B_1\right]\psi_1 - \left[(a+A_1)\,\mathcal{D}+B_1\right]\psi_1 = 0 \qquad \text{(A-7)}$$

$$\left[(bA_2+1)\,\mathcal{D}+bB_2\right]\psi_2 - \left[(aA_2+1)\mathcal{D}+aB_2\right]\psi_2 = 0 \qquad \text{(A-8)}$$

Multiplying equation (A-7) by (aA_2+1), and equation (A-8) by $(a+A_1)$, and adding the results gives,

$$\left\{\mathcal{D} + \left[\frac{(a+A_1)B_2}{(A_1A_2-1)}\right]\right\}\psi_1 = 0 \qquad \text{(A-9)}$$

where the following condition has been assumed,

$$(aA_2+1)B_1 - (a+A_1)aB_2 = 0 \qquad \text{(A-10)}$$

a similar manipulation results in,

$$\left\{\mathcal{D} + \left[\frac{(b+A_1)B_2}{(A_1A_2-1)}\right]\right\}\psi_2 = 0 \qquad \text{(A-11)}$$

where the following condition has been assumed,

$$(bA_2+1)B_1 - (b+A_1)bB_2 = 0 \qquad \text{(A-12)}$$

It is clear from a comparison of equations (A-10) and (A-12) that the assumed constant a and b have been made to satisfy the same equation. In particular,

$$a, b = \frac{A_2B_1 - A_1B_2 \pm \sqrt{(A_1B_2 - A_2B_1)^2 + 4B_1B_2}}{2B_2} \qquad \text{(A-13)}$$

Substitution of equations (A-1) into equation (A-13) gives,

$$\frac{k_{om}^2 - k_{sm}^2}{q_o\omega_c} \qquad \text{(A-14)}$$

$$b = \frac{k_{om}^2 - k_{\ell m}^2}{q_o\omega_c} \qquad \text{(A-15)}$$

Finally, substitution of equations (A-14) and (A-1) into equation (A-9); and equations (A-15) and (A-7) into equation (A-11) yields the following Bessel equations,

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi_1}{d\rho}\right)+\left(k_{\ell m}^2-\frac{m^2}{\rho^2}\right)\psi_1 = 0 \qquad \text{(A-16)}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi_2}{d\rho}\right)+\left(k_{sm}^2-\frac{m^2}{\rho^2}\right)\psi_2 = 0 \qquad \text{(A-17)}$$

The specific solutions of equations (A-16) and (A-17), applicable to the case of the charge distribution rotating outside the plasma column, are,

$$\left.\begin{aligned}\psi_1 &= b_{im}J_m(k_{\ell m}\rho)\\ \psi_2 &= a_{im}J_m(k_{sm}\rho)\end{aligned}\right\} ; \quad \rho < \rho_1 \qquad \text{(A-18)}$$

since the Neumann function must be rejected because it becomes infinite at $\rho = 0$. Moreover, the specific solution of equations (A-16) and (A-17), for the case of the charge distribution moving inside the plasma column in the specified regions, are,

$$\psi_1 = \begin{cases} b_{im}J_m(k_{\ell m}\rho) & ; \quad \rho < \rho_0 \\ b'_{im}J_m(k_{\ell m}\rho) + b''_{im}Y_m(k_{\ell m}\rho) & ; \quad \rho_0 < \rho_1 < \rho_1 \end{cases}$$

$$\psi_2 = \begin{cases} a_{im}J_m(k_{sm}\rho) & ; \quad \rho < \rho_0 \\ a'_{im}J_m(k_{sm}\rho) + a''_{im}Y_m(k_{sm}\rho) & ; \quad \rho_0 < \rho < \rho_1 \end{cases} \qquad \text{(A-19)}$$

The longitudinal field components, corresponding to the cases of (A-18) and (A-19), are found from equation (A-6), and the results are in the body of this paper. [Equations (2-35), (2-36), (4-1), and (4-2)].

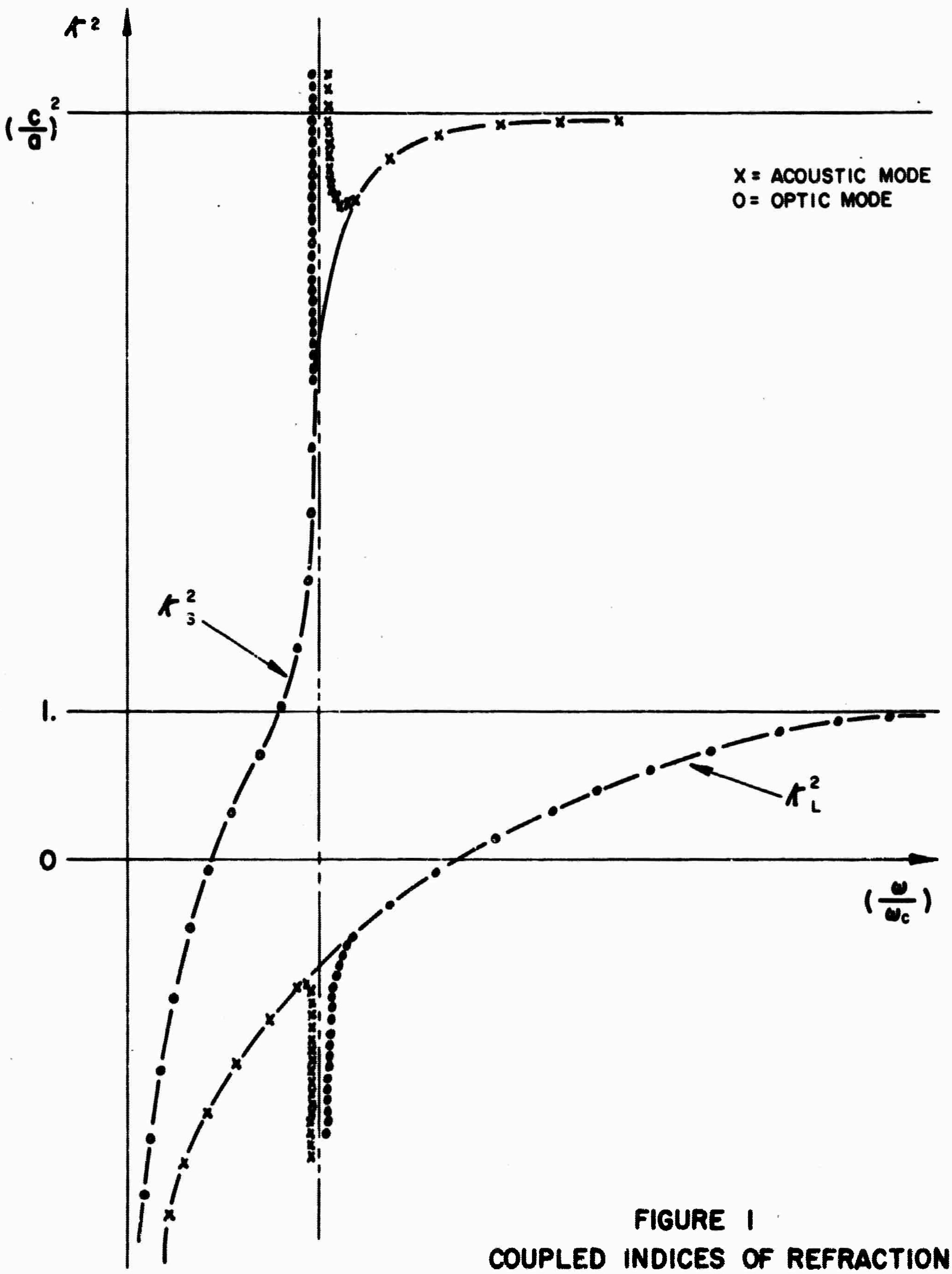

FIGURE I

COUPLED INDICES OF REFRACTION
OF A COMPRESSIBLE, MAGNETO-PLASMA

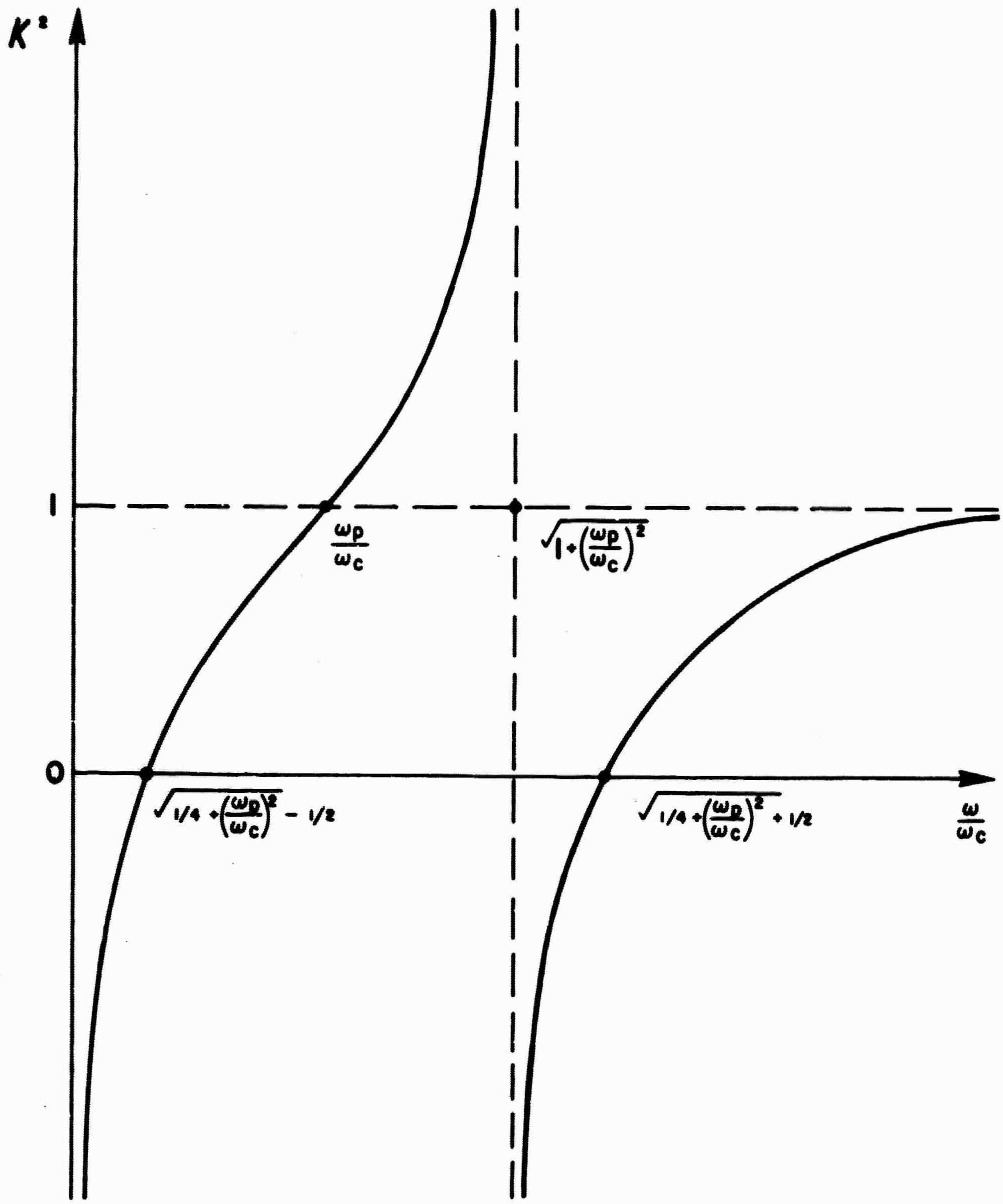

FIGURE 2
INDEX OF REFRACTION VERSUS FREQUENCY
FOR A PARTICULAR VALUE OF $\frac{\omega_p}{\omega_c}$

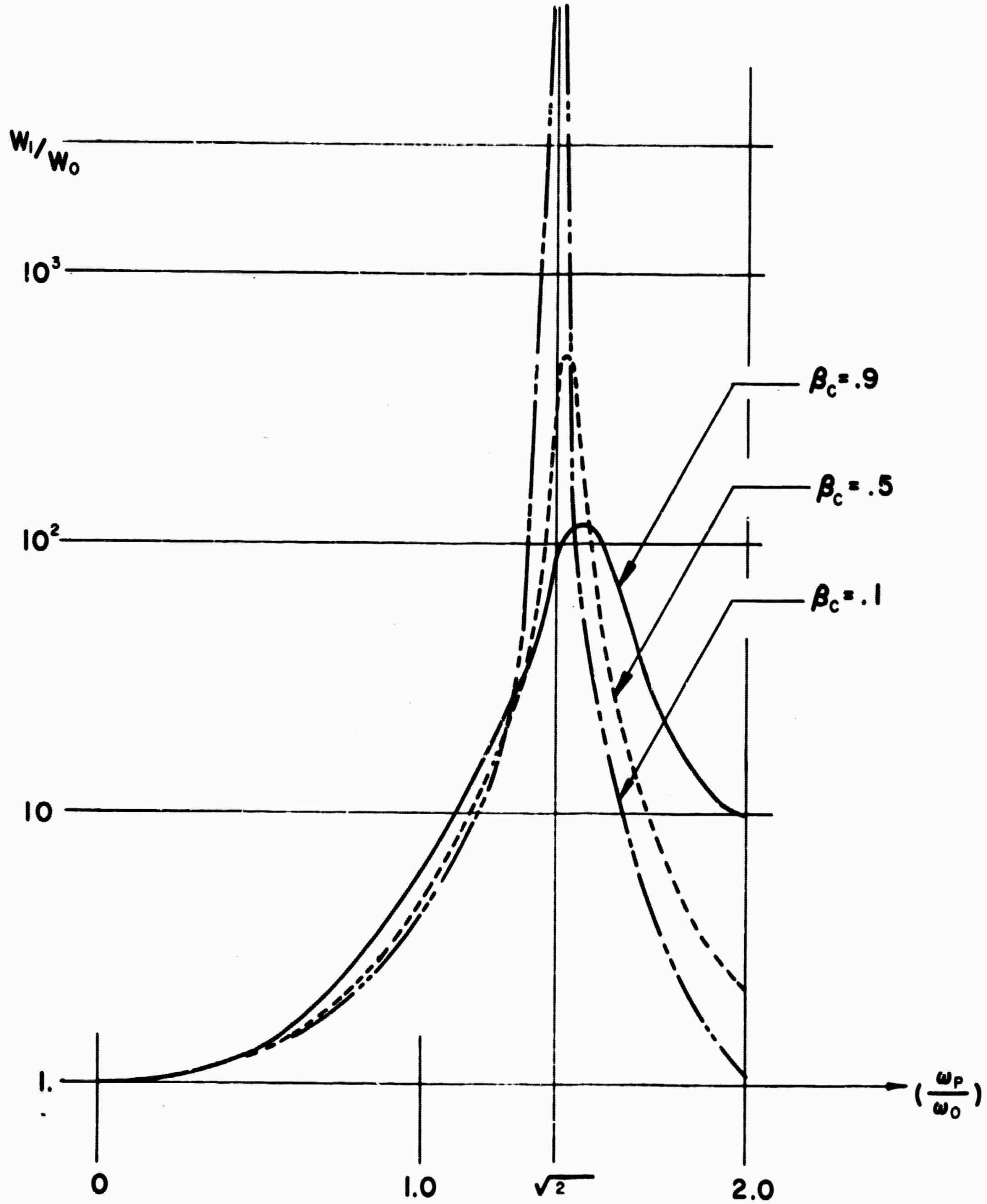

FIGURE 3

RADIATION BY LINE CHARGE INSIDE PLASMA COLUMN
(WHERE $\rho_1 \; \rho_0 = 1/2$, $\omega_c = 0$)

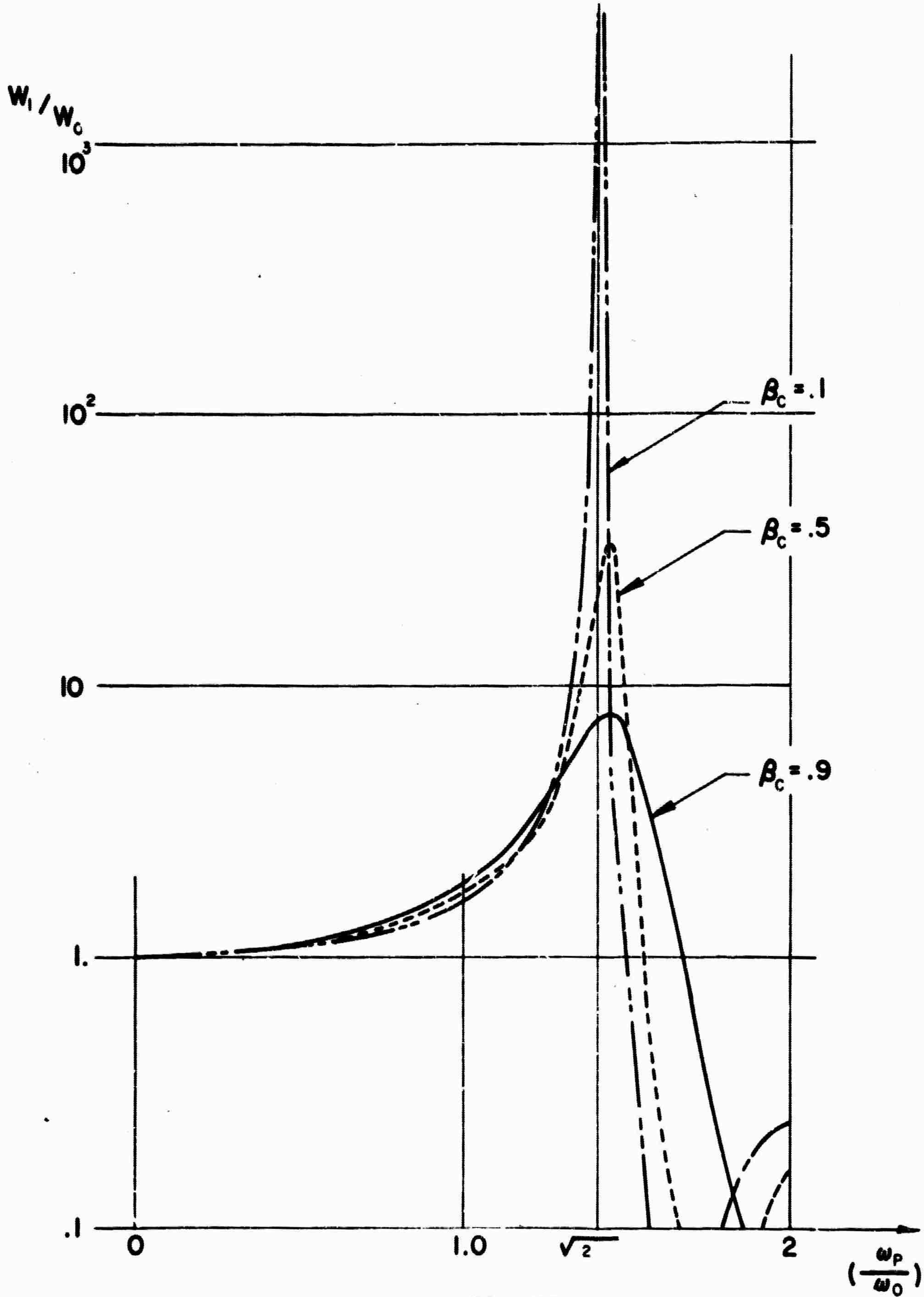

FIGURE 4

RADIATION BY LINE CHARGE OUTSIDE PLASMA COLUMN

(WHERE $\rho_1 / \rho_0 = 1/2$, $\omega_c = 0$)