STOCHASTIC WEAR PROCESSES

by

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ABSTRACT

A new class of non-decreasing stochastic processes is characterized. These processes satisfy a generalization of the notion of an increasing failure rate. From physical considerations, these processes seem suitable for describing the process of cumulative wear or damage. The main interest with the model is an investigation of the first time until the process exceeds a random barrier.

For this class of processes, it is shown that the first passage time random variable across a random barrier has an increasing failure rate, regardless of the distribution of the barrier. In addition, by the use of certain intuitive, non-parametric assumptions, tight bounds on the moments of this first passage time random variable are obtained.
INTRODUCTION

Few mechanistic models have been developed for processes which result in the failure of components. This is largely due to the great number of difficulties involved in trying to infer something about the underlying physical mechanism of wear from observations of the failure times.

H.E. Daniels (1945) and Z.W. Birnbaum and S.C. Saunders (1958) have proposed mathematical models justifying the assumption of the normal and gamma families respectively, for the distribution of the failure times in certain situations. A. Mercer (1961) considered the case in which the wear process consisted of blows occurring according to a Poisson process, the amount of wear per blow having a gamma distribution. He assumes further that the probability that the component fails in the interval \((t, t + dt)\), given the wear is \(x\), is of the form \(\lambda_1(t) + \lambda_2 x\). He obtains reasonably simple results when the amount of wear per blow is constant or an exponential random variable. E. Parzen (1959) considers the cumulative damage to a component subject to a large number of blows at a finite number of stress levels.

The model of wear we shall consider is one in which a component wears out according to a non-decreasing stochastic process \(Z_t \mid t > 0\), where \(Z_t\), a random variable, represents the cumulative damage at time \(t\). Now, rather than focusing our main attention on certain selected wear processes governed by mathematically convenient families of distributions, we shall take the perhaps more realistic approach of studying certain non-parametric classes of processes based on physical
considerations. We shall further assume that the component functions until the cumulative wear exceeds some critical threshold, in general, a random variable, at which time it breaks or otherwise becomes useless. It is this first passage time until breakdown which is of special interest in this investigation.

The wear of railroad tracks appears to be well described by this sort of model. One reason for replacing the track is that the weight per foot of the track decreases after use, and the track is considered too dangerous to be used if the weight per foot drops below some predetermined level. The wear in the track is due to a number of causes. When a train goes around the curve of a track, the flanges of the wheel repeatedly hit the rails. The force and frequency of these blows depend on the weight and speed of the train. A second cause of wear is that adjacent rails may have settled to slightly different levels, so that the train drops from one level to the next. A third and less destructive type of wear is the nearly continuous abrasive action due to nature. These three wear mechanisms will be of different degrees of importance in different sections of the track.

Summary:

In this paper, we characterize a new class of non-decreasing stochastic processes. These processes satisfy a generalization of the notion of an increasing failure rate, and from a physical point of view seem very suitable for describing the process of cumulative wear or damage. Our main interest with this model of wear will be an investigation of the random first time until breakdown. Since we
shall not assume that the wear process is known exactly, our results deal in the main with properties of the failure rate of the time until breakdown and with bounds on the moments of this random variable.

We shall now summarize partially the content and organization of this paper. In Section C, we state the definitions of certain fundamental mathematical concepts we will need such as total positivity of order two, monotone failure rate and Pólya frequency density of order two. These definitions will be referred to throughout the paper. In Section 1, we characterize the class of processes which we shall be concerned with throughout this paper. More exactly we shall consider those non-decreasing stochastic processes \( \{Z_t | t > 0\} \) such that

\[
P(Z_{t+h} > x | Z_t \leq x)
\]

be non-increasing in \( x \), for all \( t > 0 \), and \( h > 0 \) whenever defined. Such processes we shall call stochastic wear processes. We shall show that the compound-renewal process under certain non-parametric assumptions, as well as the translated Poisson process and the gamma process, are examples of stochastic wear processes.

In Section 2, we apply the concept of a stochastic wear process to an investigation of the failure rate of the random first time for the process to cross a random threshold. The key result here is that for stochastic wear processes with stationary, independent increments, the first passage time random variable has an increasing failure rate, regardless of the distribution of the random threshold. To show that this is not true in general, we provide a counter-example where the first passage time has first a decreasing failure rate, and then a constant failure rate for two successive intervals of
known threshold levels. We also remark in this section that the assumption that \( Z_t \mid t > 0 \) is a stochastic wear process is equivalent to imposing a structuring or ordering of the failure rate function of the first passage time across a known threshold. By the above, we mean if we increase the known threshold level, then the failure rate function of the first passage time uniformly decreases.

In Section 3, we consider the first passage problem across a random threshold for the special important case of the compound renewal process. Assuming that the blows have an increasing failure rate, we obtain tight upper and lower bounds on the mean and variance of this first passage time. We also obtain for any non-decreasing stochastic process the Laplace-transform of this first passage time across a random threshold.

In Section 4, we investigate the operations on stochastic processes under which the structuring of the failure rate mentioned above is preserved. We find that this structuring is preserved under formation of a "k out of n" system and, with certain conditions, under convolution. We also investigate a strengthening of the concept of a stochastic wear process and, as an application, consider the case where the threshold is known probabilistically.

0. Preliminaries

The following definitions will be referred to frequently in this investigation of stochastic wear processes.

Definition 1:

A function \( g(x,y) \geq 0 \) of two variables ranging over linearly-ordered one dimensional sets \( X \) and \( Y \) respectively, is said to be totally positive of order 2 \( (TP_2) \) if for all \( x_1 \leq x_2 \), and
\[ y_1 \leq y_2 \text{, then} \]

\[
\begin{vmatrix}
  g(x_1, y_1) & g(x_1, y_2) \\
  g(x_2, y_1) & g(x_2, y_2)
\end{vmatrix} \geq 0.
\]

**Definition 2:**

Let \( X \) be a random variable. Then \( X \) is said to have an increasing (decreasing) failure rate, IFR (DFR), if and only if

\[ P(X < x + h \mid X > x) \text{ is non-decreasing (non-increasing) in } x \text{ where defined and for all } h > 0. \]

**Definition 3:**

Let \( X \) be a random variable. Then \( X \) is said to possess a Pólya frequency density of order 2 (PF\(_2\)) if and only if

\[ \frac{f(x)}{F(x + \Delta) - F(x)} \text{ is non-decreasing in } x \text{, for all real } \Delta \text{ where defined and where } F(x) = P(X \leq x); f(x) = \frac{d}{dx} F(x). \]

The concepts of total positivity, monotone failure rate, and Pólya-frequency density are related in the following way:

**Lemma 0**

Let \( \bar{F}(x) = P(X > x) \) and \( f(x) = \frac{-d}{dx} \bar{F}(x). \)

Then

i) The random variable \( X \) is IFR if and only if \( \bar{F}(x-y) \) is TP\(_2\) in \( x \) and \( y \).

ii) The random variable \( X \) is DFR if and only if \( \bar{F}(x+y) \) is TP\(_2\) in \( x \) and \( y \) where \( x+y \geq 0 \).

iii) The density function of the random variable \( X \) is a \( \text{PF}_2 \) density function if and only if \( f(x-y) \) is TP\(_2\) in \( x \) and \( y \).

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iv) If the random variable $X$ possesses a $PF_2$ density, then $X$ is an IFR random variable. The converse is not true.

(See Barlow, Proschan, 1965)

1. A Generalization of the Failure Rate Concept to Stochastic Processes

Let $(Z_t \mid t > 0)$ be a non-decreasing stochastic process where the values at $t$, denoted $Z_t$, are finite, measurable functions on a probability space $(\Omega, \mathcal{A}, P)$ to the real line. Let $Z_0 = 0$ and the process be separable whenever necessary.

If $Z_t$ is thought of as the cumulative wear on some component at time $t$, and failure occurs whenever the process exceeds $x$, then

$$P(Z_{t+h} > x \mid Z_t \leq x)$$

is a natural generalization for stochastic processes of the failure rate (see definition 2). This is the probability that given that the component has not worn out at time $t$, then $h$ units of time later, it will have worn out. We consider the assumption that (1) is non-increasing in $x$, whenever defined, for all $h > 0$, and $t > 0$.

Perhaps a more intuitive way of viewing condition (1) is to require

$$P(Z_{t+h} \leq x \mid Z_t \leq x)$$

to be non-decreasing in $x$.

Definition 4:

A non-decreasing stochastic process $(Z_t \mid t > 0)$ will be called a stochastic wear process if (1) is non-increasing in $x$, where defined, for all $h > 0$, and $t > 0$.
Lemma 1

The stochastic process \( \{Z_t \mid t > 0\} \) is a stochastic wear process if and only if \( F(Z_t \leq x) \) is \( TP_2 \) in \( x \) and \( t \).

Proof: Write (1) as

\[
\frac{F(x,t) - F(x,t + h)}{F(x,t)}
\]

where \( F(x,t) = P(Z_t \leq x) \). By noting that (1) or equivalently (3) is non-increasing in \( x \) if and only if \( \frac{F(x,t + h)}{F(x,t)} \) is non-decreasing in \( x \), we see \( F(x,t) \) must be \( TP_2 \) in \( x \) and \( t \).

Remark: Suppose \( f(x,t) = \frac{\partial}{\partial x} F(x,t) \) exists. Then it can be shown as an easy consequence of the fact that \( F(x,t) \) is \( TP_2 \) that

\[
\frac{f(x,t)}{F(x,t)}
\]

is non-decreasing in \( t \), for all \( x \). Now \( \frac{f(x,t)}{F(x,t)} \) d\( x \) has the probabilistic interpretation that \( P(Z_t \in (x-dx,x) \mid Z_t \leq x) \) is non-decreasing in \( t \), again quite intuitive for non-decreasing processes.

As examples of processes which are stochastic wear processes (according to Definition 4) we offer the following:

Example 1: Let \( Z_t \) be the compound renewal process, that is,

\[
Z_t = \begin{cases} \sum_{i=0}^{N_s(t)} Y_i & \text{if } N_s(t) > 0 \\ 0 & \text{otherwise} \end{cases}
\]

where \( N_s(t) \) denotes the number of renewals in \((0,t]\) with respect to a spacing renewal process \( \{S_k\}_{k=1}^{\infty} \) and \( \{Y_i\}_{i=1}^{\infty} \) are...
independent, identically distributed samples from a non-negative random variable $Y$, independent from the spacing random variable $S$.

This model of wear is appropriate if a component is subject to wear produced by a series of blows, the blows occurring according to a renewal process.

**Lemma 2**

If both the random variables $S$ and $Y$ possess PF densities, then the compound renewal process is a stochastic wear process according to Definition 4.

**Proof:** By Lemma 1, it is equivalent to prove $F(x,t) = P(Z_t < x)$ is TP in $x$ and $t$. It can also be shown easily that if $f(x,t) = \frac{d}{dx} F(x,t)$ is TP in $x$ and $t$, then $F(x,t)$ is TP in $x$ and $t$. Hence, it suffices to show $f(x,t)$ is TP in $x$ and $t$.

Now $f(x,t) = \sum_{n=0}^{\infty} P(N_s(t) = n) f_Y^n(x)$ where $*$ represents convolution. However, since both $S$ and $Y$ have PF densities, it follows that $P(N_s(t) = n)$ is TP in $n$ and $t$ and $f_Y^n(x)$ is TP in $n$ and $x$ (Karlin, Proschan 1960). Hence, since total positivity is preserved under convolution (Pólya-Szego, 1925), we have $f(x,t)$ is TP in $x$ and $t$.

**Example 2:** Let $Z_t = N_s(t)$ where $N_s(t)$ denotes the number of renewals in $(0,t]$ where $S$, the spacing random variable, has an increasing failure rate. This is a weaker condition than requiring $S$ to have a PF density. If $[x]$ denotes the greatest integer contained in $x$, then

$$F(x,t) = 1 - F_s([x]+1)(t)$$
is $TP_2$ in $x$ and $t$ if $F_s(x) = P(S < X)$ is IFR (Barlow, Proschan, 1965). Hence $F(x) = IFR$ is a stochastic wear process.

Remark: $F(x,t)$ is also $TP_2$ in translation of $t$, for each fixed $x$ since the IFR property is preserved under convolutions (Barlow and Proschan, 1965).

Example 3: Let $Z_t = Y_t + \alpha t$, where $\alpha$ and $\alpha$ are constants greater than zero, and $Y_t$ the compound Poisson with constant blows, i.e.,

$$Y_t = \left\{ \begin{array}{ll} N(t) & \text{if } N(t) > 0 \\ c & \text{if } N(t) = 0 \\ 0 & \text{elsewhere} \end{array} \right.$$  

where $P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$  
$n = 0, 1, 2, \ldots$

This model of wear is appropriate for a component which, in addition to undergoing wear due to a series of blows, is undergoing an abrasive continuous wear. We shall show in Section 4 as a consequence of Theorem 4 that $Z_t$ is a stochastic wear process.

Remark: In this case, the random variable $Z_t$ has a discrete $PP_2$ density for each $t$ and hence possesses an increasing failure rate for each $t$.

Example 4: Let $Z_t | t > 0$ be the gamma process, that is,

$$f(x,t) = \left\{ \begin{array}{ll} \frac{e^{-x \cdot t - 1}}{\Gamma(t)} & x \geq 0, \ t > 0 \\ 0 & \text{otherwise} \end{array} \right.$$
In the manner of Smith (1958), one can consider $Z_t$ as the limiting form of a compound renewal process where the blows, in general, are very small in magnitude, but occur more and more frequently. Since it is well known that $f(x,t)$ is $TP_2$ in $x$ and $t$, then we see that $Z_t$ is a stochastic wear process. However $Z_t$ is not IFR for every $t$. In fact, if $t < 1$, $Z_t$ is a DFR random variable; if $t > 1$, $Z_t$ is an IFR random variable, and $Z_t$ is both IFR and DFR since the failure rate is constant.

Not all non-decreasing stochastic processes are stochastic wear processes, i.e., it is not true in general that is, $P(Z_{t+h} > x \mid Z_t \leq x)$ is non-increasing in $x$, for consider the process $\{Z_t \mid t > 0\}$ where

$$Z_t = \begin{cases} Y_t & \text{if } t \leq 1 \\ Y_1 & \text{if } t > 1 \text{ and } Y_1 \leq 1 \text{ or } Y_1 > 2 \\ Y_{1+3} & \text{if } t > 1 \text{ and } 1 < Y_1 \leq 2 \end{cases}$$

where $\{Y_t \mid t > 0\}$ is the gamma process. Then it can be shown that

$$P(Z_2 > 2 \mid Z_1 \leq 2) > P(Z_2 > 1 \mid Z_1 \leq 1)$$

and hence $\{Z_t \mid t > 0\}$ is not a stochastic wear process.

2. **Failure Rate of First Passage Random Variable**

Let 

$$U(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, since $U(x)$ is Borel-measurable, the process $\{U(x-Z_t) \mid t > 0\}$ will be a measurable process for any fixed $x$ (Halmos 1950). Then, by Fubini's Theorem, and in the manner of Smith (1958) we can introduce the
random variable

\[ T_x = \int_0^\infty U(x - Z_t) \, dt \]

Clearly, \( T_x \) is the first passage time of the non-decreasing process \( \{Z_t \mid t > 0\} \) across the barrier or threshold at a known height of \( x \).

We shall now investigate the properties of the failure rate of \( T_x \), under the assumption that \( \{Z_t \mid t > 0\} \) be a stochastic wear process.

**Theorem 1.** Let \( \{Z_t \mid t > 0\} \) be a stochastic wear process with stationary, independent increments. Then for every \( x > 0 \), \( T_x \) is an IFR random variable.

Proof: We use the following lemma, a proof of which can be found in Barlow, Proschan (1965).

**Lemma 3**

If \( F_1(0) = F_2(0) \) and \( F_1(y) \geq F_2(y) \) for all \( 0 \leq y \leq x \) and if \( Q(y) \geq 0 \) is non-increasing on \([0,x]\), then

\[ \int_0^x Q(y) \, dF_1(y) \geq \int_0^x Q(y) \, dF_2(y) \]

whenever the integrals exist.

Proof of Theorem 1: We must show for every \( x > 0 \) and \( h > 0 \), that

\[ P(T_x \leq t + h \mid T_x > t) \]

is non-decreasing in \( t \) where defined. Now, since

\[ P(T_x \leq t) = P(Z_t > x) \]

then (5) can be expressed as...
\[ \int_0^X \frac{\int_0^{x-y} h \cdot F(y, t)}{F(x, t)} \, dy \]

where \( F(x, t) = P(Z_t \leq x) \).

Now (6) can be rewritten as

\[ 1 - \int_0^X F(x-y, h) d_y \left( \frac{F(y, t)}{F(x, t)} \right) \]

By Lemma 1, \( F(x, t) \) is TP \( 2 \) in \( x \) and \( t \). Hence

\[ \frac{F(y, t_1)}{F(x, t_1)} \geq \frac{F(y, t_2)}{F(x, t_2)} \]

whenever \( t_1 \leq t_2 \) and \( 0 \leq y \leq x \).

Also, we note \( F(x-y, h) \geq C \) is non-increasing in \( y \) on \([0, x]\). Hence, applying Lemma 3, we obtain

\[ \int_0^X F(x-y, h) d_y \left( \frac{F(y, t_1)}{F(x, t_1)} \right) \geq \int_0^X F(x-y, h) d_y \left( \frac{F(y, t_2)}{F(x, t_2)} \right) \]

Hence, by (7) and (8), we obtain (5) is non-decreasing in \( t \), for all \( x > 0 \).

**Corollary**

Let \( \{Z_t \mid t \geq 0\} \) be a stochastic wear process with stationary, independent increments. Let \( X \) be an arbitrary non-negative random variable. Let \( T_X \), a random variable, denote the first time the process exceeds a barrier at a height of \( X \). Then \( T_X \) is an IFR random variable.

**Proof:** The proof follows trivially since \( T_X \) is IFR for all \( x > 0 \).

**Remark concerning Theorem 1:** The fact that the random variable \( T_X \) has an increasing failure rate implies that the random variable \( T_X \)
has the following, useful, inherited properties (for a proof of these properties and others, see Barlow and Proschan, 1965).

1) \( E(T_X^r) \leq \Gamma(r+1) [E(T_X)]^r \) if \( r \geq 1 \)
\[ E(T_X^r) \geq \Gamma(r+1) [E(T_X)]^r \] if \( 0 < r < 1 \)

where \( \Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du \)

Note:

Taking \( r = 2 \) we see that (i) implies that the random variable \( T_X \) has a coefficient of variation less than or equal to 1.

ii) \( P(T_X \geq ET_X) = P(Z_{ET_X} \leq X) \geq e^{-1} \)

that is, the probability that at time \( ET_X \), the process has not yet crossed the barrier at a height of \( X \), is greater than or equal to \( e^{-1} \).

iii) If components are replaced upon failure where \( T_X \) is the life of the component, then

\[ \text{Var} N(t) \leq \text{EN}(t) \leq t/ET_X \]

and \( P(N(t) \leq n) \geq \sum_{j=0}^{n} \left[ (t/ET_X)^j \exp \left(-t/ET_X\right) \right] \) where \( N(t) \)

is the number of spares needed to last a length of time \( t \).

A natural question at this point is, "Will \( T_X \) have an increasing failure rate, in general, if \( \{Z_t \mid t > 0\} \) is a non-decreasing process?"

The answer is no as the following counter-example demonstrates.

**Counter-Example:** Let \( Z_t = N_S(t) \) denote the number of renewals in \((0,t]\) where \( S \), the spacing random variable, is the gamma random variable with parameter \( \alpha < 1 \); that is, let
\[ f_s(x) = \begin{cases} \frac{\lambda x^{\alpha-1} - \lambda x}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

where \( f_s(x) \) is the density functions of the random variable \( S \).

Then, since
\[
P(T_x \leq t) = F([x]+1)^*(t), \quad x > 0
\]

it can quickly be shown that

i) \( T_x \) has a decreasing failure rate if \( ([x]+1) \alpha < 1 \)

ii) \( T_x \) has a constant failure rate if \( ([x]+1) \alpha = 1 \)

iii) \( T_x \) has an increasing failure rate if \( ([x]+1) \alpha > 1 \).

Hence we have an example where the random variable \( T_x \) does not have an increasing failure rate function for every value of the threshold index \( x \).

**Lemma 4**

Let \( \{Z_t \mid t > 0\} \) have stationary, independent, non-negative increments. Then

\[
E T_x \geq \frac{x}{EZ_t} \quad \text{for all } x > 0.
\]

The inequality is sharp.

**Proof:**

\[
E T_x = E \left( \int_0^\infty U(x-Z_t)dt \right) = \int_0^\infty E(U(x-Z_t))dt = \int_0^\infty E(U(x-Z_t))dt
\]

Now, since \( \{Z_t \mid t > 0\} \) is a non-decreasing process, we can say
\[
ET_x = \int_0^\infty F(x,t)dt \geq h \sum_{n=1}^\infty F(x,n,h) \text{ for all } h > 0.
\]

But, since \( \{Z_t \mid t > 0\} \) has stationary, independent increments,

\[
F(x,n,h) = F(x,h)^n
\]

Hence

\[
\sum_{n=1}^\infty F(x,nh) = E_{\frac{x}{h}}(x)
\]

where \( E_{\frac{x}{h}}(x) \) denotes the number of renewals in \((0,x]\) where the spacing random variable is \( \frac{x}{h} \). Hence, we obtain

\[
ET_x = \int_0^\infty F(x,t)dt \geq h E_{\frac{x}{h}}(x) \geq h \frac{x}{EZ_h} - 1
\]

and since \( EZ_h = \frac{x}{EZ_1} \), we obtain the desired result by letting \( h \) go to zero.

The lower bound is attained by the trivial process \( \{Z_t = t \frac{x}{EZ_1} \mid t > 0\} \).

**Corollary**

There is no non-trivial stochastic process \( \{Z_t \mid t > 0\} \) with stationary, independent, non-negative increments for which \( Z_t \), a non-discrete random variable, is IFR for each \( t \).

**Proof:**

\[
ET_x = \int_0^\infty F(x,t)dt \geq h + h \sum_{n=1}^\infty F(x,nh).
\]

Hence, \( ET_x \leq h + h E_{\frac{x}{h}}(x) \) for all \( h > 0 \). Now, if \( E_{\frac{x}{h}}(x) \) were IFR for all \( h > 0 \), then by Barlow, Proschan (1965), we would have

\[
E_{\frac{x}{h}}(x) \leq \frac{x}{EZ_h} = \frac{x}{hEZ_1}
\]

for all \( h > 0 \).

Hence, letting \( h \) go to zero, we get
But from Lemma 4, we have
\[ ET_x \leq \frac{x}{EZ_1} \]
Hence
\[ ET_x \geq \frac{x}{EZ_1} \]

Hence, taking Laplace transforms on both sides with respect to \( x \), we obtain
\[ ET_x = \int_0^\infty F(x,t) dt = \frac{x}{EZ_1} \]

Hence, taking Laplace transforms on both sides with respect to \( x \), we obtain
\[ \frac{-1}{s \log E(e^{-sZ_1})} = \frac{1}{s^2 EZ_1} \]
or equivalently \( E(e^{-sZ_1}) = e^{-sEZ_1} \) i.e., \( (Z_t | t > 0) \) is the trivial degenerate process.

Remark: Smith (1958) in his investigations of arbitrary stochastic processes with stationary, independent increments demonstrates that
\[ ET_x = \frac{x}{EZ_1} + O(x) \]
and \( ET_{x+h} - ET_x = \frac{h}{EZ_1} + O(x) \) for all \( h > 0 \).

Hence, even though \( (Z_t | t > 0) \) is an unknown stochastic wear process with stationary, independent increments, many of the properties of the random variable \( T_x \) are known.

Lemma 5

Let \( (Z_t | t > 0) \) be an arbitrary stochastic wear process. Then, for every \( L > 0 \), the ratio of the mean time the process spends under \( x \) in the interval \((0,L)\) to the mean time the process spends under \( x \) in the interval \((0, \infty)\) is non-increasing in \( x \).
Proof: By Lemma 1, we have

\[
\begin{bmatrix}
F(x_1, u) & F(x_1, v) \\
F(x_2, u) & F(x_2, v)
\end{bmatrix} \geq 0
\]

whenever \( x_1 \leq x_2 \), \( u \leq v \).

Integrating (9) on \( u \) from 0 to \( L \), and on \( v \) from \( L \) to \( \infty \), we obtain

\[
\begin{bmatrix}
\int_0^L F(x_1, u) du & \int_0^\infty F(x_1, v) dv \\
\int_0^L F(x_2, u) du & \int_0^\infty F(x_2, v) dv
\end{bmatrix} \geq 0
\]

Adding column one to column two of (10), and noting that

\[
E\left( \int_0^L U(x-Z_t) dt \right) = \int_0^L E(U(x-Z_t)) dt = \int_0^L F(x, t) dt
\]

is the mean time the process spends under \( x \) in the interval \((0,L)\), we obtain the desired result.

Definition 5:

A process \( \{Y_t \mid t > 0\} \) will be said to possess the structured failure rate property if

\[
P(Y_t \leq y + h \mid Y_t > y)
\]

is non-increasing in \( t \), whenever (11) is defined, for all \( y \) and \( h > 0 \).
Lemma 6

The stochastic process \( \{Z_t \mid t > 0\} \) is a stochastic wear process if and only if the process \( \{T_x \mid x > 0\} \) possesses the structured failure rate property. Proof is obvious.

Remark: Lemma 6 agrees with our intuition since we would expect that as the threshold level \( x \) increases, the failure rate function of the random variable \( T_x \) should decrease. We also observe that under the conditions of Theorem 1, that the failure rate function of the random variable \( T_x \), that is,

\[
P(T_x \leq t + h \mid T_x > t)
\]

is not only monotone in \( t \), but also monotone in \( x \).

We shall have more to say about this structuring of the failure rate property in Section 4. Now we shall turn our attention towards the important compound-renewal wear process.

3. The First Passage Time Over A Random Threshold for the Compound Renewal Process

Let

\[
Z_t = \begin{cases} 
  \sum_{i=1}^{N_s(t)} Y_i & \text{if } N_s(t) > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

where \( N_s(t) \) denotes the number of renewals in \((0,t]\) with respect to a spacing renewal process \( \{s_k\}_{k=1}^{\infty} \). Let \( \{Y_i\}_{i=1}^{\infty} \) be independent, identically distributed observations of a non-negative random variable \( Y \), independent from the spacing random variable \( S \).
As noted before, the compound renewal process is appropriate whenever a component undergoes wear produced by a series of blows, the blows occurring according to a renewal process. In sections one and two, we have seen under certain non-parametric assumptions concerning the spacings between blows and the size of the blows, that the first passage time over a random threshold has an increasing failure rate. This is true regardless of how the random threshold values are distributed.

Now we shall obtain tight bounds on the mean and variance for the first time for a compound renewal process to cross the random threshold. We shall assume that only the mean of the random threshold is known, and that the random magnitude of the blows has a non-decreasing failure rate, i.e., \( P(Y \leq y+h \mid Y > y) \) is non-decreasing in \( y \) whenever defined. This includes the gamma and Weibull distributions for some range of the parameters involved, as well as the exponential and degenerate distributions. The following lemma is pertinent to this section.

**Lemma 7**

Let \( Z_t \mid t > 0 \) be an arbitrary non-decreasing process where \( Z_0 = 0 \). If \( T_X \) is the first time for the process to cross a random threshold \( X \), then

\[
E(e^{-sT_X}) = P(Z_Y > X)
\]

where \( V \) is the exponential random variable with mean \( \frac{1}{s} \).

**Proof:**

\[
E(e^{-sT_X}) = -\int_0^\infty e^{-st}dG(t)
\]
where \( G(t) = P(T_X > t) \).

Integrating (14) by parts, we obtain

\[
E(e^{-st}) = 1 - s \int_0^\infty e^{-st} G(t) dt
\]

(15)

Now \( G(t) = \int_0^\infty P(T_X > t) dF(x) \) where \( F(x) = P(X \leq x) \). Hence,

\[
G(t) = \int_0^\infty P(Z_t \leq x) dF(x) = P(Z_t \leq X)
\]

(16)

Hence, by (15) and (16), we obtain (13).

**Corollary**

\[
\int_0^\infty e^{-qx} e^{sT} dx = \frac{1}{q} - \frac{s}{q} \int_0^\infty e^{-su} e^{-qZ} du
\]

Proof: In Lemma 7, let \( X \) denote the exponential random variable with mean \( \frac{1}{q} \). Then from (13), we have

\[
E(e^{-st}) = \int_0^\infty q e^{-qx} e^{sT} dx
\]

(17)

\[
= 1 - \int_0^\infty \int_0^\infty P(Z_u \geq x) e^{-su} e^{-qZ} dx \ du
\]

(18)

Hence, dividing (18) by \( q \), we have the desired result.

**Corollary**

For the case of the compound-renewal process defined in Section 5, we obtain

\[
\int_0^\infty e^{-qx} e^{pT} dx = \frac{f_s(p)(1-f(q))}{q(1-f_Y(q)f_S(p))}
\]

(19)

where \( f_Y(q) = E(e^{-qY}) \); \( f_S(p) = E(e^{-PS}) \).
Proof: Since the double Laplace transform of $Z_t$ for the compound-renewal process is well known, the result follows easily.

Formula (19) is a very attractive general formula, but unfortunately does not yield explicit solutions except to rather simple problems. Alternatively we will calculate the first and second moments of $T_x$ directly. For these calculations, the following lemma is required.

Lemma 8

Let $N_Y(t)$ denote the number of renewals in $(0,t]$ where the spacing random variable $Y$ is distributed according to $F_Y(x)$.

If

$$H_Y(\ell)(t) = \sum_{n=1}^{\infty} n \ell^n P_Y(t) \quad \ell = 1, 2, \ldots$$

then

$$E(N_Y(t)^n) = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} H_Y^{n-j}(t) \quad n = 1, 2, \ldots$$

Proof:

(21)\[ E(N_Y(t)^n) = \sum_{k=0}^{\infty} k^n P(N_Y(t) = k) \]

$$= \sum_{k=0}^{\infty} k^n (F_Y(t) - F_Y^{k+1}(t))$$

$$= \sum_{k=1}^{\infty} (k^n - (k-1)^n) F_Y^{k+1}(t).$$

But

(22)\[ k^n - (k-1)^n = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} k^{n-j}. \]

Hence, (21) and (22) yield
Note: it can also be shown

\[ H_Y(\ell)(t) = E(Y(t)^{\ell}) = \int_0^t F_Y(t-u)dH_Y(\ell)(u) \quad \ell = 1, 2, \ldots \]

Also, we see \( H_Y(0)(t) = E Y(t) \).

**Theorem 2.** Let \( \{Z_t : t > 0\} \) be the compound-renewal process defined in Section 3. Let \( T_X \) be the first time the process exceeds the random threshold \( X \). If \( Y \), the random magnitude of the blows, has a non-decreasing failure rate, then

\[
\frac{\mu}{\mu_Y} \leq ET_X \leq E\frac{\mu}{\mu_Y} + \mu_Y
\]

\[
\frac{\sigma^2}{\mu_Y} \leq \text{Var} T_X \leq \sigma^2 + \frac{(\sigma^2 + \mu^2)}{\mu_Y}
\]

where

\( \mu_s = E(S) \) and \( \sigma_s^2 = \text{Var}(S) \) where \( S \) is the random spacings between blows.

\( \mu_Y = E(Y) \) where \( Y \) is the random magnitude of the blows and

\( \mu = E(X) \) where \( X \) is the random height of the barrier.

The above upper bounds are tight.
Proof:

\[ E(T_X | X = x) = \int_0^\infty F(x,t) \, dt \]

Now since \( F(x,t) = \sum_{n=0}^\infty P(N_s(t) = n) F^n_Y(x) \) where \( F^0_Y(x) = 1 \), we have

\[ E(T_X | X = x) = \sum_{n=0}^\infty F^n_Y(x) \int_0^\infty P(N_s(t) = n) \, dt \]

But

\[ \int_0^\infty P(N_s(t) = n) \, dt = \int_0^\infty (F(n+1)_s(t) - F^n_s(t)) \, dt 
\]

\[ = (n+1)\mu_s - n\mu_s = \mu_s . \]

Hence

(23) \[ E(T_X | X = x) = \mu_s (1 + EN_Y(x)) \]

since

\[ EN_Y(x) = \sum_{n=1}^\infty F^n_Y(x) . \]

Similarly, we obtain \( E(T_X^2 | X = x) \) by noting

\[ E(T_X^2 | X = x) = 2 \int_0^\infty t F(x,t) \, dt 
\]

\[ = 2 \sum_{n=0}^\infty F^n_Y(x) \int_0^\infty t(F(n+1)_s(t) - F^n_s(t)) \, dt 
\]

\[ = \sum_{n=0}^\infty [\sigma_s^2 + 1 + 2n \mu_s^2] \]

Hence

(24) \[ E(T_X^2 | X = x) = 2 \mu_s^2 H_Y^{(1)}(1) + (\sigma_s^2 + \mu_s^2) H_Y^{(0)}(x) \]

By Lemma 8

(25) \[ 2H_Y^{(1)}(x) = E(N_Y^2(x)) + H_Y^{(0)}(x) \]
Hence, substituting (25) into (24), together with (23), we obtain

\[
\text{Var}(T_x | X = x) = \mu_\beta (\text{Var} N_Y(x)) + \sigma_s^2 (\text{EN}_Y(x) + 1)
\]

Now, as shown by Barlow and Proschan (1965), if \( Y \) has an increasing failure rate, then

\[
\text{Var} N_Y(x) \leq \text{EN}_Y(x) \leq \frac{x}{\mu_Y}
\]

Hence, using (23), (26), and (27), and unconditioning on \( X \), we obtain the upper bounds in Theorem 2. To obtain the lower bounds, we use the fact that

\[
\text{Var} N_Y(x) \geq 0 ; \text{EN}_Y(x) \geq \frac{x}{\mu_Y} - 1
\]

and uncondition on \( X \).

The upper bound of Theorem 2 is attained when \( Y \) is the exponential random variable.

Remark: Smith (1955) has shown if \( \{Z_t | t > 0\} \) is the compound renewal process, then \( Z_t \) is asymptotically normal with mean \( \mu Z_1 \), and Variance \( \frac{\mu \sigma^2}{\mu^2} \), where \( EZ_1 = \frac{\mu X}{\mu_s} \), and Variance

\[
\sigma^2 = \frac{\sigma_Y^2}{\mu_s^2} + \frac{\sigma_s^2}{\mu_s^2} \cdot \frac{y}{3}.
\]

Also, it is clear that if a large number of steps are necessary to reach the barrier, then the distribution of \( T_x \) will be nearly normal. In fact, since

\[
P(T_x < t) = P(Z_t > x),
\]

it can be shown that \( T_x \) is asymptotically normal with mean \( \frac{x}{EZ_1} \) and Variance \( \frac{x \text{ Var } Z_1}{(EZ_1)^2} \), in complete analogy with the asymptotic distribution of the renewal random variable \( N(x) \), for renewal process (Feller, 1957).
4. Preservation of Structural Failure Rate Property and An Extension

We noted in Lemma 6, that the assumption that \( \{Z_t \mid t > 0\} \) be a stochastic wear process was equivalent to the process \( \{T_x \mid x > 0\} \) possessing a structuring of its failure rate as defined by Definition 5.

It is natural to inquire under what operations with stochastic processes this structuring of the failure rates is preserved. This is important in investigating structures such as parallel and series systems where the wear or damage to a structure is actually the minimum or the maximum of a sequence of different wear processes. Also, since the total wear on a structure is often composed of two distinct types of wear which add together, it is important to investigate conditions for which this structuring is preserved under convolution.

**Theorem 3.** Let \( \{Y_t^{(i)} \mid t > 0\} , i = 1,2,\ldots,n \) be a sequence of stochastic processes such that for each \( t \), the set of random variables \( \{Y_t^{(i)} \mid i = 1,2,\ldots,n \} \) are independent and identically distributed. Define \( \{M_t^{(k)} \mid t > 0\} , k = 1,2,\ldots,n \) as a sequence of stochastic processes where the random variable \( M_t^{(k)} \) denotes the \( k \)th largest over \( i = 1,2,\ldots,n \) of the set \( \{Y_t^{(i)}\} \) . If each process \( \{Y_t^{(i)} \mid t > 0\} , i = 1,2,\ldots,n \) possesses the structured failure rate property of Definition 5, then the stochastic processes \( \{M_t^{(k)} \mid t > 0\} , k = 1,2,\ldots,n \) , do likewise.

**Proof:** We shall need the following lemma.

**Lemma 9**

If \( \{Y_t \mid t > 0\} \) possesses the structured failure rate property of Definition 5, then

\[ P(Y_t \leq x) \]
is non-increasing in \( t \) for all \( x \), and is TP\(_2\) in \( x \) and \( t \).

Proof of Lemma 9: Let \( F(x,t) = F(Y \leq x) \); \( f(x,t) = \frac{\partial}{\partial x} F(x,t) \).
Then directly from Definition 5, we have \( 1 - F(x,t) \) is TP\(_2\) in \( x \) and \( t \) since

\[
P(Y \leq x + h) \mid Y > x) = \frac{F(x+h, t) - F(x, t)}{1 - F(x,t)}
\]

As an easy consequence, we get \( \frac{f(x,t)}{1-F(x,t)} \) is non-increasing in \( t \).

Hence, since \( 1-F(x,t) = \exp(- \int_x^\infty \left( \frac{f(u,t)}{1-F(u,t)} \right) du) \) we have that \( F(x,t) \) is non-increasing in \( t \).

Proof of Theorem 3. Let

\[
P(Y_{i}^{(1)} > x) = \overline{G}(x,t) \qquad \frac{d}{dx} \overline{G}(x,t) = g(x,t) \quad i = 1, 2, \ldots, n
\]

\[
P(M_{1}^{(k)} > x) = \overline{F}_{k}(x,t) \qquad \frac{d}{dx} \overline{F}_{k}(x,t) = f_{k}(x,t) \quad k = 1, 2, \ldots, n
\]

Then,

\[
\overline{F}_{k}(x,t) = \sum_{i=k}^{n} \binom{n}{i} (\overline{G}(x,t))^i (G(x,t))^{n-i}.
\]

Now, in the manner of Barlow, Proschan (1965), we have

\[
\overline{F}_{k}(x,t) = \left( \frac{\Gamma(n+1)}{\Gamma(k) \Gamma(n+1-k)} \right) \left( \int_0^1 \overline{G}(x,t)^{k-1} (1-u)^{n-k} du \right)
\]

Hence, we obtain

\[
\frac{\overline{F}_{k}(x,t)}{f_{k}(x,t)} = \frac{\overline{G}(x,t)}{g(x,t)} \int_0^1 v^{k-1} \left[ \frac{(1-v)}{G(x,t)} + v \right]^{n-k} dv
\]

Now, since \( \{Y_{i}^{(1)} \mid t > 0\} \) has the structured failure rate property, it is easily seen \( \overline{G}(x,t) \) is non-decreasing in \( t \). Hence, using Lemma 9, we obtain \( \frac{\overline{F}_{k}(x,t)}{f_{k}(x,t)} \) non-decreasing in \( t \), i.e., \( \{M_{1}^{(k)} \mid t > 0\} \) possesses a structured failure rate \( k = 1, 2, \ldots, n \).
Remark: If $k=n$, we can weaken Theorem 3 by eliminating the condition that $(Y^{(1)}_t \mid t > 0)$ for each $t$. To see this, note that if $k = n$, then

$$F_n(x,t) = \prod_{i=1}^{n} G_i(x,t)$$

where $G_i(x,t) = P(Y^{(1)}_t > x)$. Hence,

$$\frac{d}{dx} \frac{d}{dt} \log F_n(x,t) = \sum_{i=1}^{n} \frac{d}{dx} \frac{d}{dt} \log G_i(x,t)$$

Now, $p(x,y) > 0$ is TP in $x, y$ if and only if $\frac{d}{dx} \frac{d}{dy} \log p(x,y) > 0$ (Karlin 1957). Hence, $F_i(x,t) \text{ TP}_2$ in $x, t$ for each $i = 1, 2, ..., n$ implies $F_n(x,t)$ is TP in $x, t$.

**Theorem 4.**

a) Let $(Y^{(1)}_t \mid t > 0)$ and $(Y^{(2)}_t \mid t > 0)$ be arbitrary stochastic process where $Y^{(1)}_t$ and $Y^{(2)}_t$ are independent random variables for each $t$.

b) Let $(Y^{(1)}_t \mid t > 0)$ possess the structured failure rate property of Definition 5 and each random variable $Y^{(1)}_t$, $t > 0$ possess an increasing failure rate.

c) Let $g_2(x,t)$ be TP in $x$ and $t$ where $g_2(x,t)$ denotes the density function of the random variable $Y^{(2)}_t$. This, as can be easily shown, implies $(Y^{(2)}_t \mid t > 0)$ possesses a structured failure rate. Finally, assume that each random variable $Y^{(2)}_t$, $t > 0$ possesses a PF$_2$ density. Then under conditions (a), (b), (c), the stochastic process

$$(Y^{(1)}_t + Y^{(2)}_t \mid t > 0)$$

possesses the structured failure rate property of Definition 5 and for each $t$, the random variable $Y^{(1)}_t + Y^{(2)}_t$ if IFR.
Note:

We have seen from Theorem 1 and Lemma 5 that the process 
\( \{T_x \mid x > 0\} \) satisfies condition (b) if the underlying wear 
process is a stochastic wear process with stationary, independent 
increments. Other examples of processes satisfying conditions 
(b), and (c) are the compound Poisson with constant blows, 
and the degenerate process. Before we can prove Theorem 1, we 
must state the following lemma.

**Lemma 10**

Let 
\[
\int_{-\infty}^{\infty} f(x-u, y) h(u, y) du
\]

If both \( f(x, y) \) and \( h(x, y) \) are TP in \( x \) and \( y \) and TP 
in translations of \( x \) for each \( y \), then \( g(x, y) \) is TP in \( x \) and \( y \) and TP in translations of \( x \) for each \( y \). Proof is due to 
Ghurye and Wallace (1959).

**Proof of Theorem 1:**

Let 
\[
\bar{H}(x, t) = P(Y^{(1)}_t + Y^{(2)}_t > x)
\]

and 
\[
G_t(x, t) = P(Y^{(i)}_t \leq x) \quad i = 1, 2.
\]

Then it is clear that 
\[
\bar{H}(x, t) = \int_{-\infty}^{\infty} (1 - G_1(x-y, t)) d G_2(y, t)
\]

From condition (b) and Lemma 9, we have \( 1 - G_1(x, t) \) is TP in \( x \) 
and \( t \). From condition (b) and Lemma 0, we have \( 1 - G_1(x-y, t) \) is 
TP in \( x \) and \( y \) for each \( t \). From (c) and Lemma 0 we have 
\( d_y G_2(y, t) \) is TP in \( y \) and \( t \) and TP in translations of \( y \) 
for each \( t \). Hence, by an application of Lemma 10, we have \( \bar{H}(x, t) \)
is TP\(_2\) in \(x\) and \(t\). Hence by Lemma 9,

\[ \{Y_t^{(1)} + Y_t^{(2)} \mid t > 0\} \]

possesses the structured failure rate property. Also, since

\(\overline{H}(x,t)\) is TP\(_2\) in translations of \(x\) for each \(t\), then by Lemma 0, we obtain that for each \(t > 0\), the random variable \(Y_t^{(1)} + Y_t^{(2)}\) is IFR.

An extension of the stochastic wear process characterization of Definition 4 is to require \(q(t, x)\), the density function of the random variable \(T_x\), to be TP\(_2\) in \(x\) and \(t\). This indeed is a strengthening assumption since \(q(t, x)\) TP\(_2\) implies that \(P(T_x > t)\) is TP\(_2\) in \(x, t\). This stronger condition is of interest in the problem of estimating the threshold level, having observed the time to failure. We note that this stronger assumption is equivalent to the failure time statistic possessing a monotone likelihood ratio (for a definition and applications, see Karlin, Rubin 1956).

As examples of processes possessing this stronger property, consider the following:

**Example 1:** Compound Poisson, with PF\(_2\) blows, i.e., let

\[ F(x, t) = P(Z_t < t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} F_{Y}^*(x) \]

where \(Y\) has a PF\(_2\) density. Then

\[ P(t < T_x < t + dt) = q(t, x) dt = \lambda \int_0^x (1 - F_Y(x, y)) dF(y, t) dt \]

Now, by example 1 of section 1, \(dF(y, t)\) is TP\(_2\) in \(y, t\). Also, it is well-known that \(1 - F_Y(x, y)\) is TP\(_2\) in \(x, y\) if \(Y\) is PF\(_2\).
Hence, by Pólya-Szegő (1925) $q(t,x)$ is TP in $t$, $x$.

**Example 2:** Let $(Z_t | t > 0)$ be a temporarily homogeneous strong Markoff process with a non-negative drift. Let $Z_0 = 0$. Let the process possess a realization with almost every path function continuous, where continuity is with respect to a suitable order topology. If $q(t,x)dt = P(t < T_x < t + dt)$, then $q(t,x)$ is TP in $t$ and $x$ (see Karlin, 1964). Examples include the Poisson, Wiener and degenerate processes.

As an application of the assumption that $q(t,x)$ is TP in $t$ and $x$, we consider the case when the threshold $X$ is known probabilistically, i.e.,

$$P(X \leq x) = F(x).$$

In addition, suppose we have a collection of replacement strategies, say $(\varphi_i(t))_{i=1,2,...,n}$, depending on the observed time or age at failure of the component. Such strategies might be to use a block replacement policy, age replacement policy, to replace more or less often, or perhaps not to replace at all. Let $L_i(x)$ be the loss incurred if strategy $i$ is used and $x$ is the true value of the threshold.

Consider

$$\lambda_{ij}(t) = \int_0^\infty (L_i(x) - L_j(x))q(t,x)dF(x).$$

We notice if $\lambda_{ij}(t) < 0$, then action $i$ is to be preferred over action $j$ when $t$ is observed; similarly if $\lambda_{ij}(t) > 0$, action $j$ is preferred over action $i$.

Now if $L_i(x) - L_j(x)$ has at most one sign change, say from negative to positive, then by the variation diminishing property (Karlin (1964)) of totally positive function, there exists a $t_0$ such
that

1) if $t \leq t_0$, action $i$ is preferred over action $j$

2) if $t > t_0$, action $j$ is preferred over action $i$. 
REFERENCES

Barlow, R.E. and Proschan, R. (1965), Mathematical Theory of Reliability, John Wiley and Sons, Inc.


Parzen, E. (1959), "On Models for the Probability of Fatigue Failure
Nonr-25140, Office of Naval Research.

Pólya, G. and Szegö, G. (1925), Aufgaben und Lehrsätze aus der Analysis,


Smith, W. L. (1960), "Infinitesimal Renewal Processes," in Contributions
to Probability and Statistics, edited by Olkin et. al., Stanford
University Press.
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