ESD-TDR-65-262

Technical Note 1965-26

M. Athans

Solution of the Matrix Equation

$$\frac{d}{dt} X(t) = A(t)X(t) + X(t)B(t) + U(t)$$

ESRL

29 June 1965

Prepared under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

SOLUTION OF THE MATRIX EQUATION $\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)$

*MICHAEL ATHANS**

DEPARTMENT OF ELECTRICAL ENGINEERING
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

** Consultant, M.I.T. Lincoln Laboratory*

TECHNICAL NOTE 1965-26

29 JUNE 1965

LEXINGTON MASSACHUSETTS

ABSTRACT

The purpose of this note is to state the solution to the inhomogeneous matrix differential equation $\frac{d}{dt} X(t) = A(t)\, X(t) + X(t)\, B(t) + U(t)$.

Accepted for the Air Force
Stanley J. Wisniewski
Lt Colonel, USAF
Chief, Lincoln Laboratory Office

I. TERMINOLOGY

Suppose that we are given the time varying $n \times n$ matrices $A(t)$, $B(t)$, $U(t)$. We shall assume that

a. the elements of $A(t)$ and $B(t)$ are continuous functions of the time t

b. the elements of $U(t)$ are piecewise continuous functions of t.

We shall seek the solution of the matrix differential equation

$$\frac{d}{dt} X(t) = A(t)\, X(t) + X(t)\, B(t) + U(t) \tag{1}$$

subject to the initial condition

$$X(t_o) = X_o \tag{2}$$

where $X(t)$ is an $n \times n$ matrix.

II. THE HOMOGENEOUS CASE

Bellman in Reference [1] (page 175) considers the homogeneous equation

$$\frac{d}{dt} X(t) = A(t)\, X(t) + X(t)\, B(t) \tag{3}$$

subject to the initial condition

$$X(t_o) = X_o \tag{4}$$

His result is that the solution of (3) is given by the relation

$$X(t) = \Phi(t;t_o)\, X_o\, \Psi(t;t_o) \tag{5}$$

where $\Phi(t;t_o)$ is a nonsingular fundamental matrix which satisfies the differential equation

$$\frac{d}{dt}\,\Phi(t;t_o) = A(t)\,\Phi(t;t_o);\quad \Phi(t_o;t_o) = I \tag{6}$$

and where $\Psi(t;t_o)$ is a nonsingular fundamental matrix which satisfies the differential equation

$$\frac{d}{dt}\,\Psi(t;t_o) = \Psi(t;t_o)\,B(t);\quad \Psi(t_o;t_o) = I \tag{7}$$

III. THE INHOMOGENEOUS CASE

We claim that the solution of the differential equation

$$\frac{d}{dt}\,X(t) = A(t)\,X(t) + X(t)\,B(t) + U(t) \tag{8}$$

with $X(t_o) = X_o$ is given by

$$X(t) = \Phi(t;t_o)\left[X_o + \int_{t_o}^{t} \Phi^{-1}(\tau;t_o)\,U(\tau)\,\Psi^{-1}(\tau;t_o)\,d\tau\right]\Psi(t;t_o) \tag{9}$$

To see this, differentiate (9) with respect to t to obtain (we use dots to indicate differentiation) the relations

$$\begin{aligned}
\dot{X}(t) &= \dot{\Phi}(t;t_o)\left[X_o + \int_{t_o}^{t} \Phi^{-1}(\tau;t_o)\,U(\tau)\,\Psi^{-1}(\tau;t_o)d\tau\right]\Psi(t;t_o) \\
&\quad + \Phi(t;t_o)\,\Phi^{-1}(t;t_o)\,U(t)\,\Psi^{-1}(t;t_o)\,\Psi(t;t_o) \\
&\quad + \Phi(t;t_o)\left[X_o + \int_{t_o}^{t} \Phi^{-1}(\tau;t_o)\,U(\tau)\,\Psi^{-1}(\tau;t_o)\,d\tau\right]\dot{\Psi}(t;t_o) \\
&= A(t)\,\underbrace{\Phi(t;t_o)\left[X_o + \int_{t_o}^{t} \Phi^{-1}(\tau;t_o)\,U(\tau)\,\Psi^{-1}(\tau;t_o)\,d\tau\right]\Psi(t;t_o)}_{X(t)} \\
&\quad \underbrace{+ \Phi(t;t_o)\left[X_o + \int_{t_o}^{t} \Phi^{-1}(\tau;t_o)\,U(\tau)\,\Psi^{-1}(\tau;t_o)\,d\tau\right]\Psi(t;t_o)}_{X(t)}B(t) + U(t)
\end{aligned} \tag{10}$$

and, so,

$$\dot{X}(t) = A(t)\, X(t) + X(t)\, B(t) + U(t) \tag{11}$$

IV. TIME INVARIANT CASE

If A and B are constant matrices, then

$$\Phi(t;t_o) = e^{A(t-t_o)} \tag{12}$$

$$\Psi(t;t_o) = e^{B(t-t_o)} \tag{13}$$

and, so, the solution reduces to

$$X(t) = e^{A(t-t_o)} \left[X_o + \int_{t_o}^{t} e^{-A(\tau-t_o)}\, U(\tau) e^{-B(\tau-t_o)}\, d\tau \right] e^{B(t-t_o)} \tag{14}$$

REFERENCE

1. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill Book Company, New York, 1960.

DISTRIBUTION LIST

Director's Office

C. R. Wieser

Division 2

F. C. Frick
S. H. Dodd
J. Karaku

Group 28

J. A. Arnow
J. F. Nolan
A. Armenti
C. R. Arnold
F. Belvin
R. N. Davis
P. L. Falb
L. A. Gardner, Jr.
H. K. Knudsen
O. A. Z. Leneman
J. B. Lewis
H. E. Meily
A. J. Morency
H. C. Peterson
F. C. Schweppe (50)
J. M. Winett
P. E. Wood

Group 41

E. Hoffstetter

Group 62

K. Jordan
I. Stiglitz

Group 64

P. Green
E. Kelly
R. Price

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

Solution of the Matrix Equation $\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)$

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Note

5. AUTHOR(S) *(Last name, first name, initial)*

Athans, Michael

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
29 June 1965	7	1

8a. CONTRACT OR GRANT NO. AF 19(628)-500 b. PROJECT NO. c. None d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1965-26 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TDR-65-262

10. AVAILABILITY/LIMITATION NOTICES

None

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Air Force Systems Command, USAF

13. ABSTRACT

The purpose of this note is to state the solution to the inhomogeneous matrix differential equation $\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)$.

14. KEY WORDS

matrix algebra
differential equations
functions

UNCLASSIFIED
Security Classification