Solution of the Matrix Equation

\[ \frac{d}{dt} X(t) = A(t)X(t) + X(t)B(t) + U(t) \]

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SOLUTION OF THE MATRIX EQUATION
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\]

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ABSTRACT

The purpose of this note is to state the solution to the inhomogeneous matrix differential equation

$$\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t).$$

Accepted for the Air Force
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Lt Colonel, USAF
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I. TERMINOLOGY

Suppose that we are given the time varying \( n \times n \) matrices \( A(t), B(t), U(t) \). We shall assume that

a. the elements of \( A(t) \) and \( B(t) \) are continuous functions of the time \( t \)

b. the elements of \( U(t) \) are piecewise continuous functions of \( t \).

We shall seek the solution of the matrix differential equation

\[
\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)
\]

subject to the initial condition

\[
X(t_0) = X_0
\]

where \( X(t) \) is an \( n \times n \) matrix.

II. THE HOMOGENEOUS CASE

Bellman in Reference [1] (page 175) considers the homogeneous equation

\[
\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t)
\]

subject to the initial condition

\[
X(t_0) = X_0
\]

His result is that the solution of (3) is given by the relation

\[
X(t) = \Phi(t; t_0) X_0 \Psi(t; t_0)
\]

where \( \Phi(t; t_0) \) is a nonsingular fundamental matrix which satisfies the differential equation

\[
\frac{d}{dt} \Phi(t; t_0) = A(t) \Phi(t; t_0)
\]
\[
\frac{d}{dt} \Phi(t; t_0) = A(t) \Phi(t; t_0); \quad \Phi(t_0; t_0) = I \tag{6}
\]

and where \( \Psi(t; t_0) \) is a nonsingular fundamental matrix which satisfies the differential equation

\[
\frac{d}{dt} \Psi(t; t_0) = \Psi(t; t_0) B(t); \quad \Psi(t_0; t_0) = I \tag{7}
\]

III. THE INHOMOGENEOUS CASE

We claim that the solution of the differential equation

\[
\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t) \tag{8}
\]

with \( X(t_0) = X_0 \) is given by

\[
X(t) = \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0) \tag{9}
\]

To see this, differentiate (9) with respect to \( t \) to obtain (we use dots to indicate differentiation) the relations

\[
\dot{X}(t) = \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0)
\]

\[
+ \Phi(t; t_0) \Phi^{-1}(t; t_0) U(t) \Psi^{-1}(t; t_0) \Psi(t; t_0)
\]

\[
+ \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \dot{\Psi}(t; t_0)
\]

\[
= A(t) \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0)
\]

\[
+ \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0) B(t) + U(t)
\]

\[
X(t)
\]
and, so,

\[ X(t) = A(t) X(t) + X(t) B(t) + U(t) \]  \hspace{1cm} (11)

IV. TIME ININVARIANT CASE

If \( A \) and \( B \) are constant matrices, then

\[ \Phi(t; t_0) = e^{A(t-t_0)} \]  \hspace{1cm} (12)

\[ \Psi(t; t_0) = e^{B(t-t_0)} \]  \hspace{1cm} (13)

and, so, the solution reduces to

\[ X(t) = e^{A(t-t_0)} \left[ X_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)} U(\tau) e^{-B(\tau-t_0)} d\tau \right] e^{B(t-t_0)} \]  \hspace{1cm} (14)

REFERENCE

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