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Some Iterative Methods Using Partial Order for Solution of Nonlinear Boundary Value Problems

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SOME ITERATIVE METHODS USING PARTIAL ORDER FOR SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT

This report surveys three iterative methods which may be used to numerically solve nonlinear boundary value problems.

Accepted for the Air Force
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1. INTRODUCTION

Both Picard's and Newton's methods have been the subject of recent investigations by Schröder, Kalaba, Collatz et al.

The results obtained seem to hold some promise for the iterative solution of the Euler-Lagrange boundary value problem in variational problems.

The present outline surveys, in moderate generality, three methods making use of the concept of partial order:

1) An extension of Picard's method in which Lipschitz constants become operators and the concept of contraction is generalized.
2) An extension of Picard's method using monotone properties and the Schauder fixpoint theorem, dispensing with Lipschitz conditions entirely.
3) An extension of Newton's method using convexity, resulting in maximum-minimum principles and dispensing with second derivatives.

2. SPACES

2.1 P-Spaces

A real Banach space π will be called a P-space if a relation $\leqslant$ is defined between some pairs of points and has the following properties for all x, y, z, u, v in π. [$\geqslant$ denotes the converse of relation $\leqslant$]

$$x \leqslant x \tag{1}$$

$$x \leqslant y,\ y \leqslant z \Longrightarrow x \leqslant z \tag{2}$$

$$x \leqslant y,\ y \leqslant x \Longrightarrow x = y \tag{3}$$

$$x \geqslant O_\pi \Longrightarrow c\,x \geqslant O_\pi \quad \text{for c real positive} \tag{4}$$

$$x \geqslant y,\ u \geqslant v \Longrightarrow x + u \geqslant y + v \tag{5}$$

$$O_\pi \leqslant x \leqslant y \Longrightarrow ||x|| \leq ||y|| \tag{6}$$

$$\lim_{n \to \infty} x_n = x,\ x_n \geqslant O_\pi \Longrightarrow x \geqslant O_\pi \tag{7}$$

Positive cone: These conditions amount to saying that $C \triangleq \{x : x \succ O_\pi\}$ is a closed convex cone in π which remains convex when O_π is removed and such that for all x in π

$$\{y : ||y|| < ||x||\} \cap \{x + z : z \in C\} = \phi \quad \text{(the empty set)} \tag{8}$$

then

$$x \prec y \Longleftrightarrow y - x \in C$$

Intervals: $[x, y] \triangleq \{z : x \prec z \prec y\}$ is called the interval between x and y. Then

$$[x, y] \text{ is closed, convex and bounded} \tag{9}$$

$$x \prec y \Longleftrightarrow [x, y] \neq \phi \tag{10}$$

$$[x, y] = \{x + z : z \in C\} \cap \{y - u : u \in C\} \tag{11}$$

2.2 Q-Spaces

A real vector space S will be called a Q-space if there are, associated to S

a) a P space π

b) a function N, called partial norm, mapping S into π and such that, for $x, y \in S$ the following four conditions be satisfied:

$$N(x) = O_\pi \Longleftrightarrow x = O_S \tag{12}$$

$$N(cx) = |c| \, N(x) \quad \text{for all real } c \tag{13}$$

$$N(x + y) \prec N(x) + N(y) \tag{14}$$

Let a norm in S be defined by

$$||x||_S = ||N(x)||_\pi \tag{15}$$

then S is a Banach space under this norm. Comment: then N is a continuous mapping of S into the positive cone C of π .

Examples of P-spaces:

a) PR^n space of vectors $x = (\xi_1, \ldots, \xi_n)$ with some norm.

$(\xi_1, \ldots, \xi_n) \prec (\eta_1, \ldots, \eta_n)$ is defined by

$$\xi_i \leq \eta_i \qquad i = 1, \ldots, n \tag{16}$$

b) $PC^n[0,1]$ space of continuous functions f from $[0,1]$ into PR^n, with norm

$$||f|| = \max_{0 \le t \le 1} ||f(t)||_{\text{in } R^n}$$

$f \lessdot g$ defined by

$$f(t) \lessdot g(t) \text{ in } PR^n \quad 0 \le t \le 1 \tag{17}$$

Examples of Q-spaces:

c) QR^n space of vectors $x = (\xi_1, \ldots, \xi_n)$ Associated P-space: PR^n Partial norm:

$$N(x) = (|\xi_1|, \ldots, |\xi_n|) \tag{18}$$

d) $QC^n[0,1]$ space of continuous functions from $[0,1]$ into R^n $f = (f_1(t), \ldots, f_n(t))$ Associated P-space: $PC^n[0,1]$ Partial norm:

$$N(f) = (|f_1(t)|, \ldots, |f_n(t)|) \tag{19}$$

e) $QC^{n/1}[0,1]$ as in (d)
Associated P-space: $PC^1[0,1]$ Partial norm

$$N(f) = ||f(t)||_{\text{in } R^n} \quad \text{a function of } t \tag{20}$$

f) $QC^nR^n[0,1]$ as in (d) Associated P-space: PR^n Partial norm: the vector

$$N(f) = (\max_{0 \le t \le 1} |f_1(t)|, \ldots, \max_{0 \le t \le 1} |f_n(t)|) \tag{21}$$

g) any real Banach space with π as the real line and N as the norm.
Comment: The structure of π can be as rich as that of S but may go down as far as R^1.

3. FUNCTIONS

3.1 Functions from a P-space into a P-space

domain $D \subset \pi_1$ $\qquad$ π_1, π_2 P-spaces
function $T : D \to \pi_2$

Since P-spaces are Banach, the concepts of linearity, continuity, complete continuity and (Frechet-) differentiability are defined ipso-facto.

Other concepts are:

$$\text{Positivity:} \quad x \geq O_{\pi_1} \implies Tx \geq O_{\pi_2} \tag{22}$$

$$\text{Isotony:} \quad x \geq y \implies Tx \geq Ty \tag{23}$$

$$\forall x, y \in D$$

$$\text{Antitony:} \quad x \geq y \implies Tx \leq Ty \tag{24}$$

$$\text{Monotony:} \quad Tx \geq Ty \implies x \geq y \tag{25}$$

For linear T positivity and isotony are equivalent.

The usual definition of a convex function requires that the domain D be convex and that the range space π_2 be the set of real numbers. To relax the second requirement define

Convexity: T is convex if

a) D is convex

b) T has a Fréchet derivative at every point of D.

Let $T'_{(y)}$ denote the Fréchet derivative at y, a linear function from π_1 into π_2.

c) for all x, y in D

$$T'_{(y)}(x - y) \leq Tx - Ty \tag{26}$$

Extremization: For a function F from an arbitrary set A into a P-space, define

$$m = \operatorname*{pmax}_{a \in A} Fa \iff \exists\, b \in A,\ \forall\, a \in A: Fa \leq Fb = m \tag{27}$$

and*

$$\operatorname*{pmin}_{a \in A} Fa = - \operatorname*{pmax}_{a \in A} (-Fa)$$

Then (26) can be written

$$Tx = \operatorname*{pmax}_{y \in D} (Ty + T'_{(y)}(x - y)) \tag{28}$$

the maximum being attained for x = y.

3.2 Functions from Q-spaces into Q-spaces

Let T be a function mapping a domain D of a Q-space S_1 (with partial norm N_1 taking its values in P-space π_1) into a Q-space S_2 (with partial norm N_2

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* Note that if a pmax exists its value m is unique, just as in the real case.

taking its values in P-space π_2).

Since S_1 and S_2 are Banach spaces under the induced norms $||x||_{S_1} = ||N_1 x||_{\pi_1}$ $||y||_{S_2} = ||N_2 y||_{\pi 2}$ the concepts of linearity, continuity, complete continuity and Frechet differentiability are defined.

Generalized Lipschitz Condition

The function T is said to be Lipschitz continuous-K on D iff there exists a continuous linear positive function $K : \pi_1 \rightarrow \pi_2$ such that for all x, y in D

$$N_2(Ty - Tx) \leq K N_1(y - x) \tag{29}$$

This implies

$$||Ty - Tx||_{S_2} \leq ||K|| \cdot ||y - x||_{S_1} \tag{30}$$

so that T is then Lipschitz continuous with the constant $||K||$ also.

Generalized Contraction:

A function T mapping a domain D of a Q-space S into the same space S is a K-contraction on D if it is Lipschitz continuous K on D and the linear function K on the associated P-space π satisfies the condition

$$x + Kx + K^2 x + \ldots \quad \text{converges for all } x \text{ in } \pi \tag{31}$$

It is sufficient to this effect that $||K|| < 1$, then T is a contraction in the ordinary sense under $||\cdot||_S$. It is also sufficient (and necessary) that the spectral radius of the linear operator K satisfy

$$\rho(K) < 1 \tag{32}$$

4. THE GENERALIZED PICARD ITERATION

Problem: Let T be a function mapping a domain D of a Q-space S (with partial norm N taking its values in P-space π) into S. Find an element x of D for which x = Tx (a fixpoint).

Contraction Theorem:

Let T be a K-contraction on D

Let x_0 be an element of D

Let $x_1 = Tx_0$ and $r_0 = N(x_1 - x_0)$

Then the equation

$$r - Kr = Kr_0 \tag{33}$$

has a unique solution r^* in π.
If furthermore

$$\{x : N(x - x_1) \leq r^*\} \subset D \tag{34}$$

then the sequence x_n obtained by

$$x_{n+1} = Tx_n \qquad n = 0, 1, 2, \ldots \tag{35}$$

(a) exists in D

(b) converges to a limit x^* in D

(c) x^* is the unique solution of $x = Tx$ in D

(d) an error estimate is given by

$$n(x^* - x_1) \leq r^* \tag{36}$$

Application: If the iteration has been carried out up to the calculation of x_n, apply the theorem with x_{n-1} as x_0 and x_n as x_1.

Equation (33) need not be solved, it is sufficient to find b in π such that $r^* \leq b$ and use b in (36). The bound b can be found from the condition

$$b - Kb \geq Kr_0 \tag{37}$$

For instance if $c \geq 0$ and α, β real satisfy

$$Kc \leq \beta c \qquad r_0 \leq \alpha c \qquad \beta < 1$$

then

$$r^* \leq \frac{\alpha}{1 - \beta} Kc \leq \frac{\alpha\beta}{1 - \beta} c \tag{38}$$

5. THE MONOTONE PICARD ITERATION

Problem: Let T be a function from a convex domain D in a P-space π into π. Assume $T = T_1 + T_2$ with T_1 isotone on D, T_2 antitone on D. Find a fixpoint $x = Tx$ of T in D.

Method: Start from 2 elements x_0, y_0 of D and proceed by

$$\begin{aligned} x_{n+1} &= T_1 x_n + T_2 y_n \\ y_{n+1} &= T_1 y_n + T_2 x_n \end{aligned} \tag{39}$$

Nesting Theorem:

If $[x_0, y_0] \subset D$ and $x_0 \leq x_1 \leq y_1 \leq y_0$ (40)

then (a) $x_0 \leq x_1 \leq x_2 \leq \ldots \leq y_2 \leq y_1 \leq y_0$ i.e., the iteration defines a sequence of nested non-empty intervals $[x_n, y_n]$

(b) The image of $[x_n, y_n]$ under T is a subset of $[x_{n+1}, y_{n+1}]$ and a fortiori of itself. Thus any fixpoint in $[x_0, y_0]$ belongs to $[x_n, y_n]$ for all n.

Fixpoint Theorems:

A. Finite dimensional case.

If π is finite dimensional then (39), (40) imply

(a) $x_n \to x^*$ and $y_n \to y^*$ with $x^* \leq y^*$

(b) if T_1 and T_2 are continuous then there exists a fixpoint in $[x^*, y^*]$ and a fortiori in $[x_n, y_n]$.

(c) if T_1, T_2 are linear then $\frac{1}{2}(x^* + y^*)$ is a fixpoint.

B. Infinite dimensional case.

(a) If the image of $[x_n, y_n]$ under T is totally bounded it con-contains a fixpoint of T.

(b) If T is completely continuous on $[x_n, y_n]$ then statement (a) applies.

Since intervals are closed, bounded and convex these properties result essentially from the Schauder fixpoint theorem.

Application: Solutions of differential equations can often be considered as fixpoints of completely continuous integral transformations.

Note that no Lipschitz constant or operator K need be determined.

6. THE MONOTONE NEWTON ITERATION

Problem: Let S and S_1 be P-spaces and V a real vector space. Let L be a linear and F a non-linear function from a domain D in S into S_1 and G a linear function from D into V. Let c be a given element of V. Find x in D such that

$$\begin{aligned} Lx &= Fx \\ Gx &= c \end{aligned} \tag{41}$$

Assumptions:

a) D is convex

b) there exists a linear function R from S_1 to S such that $||R|| \leq \gamma$ and $RLx = x$ for $x \in D \cap \{y : Gy = c\}$

c) F has a bounded Fréchet derivative on D

$$||F'_{(x)}|| \leq \mu \quad \text{for } x \in D$$

d) F is convex on D

e) $||Fx|| \leq \alpha + \beta||x||$ for $x \in D$

f) $\gamma(\beta + 2\mu) < 1$

g) $\sum \triangleq \{x : ||x|| \leq ||x_0||,\ ||x|| \leq \frac{\alpha\gamma}{1 - \gamma(\beta + 2\mu)}\} \subset D$

h) $Lx_0 \geq Fx_0$

i) $Gx_0 = c$

j) $Lx \geq F'_{(y)}x$ and $Gx = O_V \Longrightarrow x \geq O_s$

k) For all $x \in D$ there exists a unique solution y of the linear equations

$$\begin{aligned} Ly &= Fx + F'_{(x)}(y - x) \\ Gy &= c \end{aligned} \tag{42}$$

This defines a function T from D into S by $y = Tx$.

l) (41) has a solution x^* in D.

m) x_0 and $x_1 = Tx_0$ are in D.

Newton's Method:

Under assumptions (c) and (k) starting from an element x_0 in D attempt the iteration

$$x_{n+1} = Tx_n$$

which is possible as long as the iterates stay in D.

Monotone Property:

Under assumptions (a) (c) (d) (j) (k)

$$x_{n+1} \geq x_n \quad \text{for} \quad n \geq 1$$

Maximum Principle:

Under assumptions (a) (c) (d) (j) (k) (l) (m)

$$x^* = \underset{x \in D}{\operatorname{p max}}\ Tx \tag{43}$$

Interval Property:

Under assumptions (a) (c) (d) (h) (j) (k) (l) (m)

$$x_1 \leq x^* \leq x_0 \tag{44}$$

Minimum Principle:

Under assumptions (a) (c) (d) (j) (k) (l) (m)

$$\begin{aligned} & x^* = \text{pmin}\ x_0 \\ & \text{subject to } x_0 \in D \text{ and } Lx_0 \geq Fx_0 \end{aligned} \tag{45}$$

Boundedness Property

Under assumptions (a) (b) (c) (d) (e) (f) (g) (j) (k) and $x_1 \geq x_0$ the sequence x_0, x_1, x_n exists, is monotone increasing and contained in Σ .

In finite dimension this implies convergence to a solution of the problem.

In infinite dimension convergence follows only if the sequence can be shown to be sequentially compact (as when T is completely continuous). Otherwise, for instance in PC $[0, 1]$ only pointwise convergence follows.

7. BOUNDARY VALUE PROBLEMS

$$\begin{aligned} & \dot{x} = A(t)x + F(x,t) \qquad && F,\ c,\ x \ \text{ n-vectors} \\ & Nx(0) + Mx(1) = c \qquad && A(t),\ N,\ M \ \text{ nbyn matrices} \end{aligned} \tag{46}$$

Let $\Phi(t)$ be a fundamental matrix, a non-singular solution of $\dot{\Phi}(t) = A(t)\ \Phi(t)$

Assume $N\Phi(0) + M\Phi(1)$ is non-singular. Then the solution of

$$\begin{aligned} & \dot{x} = A(t)\ x + f(t) \\ & Nx(0) + Mx(1) = c \end{aligned} \tag{47}$$

is for $0 \leq t \leq 1$

$$x(t) = G_1(t)\ c + \int_0^1 G(t,\tau)\ f(\tau)\ d\tau \tag{48}$$

where

$$G_1(t) = \Phi(t)[N\Phi(0) + M\Phi(1)]^{-1}$$

$$G(t, \tau) = \begin{cases} G_1(t)\, N\Phi(0)\, \Phi^{-1}(\tau) & \text{for } \tau < t \\ -G_1(t)\, M\Phi(1)\, \Phi^{-1}(\tau) & \text{for } \tau > t \end{cases} \tag{49}$$

(the value for $t = \tau$ is immaterial)

The solution of the non-linear problem, if one exists, is a continuous function $x(t)$ satisfying

$$x(t) = G_1(t)\, c + \int_0^1 G(t, \tau)\, F(x(\tau), \tau)\, d\tau \tag{50}$$

i.e., it is a fixpoint $x = Tx$ of the mapping T from the space of continuous R^n-valued functions on $[0, 1]$ into itself defined by (50).

Taking any norm in R^n and the maximum over $0 \leq t \leq 1$ of this norm as the function space norm the function T is completely continuous in many cases.

Thus the iterative methods can often be applied.

Example: (Collatz)

$$\ddot{x} = -t - \sqrt{x} \qquad x(0) = 0 \qquad x(1) = 1$$

There is no Lipschitz constant for this problem but the iteration,

$$\ddot{x}_{n+1} = -t - \sqrt{x_n} \qquad x_{n+1}(0) = 0 \qquad x_{n+1}(1) = 1$$

$$\ddot{y}_{n+1} = -t - \sqrt{y_n} \qquad y_{n+1}(0) = 0 \qquad y_{n+1}(1) = 1$$

is isotone with

$$x(t) \prec y(t) \Longleftrightarrow x(t) \leq y(t) \;\; \forall \;\; t \in [0, 1]$$

and is completely continuous

Taking $\quad x_0 = t^2 \qquad x_1 = \frac{4}{3} t - t^{3/3}$

$$y_0 = (2\sqrt{t} - t)^2 \qquad y_1 = \frac{23}{15} t - \frac{8}{15} t^{5/2}$$

Then $x_0 \prec x_1 \prec y_1 \prec y_0$ and $x_1(t) \leq x^*(t) \leq y_1(t) \;\; \forall \;\; t \in [0, 1]$ for a solution x^*.

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* About 150 further references are given in [2].

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