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General Quantum Field Theories and Strict Locality

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With 3 Figures in the Text

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A class of general quantum field theories without explicit use of fields $A(x)$ is defined by a set of postulates for the von Neumann algebras $\mathfrak{R}_C$ of local observables. The vacuum state is cyclic with respect to any $\mathfrak{R}_C$, and the Borchers tube theorem is shown to hold. Some $\mathfrak{R}_C$ are factors and not of finite type. A property of local observables called strict locality is formulated and expressed by means of a necessary and sufficient condition for the algebras $\mathfrak{R}_C$. It is proved for finite regions.

I. Introduction

In this paper we want to discuss general quantum field theories in connection with a kind of independence of physical measurements at spacelike separated points, called (following KNIGHT[1] and LICHT[2]) strict locality. Formulated as a postulate strict locality would be closely related to usual locality (commutativity of spacelike separated field observables). The latter implies (section II) that we cannot destroy properties P of a quantum mechanical ensemble measurable in a space-time region B by selecting subensembles which have a certain property Q measurable in a spacelike separated region C. Strict locality requires (as will be formulated more precisely in section IV) that, on the other hand, we cannot gain information about the properties measurable in B by measuring observables belonging to C only.

By a "general quantum field theory" we mean any theory fulfilling a certain set of postulates enumerated below (section II). These postulates are, as suggested a few years ago by LUDWIG[3], formulated in the language of von Neumann algebras of bounded operators (compare, e.g., HAAG and SCHROER[4], ARAKI[5]; the mathematical theory can be found in the books of VON NEUMANN, DIXMIER, and NEUMARK[6]) without reference to

[1] KNIGHT, J. M.: J. Math. Phys. 2, 459 (1961).

[2] LICHT, A. L.: J. Math. Phys. 4, 1443 (1963). — Equivalence of states (preprint 1964).

[3] LUDWIG, G.: Vorlesungen über Quantenfeldtheorie II. Berlin 1959 (unpublished).

[4] HAAG, R., and B. SCHROER: J. Math. Phys. 3, 248 (1962).

[5] ARAKI, H.: J. Math. Phys. 5, 1 (1964).

[6] NEUMANN, J. v.: Collected Works, vol. III. Pergamon Press 1961. — DIXMIER, J.: Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann). Paris: Gauthier-Villars 1957. — NEUMARK, M. A.: Normierte Algebren. Berlin: VEB Deutscher Verlag der Wissenschaften 1959.

the usual distribution valued operator fields $A(x)$. Thus the theory may be somewhat more general than usual axiomatic field theory ($A(x)$ need not exist). We hope that systems of von Neumann algebras derived from local fields $A(x)$ will satisfy all our postulates. For the free scalar field ARAKI[5] has shown this [with the exception of postulate 1c)], but for other fields — whenever such exist — the postulates are very plausible too. On the other hand, the frame defined by the postulates will be narrow enough to allow a series of conclusions (section III) going parallel to analogous developments in usual axiomatic field theory, so that we can hope to retain much of the physical contents of field theory. Thus we can look at our postulates from two points of view: We can consider them as a possible modification (most likely a generalization) of axiomatic field theory with similar physical content, or we can use them only as a (possibly incomplete) collection of features of axiomatic field theory expressed in a language convenient for technical purposes and needing a rigorous derivation from the usual (e.g. WIGHTMAN'S[7]) axioms.

In section IV we formulate strict locality and examine the consequences which it would have as a postulate together with the postulates of section II. We give a necessary and sufficient condition for strict locality in terms of VON NEUMANN'S relative dimension function. Contrary to LICHT[2] we do not suppose throughout the "duality theorem"[4] and require strict locality for finite regions only. So our conclusions will be somewhat weaker: Strict locality can be fulfilled even for factors of type I, as is shown by an example of KADISON[8], whereas LICHT[2] derives type III in all cases.

Section V is devoted to a proof of strict locality for a large class of finite regions using a lemma due to MISRA[9]. Thus strict locality is not independent on the other postulates, but nevertheless cannot yet be derived in its strongest form.

II. The Postulates

Postulate 1. (existence of rings of local observables): There is a unique mapping $C \to \mathfrak{R}_C$ of all space-time regions C* onto a set of von Neumann algebras (rings) $\mathfrak{R}_C$ of bounded operators in a separable Hilbertspace $\mathfrak{H}$ with the following properties:

* We suppose throughout the paper C to be open and equal to int $\bar{C}$, the interior of the closure $\bar{C}$ of C.

[7] WIGHTMAN, A. S.: Phys. Rev. **101**, 860 (1956).

[8] KADISON, R. V.: J. Math. Phys. **4**, 1511 (1963).

[9] MISRA, B.: On the algebra of quasi-local operators of quantum field theory (preprint 1963).

1a) $\mathfrak{R}_{B\cup C}=\{\mathfrak{R}_B, \mathfrak{R}_C\}''$, from which follows: for $B\subset C$ is $\mathfrak{R}_B\subset\mathfrak{R}_C$. (As to the notation, compare e.g. HAAG[10]). –

1b) For the whole Minkowski space M we have $\mathfrak{R}_M=\mathfrak{B}$, the von Neumann algebra of all bounded operators. –

Physically, the projection operators $P\in\mathfrak{R}_C$ are interpreted as the properties (VON NEUMANN[11], LUDWIG[12]) which can be decided by measuring devices located inside the region C; or (equivalently) the hermitean $A\in\mathfrak{R}_C$ should be possible observables in C. As the set of projections $P\in\mathfrak{R}_C$ generates $\mathfrak{R}_C$, $\{P\,|P\in\mathfrak{R}_C\}''=\mathfrak{R}_C$ (for all theorems on von Neumann algebras used here and in the following we refer to[6]), the properties measurable inside C and the ring $\mathfrak{R}_C$ mutually fix each other. With this interpretation 1a) says that we can measure in $B\cup C$ all functions of observables from B and C and nothing else, whereas 1b) means that we have a coherent Hilbertspace (no superselection).

Let us add a continuity requirement. The sequence of regions $\{C_i\,|\,i=1,2\ldots\}$ is said to converge towards the point set $\hat{C}$, if the point sets

$$\liminf C_i=\bigcup_{k=1}^{\infty}\bigcap_{i=k}^{\infty}C_i \quad\text{and}\quad \limsup C_i=\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}C_i$$

both coincide with $\hat{C}$. The point set

$$\hat{C}=\lim_{i\to\infty}C_i$$

is not necessarily a region C (i.e. open, with $C=\operatorname{int}\bar{C}$), but we will restrict ourselves to this case and require:

1c) (continuity of the mapping $C\to\mathfrak{R}_C$):

$$\mathfrak{R}_C=\{\mathfrak{R}_{C_i}\,|\,i=1,2\ldots\}'',$$

if C is the limit of an increasing sequence $\{C_i\,|\,i=1,2\ldots,\,C_{i+1}\supset C_i\}$, and

$$\mathfrak{R}_C=\bigcap_{i=1}^{\infty}\mathfrak{R}_{C_i},$$

if the decreasing sequence $\{C_i\,|\,i=1,2\ldots,\,C_{i+1}\subset C_i\}$ tends to C. –

We do not intend to discuss this postulate, but hope it is plausible too. With the help of 1c) we can generalize 1a) to unions $\bigcup_{i\in I}C_i$, if I is any (possibly even non denumerable) index set, but we could have

[10] HAAG, R.: Ann. Physik (Lpzg.) (7) **11**, 29 (1963).
[11] NEUMANN, J. v.: Mathematische Grundlagen der Quantenmechanik. Berlin: Springer 1932.
[12] LUDWIG, G.: Die Grundlagen der Quantenmechanik. Berlin-Göttingen-Heidelberg: Springer 1954.

required as well from the beginning

1a') $\mathfrak{R}_{\bigcup_{i\in I} C_i}=\{\mathfrak{R}_{C_i}\,|\,i\in I\}''$. —

In customary field theory the mapping $C\rightarrow\mathfrak{R}_C$ is mediated by an operator distribution $A(x)$, and $\mathfrak{R}_C$ is generated by all bounded functions of the operators

$$A(f)=f_0\cdot 1+\int f_1(x_1)\,A(x_1)\,d^4x_1+ \\ +\int f_2(x_1,x_2)\,A(x_1)\,A(x_2)\,d^4x_1\,d^4x_2+\cdots,$$

where the $f_i(x_1\ldots x_i)$ are testing functions in $\mathscr{D}_{4i}$ with compact support contained in $C\times C\times\cdots C$. As the $A(f)$ themselves are unbounded, domain questions will arise which are avoided here by considering the $\mathfrak{R}_C$ from the beginning.

Let us call $\mathfrak{R}_C$ the algebra (or ring) of local observables belonging to C, and C the support of $\mathfrak{R}_C$.

Postulate 2. (invariance): There is in $\mathfrak{H}$ a unitary representation $U(a,\Lambda)$ of the inhomogeneous Lorentz group with the property

$$U(a,\Lambda)\,\mathfrak{R}_C\,U^{-1}(a,\Lambda)=\mathfrak{R}_{(a,\Lambda)C},$$

where the region $(a,\Lambda)\,C$ is generated from C by the Lorentz transformation (a,Λ):

$$(a,\Lambda)\,C\equiv\{x\,|\,\Lambda^{-1}(x-a)\in C\}. \text{ —}$$

Postulate 3. (spectral condition): The representation $U(a,1)=e^{-iaP}$ of the translation group defines the energy momentum operator P_μ, the spectrum of which shall lie in the closed forward cone: $P^2\geqq 0$, $P_0\geqq 0$.

There exists one and only one translationally invariant state (eigenstate of P_μ with eigenvalue 0), the vacuum Ω_0. —

We need not discuss postulates 2 and 3; they are immediately carried over from field theory [7]. The same is true for

Postulate 4. (usual locality): If B, C are spacelike separated regions (i.e. $(x-y)^2<0$ for all pairs $x\in B$, $y\in C$), then $\mathfrak{R}_B$ and $\mathfrak{R}_C$ commute: $\mathfrak{R}_B\subset\mathfrak{R}'_C$. —

For later use (comparison with strict locality) we recall shortly the intuitive meaning of postulate 4. Suppose we have decided positively some property $P\in\mathfrak{R}_B$ and therefore reduced the statistical operator [11,12] from W to αPWP, with the normalization factor $\alpha=(\text{trace}\,(PW))^{-1}$. Then no subsequent measurement in C can destroy this property, because if any property $Q\in\mathfrak{R}_C$ turns out to be true, we have further reduced the statistical operator from αPWP to $\beta QPWPQ=\beta PQWQP$ [with

$\beta = (\text{trace}\,(QPW))^{-1}]$, for which the property P remains true:

$$\text{trace}(P \cdot \beta\, QPWPQ) = \beta \cdot \text{trace}(QPWPQ) = 1 .$$

On the other hand, this "compatibility" of any pair of properties $P \in \mathfrak{R}_B$, $Q \in \mathfrak{R}_C$ implies[12] their commutativity $PQ = QP$ and so

$$\{P \mid P \in \mathfrak{R}_B\} \subset \{Q \mid Q \in \mathfrak{R}_C\}'$$

from which follows[6]

$$\mathfrak{R}_B = \{P \mid P \in \mathfrak{R}_B\}'' \subset \{Q \mid Q \in \mathfrak{R}_C\}''' = \mathfrak{R}'_C .$$

Postulate 5. (primitive causality): For any region T containing a complete spacelike hypersurface Σ (shortly called a T-region) is $\mathfrak{R}_T = \mathfrak{R}_M = \mathfrak{B}$. –

This means: All that can be measured anywhere can as well be measured in any T-region, or: any state Ψ is fixed by its behavior with respect to measurements in a T-region (compare HAAG and SCHROER[4]).

III. Some immediate consequences

From postulates 1–5 we can draw easily some conclusions, which will be formulated as lemmas 1–4.

Lemma 1. (generalized Reeh-Schlieder theorem[13]): For any C (C is open, as assumed in postulate 1) the algebra $\mathfrak{R}_C$ is cyclic with respect to the vacuum state Ω_0. –

The proof makes no use of postulates 4 and 5 and consists in almost literally translating the proof of REEH and SCHLIEDER[13] from usual to "quasi-local" fields (see below). It can be found in ARAKI's paper[5].

Lemma 2. (generalized Borchers theorem[14]): Let be Z the cylinder region $|x_0| < t_0$, $|\vec{x}| < r_0$. Call D the double conic frustrum generated by Z:

$$D = \{x \mid |x_0| < t_0,\ |\vec{x}| + |x_0| < r_0 + t_0\}$$

(see Fig. 1). Then $\mathfrak{R}_D = \mathfrak{R}_Z$. –

The proof from postulates 1–3 is based on the same facts as the original proof of BORCHERS[14]. But because we found no derivation in the literature, we will sketch it shortly.* The relation $\mathfrak{R}_Z \subset \mathfrak{R}_D$ is clear, we must prove only $\mathfrak{R}_Z \supset \mathfrak{R}_D$, or $\mathfrak{R}'_Z \subset \mathfrak{R}'_D$. Let $C \subsetneqq Z$ be a cylinder region

* *Note added in proof.* Meanwhile we got knowledge of a proof due to ARAKI (Einführung in die Axiomatische Quantenfeldtheorie, mimeographed lecture notes, Zürich 1961/62).

[13] REEH, H., and S. SCHLIEDER: Nuovo Cimento **22**, 1051 (1961).

[14] BORCHERS, H. J.: Nuovo Cimento **19**, 787 (1961).

$|x_0|<\tau<t_0$, $|\vec{x}|<\rho<r_0$, A any operator from $\mathfrak{R}_C$, $A(x)$ the "quasi-local field" defined by $A(x)=U(x,1)\,A\,U^{-1}(x,1)$, i.e. by translating A by x. Any $A'\in\mathfrak{R}'_Z$ commutes with $A(x)$ as long as $x\in\bar{Z}=\{x\,\big|\,|x_0|<t_0-\tau,\ |\vec{x}|<r_0-\rho\}$ (Fig. 2), so the function $F(x)=\langle p|\,[A(x),A']\,|q\rangle$ with two arbitrary energy-momentum eigenstates $|p\rangle, |q\rangle$ vanishes for any $x\in\bar{Z}$.

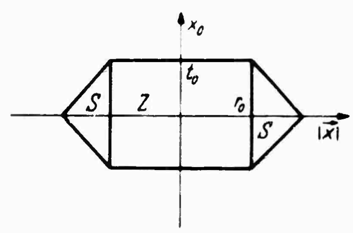

Fig. 1. The regions Z and $D=Z\cup S$ with $\mathfrak{R}_D=\mathfrak{R}_Z$

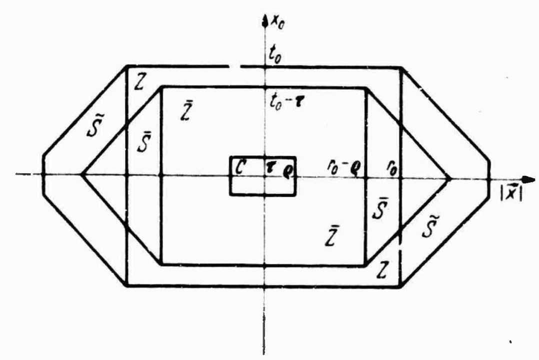

Fig. 2. Regions used in the proof of lemma 2 ($\tilde{D}=Z\cup\tilde{S}$, $\bar{D}=\bar{Z}\cup\bar{S}$)

By invariance and spectral conditions $F_+(x)=\langle p|\,A(x)\,A'\,|q\rangle$ and $F_-(x)=\langle p|\,A'\,A(x)\,|q\rangle$ have analytic continuations $F_+(z)$, $F_-(z)$ into the regions $\operatorname{Im} z\in V_+$ resp. $\operatorname{Im} z\in V_-$ (V_+ and V_- are the open forward resp. backward light cone) and coincide for real $z\in\bar{Z}$. By the generalized edge of the wedge theorem[14] they coincide for real $z\in\bar{D}$ too, where $\bar{D}$ is the double conic frustrum generated by $\bar{Z}$: $\bar{D}=\{x\,\big|\,|x_0|<t_0-\tau,\ |\vec{x}|+|x_0|<r_0-\rho+t_0-\tau\}$. That means: From $A'\in\mathfrak{R}'_Z$ follows $[A(x),A']=0$ for $x\in\bar{D}$ too. But as is easily seen all translated regions $C_y=\{x|\,x-y\in C$, with $y\in\bar{D}$ cover D up to an edge; their union is the region $\tilde{D}$ of Fig. 2. As we have [postulate 1a')] $\mathfrak{R}'_{\tilde{D}}=\{\mathfrak{R}_{C_y}|\,y\in\bar{D}\}'$ and any $A(y)\in\mathfrak{R}_{C_y}$ is of the form $A(y)=U(y,1)\,A\,U^{-1}(y,1)$ with some $A\in\mathfrak{R}_C$ (postulate 2), we have indeed shown $A'\in\mathfrak{R}'_{\tilde{D}}$, consequently $\mathfrak{R}'_Z\subset\mathfrak{R}'_{\tilde{D}}$, or equivalently (the property $\mathfrak{R}_Z\subset\mathfrak{R}_{\tilde{D}}$ is obvious) $\mathfrak{R}_Z=\mathfrak{R}_{\tilde{D}}$.

Now we repeat the whole procedure by using instead of C the double conic frustrum $\tilde{C}$ generated by C: $\tilde{C}=\{x\,\big|\,|x_0|<\tau,\ |\vec{x}|+|x_0|<\rho+\tau\}$. The translated $\tilde{C}_y$ lie in $\tilde{D}$ for $y\in Z$, and we cover the whole region D by the $\tilde{C}_y$ with $y\in\tilde{D}$. So we conclude as above $\mathfrak{R}'_{\tilde{D}}\subset\mathfrak{R}'_D$ and finally $\mathfrak{R}_D=\mathfrak{R}_{\tilde{D}}=\mathfrak{R}_Z$.

Corollary (generalized Borchers tube theorem): For any tube, i.e. any open region R containing an infinite timelike curve $\vec{x}=\vec{z}(t)$, $x_0=t$ with $|\vec{z}(t)|<\infty$ for all t, we have $\mathfrak{R}_R=\mathfrak{R}_M=\mathfrak{B}$. –

The deduction from Lemma 2 follows literally the proof of an analogous corollary in ref.[14].* The generalization of BORCHERS' tube theorem due to ARAKI[15] is also true here, as may be shown similarly. From Lemma 2 we can construct for every region C an extended region $\tilde{C} \supset C$ with $\mathfrak{R}_{\tilde{C}} = \mathfrak{R}_C$, if we define recursively $\tilde{C}_0 = C$ and $\tilde{C}_{i+1}$ as union of $\tilde{C}_i$ and all double conic frustra generated by cylinder regions in $\tilde{C}_i$ and take

$$\tilde{C} = \bigcup_{i=1}^{\infty} \tilde{C}_i = \lim_{i\to\infty} \tilde{C}_i .$$

An example shows Fig. 3. We define the spacelike complement C' of C as the set of all points lying spacelike to the closure $\bar{C}$ of C:

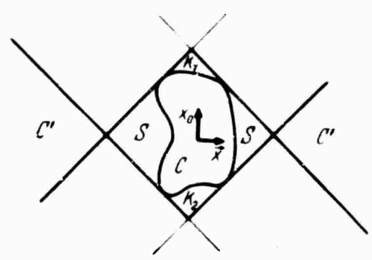

Fig. 3. A normal region C and the related regions C', $\tilde{C} = C \cup S$ and $C'' = \tilde{C} \cup K_1 \cup K_2$

$$C' = \{x \mid (x-y)^2 < 0 \text{ for all } y \in C\}.$$

A region C shall be called normal, if the closure of $\tilde{C} \cup C'$ contains a whole spacelike hypersurface Σ. (For instance, the C of Fig. 3 is normal. As a counterexample, the union of two finite regions separated by a timelike distance is certainly not normal). We can then prove:

Lemma 3. For a normal region C the von Neumann algebra $\mathfrak{R}_C$ is a factor[6], i.e. $\{\mathfrak{R}_C, \mathfrak{R}'_C\}'' = \mathfrak{B}$, or equivalently $\mathfrak{R}_C \cap \mathfrak{R}'_C = \{\lambda\, 1\}$. –

Proof: From postulate 4 we have $\mathfrak{R}_{C'} \subset \mathfrak{R}'_C$ and thus

$$\{\mathfrak{R}_C, \mathfrak{R}'_C\}'' \supset \{\mathfrak{R}_C, \mathfrak{R}_{C'}\}'' = \{\mathfrak{R}_{\tilde{C}}, \mathfrak{R}_{C'}\}'' = \mathfrak{R}_{\tilde{C} \cup C'} .$$

But because C is normal, we can find a decreasing sequence $\{T_i\}$ of T-regions converging towards $\tilde{C} \cup C'$ (see Fig. 3). From postulates 1c) and 5 we then have

$$\mathfrak{R}_{\tilde{C} \cup C'} = \bigcap_{i=1}^{\infty} \mathfrak{R}_{T_i} = \mathfrak{B} \cap \mathfrak{B} \cap \cdots = \mathfrak{B} .$$

Thus we have proven $\{\mathfrak{R}_C, \mathfrak{R}'_C\}'' \supset \mathfrak{R}_{\tilde{C} \cup C'} = \mathfrak{B}$, which means $\{\mathfrak{R}_C, \mathfrak{R}'_C\}'' = \mathfrak{B}$ because $\mathfrak{B}$ is maximal.

Lemma 4. (KADISON[8], GUENIN and MISRA[16]): If $\mathfrak{R}_C$ is a factor, it cannot be of finite type. (For the classification of factors, we again refer to[6]; a short account can be found in ref.[16]). –

The proof of GUENIN and MISRA[16] uses nothing but the facts collected here in the postulates and lemma 1 and need not be repeated here.

* *Note added in proof.* With similar methods, we can also get the following corollary: For any region G containing the origin and invariant under the homogeneous Lorentz group there exists a T-region $T_\varepsilon = \{x \mid |x_0| < \varepsilon\}$ with $\mathfrak{R}_G \supset \mathfrak{R}_{T_\varepsilon}$. (Postulate 5 then implies $\mathfrak{R}_G = \mathfrak{B}$.)

[15] ARAKI, H.: Helv. Phys. Acta **36**, 132 (1963).

[16] GUENIN, M., and B. MISRA: Nuovo Cimento **30**, 1272 (1963).

IV. Strict locality

As a counterpart to usual locality (postulate 4), we could require the following: Let be B, C two spacelike separated regions, P any property measurable at B. If we consider any state vector in $\mathfrak{H}$, we should not be able by examining this state in the region C only to predict anything about the outcome of a subsequent measurement of P in B. There should be all possibilities left open: P can possibly be found identically true, identically wrong, or true with any probability μ between 0 and 1 in this state*. We can formulate this in mathematical terms as follows:

Postulate 6. (strict locality): If B, C are spacelike separated regions, P any nontrivial projection ($P \neq 0, 1$) in $\mathfrak{R}_B$, Φ any state from $\mathfrak{H}$, there exists a state Ψ from $P\mathfrak{H}$ which is equivalent to Φ with respect to measurements in C, i.e. $(\Phi, A\Phi) = (\Psi, A\Psi)$ for all hermitean $A \in \mathfrak{R}_C$ (or, equivalently by the spectral theorem, for all projections $P \in \mathfrak{R}_C$). —

From $A = \frac{1}{2}(A + A^+) + \frac{1}{2}(A - A^+)$, $(A + A^+)$ and $i(A - A^+)$ hermitean, we then find $(\Phi, A\Phi) = (\Psi, A\Psi)$ for all $A \in \mathfrak{R}_C$.

Clearly postulate 6 follows from a part of our above requirement: We should be unable by measuring in a given state at C only, to exclude the possibility of P being identically true in this state**. But if we replace P by $1 - P \in \mathfrak{R}_B$, we deduce from postulate 6 the existence of a Ψ_1 equivalent to Φ with respect to C, for which P is identically wrong, and the state $\mu^{\frac{1}{2}}\Psi + (1-\mu)^{\frac{1}{2}}\Psi_1$ is equivalent to Φ too and gives the probability μ for the property P. So our intuitive requirements are fully contained in postulate 6.

To be more carefully, we should formulate strict locality by considering general statistical operators W instead of pure states $W = P_\Phi$, for we cannot distinguish pure states from mixtures by measuring expectation values at C only. (It would be possible if $\mathfrak{R}_C$ were irreducible but this is not the case.) Then we would have.

Postulate 6′. Let be B, C, P as in postulate 6, W any statistical operator ($W^+ = W$, $W > 0$, trace $(W) = 1$). Then there exists a statistical operator $\tilde{W}$ with trace $(A\tilde{W}) =$ trace (AW) for all hermitean $A \in \mathfrak{R}_C$ and trace $(P\tilde{W}) = 1$. —

* Such a requirement was first proposed by LUDWIG[3]; a formulation and detailed discussion was independently given by LICHT[2]. At first sight LICHT's notion of "primitive locality" (ref.[2], second paper) looks quite different from our postulate 6, but the equivalence of postulate 6 and LICHT's theorem 4 can easily be shown.

** As is well known[12], we can in principle measure all expectation values $(\Phi, A\Phi)$ of observables A in a state Φ simultaneously, even if the A are not commuting. This should not be confused with the fact that we can decide properties P, Q simultaneously, i.e. construct ensembles for which P as well as Q or $1 - Q$ are true, only if the properties are compatible, i.e. $[P, Q] = 0$.

If we specialize to pure states $W=P_\Phi$, we now cannot know whether the equivalent $\tilde{W}$ is a pure state P_Ψ (as above) or some mixture. Thus the requirement 6′ seems to be weaker than postulate 6.

In the rest of this section we discuss the implications which the addition of postulate 6 to postulates 1–5 would have. A characterization of strict locality equivalent to postulate 6 for factors $\mathfrak{R}_C$ gives

Theorem 1. Postulate 6 is fulfilled for the (ordered) pair of spacelike separated regions B, C with $\mathfrak{R}_C$ a factor if and only if every nontrivial projection $P\in\mathfrak{R}_B$ is infinite with respect to the factor $\mathfrak{R}'_C$. –

The proof of Theorem 1 follows mainly the procedure of LICHT[2], partly suggested to him by ARAKI.

The sufficiency is obvious: P infinite with respect to $\mathfrak{R}'_C$ together with the separability of $\mathfrak{H}$ implies[5] the equivalence of P and 1, or the existence of a partially isometric operator $V\in\mathfrak{R}'_C$ which maps $\mathfrak{H}$ onto $P\mathfrak{H}$ $(V^+V=1, VV^+=P)$. For any $\Phi\in\mathfrak{H}$ take $\Psi=V\Phi\in P\mathfrak{H}$; then from $V\in\mathfrak{R}'_C$ it follows

$$(\Psi, A\Psi)=(V\Phi, AV\Phi)=(\Phi, V^+AV\Phi)=(\Phi, AV^+V\Phi)=(\Phi, A\Phi)$$

for every $A\in\mathfrak{R}_C$.

To prove the necessity, we will construct a partially isometric operator $V\in\mathfrak{R}'_C$ with $V^+V=1$, $VV^+=P_1\subset P$, which shows the equivalence of $\mathfrak{H}$ and $P_1\mathfrak{H}\subset P\mathfrak{H}$ with respect to $\mathfrak{R}'_C$. Thus $\mathfrak{H}$ is equivalent to a subspace of it and consequently infinite. (The same fact follows[6,16] from the existence of a vector Ω_0 cyclic for the infinite factor $\mathfrak{R}_C$ and for $\mathfrak{R}'_C$.) $P\mathfrak{H}$ is then infinite because a subspace $P_1\mathfrak{H}$ of it is equivalent to $\mathfrak{H}$ and thus infinite too.

The above mentioned V can be constructed explicitly as follows: From postulate 6, there exists a vector $\Psi_0\in P\mathfrak{H}$ equivalent to the vacuum state Ω_0: $(\Psi_0, A\Psi_0)=(\Omega_0, A\Omega_0)$ for $A\in\mathfrak{R}_C$. (From $1\in\mathfrak{R}_C$ we have $\|\Psi_0\|=\|\Omega_0\|=1$.) The set $\mathfrak{R}_C\Omega_0=\{A\Omega_0 | A\in\mathfrak{R}_C\}$ is dense in $\mathfrak{H}$ (lemma 1), and we define on $\mathfrak{R}_C\Omega_0$ an operator $\tilde{V}$ by $\tilde{V}A\Omega_0=A\Psi_0$, which is obviously linear. Furthermore, $\tilde{V}$ leaves invariant inner products in $\mathfrak{R}_C\Omega_0$.

$$\begin{aligned}(\tilde{V}A_1\Omega_0, \tilde{V}A_2\Omega_0)&=(A_1\Psi_0, A_2\Psi_0)=(\Psi_0, A_1^+A_2\Psi_0)=(\Omega_0, A_1^+A_2\Omega_0)\\&=(A_1\Omega_0, A_2\Omega_0).\end{aligned}$$

Therefore, $\tilde{V}$ has a partially isometric closure V with $V^+V=1$. The range of $\tilde{V}$ and consequently, because $P\mathfrak{H}$ is closed and V continous, the range of V is contained in $P\mathfrak{H}$:

$$P\tilde{V}A\Omega_0=PA\Psi_0=AP\Psi_0=A\Psi_0=\tilde{V}A\Omega_0.$$

Every $\Phi \in \mathfrak{H}$ is (strong) limit of $\mathfrak{R}_C \Omega_0$,

$$\Phi = \lim_{i \to \infty} A_i \Omega_0, \quad A_i \in \mathfrak{R}_C,$$

and

$$VA\Phi = \lim_{i \to \infty} \tilde{V} A A_i \Omega_0 = \lim_{i \to \infty} A A_i \Psi_0 = \lim_{i \to \infty} A \tilde{V} A_i \Omega_0 = AV \lim_{i \to \infty} A_i \Omega_0 = AV\Phi$$

for any $A \in \mathfrak{R}_C$, thus $V \in \mathfrak{R}'_C$.

The possibility of constructing V using the existence of an equivalent state $\Psi_0 \in P\mathfrak{H}$ for the vacuum Ω_0 only (Ψ_0 is then eigenstate of P "strictly localized outside C" in the terminology of KNIGHT[1] and LICHT[2]) allows to restrict the $\Phi \in \mathfrak{H}$ in postulate 6 to the vacuum Ω_0. But because the vacuum Ω_0 is not distinguished with respect to local measurements, such a restricted postulate retains the old intuitive meaning.

With postulate 6 we can give a short proof of Lemma 4:

Corollary 1. Strict locality for regions B, C with $\mathfrak{R}_C$ a factor implies $\mathfrak{R}_C$ infinite. –

As $I \in \mathfrak{R}_B \subset \mathfrak{R}'_C$ is infinite, $\mathfrak{R}'_C$ is infinite. As Ω_0 is cyclic with respect to $\mathfrak{R}_B$, a fortiori for $\mathfrak{R}'_C$, and for $\mathfrak{R}_C$, it follows[16] $\mathfrak{R}_C$ is infinite too.

Corollary 2. (LICHT[2]): Suppose $\mathfrak{R}_{C'} = \mathfrak{R}'_C$ ("duality theorem", as a hypothesis proposed by HAAG and SCHROER[4], proved for one and disproved for another kind of regions for free scalar fields by ARAKI[5]). Then strict locality holds for the pair C', C of regions if and only if $\mathfrak{R}_C$ is a factor of type III. –

Necessity: $\mathfrak{R}_C$ is a factor, i.e. $\mathfrak{R}_C \cap \mathfrak{R}'_C = \{\lambda 1\}$, because the existence of a nontrivial $P \in \mathfrak{R}_C \cap \mathfrak{R}'_C = \mathfrak{R}_C \cap \mathfrak{R}_{C'}$ clearly would violate postulate 6. Furthermore, every $P \in \mathfrak{R}_{C'} = \mathfrak{R}'_C$ must be infinite with respect to $\mathfrak{R}'_C$, which means $\mathfrak{R}'_C$ of type III and thus[6] $\mathfrak{R}_C$ of type III too.

Sufficiency: $\mathfrak{R}_C$ factor of type III implies $\mathfrak{R}'_C = \mathfrak{R}_{C'}$ of type III, i.e. every $P \neq 0$ from $\mathfrak{R}_{C'}$ is infinite with respect to $\mathfrak{R}'_C$, which is strict locality.

We needed the "duality theorem" together with postulate 6 for the pair of regions C', C to derive type III. Note that postulate 6 for the pair of regions C', C is the strongest possible strict locality requirement: From it strict locality follows for every other pair $B \subset C'$, $C_1 \subset C$ as is evident from $\mathfrak{R}_B \subset \mathfrak{R}_{C'}$, $\mathfrak{R}'_C \subset \mathfrak{R}'_{C_1}$. It seems to us that requirements on the local rings $\mathfrak{R}_C$ make sense operationally only if restricted to finite C. (It is precisely this case of finite regions for which strict locality will be derived in section V.) Conversely, to derive strict locality for C', C (one of which is necessarily infinite) from strict locality for all finite pairs $B \subset C'$, $C_1 \subset C$ only we did not succeed. So we are not as convinced as LICHT[2] is that for regions C for which the duality $\mathfrak{R}_{C'} = \mathfrak{R}'_C$ is valid strict locality inevitably requires factors $\mathfrak{R}_C$ of type III.* We can even imagine finite regions C with factors $\mathfrak{R}_C$ of type I without violating lemma 1, the duality equation $\mathfrak{R}_{C'} = \mathfrak{R}'_C$, and postulate 6 at least for pairs B, C with $B \subset C'$ finite, if we remember a theorem of KADISON[8]:

Lemma. There exist pairs of factors $\mathfrak{R}'_C, \mathfrak{R}_B \subsetneq \mathfrak{R}'_C$ of type I_∞ with joint cyclic and separating vector Ω_0 such that every projector $P \neq 0$ from $\mathfrak{R}_B$ is infinite with respect to $\mathfrak{R}'_C$. –

On the other hand, strict locality for infinite regions, especially for the pair C', C, without the duality equation restricts the possible factor types as follows: If for a normal region C the factor $\mathfrak{R}_C$ were of type I, postulate 5 would imply[6] the duality theorem $\mathfrak{R}_{C'} = \mathfrak{R}'_C$, and together with corollary 2 this means:

Corollary 3. Strict locality for C', C with normal C excludes factor type I for $\mathfrak{R}_C$. –

If, however, $\mathfrak{R}_C$ is of type II_∞ and the duality $\mathfrak{R}_{C'} = \mathfrak{R}'_C$ does not hold, the factorization $\mathfrak{R}_C$, $\mathfrak{R}_{C'}$ is not a coupled one, and $\mathfrak{R}_{C'}$ can even be of type III (VON NEUMANN[6]), which implies strict locality for C', C. Thus we expect no analogue to corollary 3 for type II.

V. A proof of strict locality

In this last section we will show that for finite regions B, C postulate 6 is not an independent one, but can be derived from postulates 1–5. Thus any general quantum field theory has automatically the strict locality property, and it is unnecessary to discuss possible other formulations like postulate 6′ any further. We first demonstrate:

Theorem 2. Let be B, C finite normal spacelike separated regions. Then B, C are strictly local with respect to each other. –

The clue in proving this is the following result due to MISRA[9]:

Lemma. Let be $\mathfrak{R}_B$ a factor, x any spacelike translation which transforms its support B into $B_x \subset B'$, and let there exist a region $B_1 \supset B \cup B_x$ with $\mathfrak{R}_{B_1}$ a factor. Then every projection $P \in \mathfrak{R}_B$ is infinite with respect to $\mathfrak{R}_{B_1}$. –

If we take this lemma for granted, the remainder of the proof is very easy. As we immediately convince ourselves, the union $B \cup B_x$ of the normal region B and the spacelike separated translated region $B_x \subset B'$ is a normal region too, thus $\mathfrak{R}_B$ and $\mathfrak{R}_{B \cup B_x}$ are factors, and we can put

* *Note added in proof.* Recently, ARAKI has found the factors $\mathfrak{R}_C$ for the free scalar field to be type III for most regions of physical interest. The author thanks Professor H. ARAKI for a copy of his paper (Type of von Neumann algebras associated with free scalar field, preprint 1964).

$B_1 = B \cup B_x$ in the above lemma. Furthermore, because C is finite, we can always choose a translation x big enough to make B_x and C spacelike separated, $\mathfrak{R}_{B \cup B'} \subseteq \mathfrak{R}'_C$. But every $P \in \mathfrak{R}_B$ is by the above lemma infinite with respect to $\mathfrak{R}_{B \cup B_x}$ and thus also with respect to $\mathfrak{R}'_C$. – Theorem 2 immediately leads to the following slight generalization:

Corollary. Any spacelike separated pair of finite "unpathological" regions B, C fulfills the strict locality condition. –

Proof: For any not too pathological finite region C the double cone C'' generated by C, i.e. the set of points spacelike separated from the closure of C' (Fig. 3), is certainly a normal region. If B, C are spacelike separated the same holds true for B'', C''. Thus from theorem 2 we have strict locality for B'', C'', which implies strict locality for B, C.

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