AD 281 629

TECHNICAL MEMORANDUM FRL-TM-25

TRAJECTORY EQUATIONS FOR A SIX-DEGREE-OF-FREEDOM MISSILE

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MAY 1962

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PICATINNY ARSENAL
DOVER, N. J.

COPY 92

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TRAJECTORY EQUATIONS FOR A SIX-DEGREE-OF-FREEDOM MISSILE

by

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May 1962

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ABSTRACT

The purpose of this report is to explicitly derive two sets of six-degree-of freedom equations of motion for a symmetric missile.

One set is based upon a coordinate system that is rigidly attached to the missile (body-fixed system), while in the second set (fixed-plane system) a coordinate system with one axis constrained to lie in a given plane is employed to derive the equations of motion.

Both sets of equations assume the earth to be spherical, include the effect of the earth's rotation, and consider variable wind. In addition, for the body-fixed system, discussion of initial conditions and singularities of the differential equations is presented.

TABLE OF SYMBOLS

$\vec{i}_I, \vec{j}_I, \vec{k}_I$ $\vec{i}_E, \vec{j}_E, \vec{k}_E$ $\vec{i}_M, \vec{j}_M, \vec{k}_M$ $\vec{i}_F, \vec{j}_F, \vec{k}_F$	Coordinate systems
$\vec{A}, \vec{B}, \vec{C}$	Arbitrary vector quantities
$\vec{\Omega}_E$ $\vec{\Omega}$ $\vec{\omega}$ $\vec{\omega}_F$	Angular velocities
$\vec{V}$	Missile velocity, independent of wind velocity
$\vec{V}_W$	Wind velocity
$\vec{V}_r$	Missile velocity relative to the wind
$\Sigma\vec{F}$	Summation of forces acting on missile
$\Sigma\vec{L}$	Summation of torques acting on missile
$\vec{J}$	Total angular momentum of missile
I_X, I_Y, I_Z	Moments of inertia about the three axes of the missile
θ, ψ, ϕ	Euler angles
$\vec{R}$	Position of missile center of gravity relative to reference coordinate system
Δt	Increment of time
$\vec{Y}$	Unit vector indicating direction of gravitational force
$\vec{h}_o$	Vector indicating altitude of missile

$R_{\oplus}$	Mean radius of the earth
g	Gravitational factor
m	Mass of rocket
ρ	Density of air
d	Diameter of missile
λ'	Distance from nose of missile to center of pressure, expressed in calibers
λ_F	Distance from nose of missile to magnus center of pressure, expressed in calibers
ℓ	Distance from nose of missile to missile center of gravity, expressed in calibers
ℓ_t, ℓ_u, ℓ_l	Thrust malalignment distances
T	Thrust of missile
k_{DA}	Axial drag coefficient
k_N	Normal force coefficient
k_F	Magnus force coefficient
k_H	Yaw damping moment coefficient
k_ϕ, $k_{\phi\rho}$	Rolling moment coefficients
K	Proportionality factor of thrust applied to jet torque
γ	Angle between yaw plane and $\vec{j}_M$ axis
δ_T, δ_A	Angles giving position of thrust relative to missile coordinate system

A° Longitude of missile launch position

B° Latitude of missile launch position

H° Missile angle of declination relative to geographic system

G° Angle measured from positive $\vec{k}''_c$ axis relating lateral missile position in geographic system

$\frac{d}{dt}$ Denotes time derivative

Notation

A general vector $\vec{V}$ will be represented in a particular coordinate system by $(\vec{V})_i$ where i denotes the referencing coordinate system.

A component of a vector will be denoted by two subscripts, the first giving the component, the second denoting the coordinate system used. Here X, Y, and Z represent components along the $\vec{i}$, $\vec{j}$, $\vec{k}$ axes respectively. For example $(V_W)_{YE}$ is the component of the wind velocity along the $\vec{j}_E$ axis.

Finally $\frac{d_i V}{dt}$ denotes differentiation relative to the i^{th} coordinate system.

INTRODUCTION

This report is part of a continuing program to give Picatinny Arsenal a complete capability in the flight simulation of all types of projectiles and missiles, whether ballistic or rocket-boosted, guided or unguided. This capability is important to the development of both conventional and special weapons. It provides necessary information for aerodynamic design, range calculations, error analyses, fuzing systems, and complete weapon systems evaluations. The mathematical model defined in this report forms the basis for a corresponding digital computer program which has been developed for the IBM 709/1401 system.

Contained herein are two derivations of the six-degree-of-freedom equations of motion of a missile. The two derivations differ by the fact that, in one case, forces and moments acting on a missile are referred to a coordinate system that is rigidly attached to a missile (body-fixed system), while, for the second set, the forces and moments are referred to a coordinate system that has one axis constrained to lie in a given plane (fixed-plane system). It is believed that the first set is more appropriate for asymmetric missiles with the missile coordinate system chosen to coincide with the principal axes of the body. This system, however, seems to have a disadvantage for high spin rates (spin-stabilized rockets) in that it may be necessary to take very small time increments to obtain an accurate trajectory by numerical methods (this will be indicated in RESULTS AND DISCUSSION). For this case the second set of equations seems more appropriate.

It should be noted that it may be advantageous to choose other coordinate systems than considered here, either to suit a particular missile or because of the type of resultant data desired. Stability considerations and terminal effects are examples for which specialized results are required.

Both sets of equations in this report include the effect of the earth's rotation about its axis, gravitational expressions for both the flat earth (the earth is assumed to be a plane, valid for short trajectories) and the spherical earth (the earth is assumed to be a perfect sphere), variable air density, and the effect of wind. It is also well to state specifically that the following effects are not included in the derivations:

1. Guidance factors
2. The motion of the earth along its orbit
3. Launcher effects
4. Asymmetric missiles
5. Stability criteria

These represent areas of extension to the present equations. The reader is referred to References 1, 2, and 3, which treat in some detail some of these additional factors.

To introduce the reader to the equations of motion, a brief derivation is presented in Equations 1 to 11. The definitions of the various symbols are given on pages 4, 5, and 6. Equations 1 and 2 provide the foundation for the equations of motion. Arrows over symbols indicate vector quantities.

$$\Sigma\vec{F} = m\frac{d^2\vec{R}}{dt^2} \qquad (1)$$

$$\Sigma\vec{L} = \frac{d\vec{J}}{dt} \qquad (2)$$

Here the derivatives are taken with respect to a fixed coordinate system $(\vec{i}_I, \vec{j}_I, \vec{k}_I)$. Now

$$\frac{d\vec{R}}{dt} = \vec{V} = V_{XI}\vec{i}_I + V_{YI}\vec{j}_I + V_{ZI}\vec{k}_I = V_{XM}\vec{i}_M + V_{YM}\vec{j}_M + V_{ZM}\vec{k}_M \qquad (3)$$

$\vec{i}_M, \vec{j}_M, \vec{k}_M$ indicate a moving coordinate system.

Equation 1 can be written in terms of the moving coordinates as

$$\Sigma\vec{F} = m\frac{d_I\vec{V}}{dt} = m\left[\frac{d_M\vec{V}}{dt} + (\vec{\omega} \times \vec{V})\right] \qquad (4)$$

where $\frac{d_M(\;)}{dt}$ indicates a derivative relative to the moving coordinate system. In addition,

$$\vec{\omega} = \omega_{XI}\vec{i}_I + \omega_{YI}\vec{j}_I + \omega_{ZI}\vec{k}_I = \omega_{YM}\vec{i}_M + \omega_{YM}\vec{j}_M + \omega_{ZM}\vec{k}_M \qquad (5)$$

which upon expansion yields

$$\Sigma\vec{F} = m\left[\frac{dV_{XM}}{dt}\vec{i}_M + \frac{dV_{YM}}{dt}\vec{j}_M + \frac{dV_{ZM}}{dt}\vec{k}_M + (\omega_{YM}V_{ZM} - \omega_{ZM}V_{YM})\vec{i}_M + (\omega_{ZM}V_{XM} - \omega_{XM}V_{ZM})\vec{j}_M + (\omega_{XM}V_{YM} - \omega_{YM}V_{XM})\vec{k}_M\right] \qquad (6)$$

in terms of the moving coordinate system.

Similarly for Equation 2

$$\Sigma\vec{L} = \frac{d_I\vec{J}}{dt} = \frac{d_M\vec{J}}{dt} + (\vec{\omega} \times \vec{J}) \tag{7}$$

where now

$$\vec{J} = I_{XM}\omega_{XM}\vec{i}_M + I_{YM}\omega_{YM}\vec{j}_M + I_{ZM}\omega_{ZM}\vec{k}_M \tag{8}$$

which upon expansion yields

$$\Sigma\vec{L} = I_{XM}\frac{d\omega_{XM}}{dt}\vec{i}_M + I_{YM}\frac{d\omega_{YM}}{dt}\vec{j}_M + I_{ZM}\frac{d\omega_{ZM}}{dt}\vec{k}_M + \omega_{YM}\omega_{ZM}(I_{ZM} - I_{YM})\vec{i}_M$$

$$+ \omega_{XM}\omega_{ZM}(I_{XM} - I_{ZM})\vec{j}_M + \omega_{XM}\omega_{YM}(I_{YM} - I_{XM})\vec{k}_M \tag{9}$$

also in terms of the moving coordinate system.

The forces and moments are:

$$\Sigma\vec{F} = (F_{X\,\text{drag}} + F_{X\,\text{gravitational}} + F_{X\,\text{thrust}})\vec{i}_M$$

$$+ (F_{Y\,\text{normal}} + F_{Y\,\text{magnus}} + F_{Y\,\text{gravitational}} + F_{Y\,\text{thrust}})\vec{j}_M$$

$$+ (F_{Z\,\text{normal}} + F_{Z\,\text{magnus}} + F_{Z\,\text{gravitational}} + F_{Z\,\text{thrust}})\vec{k}_M \tag{10}$$

$$\Sigma\vec{L} = (L_{X\,\text{roll damping}} + L_{X\,\text{jet torque}} + L_{X\,\text{spin deceleration}}$$

$$+ L_{X\,\text{thrust malalignment}})\vec{i}_M$$

$$+ (L_{Y\,\text{restoring}} + L_{Y\,\text{yaw damping}} + L_{Y\,\text{magnus}}$$

$$+ L_{Y\,\text{thrust malalignment}})\vec{j}_M$$

$$+ (L_{Z\,\text{restoring}} + L_{Z\,\text{yaw damping}} + L_{Z\,\text{magnus}}$$

$$+ L_{Z\,\text{thrust malalignment}})\vec{k}_M \tag{11}$$

In succeeding sections of this report, these equations of motion, along with all the necessary supplementary equations for their solution, will be explicitly derived. In particular, Part A of the section entitled PROCEDURE covers principles of rotating coordinate systems and Euler angles. In Part B, these principles are used in deriving the equations of motion. A complete tabulation of the resulting equations is presented at the end of the PROCEDURE.

PROCEDURE

Several coordinate systems are used in deriving the equations of motion. The reasons can be summarized as follows:

1. Forces and moments acting on a missile are commonly resolved along the axes of a "missile coordinate system," that is, a coordinate system with a conveniently specified orientation relative to the missile itself. It should be noted that this coordinate system will usually travel in some manner along the missile trajectory.

2. Newton's laws of motion are strictly valid in an inertial coordinate system, that is, a coordinate system fixed in space. Because of this, any forces and moments that are resolved along the axes of a moving coordinate system must eventually be referred to fixed (or inertial) coordinates to correctly relate forces and accelerations.

3. It is often convenient to refer the motion of a missile to coordinates that are neither inertial nor dependent upon missile orientation. An example is the case when it is desired to refer the motion of a missile to coordinates fixed on the earth's surface.

For these reasons the basic equations governing moving coordinate systems will now be derived.

A. Mathematical Preliminaries

1. Rotating Coordinate Systems

Relations that exist between rotating coordinate systems will now be derived. Consider first two non-rotating right-handed coordinate systems $(\vec{i}_A, \vec{j}_A, \vec{k}_A)$ and $(\vec{i}_B, \vec{j}_B, \vec{k}_B)$ as in Figure 1 (p 10). A vector $\vec{A}$ can be represented in either coordinate system as follows:

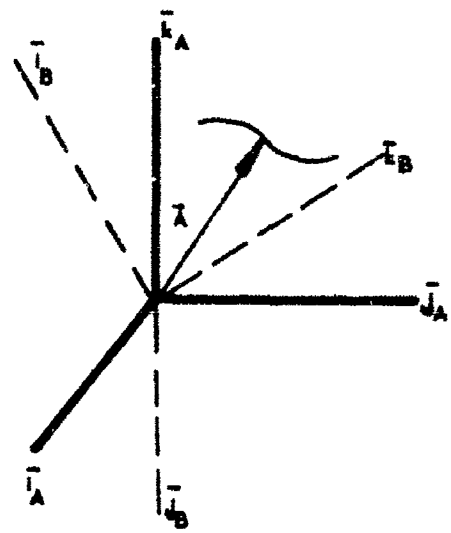

Figure 1

$$\vec{A}_A = A_{XA}\vec{i}_A + A_{YA}\vec{j}_A + A_{ZA}\vec{k}_A \qquad (12)$$

$$\vec{A}_B = A_{XB}\vec{i}_B + A_{YB}\vec{j}_B + A_{ZB}\vec{k}_B \qquad (13)$$

where, since a vector is independent of the coordinate system representing it,

$$\vec{A}_A = \vec{A}_B = \vec{A} \qquad (14)$$

If $\vec{A}$ is moving, then, relative to each coordinate system

$$\vec{V}_A = \frac{d_A\vec{A}_A}{dt} = \frac{dA_{XA}}{dt}\vec{i}_A + \frac{dA_{YA}}{dt}\vec{j}_A + \frac{dA_{ZA}}{dt}\vec{k}_A \qquad (15)$$

$$\vec{V}_B = \frac{d_B\vec{A}_B}{dt} = \frac{dA_{XB}}{dt}\vec{i}_B + \frac{dA_{YB}}{dt}\vec{j}_B + \frac{dA_{ZB}}{dt}\vec{k}_B \qquad (16)$$

Since the time rate of change of $\vec{A}$ ($= \vec{V}$) is another vector which again is independent of the representing coordinate system,

$$\vec{V}_A = \vec{V}_B = \vec{V} \tag{17}$$

Because of Equation 17 it is not necessary to actually specify $\frac{d_A\vec{A}}{dt}$, $\frac{d_B\vec{A}}{dt}$, or to which coordinate the motion of A is calculated, or mathematically

$$\frac{d_A\vec{A}}{dt} = \frac{d_B\vec{A}}{dt} = \frac{d\vec{A}}{dt} \tag{18}$$

This is not true if the $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$ coordinate system is rotating relative to $\vec{i}_A$, $\vec{j}_A$, $\vec{k}_A$. As a trivial example, it is possible that $\frac{d_B\vec{A}}{dt} = 0$ while $\frac{d_A\vec{A}}{dt} \neq 0$, for the case of a vector fixed in the ($\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$) system. To calculate the rate of change of $\vec{A}$ relative to the fixed coordinates, but in terms of the rotating coordinates $\left(\frac{d_A\vec{A}_B}{dt}\right)$, it is necessary to account also for the rotation of $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$, namely,

$$\frac{d_A\vec{A}_B}{dt} = \frac{dA_{XB}}{dt}\vec{i}_B + \frac{dA_{YB}}{dt}\vec{j}_B + \frac{dA_{ZB}}{dt}\vec{k}_B + A_{XB}\frac{d_A\vec{i}_B}{dt} + A_{YB}\frac{d_A\vec{j}_B}{dt} + A_{ZB}\frac{d_A\vec{k}_B}{dt} \tag{19}$$

To determine expressions for $\frac{d_A\vec{i}_B}{dt}$, $\frac{d_A\vec{j}_B}{dt}$, and $\frac{d_A\vec{k}_B}{dt}$, imagine a vector $\vec{B}$ of fixed length rotating with angular velocity $\vec{\omega}$ as shown in Figure 2 (p12).

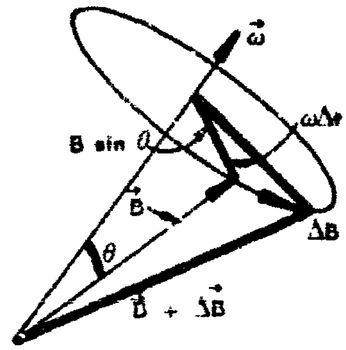

Figure 2

The time derivative of $\vec{B}$ with respect to any fixed system is as follows:

$$\frac{d\vec{B}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{B}(t + \Delta t) - \vec{B}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{B}}{\Delta t}$$

Referring to Figure 2, the magnitude is

$$\left| \frac{\Delta \vec{B}}{\Delta t} \right| = \omega B \sin \theta \tag{20}$$

in a direction perpendicular to the plane containing $\vec{B}$ and $\vec{\omega}$. This vector, in the limit, is precisely $\vec{\omega} \times \vec{B}$.

Consequently

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{B}}{\Delta t} = \frac{d\vec{B}}{dt} = \vec{\omega} \times \vec{B} \tag{21}$$

For our particular case $\vec{B}$ can represent any one of the unit vectors; hence,

$$\frac{d_A \vec{i}_B}{dt} = \vec{\omega}_B \times \vec{i}_B; \quad \frac{d_A \vec{j}_B}{dt} = \vec{\omega}_B \times \vec{j}_B; \quad \frac{d_A \vec{k}_B}{dt} = \vec{\omega}_B \times \vec{k}_B \tag{22}$$

where $\vec{\omega}_B$ is the angular velocity of $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$

Equation 19 becomes

$$\frac{d_A\vec{A}_B}{dt} = \frac{d_B\vec{A}_B}{dt} + A_{XB}(\vec{\omega}_B \times \vec{i}_B) + A_{YB}(\vec{\omega}_B \times \vec{j}_B) + A_{ZB}(\vec{\omega}_B \times \vec{k}_B) \qquad (23)$$

where $\frac{d_B\vec{A}_B}{dt}$ denotes the time rate of change of $\vec{A}$ relative to the rotating coordinate system $(\vec{i}_B, \vec{j}_B, \vec{k}_B)$. This derivative shows how $\vec{A}$ will change in time to an observer situated on the rotating coordinate system. Rewriting Equation 23 we obtain

$$\frac{d_A\vec{A}_B}{dt} = \frac{d_B\vec{A}_B}{dt} + (\vec{\omega}_B \times A_{XB}\vec{i}_B) + (\vec{\omega}_B \times A_{YB}\vec{j}_B) + (\vec{\omega}_B \times A_{ZB}\vec{k}_B)$$

or finally

$$\frac{d_A\vec{A}_B}{dt} = \frac{d_B\vec{A}_B}{dt} + \vec{\omega}_B \times \vec{A}_B \qquad (24)$$

This is the basic equation governing rotating coordinate systems.

To determine the acceleration of $\vec{A}$ relative to a fixed system, again in terms of the rotating coordinates, we have

$$\frac{d_A{}^2\vec{A}_B}{dt^2} = \frac{d_A}{dt}\left(\frac{d_A\vec{A}_B}{dt}\right) \qquad (25)$$

Following rules just established:

$$\frac{d_A{}^2\vec{A}_B}{dt^2} = \frac{d_B}{dt}\left(\frac{d_A\vec{A}_B}{dt}\right) + \left(\vec{\omega}_B \times \frac{d_A\vec{A}_B}{dt}\right) \qquad (25a)$$

$$= \frac{d_B}{dt}\left[\frac{d_B\vec{A}_B}{dt} + (\vec{\omega}_B \times \vec{A}_B)\right] + \vec{\omega}_B \times \left(\frac{d_B\vec{A}_B}{dt} + (\vec{\omega}_B \times \vec{A}_B)\right)$$

$$= \frac{d_B{}^2\vec{A}_B}{dt^2} + \frac{d_B\vec{\omega}_B}{dt} \times \vec{A}_B + \vec{\omega}_B \times \frac{d_B\vec{A}_B}{dt}$$

$$+ \vec{\omega}_B \times \frac{d_B\vec{A}_B}{dt} + \vec{\omega}_B \times (\vec{\omega}_B \times \vec{A}_B)$$

Finally we have the result

$$\frac{d_A{}^2\vec{A}_B}{dt^2} = \frac{d_B{}^2\vec{A}_B}{dt^2} + \frac{d_B\vec{\omega}_B}{dt} \times \vec{A}_B + 2\vec{\omega}_B \times \frac{d_B\vec{A}_B}{dt} + \vec{\omega}_B \times (\vec{\omega}_B \times \vec{A}_B) \quad (26)$$

We note in passing that

$$\frac{d_A\vec{\omega}_B}{dt} = \frac{d_B\vec{\omega}_B}{dt} + (\vec{\omega}_B \times \vec{\omega}_B) \qquad (27)$$

or since $\vec{\omega}_B \times \vec{\omega}_B = 0$

$$\frac{d_A\vec{\omega}_B}{dt} = \frac{d_B\vec{\omega}_B}{dt} \qquad (28)$$

or the time derivative of $\vec{\omega}$ can be taken relative to either coordinate system.

We can now remove the restriction that the two origins are coincident. Assume that the $(\vec{i}_B, \vec{j}_B, \vec{k}_B)$ system is translating, as well as rotating, relative to the $(\vec{i}_A, \vec{j}_A, \vec{k}_A)$ system. We have, referring to Figure 3,

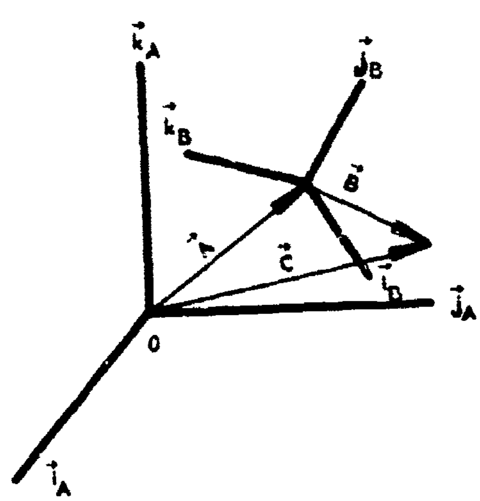

Figure 3

$$\vec{C} = \vec{A} + \vec{B} \tag{29}$$

$$\frac{d_A\vec{C}}{dt} = \frac{d_A\vec{A}}{dt} + \frac{d_A\vec{B}}{dt} \tag{30}$$

$$\frac{d_A\vec{C}}{dt} = \frac{d_A\vec{A}}{dt} + \frac{d_B\vec{B}}{dt} + \vec{\omega}_B \times \vec{B} \tag{31}$$

and finally

$$\frac{d_A{}^2\vec{C}}{dt^2} = \frac{d_A{}^2\vec{A}}{dt^2} + \frac{d_B{}^2\vec{B}}{dt^2} + 2\vec{\omega}_B \times \frac{d_B\vec{B}}{dt} + \vec{\omega}_B \times (\vec{\omega}_B \times \vec{B}) + \frac{d_B\vec{\omega}_B}{dt} \times \vec{B} \tag{32}$$

This discussion is concluded by a theorem that will be of considerable use in deriving the ballistic equations of motion when more than two coordinate systems are involved.

Given a primed coordinate system rotating with angular velocity $\vec{\omega}_A$ with respect to an unprimed coordinate system, and a starred coordinate system rotating with angular velocity $\vec{\omega}_B$ relative to the primed coordinate system, then the angular velocity of the starred coordinate system relative to the unprimed system is $\vec{\omega}_A + \vec{\omega}_B$. The proof is as follows: The velocity of any vector $\vec{C}$, fixed in the starred coordinate system, relative to the primed coordinate system (denoted by $d'\vec{C}/dt$) is

$$\frac{d'\vec{C}}{dt} = \vec{\omega}_B \times \vec{C} \tag{33}$$

and the velocity of $\vec{C}$ relative to the unprimed coordinate systems becomes

$$\frac{d\vec{C}}{dt} = \frac{d'\vec{C}}{dt} + \vec{\omega}_A \times \vec{C} = (\vec{\omega}_B \times \vec{C}) + (\vec{\omega}_A \times \vec{C}) = (\vec{\omega}_B + \vec{\omega}_A) \times \vec{C} \tag{34}$$

Before deriving the equations of motion a second mathematical preliminary must be disposed of, namely, the Euler angles and their derivatives. It is presumed that the reader has a knowledge of the elementary operations of matrices.

II. Euler Angles and Their Derivatives

The purpose of this development is to determine what relationships exist between the axes of two differently oriented coordinate systems. These relationships are important since we may know the components of a vector (e.g. velocity) in one coordinate system, and wish to know the components of the same vector in another coordinate system. These relations will be handled by what are known as Euler angles, although one may use other techniques such as direction cosines. To fix ideas, three arbitrary rotations of a (right-handed) coordinate system will be performed about selected axes. This is sufficient to orient a given coordinate system into any other desired (right-handed) position.

In particular, consider a fixed coordinate system $\vec{i}_A$, $\vec{j}_A$, $\vec{k}_A$, and a second coordinate system ($\vec{i}$, $\vec{j}$, $\vec{k}$) initially coincident with it, and whose final position will be denoted by $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$. Intermediate positions of $\vec{i}$, $\vec{j}$, $\vec{k}$, will be denoted by a sequence of primes, the number of which denotes the number of rotations already performed, $\vec{i}$, . . . , $\vec{k}$ of course, all being unit vectors.

Now first rotate $\vec{i}$, $\vec{j}$, $\vec{k}$ about $\vec{k}_A$ by an angle of magnitude ψ, as shown in Figure 4, where each axis of the triad $\vec{i}$, $\vec{j}$, $\vec{k}$ becomes $\vec{i}'$, $\vec{j}'$, $\vec{k}'$, respectively.

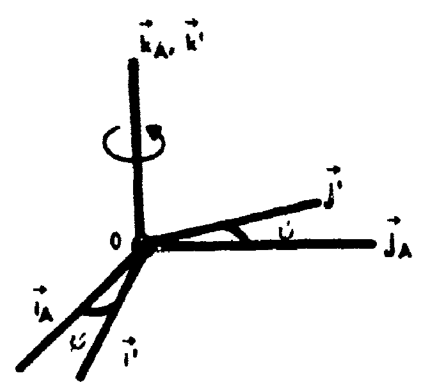

Figure 4

By trigonometry, the projections of the primed axis on the ''A'' subscripted or fixed axis can be determined. The result of this computation can be conveniently written in matrix form as follows:

$$\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} = \begin{bmatrix} C\psi & S\psi & 0 \\ -S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}_A \\ \vec{j}_A \\ \vec{k}_A \end{bmatrix} \qquad (35)$$

It is important to note that this matrix equation (and all similar to this) is a shorthand notation for writing three equations at once. Each equation expresses one of the unit vectors, along the coordinate axis of one coordinate system in terms of the other (rotated) coordinates.

For the sake of brevity we have written

SA for sin A and

CA for cos A

where A is any angle.

Similarly, the projections of the unprimed coordinates on the primed coordinates can be written as the inverse (also the transpose) of the above coefficient matrix, namely,

$$\begin{bmatrix} \vec{i}_A \\ \vec{j}_A \\ \vec{k}_A \end{bmatrix} = \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} \qquad (36)$$

If next a rotation of magnitude θ about $\vec{j}'$ is performed we obtain

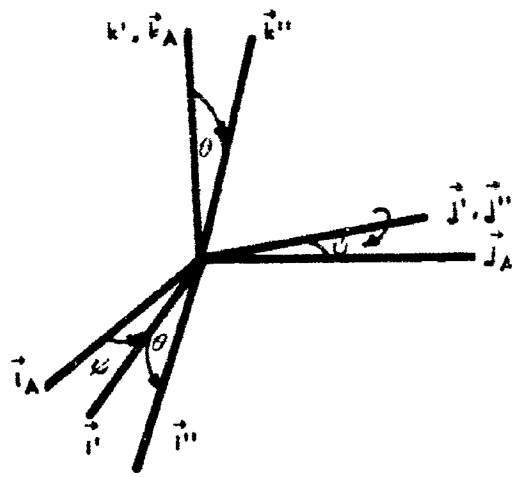

Figure 5

$$\begin{bmatrix} \vec{i}'' \\ \vec{j}'' \\ \vec{k}'' \end{bmatrix} = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} \tag{37}$$

and the inverse

$$\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} \vec{i}'' \\ \vec{j}'' \\ \vec{k}'' \end{bmatrix} \tag{38}$$

The double primes are used to denote the new position of the primed coordinates.

Finally we perform a rotation about $\vec{i}''$ of angle ϕ, obtaining

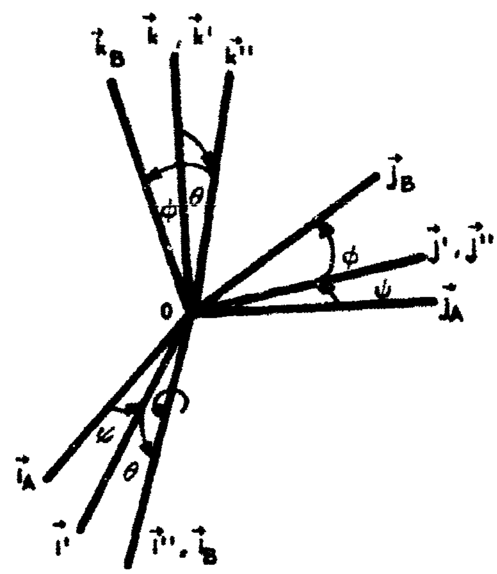

Figure 6

$$\begin{bmatrix} \vec{i}_B \\ \vec{j}_B \\ \vec{k}_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & S\phi \\ 0 & -S\phi & C\phi \end{bmatrix} \begin{bmatrix} \vec{i}'' \\ \vec{j}'' \\ \vec{k}'' \end{bmatrix} \tag{39}$$

and

$$\begin{bmatrix} \vec{i}'' \\ \vec{j}'' \\ \vec{k}'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \begin{bmatrix} \vec{i}_B \\ \vec{j}_B \\ \vec{k}_B \end{bmatrix} \tag{40}$$

Now the B-subscript denotes the new and final position of the double primed coordinates. Three sequential rotations of magnitude ψ, θ, and ϕ, respectively, have now been completed.

If one successively multiplies these matrices, the following is obtained:

$$\begin{bmatrix} \vec{i}_B \\ \vec{j}_B \\ \vec{k}_B \end{bmatrix} = \begin{bmatrix} C\theta C\psi & C\theta S\psi & -S\theta \\ S\theta S\phi C\psi - C\phi S\psi & S\theta S\phi S\psi + C\phi C\psi & S\phi C\theta \\ S\theta C\phi C\psi + S\phi S\psi & S\theta C\phi S\psi - S\phi C\psi & C\phi C\theta \end{bmatrix} \begin{bmatrix} \vec{i}_A \\ \vec{j}_A \\ \vec{k}_A \end{bmatrix} \qquad (41)$$

and the inverse

$$\begin{bmatrix} \vec{i}_A \\ \vec{j}_A \\ \vec{k}_A \end{bmatrix} = \begin{bmatrix} C\theta C\psi & S\theta S\phi C\psi - C\phi S\psi & S\theta C\phi C\psi + S\phi S\psi \\ C\theta S\psi & S\theta S\phi S\psi + C\phi C\psi & S\theta C\phi S\psi - S\phi C\psi \\ -S\theta & S\phi C\theta & C\phi C\theta \end{bmatrix} \begin{bmatrix} \vec{i}_B \\ \vec{j}_B \\ \vec{k}_B \end{bmatrix} \qquad (42)$$

To obtain the significance of these equations, consider a vector $\vec{R}$ written in terms of each coordinate system:

$$\vec{R} = (R)_{XA}\vec{i}_A + (R)_{YA}\vec{j}_A + (R)_{ZA}\vec{k}_A = (R)_{XB}\vec{i}_B + (R)_{YB}\vec{j}_B + (R)_{ZB}\vec{k}_B \qquad (42a)$$

Assume that the components $(R)_{XB}$, $(R)_{YB}$, and $(R)_{ZB}$ are known, and we wish to obtain values for $(R)_{XA}$, $(R)_{YA}$, and $(R)_{ZA}$. To obtain these values we write $\vec{i}_B$, $\vec{j}_B$, and $\vec{k}_B$ in terms of the $\vec{i}_A$, $\vec{j}_A$, $\vec{k}_A$ coordinates and equate like components. For example, from Equation 41 we have

$$\begin{aligned} \vec{i}_B &= C\theta C\psi\vec{i}_A + C\theta S\psi\vec{j}_A - S\theta\vec{k}_A \\ \vec{j}_B &= (S\theta S\phi C\psi - C\phi S\psi)\vec{i}_A + (S\theta S\phi S\psi + C\phi C\psi)\vec{j}_A + S\phi C\theta\vec{k}_A \\ \vec{k}_B &= (S\theta C\phi C\psi + S\phi S\psi)\vec{i}_A + (S\theta C\phi S\psi - S\phi C\psi)\vec{j}_A + C\phi C\theta\vec{k}_A \end{aligned} \qquad (42b)$$

Substituting these values in Equation 42a and equating the coefficient of $\vec{i}_A$ on the right hand side of the equation to $(R)_{XA}$, the coefficient of $\vec{i}_A$ on the left side of the equation, and the coefficients of $\vec{j}_A$ and $\vec{k}_A$ to $(R)_{YA}$ and $(R)_{ZA}$ respectively yields:

$$\begin{aligned} (R)_{XA} &= C\theta C\psi(R)_{XB} + (S\theta S\phi C\psi - C\phi S\psi)(R)_{YB} + (S\theta C\phi C\psi + S\phi S\psi)(R)_{ZB} \\ (R)_{YA} &= C\theta S\psi(R)_{XB} + (S\theta S\phi S\psi + C\phi C\psi)(R)_{YB} + (S\theta C\phi S\psi - S\phi C\psi)(R)_{ZB} \\ (R)_{ZA} &= -S\theta(R)_{XB} + S\phi C\theta(R)_{YB} + C\phi C\theta(R)_{ZB} \end{aligned} \qquad (43)$$

Thus, Equations 41 and 42 allow us to convert a vector in one coordinate system to another coordinate system.

It is also possible, and necessary, to express the angular velocity (discussed previously) of the rotating system relative to the fixed system in terms of the rotating triad with expressions containing the Euler angles and their derivatives. Using prior notations the angular velocity can be written as

$$\vec{\omega} = \frac{d\phi}{dt}\vec{i}'' + \frac{d\theta}{dt}\vec{j}' + \frac{d\psi}{dt}\vec{k}_A \qquad (44)$$

To obtain $\vec{\omega}$ in the rotating coordinate system, i.e. $\vec{\omega}_B$, determine

$\vec{i}''$ in terms of $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$

$\vec{j}'$ in terms of $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$

$\vec{k}_A$ in terms of $\vec{i}_B$, $\vec{j}_B$, $\vec{k}_B$

Using relations already established

$$\vec{i}'' = \vec{i}_B$$

$$\vec{j}' = C\phi\vec{j}_B - S\phi\vec{k}_B$$

$$\vec{k}_A = -S\theta\vec{i}_B + S\phi C\theta\vec{j}_B + C\phi C\theta\vec{k}_B \qquad (45)$$

If like components are now combined the following results are obtained:

$$(\omega)_{XB} = \frac{d\phi}{dt} - \frac{d\psi}{dt}S\theta \qquad \text{in the } \vec{i}_B \text{ direction}$$

$$(\omega)_{YB} = \frac{d\theta}{dt}C\phi + \frac{d\psi}{dt}S\phi C\theta \qquad \text{in the } \vec{j}_B \text{ direction}$$

$$(\omega)_{ZB} = -\frac{d\theta}{dt}S\phi + \frac{d\psi}{dt}C\phi C\theta \qquad \text{in the } \vec{k}_B \text{ direction} \qquad (46)$$

It is also possible to solve for $\frac{d\phi}{dt}$, $\frac{d\psi}{dt}$, and $\frac{d\theta}{dt}$ in terms of $(\omega)_{XB}$, $(\omega)_{YB}$, and $(\omega)_{ZB}$ obtaining

$$\frac{d\theta}{dt} = \omega_{YB} C\phi - \omega_{ZB} S\phi$$

$$\frac{d\phi}{dt} = \omega_{XB} + \omega_{YB} \tan\theta \, S\phi + \omega_{ZB} \tan\theta \, C\phi$$

$$\frac{d\psi}{dt} = \omega_{YB} S\phi \sec\theta + \omega_{ZB} C\phi \sec\theta \tag{47}$$

We now have sufficient mathematical tools to derive the equations of motion.

B. The Equations of Motion

1. Missile Fixed Coordinate System

In this first derivation of the equations of motion, all forces and moments are referred to a coordinate system rigidly attached to the missile. The following definitions and assignments are made for this case:

$\vec{i}_I, \vec{j}_I, \vec{k}_I$	A space-fixed (inertial) coordinate system whose origin is located at the center of the earth with the $\vec{k}_I$ axis coincident with the spin axis of the earth.
$\vec{i}_E, \vec{j}_E, \vec{k}_E$	A coordinate system that rotates with the earth, having the same origin as $\vec{i}_I, \vec{j}_I, \vec{k}_I$ with $\vec{k}_E$ coincident with $\vec{k}_I$. This system will be called the rotating earth coordinate system and is that to which the missile trajectory will be referred.
$\vec{i}_M, \vec{j}_M, \vec{k}_M$	A coordinate system rigidly attached to the missile, origin as yet unspecified, but $\vec{i}_M$ taken along the longitudinal axis of the missile.

$\vec{\Omega}_E$ The angular velocity of $(\vec{i}_E, \vec{j}_E, \vec{k}_E)$ relative to $(\vec{i}_I, \vec{j}_I, \vec{k}_I)$; that is, the earth's rotation.

$\vec{\omega}$ Angular velocity of $(\vec{i}_M, \vec{j}_M, \vec{k}_M)$ relative to $(\vec{i}_E, \vec{j}_E, \vec{k}_E)$.

$\vec{\omega}_F$ Angular velocity of $(\vec{i}_M, \vec{j}_M, \vec{k}_M)$ relative to $(\vec{i}_I, \vec{j}_I, \vec{k}_I)$.

We can note by a theorem previously proved that

$$\vec{\omega}_F = \vec{\omega} + \vec{\Omega}_E \tag{48}$$

Let now $\vec{i}_M$ be along the longitudinal axis of the missile, as previously stated. Let also $\vec{R}$ be a vector defined between the origins of the inertial coordinate system $(\vec{i}_I, \vec{j}_I, \vec{k}_I)$ and the missile coordinate system $(\vec{i}_M, \vec{j}_M, \vec{k}_M)$. Finally, let $\vec{S}_M$ be the vector from the origin of the missile coordinate system to the gravity center of the missile, and $\vec{R}_o$ be a vector from the origin of the inertial coordinate system to the terminus of $\vec{S}_M$. These vectors are shown in Figure 7.

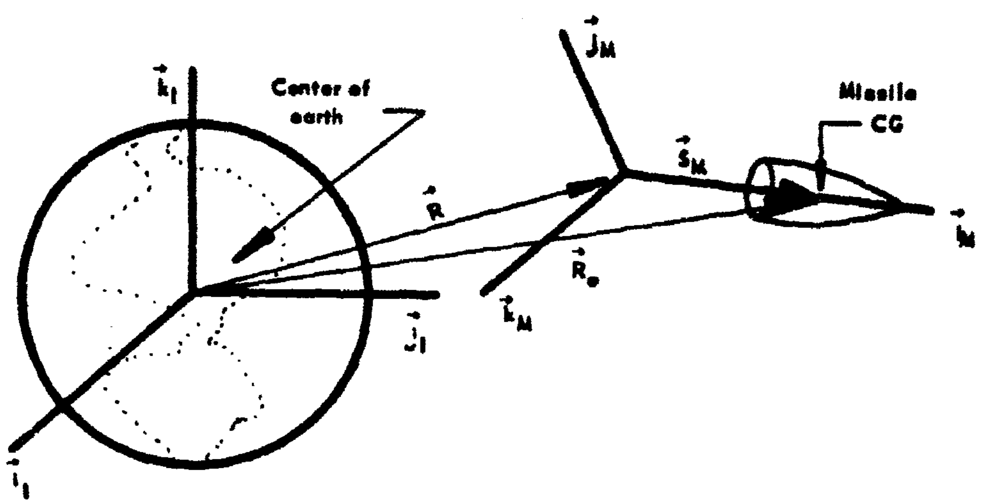

Figure 7

We have by Newton's Law for a rigid body

$$\Sigma\vec{F} = m \frac{d_I{}^2\vec{R}_o}{dt^2} \tag{49}$$

where $\Sigma\vec{F}$ is the summation of forces acting on the body and $\frac{d_I{}^2\vec{R}_o}{dt^2}$ is the acceleration of the center of gravity of the missile relative to the inertial coordinate system. In particular, the expression for $\frac{d_I{}^2R_o}{dt^2}$ is given by Equation 32 with $\vec{A}$ and $\vec{B}$ replaced by $\vec{R}$ and $\vec{S}_M$ respectively:

$$\frac{d_I{}^2\vec{R}_o}{dt^2} = \frac{d_I{}^2R}{dt^2} + \frac{d_M{}^2\vec{S}_M}{dt^2} + 2(\vec{\omega}_F)_M \times \frac{d_M\vec{S}_M}{dt} + (\vec{\omega}_F)_M \times ((\vec{\omega}_F)_M \times \vec{S}_M) + \frac{d_M(\vec{\omega}_F)_M}{dt} \times \vec{S}_M \tag{50}$$

If the origin of the missile coordinate system is located at the missile gravity center, then $\vec{S}_M$ becomes zero, and

$$\Sigma\vec{F} = m \frac{d_I{}^2\vec{R}_o}{dt^2} = m \frac{d_I{}^2\vec{R}}{dt^2} \tag{51}$$

For this assignment, the fictitious accelerations (Coriolis, etc) in Equation 50 become zero. It is for this simplification that the center of gravity of the missile is chosen as the origin of the "missile" coordinate system.

To obtain a particular form of the equations of motion we can write $\frac{d_I\vec{R}}{dt}$ in either the inertial or missile coordinate system as follows:

$$\frac{d_I\vec{R}}{dt} = (\vec{V})_I = (V)_{XI}\vec{i}_I + (V)_{YI}\vec{j}_I + (V)_{ZI}\vec{k}_I = (\vec{V})_M = (V)_{XM}\vec{i}_M + (V_{YM})\vec{j}_M + (V_{ZM})\vec{k}_M \tag{52}$$

Using Equations 24 and 34, we can express Equation 51 in the following forms

$$\Sigma\vec{F} = m\frac{d_I(\vec{V})_I}{dt} = m\frac{d_I(\vec{V})_M}{dt} \tag{53a}$$

$$\Sigma\vec{F} = m\frac{d_M(\vec{V})_M}{dt} + (\vec{\omega}_F)_M \times (\vec{V})_M \tag{53b}$$

$$\Sigma\vec{F} = m\frac{d_M(\vec{V})_M}{dt} + ((\vec{\omega})_M + (\vec{\Omega}_E)_M) \times (\vec{V})_M \tag{53c}$$

Note that Equation 53c is expressed in terms of the missile coordinate system. To use this equation, the forces also will have to be expressed in terms of the missile coordinates.

For the moment equation we have

$$\Sigma\vec{L} = \frac{d_I(\vec{J})_I}{dt} = \frac{d_I(\vec{J})_M}{dt} \tag{54}$$

Where $\Sigma\vec{L}$ is the summation of torques acting on the missile, and $(\vec{J})_I = (\vec{J})_M$ is the total angular momentum of the rocket relative to the inertial system, but in terms of the $(\vec{i}_I, \vec{j}_I, \vec{k}_I)$ or $(\vec{i}_M, \vec{j}_M, \vec{k}_M)$ coordinates, respectively.

Again, using Equation 24

$$\frac{d_I\vec{J}_M}{dt} = \frac{d_M\vec{J}_M}{dt} + (\vec{\omega}_F)_M \times (\vec{J})_M \tag{55}$$

which, when considering the earth's rotation again, becomes

$$\Sigma\vec{L} = \frac{d_I\vec{J}_M}{dt} = \frac{d_M\vec{J}_M}{dt} + ((\vec{\omega})_M + (\vec{\Omega}_E)_M) \times \vec{J}_M \tag{56}$$

If the principal axes of the missile are chosen to be coordinate axes of the missile coordinate system, we can write the angular momentum as

$$\vec{J}_M = I_{XM}(\omega_F)_{XM}\vec{i}_M + I_{YM}(\omega_F)_{YM}\vec{j}_M + I_{ZM}(\omega_F)_{ZM}\vec{k}_M \tag{57}$$

I_{XM}, I_{YM}, I_{ZM} being the moments of inertia about the $\vec{i}_M, \vec{j}_M, \vec{k}_M$ axes, respectively. Equations 53 and 56 are the general vector equations of motion of a missile.

Since the spin rate $\vec{\Omega}_E$ of the earth is known, we can appropriately select a convenient orientation of the $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ and $\vec{i}_M$, $\vec{j}_M$, $\vec{k}_M$ coordinate systems and, by use of the Euler angles, obtain Ω_E in terms of the $\vec{i}_M$, $\vec{j}_M$, $\vec{k}_M$ system, that is, $(\vec{\Omega}_E)_M$.

To accomplish this objective, set both $\vec{k}_I$ and $\vec{k}_E$ permanently along the spin axis of the earth, so that

$$\vec{\Omega}_E = \Omega_E \vec{k}_I = \Omega_E \vec{k}_E \tag{58}$$

and determine the components of $\vec{k}_I = \vec{k}_E$ in terms of $\vec{i}_M$, $\vec{j}_M$ $\vec{k}_M$. Using Equation 42 yields

$$\Omega_E \vec{k}_E = -\Omega_E S\theta \vec{i}_M + \Omega_E S\phi C\theta \vec{j}_M + \Omega_E C\phi C\theta \vec{k}_M \tag{59}$$

In component form, Equation 59 becomes

$$
\begin{aligned}
(\Omega_E)_{XM} &= -\Omega_E S\theta \\
(\Omega_E)_{YM} &= \Omega_E S\phi C\theta \\
(\Omega_E)_{ZM} &= \Omega_E C\phi C\theta
\end{aligned} \tag{60}
$$

In the vector Equations 53 and 56, (unlike $\vec{\Omega}_E$) $\vec{\omega}$ and $\vec{V}$ are not known quantities. They are unknowns which will ultimately give rise to the trajectory through solution of the differential equations of motion. From before

$$(\vec{V})_M = (V)_{XM}\vec{i}_M + (V)_{YM}\vec{j}_M + (V)_{ZM}\vec{k}_M \tag{52}$$

Similarly, writing $\vec{\omega}$ in component form yields:

$$\vec{\omega}_M = (\omega)_{XM}\vec{i}_M + (\omega)_{YM}\vec{j}_M + (\omega)_{ZM}\vec{k}_M \tag{61}$$

Now, we can immediately write the basic vector equations in component form which are valid for the spherical and rotating earth case. Two facts should be pointed out in these general vector equations: (1) Factors such as variable air density, variable wind, and the gravitational acceleration do not appear in the inertial terms (which have just been defined), but only in the force and moment part of the equations of motion (which will be derived in Section B of the PROCEDURE). (2) The general vector Equations 53 and 56 are written in terms of the missile coordinate system (as indicated by the presence of $\frac{d_M(\;)}{dt}$). As such, the unknown quantities $\vec{V}_M$, $\vec{\omega}_M$ must be related back to a coordinate system to which the motion is referred. The latter presents no additional derivations. $\vec{R}_B$ in Equation 43, which expresses the components of an arbitrary vector in the coordinate system, may be replaced by the missile velocity vector, $(\vec{V})_M$. Similarly, $(\vec{\omega})_M$ may be related to the rates of change of the Euler angles by Equation 46 or 47. When the angular velocity of the missile is known, the new Euler angles can be computed by an integration, while knowing the velocity components will yield, upon integration, the missile position ($(R)_{XE}$, $(R)_{YE}$, $(R)_{ZE}$). These equations are all tabulated at the end of the PROCEDURE.[1]

II. Fixed Plane Coordinate System

It is not necessary to specify a "missile coordinate system" that is rigidly attached to the missile, as has been done previously. In this section a "missile coordinate system" with one axis constrained to lie in a given plane is considered, although a coordinate system that is rigidly attached to the missile is temporarily used.

The following definitions and assignments are made:

$\vec{i}_I, \vec{j}_I, \vec{k}_I$ Same coordinate system as defined earlier (see p 22)

$\vec{i}_E, \vec{j}_E, \vec{k}_E$ Same coordinate system as defined earlier (see p 22)

[1] For the interested reader, Reference 5 contains the derivation of these equations of motion without the use of matrix notation.

$\vec{i}_F, \vec{j}_F, \vec{k}_F$ The missile coordinate system. The particular orientation of this system will be such that the $\vec{i}_F$ axis is coincident with the missile axis of symmetry, while the j_F axis is constrained to lie in a plane parallel to the plane determined by $\vec{i}_I, \vec{j}_I$. The origin of $\vec{i}_F, \vec{j}_F, \vec{k}_F$ is at the missile center of gravity.

$\vec{i}_M, \vec{j}_M, \vec{k}_M$ A coordinate system that is rigidly attached to the missile, with $\vec{i}_M$ along the missile axis of symmetry, and with the same origin as $\vec{i}_F, \vec{j}_F, \vec{k}_F$.

$\vec{\Omega}_E$ The angular velocity of $\vec{i}_E, \vec{j}_E, \vec{k}_E$ relative to $\vec{i}_I, \vec{j}_I, \vec{k}_I$.

$\vec{\Omega}$ Angular velocity of $\vec{i}_F, \vec{j}_F, \vec{k}_F$ relative to $\vec{i}_E, \vec{j}_E, \vec{k}_E$.

$\vec{\omega}$ Angular velocity of $\vec{i}_M, \vec{j}_M, \vec{k}_M$ relative to $\vec{i}_F, \vec{j}_F, \vec{k}_F$. Note the different meaning of this $\vec{\omega}$ compared to that defined on page 23.

$\vec{\omega}_F$ Angular velocity of $\vec{i}_M, \vec{j}_M, \vec{k}_M$ relative to $\vec{i}_I, \vec{j}_I, \vec{k}_I$.

Our basic vector equations in terms of $\vec{i}_F, \vec{j}_F, \vec{k}_F$ now become

$$\Sigma\vec{F} = m\frac{d_F\vec{V}_F}{dt} + ((\vec{\Omega})_F + (\vec{\Omega}_E)_F) \times \vec{V}_F \tag{62}$$

$$\Sigma\vec{L} = \frac{d_F\vec{J}_F}{dt} + ((\vec{\Omega})_F + (\vec{\Omega}_E)_F) \times \vec{J}_F \tag{63}$$

In these equations the quantity $((\vec{\Omega})_F + (\vec{\Omega}_E)_F)$ is the angular velocity of the missile coordinate system to which we refer our forces and moments, relative to the inertial system, i.e., $(\vec{i}_F, \vec{j}_F, \vec{k}_F)$ relative to $(\vec{i}_I, \vec{j}_I, \vec{k}_I)$.

To derive the equations of motion for the fixed plane system, it is first necessary to specify the orientation of $\vec{i}_F, \vec{j}_F, \vec{k}_F$ relative to $\vec{i}_E, \vec{j}_E, \vec{k}_E$ in terms of the Euler angles, so that the $\vec{j}_F$ axis lies in a plane parallel to the plane determined by $\vec{i}_I$, $\vec{j}_I$. This condition will also be fulfilled if the $\vec{j}_F$ axis is constrained to lie parallel to the plane determined by $\vec{i}_E$, $\vec{j}_E$, which (by definition) is coincident with the $\vec{i}_I$, $\vec{j}_I$ plane.

To accomplish this objective, consider the following two rotations: (1) Rotate the $\vec{i}_E$, $\vec{j}_E$, $\vec{k}_E$ coordinate system about $\vec{k}_E$ by an angle of magnitude ψ, obtaining the identical matrix expressed in Equation 35. (2) Rotate the resultant coordinate system $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ about the $\vec{j}'$ axis by an angle of magnitude θ, the final position being denoted as $\vec{i}_F$, $\vec{j}_F$, $\vec{k}_F$. If the resultant matrices are multiplied there is obtained

$$\begin{bmatrix} \vec{i}_E \\ \vec{j}_E \\ \vec{k}_E \end{bmatrix} = \begin{bmatrix} C\psi C\theta & -S\psi & C\psi S\theta \\ S\psi C\theta & C\psi & S\psi S\theta \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} \vec{i}_F \\ \vec{j}_F \\ \vec{k}_E \end{bmatrix} \tag{64}$$

It can be seen from the inverse that $\vec{j}_F$ has no component along $\vec{k}_E$ for all ψ and θ, which fulfills the constraint on the $\vec{j}_F$ axis.

The angular velocity $\vec{\Omega}$ of $\vec{i}_F$, $\vec{j}_F$, $\vec{k}_F$ relative to $\vec{i}_E$, $\vec{j}_E$, $\vec{k}_E$ can be written as

$$\vec{\Omega} = \frac{d\psi}{dt}\vec{k}_E + \frac{d\theta}{dt}\vec{j}' \tag{65}$$

By obtaining the components of $\vec{k}_E$ and $\vec{j}'$ on the $\vec{i}_F$, $\vec{j}_F$, $\vec{k}_F$ axes, we have

$$(\Omega)_F = -S\theta \frac{d\psi}{dt}\vec{i}_F + \frac{d\theta}{dt}\vec{j}_F + \frac{d\psi}{dt} C\theta \vec{k}_F \tag{66}$$

Solving for $\frac{d\psi}{dt}$ and $\frac{d\theta}{dt}$ yields

$$\frac{d\psi}{dt} = \frac{-(\Omega)_{XF}}{\sin\theta} = \frac{(\Omega)_{ZF}}{\cos\theta} \tag{67}$$

$$\frac{d\theta}{dt} = (\Omega)_{YF} \tag{68}$$

For Equations 62 and 63 to be of use, it is necessary to express $\vec{J}_F$, the angular momentum, in terms of the angular velocities and the moments of inertia of the missile, namely

$$\vec{J}_F = I_{XF}(\omega_F)_{XF}\vec{i}_F + I_{YF}(\omega_F)_{YF}\vec{j}_F + I_{ZF}(\omega_F)_{ZF}\vec{k}_F \tag{69}$$

where

$$\vec{\omega}_F = \vec{\omega} + \vec{\Omega} + \vec{\Omega}_E \tag{70}$$

Additional relationships between $\vec{\omega}$ and $\vec{\Omega}$ must be derived since six unknowns, instead of the usual three for the angular velocity, are introduced when $\vec{\omega}$ and $\vec{\Omega}$ are written in component form. These additional relationships will now be derived.

It will be recalled that both $\vec{i}_M$ and $\vec{i}_F$ are to be coincident at all times. Hence, the only rotation enjoyed by the $\vec{i}_M$, $\vec{j}_M$, $\vec{k}_M$ coordinates not shared by the $\vec{i}_F$, $\vec{j}_F$, $\vec{k}_F$ system is the motion about $\vec{i}_F$, or the spin of the missile about its axis of symmetry. This can be expressed as

$$(\omega)_{XF} = P \qquad (\omega)_{YF} = (\omega)_{ZF} = 0 \tag{71}$$

where P is the spin of the missile.

Since a new unknown, P, has been introduced, it becomes necessary to eliminate one of the unknown components of $\vec{\Omega}_F$. This is easily done since $(\Omega)_{XF}$ can be written in terms of $(\Omega)_{ZF}$ by use of Equation 67.

$$(\Omega)_{XF} = -(\Omega)_{ZF}\tan\theta \tag{72}$$

Finally, we can account for the earth's rotation, $\vec{\Omega}_E$, by defining $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ and $\vec{i}_E$, $\vec{j}_E$, $\vec{k}_E$ coordinate systems as in the "missile fixed" case with the result as before

$$\vec{\Omega}_E = (\Omega_E)\vec{k}_I = (\Omega_E)\vec{k}_E$$

where, by Equation 64, $\vec{k}_E$ can be expressed in terms of the $\vec{i}_F$, $\vec{j}_F$, $\vec{k}_F$ system. Therefore:

$$(\Omega_E)_{XF} = -(\Omega_E)S\theta \tag{73}$$

$$(\Omega_E)_{YF} = 0 \tag{74}$$

$$(\Omega_E)_{ZF} = (\Omega_E)C\theta \tag{75}$$

With the relations obtained from the above analysis, $(\vec{\omega}_F)_F$ can now be written as

$$(\vec{\omega}_F)_F = (P - \tan\theta(\Omega)_{ZF} - (\Omega_E)S\theta)\vec{i}_F + (\Omega)_{YF}\vec{j}_F + ((\Omega)_{ZF} + (\Omega_E)_{ZF}C\theta)\vec{k}_F \tag{76}$$

Once Equations 62 and 63 are integrated to obtain the unknowns $(V)_{XF}$, $(V)_{YF}$, $(V)_{ZF}$, P, $(\Omega)_{YF}$, $(\Omega)_{ZF}$, the inverse of the matrix in Equation 64 is used to obtain $(V)_{XE}$, $(V)_{YE}$, $(V)_{ZE}$. Equations 67 and 68 are used to determine $\frac{d\psi}{dt}$ and $\frac{d\theta}{dt}$, integration of which provide ψ and θ. These equations are tabulated in Part V of this section.

III. Forces and Moments

The forces and moments acting on a missile may be classified into four categories, as follows:

1. Gravitational
2. Aerodynamic
3. Jet
4. Guidance

Guidance terms are beyond the scope of this report and will not be considered. The remaining forces and moments will be considered in the order shown on page 31.

a. Gravitational Force

The gravitational force acts at the center of gravity of the missile and, hence, does not produce any moments. Further, when specifying this term and other altitude dependent terms, distinction must be made between the spherical earth and the flat earth cases in the equations of motion. In both cases, however, the magnitude is given by $g = g_0 R_W^2/h_0^2$ where R_W is the mean radius of the earth, and g_0 is the gravitational acceleration at sea level. To ascertain the direction of this force for the flat non-rotating earth we may take the inertial coordinate system $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ to be such that $\vec{k}_I$ is pointing vertically upwards.

The gravitational force $mg\vec{Y}$ ($\vec{Y}$ being a unit vector specifying the direction) then becomes $mg\vec{Y} = -mg\vec{k}_I$ (acts opposite to $\vec{k}_I$). With respect to the missile fixed system (by obtaining $\vec{k}_I$ in terms of $\vec{i}_M$, $\vec{j}_M$, $\vec{k}_M$), this force is

$$mg\vec{Y} = mg[S\theta\vec{i}_M - S\phi C\theta\vec{j}_M - C\phi C\theta\vec{k}_M] \tag{77}$$

and, by similar reasoning, for the fixed plane system

$$mg\vec{Y} = [S\theta\vec{i}_F - C\theta\vec{k}_F] \tag{78}$$

In both cases, the altitude (h_o) will equal $(R)_{ZE}$.

For the spherical earth case, it is necessary to define a vector from the earth's origin to the center of gravity of the missile. It is along this vector that the gravitational force acts.

Referring to Figure 8

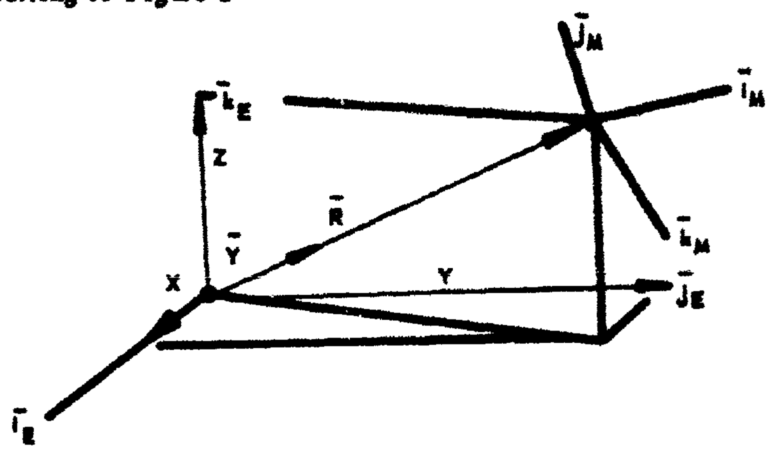

Figure 8

we may note

$$\vec{R} = (R)_{XE}\vec{i}_E + (R)_{YE}\vec{j}_E + (R)_{ZE}\vec{k}_E \tag{79}$$

$$\vec{R} = R\vec{Y} \tag{80}$$

Hence

$$\vec{Y} = \frac{\vec{R}}{R} = \frac{(R)_{XE}\vec{i}_E}{\sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}} + \frac{(R)_{YE}\vec{j}_E}{\sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}}$$

$$+ \frac{(R)_{ZE}\vec{k}_E}{\sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}} \tag{81}$$

For the missile fixed coordinate system,

$$\vec{i}_E = C\theta C\psi \vec{i}_M + (S\theta S\phi C\psi - C\phi S\psi)\vec{j}_M + (S\theta C\phi C\psi + S\phi S\psi)\vec{k}_M \tag{82}$$

with similar expressions for $\vec{j}_E$ and $\vec{k}_E$. Combining all the $\vec{i}_M$ components will yield in this direction:

$$(mg\vec{Y})_{XM} = \frac{-mg}{\sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}}\left[(R)_{XE}C\theta C\psi + (R)_{YE}C\theta S\psi - (R)_{ZE}S\theta\right] \quad (83)$$

The same procedure is used for the fixed plane coordinate system, where again a tabulation is presented in Part V of this section.

The altitude for the spherical earth case is simply

$$R = |\vec{R}| = h_o = \sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}} \quad (84)$$

b. Aerodynamic Forces and Moments

The aerodynamic forces and moments that will be included in the equations are presented in the following table.[1] For convenience, both the scalar magnitude and direction are included in the table.

Symbol	Force	Moment	Scalar Magnitude	Direction
(DF)	Axial Drag	–	$\rho d^2 V_r^2 k_{DA}$	Along missile axis
(NF)	Normal	–	$\rho d^2 V_r^2 k_N$	$\perp$ to missile axis in plane of yaw
(MF)	Magnus	–	$\rho d^3 \omega_{FX} V_r k_F$	$\perp$ to plane of yaw
(RM)	–	Restoring	$\rho d^3 V_r^2 k_N(\lambda - \epsilon)$	$\perp$ to plane of yaw
(MM)	–	Magnus	$\rho d^4 \omega_{FX} V_r k_F(\lambda_F - \epsilon)$	$\perp$ to missile axis in plane of yaw
(YDM)	–	Yaw Damping	$\rho d^4 V_r(\omega^2_{FY} + \omega^2_{FZ})^{\frac{1}{2}} k_M$	Independent of yaw plane, but $\perp$ to missile axis

[1] This table was essentially taken from Reference 6. For a more complete listing with explanations, see Reference 4.

Symbol	Force	Moment	Scalar Magnitude	Direction
(AR)	–	Roll Damping Moment	$\rho V_r k_{\phi p} \omega_{FX} d^4$	Along missile axis
	–	Roll Moment	$\rho V_r^2 k_{\phi} d^3$	Along missile axis

The velocity V_r expressed in the table is the velocity of the missile relative to the wind. For example, if the wind has a velocity component $(V_W)_{XM}$ in the same direction as the missile velocity component $(V)_{XM}$ as shown below, then the velocity of the missile relative to the wind $((V_r)_{XM})$ becomes

$$(V_r)_{XM} = (V)_{XM} - (V_W)_{XM} \tag{85}$$

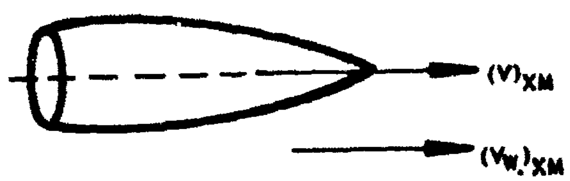

Figure 9

Similarly

$$(V_r)_{YM} = (V)_{YM} - (V_W)_{YM}$$

$$(V_r)_{ZM} = (V)_{ZM} - (V_W)_{ZM} \tag{86}$$

It should also be noted that the angular velocities ω_{FX}, ω_{FY}, and ω_{FZ} expressed in the table are the three components of $\vec{\omega}_F$ in vector Equation 48. For example, in the $\vec{i}_M$ direction, with the use of Equation 59, ω_{FX} can be written as

$$(\omega_F)_{XM} = -(\Omega_E)S\theta + (\omega)_{XM} \tag{87}$$

Finally, if the wind is given in terms of either the inertial or rotating earth coordinate system, the Euler transformations should be used to determine the wind components along the axes of the missile coordinate system.

Each of the forces and moments given in the table will now be briefly discussed. The reader is referred to other texts for a more complete discussion.

The axial drag acts along the missile longitudinal axis of symmetry, but in direction opposite to the velocity $(-(V_r)_{XM})$ of the missile along that axis. Since the line of action of this force passes through the centroid of the missile, there is no induced moment due to this force. The magnitude of this force is as given in the table.

The normal force, on the other hand, acts perpendicular to the missile axis and lies in the plane of yaw, that plane determined by the velocity vector of the rocket$(\vec{V}_r)$ and the rocket axis (See Fig 11, p 37). The components of the normal force along the $\vec{j}_M$ and $\vec{k}_M$ axes act opposite to those of the missile velocity components as shown in Figure 10, p 37. To obtain the magnitude of these components, as shown in the table (pp 34 and 35) note the normal force lies in the yaw plane but perpendicular to the missile axis; hence, if γ (in Fig 11) is defined as the angle between the yaw plane and the $\vec{j}_M$ axis, then it is easily seen that

$$|NF|_{YM} = |NF| \cos \gamma$$

$$|NF|_{ZM} = |NF| \sin \gamma \tag{88}$$

where

$$\text{Cos } \gamma = \frac{(V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}} \tag{89}$$

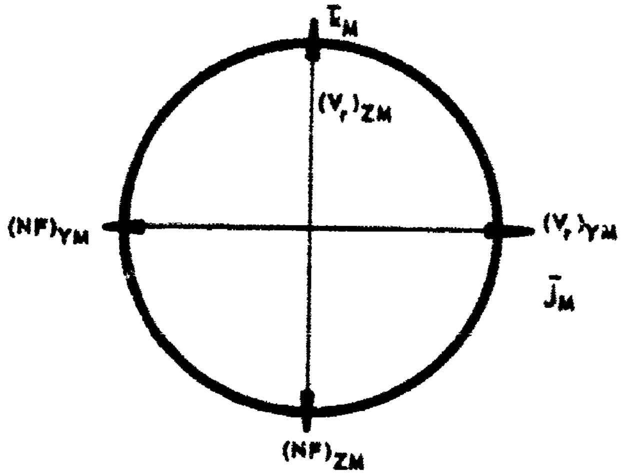

Figure 10

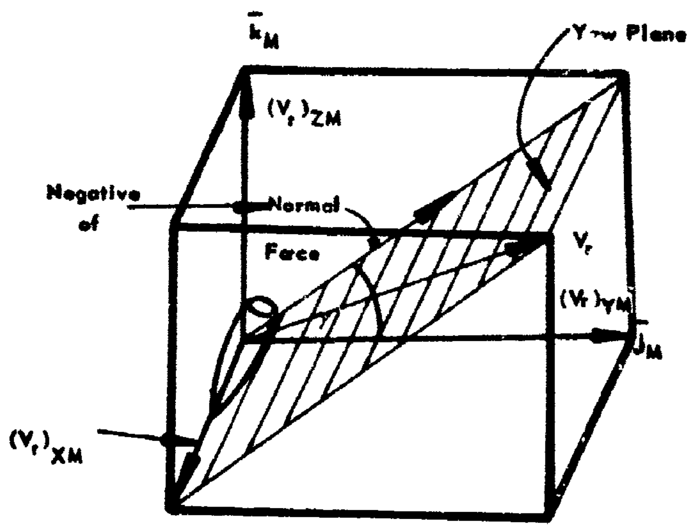

Figure 11

and

$$\operatorname{Sin} \gamma = \frac{(V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$$

This force does not act at the centroid (CG) of the missile, but at what is known as the center of pressure (CP), which is determined only by the exterior configuration of the missile. Consequently, this force gives rise to a moment, called the restoring moment. If the distance between the missile center of gravity and the center of pressure (CP) is denoted by λ' (CP is assumed behind the CG as shown in Figure 12), then the components of the restoring moment are:

$$|RM|_{YM} = |NF|_{ZM}\,\lambda'$$

$$|RM|_{ZM} = |NF|_{YM}\,\lambda' \qquad (90)$$

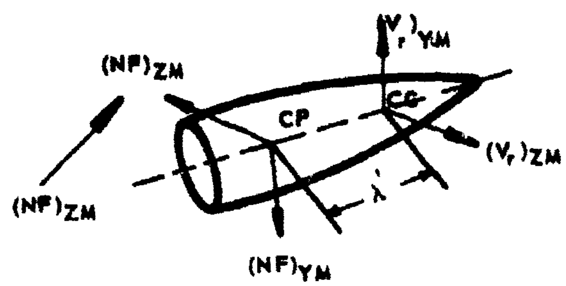

Figure 12

If the velocity components are along the positive $\vec{j}_M$ and $\vec{k}_M$ axes, then, by the right-hand rule, the components of the restoring moment will be negative along the $\vec{j}_M$ axis and positive along the $\vec{k}_M$ axis.

The normal force and axial drag as described above should not be confused with the lift and drag which are components perpendicular and parallel, respectively, to the velocity vector of the rocket.

The magnus force depends upon the angular velocity of the missile and acts perpendicular to the plane of the yaw. Consequently, to obtain the component of this force along the $\vec{j}_M$ axis, it is necessary to multiply the scalar magnitude by sin γ. Thus,

$$|MF|_{YM} = |MF| \sin\gamma \text{ along } \vec{j}_M$$

and

$$|MF|_{ZM} = |MF| \cos\gamma \text{ along } \vec{i}_M \qquad (91)$$

If $(V_r)_{YM}$ and $(V_r)_{ZM}$ are the velocity components of the missile relative to the wind (discussed previously), then the sign of the components of this force is as indicated in Figure 13, where the spin is considered to be

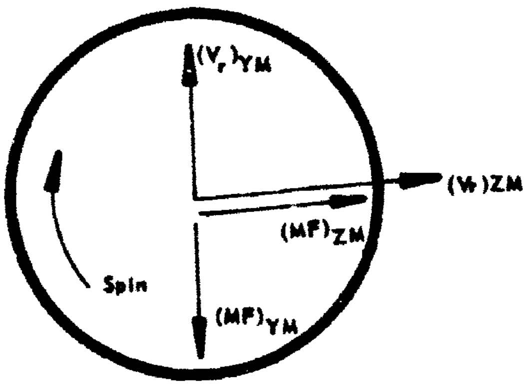

Figure 13

positive, i.e., acts in the direction of the positive $\vec{i}_M$ axis. If the direction of the spin is reversed, the magnus force will act in the opposite direction. Again, since the magnus force does not act at the centroid of the missile, but at the magnus center of pressure, this force gives rise to a moment called the magnus moment. If the magnus center of pressure is again behind the center of gravity, as indicated in Figure 14 (p 40), then, by the right-hand rule, both components of the magnus moment will be in the positive $\vec{j}_M, \vec{k}_M$ directions, the magnitude of which will be the magnitude of the scalar component multiplied by the moment arm λ_F as seen in the figure.

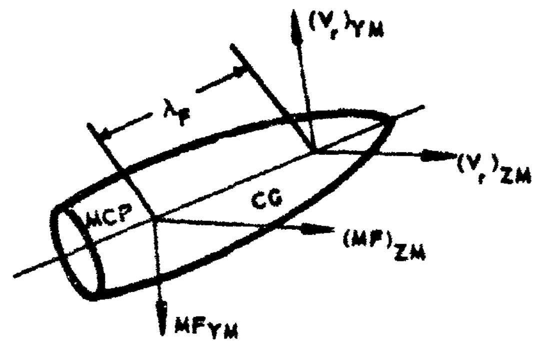

Figure 14

The yaw damping moment tends to reduce the magnitude of the yaw (assuming k_H positive) and acts opposite to the resultant of the lateral angular velocities $((\omega_F)_{YM}\ (\omega_F)_{ZM})$. This is shown in Figure 15. Note that there is no component of the yaw damping moment along the missile axis of symmetry. Noteworthy also is that a force associated with k_H exists; because it is difficult to measure this effect, however, this force was omitted in the analysis.

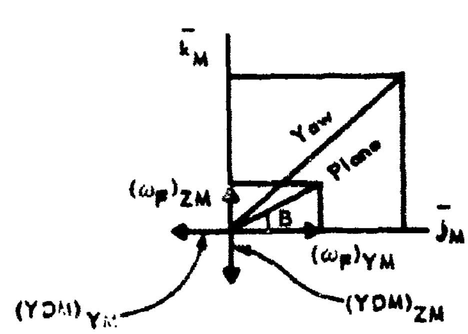

Figure 15

The final aerodynamic terms to be considered are the roll damping moment and the roll moment. Both moments act along the longitudinal axis of the missile. The former is induced by the fin cant, while the latter tends to reduce the axial spin by skin friction on the surface of the missile during flight. Both moments act in a direction opposite to that of the spin of the missile.

It should be noted that none of the aerodynamic coefficients are assumed to be linear in nature and hence are not to be taken as slopes from experimental curves. Rather, these coefficients are obtained directly from experiment as functions of Mach number and angle of attack. For example, one value of Mach number, the angle of attack, and the corresponding axial drag are sufficient to determine one value of k_{DA}.

c. Jet Forces and Moments

The jet forces and moments considered in this report are due to thrust, thrust malalignment torques, and jet torque to initially spin the missile about its axis for spin stabilization.

The jet thrust is the primary force that imparts forward motion to the missile. Although it should ideally act along the missile axis of symmetry, the thrust may have components along all three axes of the missile coordinate system. This is due to imperfections in the rocket design, the practical nature of propellants, and other factors. To obtain quantitative relationships, one may conveniently define two angles: (1) δ_T, the angle between the direction of the thrust and an axis parallel to the missile axis, and (2) δ_A, the angle between the projection of the thrust force on the $\vec{j}_M$, $\vec{k}_M$ plane and the $\vec{j}_M$ axis. These angles are illustrated in Figure 16.

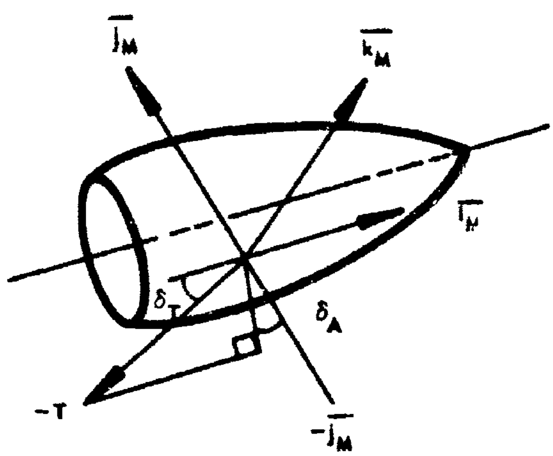

Figure 16

From geometrical considerations,

$$(T)_{XM} = T \cos \delta_T \qquad \text{along } \vec{i}_M \text{ axis}$$

$$(T)_{YM} = T \sin \delta_T \cos \delta_A \qquad \text{along } \vec{j}_M \text{ axis}$$

$$(T)_{ZM} = T \sin \delta_T \sin \delta_A \qquad \text{along } \vec{k}_M \text{ axis} \qquad (92)$$

If the direction of the thrust does not change relative to the missile, then for the missile fixed coordinate system, both δ_A and δ_T can be taken as constants during the missile flight.

To obtain the corresponding malalignment torques, reference is made to Figure 17.

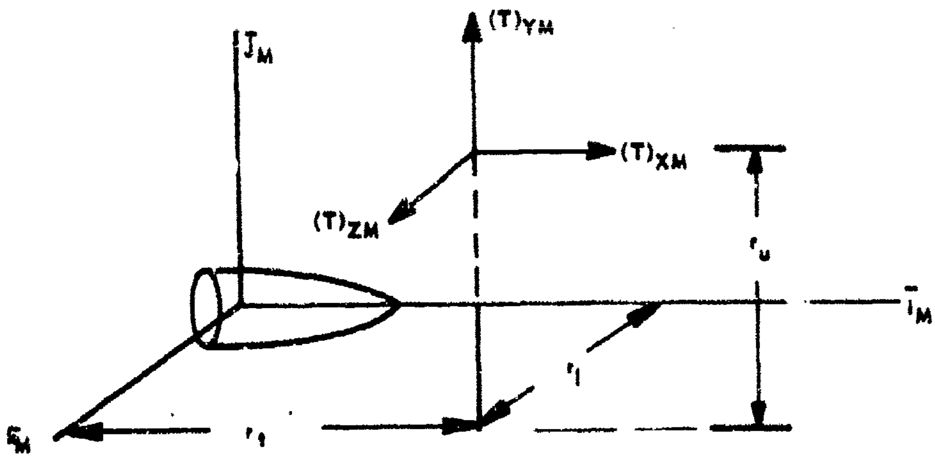

Figure 17

If $(T)_{XM}$, $(T)_{YM}$, $(T)_{ZM}$ act along the positive $\vec{i}_M$, $\vec{j}_M$, $\vec{k}_M$ axes respectively, then taking moments will yield

Thrust moment in $\vec{i}_M$ direction:

$$(TM)_{XM} = (T)_{ZM}r_u - (T)_{YM}r_l \tag{93}$$

Thrust moment in $\vec{j}_M$ direction:

$$(TM)_{YM} = (T)_{XM}r_l - (T)_{ZM}r_t \tag{94}$$

Thrust moment in $\vec{k}_M$ direction:

$$(TM)_{ZM} = (T)_{YM}r_t - (T)_{XM}r_u \tag{95}$$

where r_t, r_l, and r_u are defined in Figure 17.

Finally, the jet torque can be considered as proportional to the total thrust or KT. If the spin about the $\vec{i}_M$ axis is clockwise, then the jet torque is said to be positive by the right hand rule. Its purpose is to initially spin the missile for spin stabilization. It acts as indicated in Figure 18.

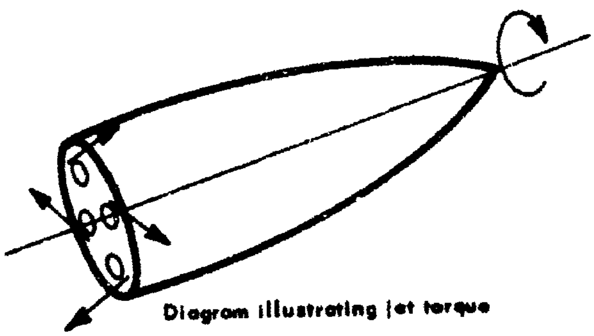

Figure 18

In summary, the forces and moments as discussed in this section are exhibited in the table shown on page 44.

IV. Initial Conditions, Conversions to Specific Cases and Singularities

a. Initial Conditions

Initial conditions for the equations of motion will consist of initial values for the following variables: $(R)_{XI}$, $(R)_{YI}$, $(R)_{ZI}$; $(V)_{XM}$, $(V)_{YM}$, $(V)_{ZM}$; θ, ψ, ϕ; $(\omega)_{XM}$, $(\omega)_{YM}$, $(\omega)_{ZM}$. Of these quantities $(R)_{XI}$, $(R)_{YI}$, $(R)_{ZI}$, and θ, ψ, ϕ are in reference to the inertial coordinate system, while $(V)_{XM}$, $(V)_{YM}$, $(V)_{ZM}$ and $(\omega)_{XM}$, $(\omega)_{YM}$, $(\omega)_{ZM}$ are referred to the missile coordinate system. In addition, it is important to know the orientation of these missile coordinates at launch, relative to the earth. Any initially directed forces (for example, thrust malalignment) will influence the particular latitude and longitude of the missile impact point at the end of the trajectory. These directed forces take on added importance in influencing the terminal point of the trajectory if an ellipsoid is used to approximate the earth's geometry.

	Components		
Name	**$\vec{i}_M$ Axis**	**$\vec{j}_M$ Axis**	**$\vec{k}_M$ Axis**
Axial Drag	$-\rho d^2 (V_r)^2_M k_{DA}$	–	–
Normal Force	–	$\dfrac{-\rho d^2 (V_r)^2_M k_N (V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$	$\dfrac{-\rho d^2 (V_r)^2_M k_N (V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$
Magnus Force	–	$\dfrac{-\rho d^3 (\omega_F)_{XM} (V_r)_M k_F (V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$	$\dfrac{\rho d^3 (\omega_F)_{XM} (V_r)_M k_F (V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$
Gravitational (Flat Earth)	$mg \sin \theta$	$-mg \cos \theta \sin \phi$	$-mg \cos \theta \cos \phi$
Thrust	$T \cos \delta_T$	$T \sin \delta_T \cos \delta_A$	$T \sin \delta_T \sin \delta_A$
Magnus Moment	–	$\dfrac{\rho d^3 (\omega_F)_{XM} (V_r)_M k_F (\lambda_F - r)(V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$	$\dfrac{\rho d^3 (\omega_F)_{XM} (V_r)_M k_F (\lambda_F - r)(V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$
Restoring Moment	–	$\dfrac{-\rho d^3 (V_r)^2_M k_N (\lambda - r)(V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$	$\dfrac{\rho d^3 (V_r)^2_M k_N (\lambda - r)(V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$
Roll Damping	$-\rho (V_r)_M k_{\phi\rho} (\omega_F)_{XM} d^4$	–	–
Yaw Damping	–	$-k_H (V_r)_M \rho d^4 (\omega_F)_{YM}$	$-k_H (V_r)_M \rho d^4 (\omega_F)_{ZM}$
Thrust Malalignment Torque	$T \sin \delta_T \sin \delta_A r_u$ $-T \sin \delta_T \cos \delta_A r_l$	$T \cos \delta_T r_l$ $-T \sin \delta_A \sin \delta_T r_t$	$T \sin \delta_T \cos \delta_A r_t$ $-T \cos \delta_A r_u$
Jet Torque	KT	–	–

To obtain the initial conditions, let us now precisely orient the inertial coordinate system. As previously stated, the origin of this coordinate system is situated at the earth's center with the $\vec{k}_I$ axis coincident with the axis of spin of the earth. Now let $\vec{i}_I$ (arbitrarily) coincide with the Greenwich Meridian (zero degrees longitude), while $\vec{j}_I$ is perpendicular to $\vec{i}_I$ and $\vec{k}_I$ in a direction given by the right-hand rule. Further, let us assume the existence of another coordinate system $\vec{i}_C$, $\vec{j}_C$, $\vec{k}_C$ initially coincident with $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$. Upon successive specified rotations, it is intended that $\vec{i}_C$, $\vec{j}_C$, $\vec{k}_C$ will become the missile coordinate system. Here the subscript C simply denotes that this coordinate system is used to determine initial conditions. As before, primes will be used to denote the intermediate rotations. Let the launch point of the missile be at A° longitude and B° latitude.

Our first objective is to orient the $\vec{i}_C$ axis so that it passes through the launch point of the missile. This is easily done by two rotations, first a rotation about $\vec{k}_I$ of magnitude A°, and second a rotation about $\vec{j}_C'$ of magnitude B°. Note that $\vec{i}_C$, $\vec{j}_C$, $\vec{k}_C$ is now labelled as $\vec{i}_C''$, $\vec{j}_C''$, $\vec{k}_C''$. Using the techniques in Part A of the Procedure, we can readily establish relationships between $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ and $\vec{i}_C''$, $\vec{j}_C''$, $\vec{k}_C''$. By matrix multiplication, we have

$$\begin{bmatrix} \vec{i}_C'' \\ \vec{j}_C'' \\ \vec{k}_C'' \end{bmatrix} = \begin{bmatrix} CACB & CBSA & -SB \\ -SA & CA & 0 \\ SBCA & SBSA & CB \end{bmatrix} \begin{bmatrix} \vec{i}_I \\ \vec{j}_I \\ \vec{k}_I \end{bmatrix} \qquad (96)$$

It will be convenient if, in principle, we now translate the $\vec{i}_C'$, $\vec{j}_C''$, $\vec{k}_C''$ coordinate system out to the surface of the earth so that the origin is at the launch point 0' of the missile. It will be noted that $\vec{i}_C''$ is still in a direction extending radially outward from the center of the earth and provides a perpendicular line to the earth's surface at the launch point, while $\vec{j}_C''$ and $\vec{k}_C''$ determine a tangent plane to the earth's surface at 0'.

Note also that the $\vec{k}_C''$ axis points towards north. In this "translated" position 0', $\vec{i}_C''$, $\vec{j}_C''$, $\vec{k}_C''$ will be called the geographic system. To completely specify the orientation of the missile, two additional angles are specified relative to this geographic system. In this analysis we will use (1) the angle of declination of H° from the perpendicular to the earth's surface, and (2) a lateral orientation angle G° measured from the negative $\vec{k}_C''$ axis in a counterclockwise direction in the tangent plane.
These angles are pictured in Figure 19.

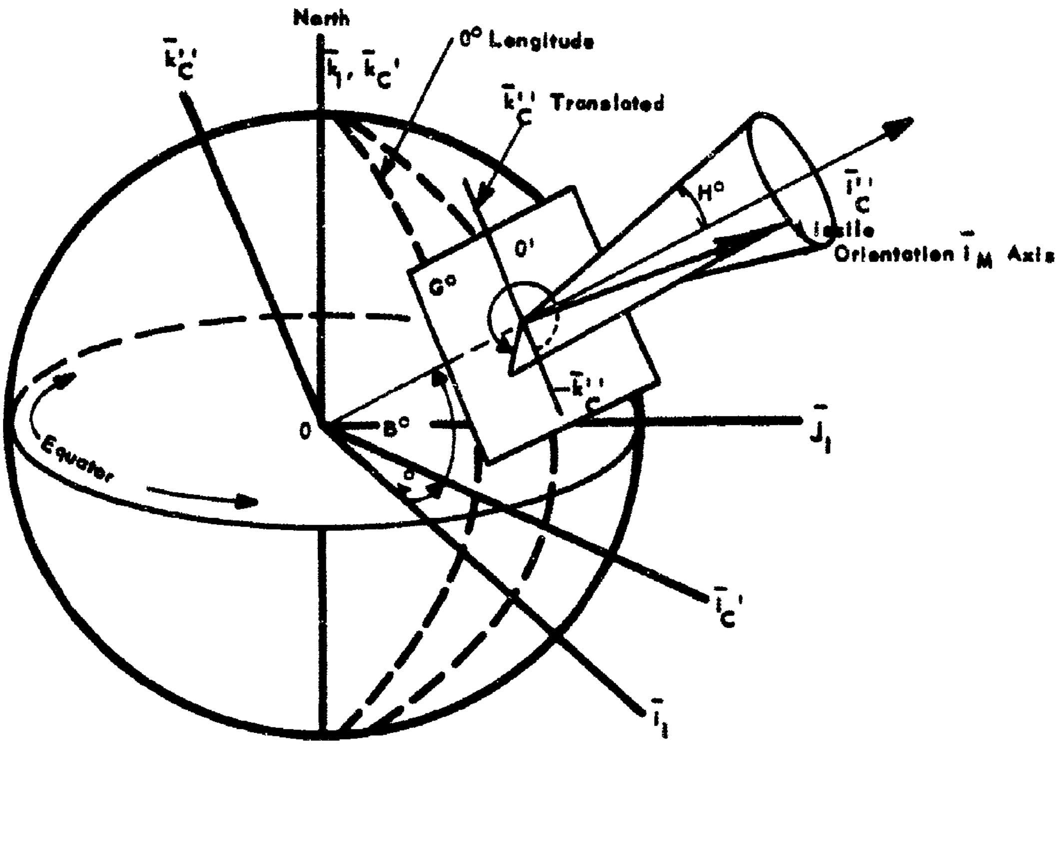

Figure 19

The object now is to perform two additional rotations using the input information of G° and H°, so that the $\vec{i}_C''$ axis finally coincides with the missile axis. We first rotate about $\vec{i}_C''$ by G° so that $-\vec{k}_C''$ coincides with the projection of the missile axis on the $\vec{j}_C''$, $\vec{k}_C''$ plane. The second rotation is about the $\vec{j}_C'''$ axis of magnitude H°, which finally causes the $\vec{i}_C'''$ axis to become coincident with the missile axis. It is important to note that the $\vec{j}_M$ axis is located in the tangent G° from the east, measured in a counterclockwise direction. Figure 20 shows the final rotation.

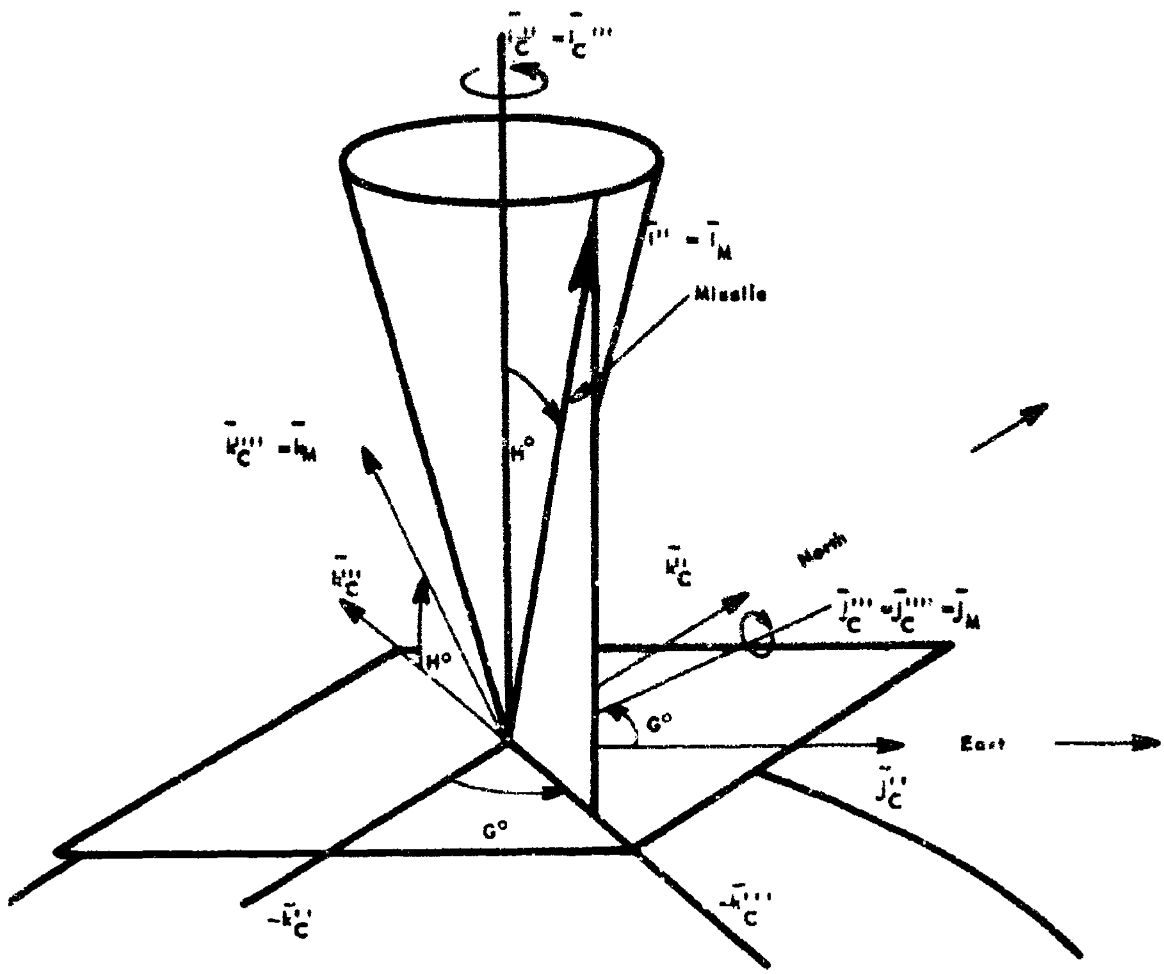

Figure 20

Mathematically, the resultant matrix for the G° and H rotations is

$$\begin{bmatrix} \vec{i}_M \\ \vec{j}_M \\ \vec{k}_M \end{bmatrix} = \begin{bmatrix} CH & SHSG & -SHCG \\ 0 & CG & SG \\ SH & -CHSG & CHSG \end{bmatrix} \begin{bmatrix} \vec{i}_C'' \\ \vec{j}_C'' \\ \vec{k}_C'' \end{bmatrix} \tag{97}$$

The matrix for all four rotations becomes, by matrix multiplication,

$$\begin{bmatrix} \vec{i}_M \\ \vec{j}_M \\ \vec{k}_M \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \vec{i}_I \\ \vec{j}_I \\ \vec{k}_I \end{bmatrix}$$

where

$$\begin{aligned} A_{11} &= CACBCH - SASGSH - CASBCGSH \\ A_{12} &= SACBCH + CASGSH - SASBCGSH \\ A_{13} &= -CHSB - CGCBSH \\ A_{21} &= -SACG + SGSBCA \\ A_{22} &= CGCA + SGSBSA \\ A_{23} &= SGCB \\ A_{31} &= SHCBCA + CHSGSA + CHCGSBCA \\ A_{32} &= SHCBSA - CHSGCA + CHCGSBSA \\ A_{33} &= -SHSB + CHCGCB \end{aligned} \tag{98}$$

We are now in a position to obtain the initial Euler angles, so that $\vec{i}_I$ becomes coincident with the missile axis. We first perform a rotation about $\vec{k}_I$ of magnitude ψ, and a rotation about $\vec{j}'$ of magnitude θ. We will now have $\vec{i}'''$ coincident in both cases. To obtain $\vec{j}'''$ we perform a rotation about $\vec{i}'''$ of magnitude ϕ. Mathematically, the three rotations expressed in matrix form are

$$\begin{bmatrix} \vec{i}_M \\ \vec{j}_M \\ \vec{k}_M \end{bmatrix} = \begin{bmatrix} C\theta C\psi & C\theta S\psi & -S\theta \\ S\theta S\phi C\psi - C\phi S\psi & S\theta S\phi S\psi + C\psi C\phi & S\phi C\theta \\ S\theta C\phi C\psi + S\phi S\psi & S\theta C\phi S\psi - S\phi C\psi & C\phi C\theta \end{bmatrix} \begin{bmatrix} \vec{i}_I \\ \vec{j}_I \\ \vec{k}_I \end{bmatrix} \quad (99)$$

This and the preceding matrix must be equivalent, since the final positions of the missile coordinate systems are to be identical. To obtain θ, ϕ, we simply compare corresponding elements of the two matrices. In particular,

$$S\theta = SBCH + CBCGSH$$

$$\frac{s\psi C\theta}{C\psi C\theta} = \tan\psi = \frac{(SACBCH + CASGSH - SASBCGSH)}{(CACBCH - SASGSH - CASBCGSH)}$$

$$\frac{C\theta S\phi}{C\theta C\phi} = \tan\phi = \frac{SGCB}{CHCGCB - SHSB} \quad (100)$$

It is to be understood that these angles are at $t = 0$, or at time of launch.

One must be careful, however, in ascertaining the correct quandrant of these angles.

For the initial position of the missile, only the latitude and longitude are required to compute the $(R)_{XI}$, $(R)_{YI}$, $(R)_{ZI}$ coordinates of the missile launch point. We can assume at $t = 0$ that the reference coordinate system $\vec{i}_E$, $\vec{j}_E$, $\vec{k}_E$ is coincident with the $\vec{i}_I$, $\vec{j}_I$, $\vec{k}_I$ coordinates, where again the $\vec{i}_I$ axis coincides with 0° longitude. From Figure 21 (p 50) it is easily seen that

$$(R)_{XI} = R_{\Psi} \cos B^{\circ} \cos A^{\circ}$$

$$(R)_{YI} = R_{\Psi} \cos B^{\circ} \sin A^{\circ} \quad (101)$$

$$(R)_{ZI} = R_{\Psi} \sin B^{\circ}$$

R_{Ψ} being the mean radius of the earth.

Similarly, knowing the coordinates of the terminal position of the missile enables the determination (by the same equation) of the latitude and longitude of the trajectory's end.

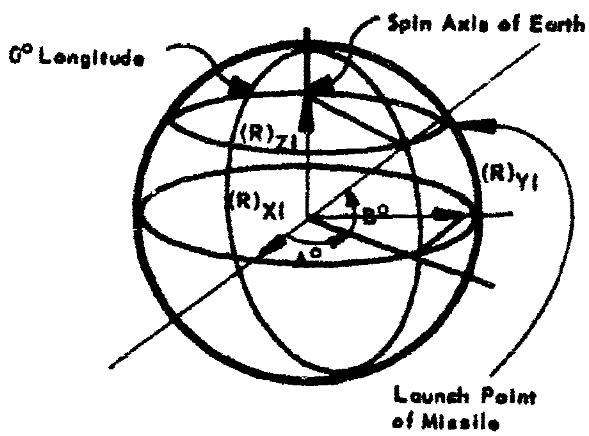

Figure 21

Finally, knowing the orientation of the $\vec{j}_M$ axis and any directional forces and moments relative to a specified compass direction enables one to compute the components of these forces and moments along the missile coordinate system. For example, if one wanted the effect on the terminal point of the trajectory due to thrust malalignment initially directed in an easterly direction, then δ_A (the lateral thrust malalignment angle) may now be given a definite value, namely the G° as we have defined it. Similarly, knowing the orientation of the missile coordinates will enable the computation of any initial lateral velocities and angular velocities. This completes the discussion on initial conditions.

b. Conversion to Specific Cases

For short trajectories, generalization to the "rotating earth" and "spherical earth" cases may not be warranted in solving the equations of motion. Converting the equations to the flat earth requires using the appropriate gravitational term as well as setting the altitude equal to Z (in place of $h_o = \sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}$). To convert to the non-rotating earth it is only necessary to set Ω_E equal to zero. Similarly, to neglect wind one sets V_W equal to zero.

c. Singularities

Care must be taken that certain quantities do not become increasingly large, without bound, in solving the equations of motion. Such a singularity is present in the missile fixed system. Equation 47 is repeated below for convenience

$$\frac{d\theta}{dt} = \omega_Y C\phi - \omega_Z S\phi$$

$$\frac{d\phi}{dt} = \omega_X + \omega_Y \tan\theta S\phi + \omega_Z \tan\theta C\phi \qquad (47)$$

$$\frac{d\psi}{dt} = \omega_Y S\phi \sec\theta + \omega_Z C\phi \sec\theta$$

When $\theta \to 90°$ both $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt} \to \infty$, which means that the above set of equations cannot be used in the limit. To overcome this difficulty we take

$$\tan\phi = -\frac{\omega_Z}{\omega_Y} \qquad (102)$$

which is obtained from Equation 46 with $\theta = 90°$. Differentiating both sides of this equation yields

$$\frac{d\phi}{dt} = \frac{\omega_Z\left(\frac{d\omega_Y}{dt}\right) - \omega_Y\left(\frac{d\omega_Z}{dt}\right)}{\omega_Y^2}\cos^2\phi \qquad (103)$$

If ω_Y also equals zero, one may use the alternate expression

$$\text{Cot}\ \phi = -\frac{\omega_Y}{\omega_Z} \qquad (104)$$

obtaining

$$\frac{d\phi}{dt} = \frac{\omega_Z\left(\frac{d\omega_Y}{dt}\right) - \omega_Y\left(\frac{d\omega_Z}{dt}\right)}{\omega_Z^2}\sin^2\phi \qquad (105)$$

Knowing $\frac{d\phi}{dt}$ we can solve for $\frac{d\psi}{dt}$ through

$$\frac{d\psi}{dt} = \frac{d\phi}{dt} - \omega_X \qquad (106)$$

which is also obtained from Equation 46 with $\theta = 90°$. If in addition to $\theta = 90°$, both ω_Y and ω_Z equal zero, recourse must be taken to higher order derivatives.

V. Tabulated Equations of Motion

Following is a summary of the six-degree-of-freedom equations of motion for body fixed system (rotating spherical earth).

Force Equations

$$T \cos \delta_T - \rho d^2 (V_r)_M^2 k_{DA} + \frac{mg}{\sqrt{(R)_{XE}^2 + (R)_{YE}^2 + (R)_{ZE}^2}}$$

$$[-(R)_{XE} \cos\theta \cos\psi - (R)_{YE} \cos\theta \sin\psi + (R)_{ZE} \sin\theta] =$$

$$m \left[\frac{d(V)_{XM}}{dt} + ((\omega)_{YM} + (\Omega_E) \sin\phi \cos\theta)(V)_{ZM} \right.$$

$$\left. - ((\omega)_{ZM} + (\Omega_E) \cos\phi \cos\theta)(V)_{YM} \right]$$

$$T \sin\delta_T \cos\delta_A - \frac{\rho d^2 (V_r)_M^2 k_N (V_r)_{YM}}{\sqrt{(V_r)_{YM}^2 + (V_r)_{ZM}^2}}$$

$$- \frac{\rho d^3 (\omega_F)_{XM} (V_r)_M k_F (V_r)_{ZM}}{\sqrt{(V_r)_{YM}^2 + (V_r)_{ZM}^2}} + \frac{mg}{\sqrt{(R)_{XE}^2 + (R)_{YE}^2 + (R)_{ZE}^2}}$$

$$[-(R)_{XE}(\sin\theta \sin\phi \cos\psi - \cos\phi \sin\psi)$$

$$- (R)_{YE}(\sin\theta \sin\phi \sin\psi + \cos\psi \cos\phi) - (R)_{ZE}(\sin\phi \cos\theta)] =$$

$$m \left[\frac{d(V)_{YM}}{dt} + ((\omega)_{ZM} + (\Omega_E) \cos\phi \cos\theta)(V)_{XM} \right.$$

$$\left. - ((\omega)_{XM} - (\Omega_E) \sin\theta)(V)_{ZM} \right]$$

$$T \sin \delta_T \sin \delta_A - \frac{\rho d^2 (V_r)^2_M k_N (V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$$

$$+ \frac{\rho d^3 (\omega_F)_{XM} (V_r)_M k_F (V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}} + \frac{mg}{\sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}}$$

$$[-(R)_{XE} (\sin \theta \cos \phi \cos \psi + \sin \phi \sin \psi)$$

$$-(R)_{YE} (\sin \theta \cos \phi \sin \psi - \sin \phi \cos \psi) - (R)_{ZE} (\cos \phi \cos \theta)] =$$

$$m \left[\frac{d(V)_{ZM}}{dt} + ((\omega)_{XM} - (\Omega_E) \sin \theta) (V)_{YM} \right.$$

$$\left. - ((\omega)_{YM} + (\Omega_E) \sin \phi \cos \theta) (V)_{XM} \right]$$

Moment Equations

$$T \sin \delta_T \sin \delta_A r_u - T \sin \delta_T \cos \delta_A r_l + kT$$

$$- \rho (V_r)_M k \phi \rho (\omega_F)_{XM} d^4 - \rho (V_r)^2_M k \phi d^3 =$$

$$I_{XM} \left[\frac{d(\omega)_{XM}}{dt} - (\Omega_E) \cos \theta \cos \phi (\omega)_{YM} + (\Omega_E) \cos \theta \sin \phi (\omega)_{ZM} \right]$$

$$+ [I_{ZM} - I_{YM}] ((\omega)_{YM} (\omega)_{ZM} + (\Omega_E) \sin \phi \cos \theta (\omega)_{ZM}$$

$$+ (\Omega_E) \cos \phi \cos \theta (\omega)_{YM} + (\Omega_E)^2 \cos^2 \theta \sin \phi \cos \phi)$$

$$T \cos \delta_T r_l - T \sin \delta_A \sin \delta_T r_t - \frac{\rho d^3 (V_r)^2_M k_N (\lambda - r) (V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$$

$$+ \frac{\rho d^4 (\omega_F)_{XM} (V_r)_M k_F (\lambda_F - r) (V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}} - k_H (V_r) \rho d^4 (\omega_F)_{YM} =$$

$$I_{YM}\left[\frac{d(\omega)_{YM}}{dt} + (\Omega_E)\cos\phi\cos\theta\,(\omega)_{XM} + (\Omega_E)\sin\theta\,(\omega)_{ZM}\right]$$

$$+ [I_{XM} - I_{ZM}]\,((\omega)_{ZM}(\omega)_{XM} - (\Omega_E)\sin\theta\,(\omega)_{ZM}$$

$$+ (\Omega_E)\cos\phi\cos\theta\,\omega_X - (\Omega_E)^2\cos\phi\cos\theta\sin\theta)$$

$$T\sin\delta_T\cos\delta_A r_t - T\cos\delta_T r_u + \frac{\rho d^3 (V_r)^2_M k_N(\lambda - \varepsilon)\,(V_r)_{YM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}}$$

$$+ \frac{\rho d^4(\omega_F^2)_{XM}(V_r)_M k_F(\lambda_F - \varepsilon)(V_r)_{ZM}}{\sqrt{(V_r)^2_{YM} + (V_r)^2_{ZM}}} - k_H(V_r)_M\rho d^4(\omega_F)_{ZM}$$

$$= I_{ZM}\left[\frac{d(\omega)_{ZM}}{dt} - (\Omega_E)\sin\phi\cos\theta\,(\omega)_{XM} - (\Omega_E)\sin\theta\,(\omega)_{YM}\right]$$

$$+ [I_{YM} - I_{XM}]\,((\omega)_{XM}(\omega)_{YM} + (\Omega_E)\sin\phi\cos\theta\,(\omega)_{XM}$$

$$- (\Omega_E)\sin\theta\,(\omega)_{YM} - (\Omega_E)^2\sin\theta\sin\phi\cos\theta)$$

Additional Equations

$$V_{XE} = \cos\theta\cos\psi\,(V)_{XM} + (\sin\theta\sin\phi\cos\psi - \cos\phi\sin\psi)\,(V)_{YM}$$
$$+ (\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi)\,(V)_{ZM}$$

$$V_{YE} = \cos\theta\sin\psi\,(V)_{XM} + (\sin\theta\sin\phi\sin\psi + \cos\psi\cos\phi)\,(V)_{YM}$$
$$+ (\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)\,(V)_{ZM}$$

$$V_{ZE} = -\sin\theta\,(V)_{XM} + \sin\phi\cos\theta\,(V)_{YM} + \cos\phi\cos\theta\,(V)_{ZM}$$

$$(V_W)_{XM} = (V_W)_{XE}\cos\theta\cos\psi + (V_W)_{YE}\cos\theta\sin\psi - (V_W)_{ZE}\sin\theta$$

$$(V_W)_{YM} = (V_W)_{XE}(\sin\theta\sin\phi\cos\psi - \cos\phi\sin\psi) + (V_W)_{YE}$$
$$(\sin\theta\sin\phi\sin\psi + \cos\psi\cos\phi) + (V_W)_{ZE}\sin\phi\cos\theta$$

$$(V_W)_{ZM} = (V_W)_{XE}(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) + (V_W)_{YE}$$
$$(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) + (V_W)_{ZE}\cos\phi\cos\theta$$

$$\frac{d\theta}{dt} = \omega_Y \cos\phi - \omega_Z \sin\phi$$

$$\frac{d\phi}{dt} = \omega_X + \omega_Y \tan\theta \sin\phi + \omega_Z \tan\theta \cos\phi$$

$$\frac{d\psi}{dt} = \omega_Y \sin\phi \sec\theta + \omega_Z \cos\phi \sec\theta$$

$$(V_r)_{XM} = (V)_{XM} - (V_W)_{XM} \qquad (\omega_F)_{XM} = (\omega)_{XM} - (\Omega_E) \sin\theta$$

$$(V_r)_{YM} = (V)_{YM} - (V_W)_{YM} \qquad (\omega_F)_{YM} = (\omega)_{YM} + (\Omega_E) \sin\phi \cos\theta$$

$$(V_r)_{ZM} = (V)_{ZM} - (V_W)_{ZM} \qquad (\omega_F)_{ZM} = (\omega)_{ZM} + (\Omega_E) \cos\phi \cos\theta$$

Equations for Initial Conditions

$$\sin\theta(0) = \cos H \sin B + \sin H \cos G \cos B$$

$$\tan\phi(0) = \frac{\sin G \cos B}{\cos H \cos G \cos B - \sin H \sin B}$$

$$\tan\psi(0) = \cos H \cos B \sin A + \sin H \sin G \cos A - \sin H \cos G \sin B \sin A$$

$$\cos\theta(0) = \frac{\cos H \cos B \sin A + \sin H \sin G \cos A - \sin H \cos G \sin B \sin A}{\sin\psi(0)}$$

$$(R)_{XI}(0) = R_W \cos B \cos A$$

$$(R)_{YI}(0) = R_W \cos B \sin A \qquad \text{if given longitude and latitude}$$

$$(R)_{ZI}(0) = R_W \sin B$$

Step Equations

$$(R)_{XE} = \int_0^t (V)_{XE}\,dt \qquad \theta = \int_0^t \frac{d\theta}{dt}\,dt$$

$$(R)_{YE} = \int_0^t (V)_{YE}\,dt \qquad \phi = \int_0^t \frac{d\phi}{dt}\,dt$$

$$(R)_{ZE} = \int_0^t (V)_{ZE}\,dt \qquad \psi = \int_0^t \frac{d\psi}{dt}\,dt$$

Equations for Singular Conditions

For $\theta = 90°$ $(\omega)_{YM} \neq 0$

$$\frac{d\phi}{dt} = \left[\frac{(\omega)_{ZM} \left(\frac{d(\omega)_{YM}}{dt} \right) - (\omega)_{YM} \left(\frac{d(\omega)_{ZM}}{dt} \right)}{(\omega)^2_{YM}} \right] \cos^2 \phi$$

For $\theta = 90°$ $(\omega)_{YM} = 0$ $(\omega)_{ZM} \neq 0$

$$\frac{d\phi}{dt} = \left[\frac{(\omega)_{ZM} \left(\frac{d(\omega)_{YM}}{dt} \right) - (\omega)_{YM} \left(\frac{d(\omega)_{ZM}}{dt} \right)}{(\omega)^2_{ZM}} \right] \sin^2 \phi$$

In both cases

$$\frac{d\psi}{dt} = \frac{d\phi}{dt} - (\omega)_{XM}$$

Following is an alternate set of more important equations for the fixed plane coordinate system:

$$\Sigma F_{XF} = m \left[\frac{dV_{XF}}{dt} + \Omega_{YF} V_{ZF} - (\Omega_{ZF} + (\Omega_E) C\theta) V_{YF} \right]$$

$$\Sigma F_{YF} = m \left[\frac{dV_{YF}}{dt} + (\Omega_{ZF} + \Omega_E C\theta) V_{XF} - (-\Omega_{ZF} \tan \theta - \Omega_E S\theta) V_{ZF} \right]$$

$$\Sigma F_{ZF} = m\left[\frac{dV_{ZF}}{dt} + (-\Omega_{ZF}\tan\theta - \Omega_E S\theta)V_{YF} - (\Omega_{YF})V_{XF}\right]$$

$$\Sigma L_{XF} = I_{XF}\left[\frac{dP}{dt} - \Omega_E C\theta S\theta\,\Omega_{YF} - \frac{d\Omega_{ZF}}{dt}\tan\theta - \Omega_{ZF}\Omega_{YF}\sec^2\theta\right]$$

$$+ \Omega_{YF}(\Omega_{ZF} + \Omega_E C\theta)I_{ZF} - (\Omega_{ZF} + \Omega_E C\theta)\Omega_{YF}I_{YF}$$

$$\Sigma L_{YF} = \frac{d\Omega_{YF}}{dt}I_{YF} + (\Omega_{ZF} + \Omega_E C\theta)(P - \Omega_{ZF}\tan\theta - \Omega_E S\theta)I_{XF}$$

$$-(-\Omega_{ZF}\tan\theta - \Omega_E S\theta)(\Omega_{ZF} + \Omega_E C\theta)I_{ZF}$$

$$\Sigma L_{ZF} = \left[\frac{d\Omega_{ZF}}{dt} - \Omega_E S\theta\,\Omega_{YF}\right]I_{ZF} + (-\Omega_{ZF}\tan\theta - \Omega_E S\theta)\Omega_{YF}I_{YF}$$

$$-(\Omega_{YF})(P - \Omega_{ZF}\tan\theta - \Omega_E S\theta)I_{XF}$$

$$V_{XE} = C\psi C\theta V_{XF} - S\psi V_{YF} + C\psi S\theta V_{ZF}$$

$$V_{YE} = S\psi C\theta V_{XF} + C\psi V_{YF} + S\psi S\theta V_{ZF}$$

$$V_{ZE} = -S\theta V_{XF} + C\theta V_{ZF}$$

$$(V_W)_{XF} = (V_W)_{XE}C\psi C\theta + (V_W)_{YE}S\psi C\theta - (V_W)_{ZE}S\theta$$

$$(V_W)_{YF} = -(V_W)_{XE}S\psi + (V_W)_{YE}C\psi$$

$$(V_W)_{ZF} = (V_W)_{YE}C\psi S\theta + (V_W)_{YE}S\psi S\theta + (V_W)_{ZE}C\theta$$

$$\frac{d\theta}{dt} = \Omega_{YF}$$

$$\frac{d\psi}{dt} = \Omega_{ZF}\sec\theta$$

Flat Earth

$$m\vec{gy} = mg\,[S\theta\vec{i}_F - C\theta\vec{k}_F]$$

Spherical Earth

$$m\vec{gy} = \frac{mg}{\sqrt{(R)^2_{XE} + (R)^2_{YE} + (R)^2_{ZE}}} \Big[(-(R)_{XE}C\psi C\theta - (R)_{YE}S\psi C\theta$$

$$+ (R)_{ZE}S\theta)\vec{i}_F + ((R)_{XE}S\psi - (R)_{YE}C\psi)\vec{j}_F$$

$$+ ((R)_{XE}C\psi S\theta - (R)_{YE}S\psi S\theta + (R)_{ZE})\vec{k}_F \Big]$$

RESULTS AND DISCUSSION

The complete set of the equations of motion are tabulated in the preceding section. Before these equations are solved, however, the user must specify other conditions of the problem. For example, the time to the cutoff point (i.e., where the thrust period terminates) must be given. Variations in mass, center of gravity, center of pressure, and magnus center of pressure during and after the thrust stage are required. The thrust malalignment distances (r_ξ , r_η, r_u) of the rocket must also be furnished. Further, a complete set of the aerodynamic coefficients must be given for the complete velocity range and for all angles of attack encountered.

If it is desired to include wind, a wind profile must be available. In addition, air density, speed of sound, and gravitational acceleration as functions of altitude must be specified. The user should also furnish information to determine the end of the trajectory such as terminal altitude, etc.

Similarly, the user must decide upon a particular numerical integration scheme in order to solve the equations of motion. Frequently used in this area is one of the Runge-Kutta techniques.

It should also be noted that in the tabulated equations of motion (for the missile fixed system) several terms contain a trigonometric function of the angle ϕ. For spin-stabilized rockets, i.e., large $(\omega)_{XM}$ and $(\omega_F)_{XM}$, the rate of change of ϕ may be quite high ($d\phi/dt$). This may require small time increments for the numerical solution of these equations to obtain suitable accuracy. This problem is not encountered when one considers a missile triad that does not spin with the missile. For this latter case, however, one seems to be confronted with time variant moments and products of inertia for asymmetric missiles.

A final word about the equations is that there is no estimate of the dispersion of the missile. This requires the computation of several trajectories, each for slightly different initial conditions, and appropriate statistical combinations of the various range deflections from the standard range.

ACKNOWLEDGEMENT

The author wishes to acknowledge the invaluable assistance of Dr. S. L. Gerhard, Mr. S. Wasserman, and Mr. L. Nichols in the preparation of this report.

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APPENDIX – REQUIRED INPUT PARAMETERS

The complexity of the six-degree-of-freedom equations of motion requires the use of a computer for solution. These equations have, therefore, been programmed for the IBM 709 of Picatinny Arsenal. This appendix tabulates the input data that the user must supply to the computing personnel for a meaningful statement of the problem.

Table of Input Data

1. Specify flat or spherical earth case.

2. Specify rotating or non-rotating earth case.

3. Specify phases of flight that are to be considered.

   Phase I. Acceleration of booster and main stage.

   Phase II. Coasting of booster and main stage.

   Phase III. Separation of booster and main stage.

   Phase IV. Coasting of main stage.

   Phase V. Acceleration of main stage.

   Phase VI. Free flight of main stage.

4. Specify the time duration of each phase considered.

5. Specify mass of missile and booster combination.

6. Specify mass of booster alone (at launch).

7. Specify rate of change of mass of booster during Phase I.

8. Specify rate of change of mass of main stage during Phase III (if considered).

9. Specify rate of change of mass of main stage during Phase V (if considered).

10. Give data relating thrust vs time for Phase I and for Phases III and V (when considered).

11. Specify initial moments of inertia of booster and main stage (at launch, Phase I) along principal axes of rocket.

12. Specify rate of change of moments of inertia of booster along principal axes during Phase I.

13. Specify rate of change of moments of inertia of main stage during Phases III and V (when considered).

14. Specify distance from missile nose to missile centroid at start of Phase I and after separation of booster.

15. Specify rate of change of missile centroid position for Phase I and for Phases III and V (when considered).

16. Specify same information as in Step 14 for missile center of pressure.

17. Specify magnus center of pressure.

18. Specify maximum diameter of missile for Phases I and III.

19. Specify magnitudes for the aerodynamic coefficients

$$k_{DA} \quad k_{\phi}$$

$$k_N \quad k_{\phi\rho}$$

$$k_F \quad k_H$$

as functions of Mach number and angle of attack for booster and main stage combination and main stage alone.

20. Specify thrust malalignment angles (δ_T, δ_A) for Phases I, III, and V.

21. Specify thrust malalignment distances and their rates of change for Phases I, III, and V.

22. Specify proportionality factor of thrust used for initi-' spin stabilization.

23. Specify latitude and longitude of launch point of missile.

24. Specify orientation angles of missile relative to geographic system.

25. Specify initial velocity components of missile $((V)_{XM}, (V)_{YM}, (V)_{ZM})$.

26. Specify initial angular velocity components of missile $((\omega)_{XM}, (\omega)_{YM}, (\omega)_{ZM})$.

27. Specify wind velocity components as function of altitude.

28. Specify altitude at which trajectory is to terminate (air burst, ground burst).

29. Specify output data desired $((R)_{XE}, (R)_{YE}, (R)_{ZE}, \theta, \phi, \psi, (V)_{XE} \ldots)$.

30. Specify time increments for each phase when above quantities are to be tabulated.

31. Specify any peculiarities of the missile system.

32. Good Luck!