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Adaptive Realizations of Optimum Detectors for Synchronous and Sporadic Recurrent Signals in Noise

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The University of Michigan

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by

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for

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ERRATA

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Page xiv, last four lines	Replace "$P_N(A)$" with "$P(A\|N)$", "$P_{SN}(B)$" with "$P(B\|SN)$", "$P_{SN}(A)$" with "$P(A\|SN)$", and "$P_N(A)$" with "$P(B\|N)$".
Page vx, line 6	$\frac{S}{N}_{in}$ should be $\left(\frac{S}{N}\right)_{in}$.
Page 16, footnote 1	Replace "assumption" with "result".
Page 32, Fig. 4.6	1 should appear on line connecting $c_{i,n-1}$ and c_{i,n_i}.
Page 51, lines 5, 11, 18, 27	Replace n_i with $(n_i + 1)$
Page 71, Eq. 5.66	Enclose entire expression following the first summation in brackets.
Page 72, last line preceding Table 5.7	Delete "and a block diagram is shown in Fig. 5.12".
Page 75, sentence preceding Table 5.8	Should read "The equations for the receiver design are presented in Table 5.8 and a block diagram is shown in Fig. 5.12".
Page 79, lines 3, 5, 6, 9 from bottom	Replace n_i with $(n_i + 1)$
Page 117, line 9	"5.12b" should read "5.13b".
Page 120, Fig. 7.21	Labelling on x-axis should be ".01", ".05", ".1", ".5", and ".8".
Page 134, Fig. 7.35	Label on y-axis is "Detectability, d'".
Page 156, line 4	Replace "waveshape." with "waveshape*" and add footnote "* Section 7.3.2.6."

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LIST OF SYMBOLS

W	receiver bandwidth
T	length of observation time
x_i	unit observation
s	signal vector
$\ell(\ )$	a likelihood ratio
SN	signal and noise hypothesis
N	noise hypothesis
$f(\ \mid\ )$	a conditional probability density function
s_i	signal samples
$p_0(s \mid SN)$	probability (prior to start of observations) of the signal, s, conditional to SN hypothesis
S	signal ensemble
τ_1	basic processing interval
X_k	observation during the time $0 \leq t \leq k\tau_1$
x_k	observation during the time $(k-1)\tau_1 \leq t \leq k\tau_1$
k	index of time
$\ell(\ \mid\ )$	a conditional likelihood ratio
t_0	time prior to the observations
$p_k(\ )$	a probability after taking the observation X_k
C^i	ith component
$c_{i,j}$	jth sample of ith component
n_i	total number of samples in ith component
$g(\ \mid\ )$	a conditional probability representing the generator process

p_i	probability of component occurence conditional to ith recurrence phenomenon
$C_{i,0}$	vector representing ith recurrence phenomenon, component absence
$C_{i,1}$	vector representing ith recurrence phenomenon, component presence
S_k	a "block" of the signal vector
b	total number of possible components in component ensemble
T_1	duration of a component
N	noise power in bandwidth W
$b_{i,j}(k)$	probability that the kth sample of the signal is jth sample of the ith component, conditional to the observation X_k and the SN hypothesis
t_k	time after taking kth observation
$b'_{i,j}(k)$	probability that kth sample of the signal is jth sample of the component, conditional to the observation X_k, ith component, and SN hypothesis
Q_{s_k}	product of $f(X_k)$ and $p_k(s_k \mid SN)$
Q_{S_k}	product of $f(X_k)$ and $p_k(S_k \mid SN)$
$Q_{i,j}(k)$	product of $f(X_k)$ and $b_{i,j}(k)$
S_i	ensemble of signals composed of ith component
$Q_i(k)$	product of $f(X_k)$ and $p_k(C^i \mid SN)$
a	pulse amplitude
$c_{i,z}$	overlap state
A	response, "signal present"
B	response, "signal absent"
$P_N(A)$	probability of false alarm
$P_{SN}(B)$	probability of a miss
$P_{SN}(A)$	probability of a correct detection
$P_N(A)$	probability of a correction rejection

LIST OF SYMBOLS (Cont.)

E	signal energy
N_0	noise power per unit bandwidth
ROC	receiver operating characteristic
d'	parameter on normal ROC
d	parameter on normal ROC equal to $(d')^2$
$\frac{S}{N}_{in}$	input signal-to-noise ratio
E_c	component energy
M	total number of possible orthogonal components in component ensemble
SKE	signal known exactly
CKE	component known exactly
CKS	component known statistically
C	component vector

LIST OF APPENDICES

ABSTRACT

The theory of signal detectability is extended to include optimum adaptive receiver designs for the detection of signals with a nonperiodic time structure. The specific problem considered is that of detecting a recurrence phenomenon in noise. This phenomenon is a fixed waveform that recurs in time. The fixed waveform is selected from a finite class of possible waveforms, and the receiver is initially uncertain as to the waveform selection. Three basic recurrence patterns are considered: (1) Sporadic-Poisson, (2) Synchronous-Poisson, and (3) Periodic. For (1) and (2), the recurrence time is a random variable. The approach to the detection problem is Bayesian and the initial uncertainties of the fixed waveform and recurrence times are expressed in terms of a priori probabilities. For the Sporadic- and Synchronous Poisson cases, the recurrence time is always uncertain, but an adaptive receiver can learn the fixed waveform.

Several realizations of the optimum receiver are presented for each of the three basic recurrence time patterns. The receivers are designed by solving an over-all optimization problem in which the likelihood ratio of the entire input observation is formed. A difficulty in the design of the optimum receiver for signals with a nonperiodic time structure is the exponentially growing memory required by the classical non-sequential realization. To obtain a receiver design with a practical memory size, a basic technique is presented in which the signal ensemble is described indirectly in terms of the fixed waveform and the time structure by which these waveforms are assembled. The receiver design is obtained by realizing the likelihood ratio in a sequential manner rather than by postulating a sequential learning model per se. Therefore, the use and proper updating of the contents of the temporary receiver memory are specified by the design procedure.

Although equivalent for detection purposes, different realizations of the same optimum receiver appear to operate in different manners. Receiver designs are

presented in which the receiver appears to "learn" the fixed waveform which is being transmitted. Such a receiver was simulated for a special but useful case to illustrate its operation. Other receiver designs are presented which, although optimum, do not incorporate this learning feature in an obvious manner. The important feature of the adaptive realizations is their fixed-size memory requirement and availability of a classification output.

The effect on detectability of the uncertainty in arrival times of the fixed waveform is investigated. The detection performance for the case of a fixed waveform, known exactly, that recurs with a Synchronous-Poisson Time Structure is presented in terms of the receiver operating characteristic (ROC) as a function of average duty factor, $\frac{2E_c}{N_o}$, and time. E_c is the energy in the fixed waveform and N_o is the noise power per unit bandwidth. This is a useful case since its performance is an upper bound on the attainable performance when the fixed waveform is uncertain or recurs with the Sporadic-Poisson Time Structure. The performance results show that even when the fixed waveform is known exactly, the uncertain arrival times can have a substantial effect on detectability.

The importance of storing and updating likelihood ratio terms in the temporary memory was investigated by comparing the performance of the optimum receiver with one that simply recirculates the input waveshape. It was found that storing and updating likelihood ratio terms rather than recirculating input waveshape becomes more important as $\frac{2E_c}{N_o}$ increases and the average duty factor decreases.

CHAPTER I

INTRODUCTION

1.1 Nature of the Problem

The problem of reception of a signal buried in noise is common to sonar, radar, and communication situations in general. In some cases, such as arise in the reception of speech, the goal is that of recovering the signal so that its waveshape is as close as possible to the original transmitted signal. However, in many applications the primary goal is often deciding whether a signal is present or not, and there is no particular need to reconstruct the original waveshape.

In the early 1950's several authors formulated a theory of signal detectability in which the making of the best possible decisions was the primary goal (Refs. 1-3). Since the noise is considered known only in a probabilistic sense and since there are uncertainties regarding the signal, one cannot decide with certainty whether or not a signal is present in the noise. The early work in signal detectability theory recognized the detection of signals in noise as a problem which could be solved by the application of statistical decision theory.

Signal detection theory encompasses receiver design and performance. The branch of signal detection theory that is given primary emphasis in this study is the design of receivers that are optimum in the sense of making the best decisions. In particular, rather general techniques of designing optimum receivers which operate in a sequential mode are considered. Such receivers frequently exhibit adaptive characteristics.

Both the design and performance of an optimum receiver depend upon the signal uncertainties and the noise. The optimum receiver usually takes on its simplest form at either of the two extremes of knowledge regarding the signal; i.e., precise knowledge of the signal on the one hand, or at the other extreme, a large amount of initial signal uncertainty in which parameters of the signal cannot be learned. The performance of the optimum receiver usually decreases as the amount of signal uncertainty increases.

Most of the literature on signal detectability has been concerned with periodic signals which may be, for example, uncertain in amplitude and phase. This is understandable since a primary application of signal processing is to active sonar and radar systems where a periodic transmission is characteristic. The knowledge that the signal is periodic or nearly periodic is definite information that a receiver designer can use to advantage.

One class of signals studied here is a type that is more likely to be encountered in a passive situation. Here, the signal emitted is beyond the control of the designer of the over-all transmitter-receiver system and is often nonperiodic. A broad class of such signals is one in which a fixed but quite unknown waveform is emitted recurrently in a non-periodic and quite unknown way. The interest is in detecting the presence or absence of the entire recurrence phenomenon rather than making a local detection of the presence or absence of an individual fixed waveform. If the signal-to-noise ratio were high, individual local detections could be made relatively easily. However, a case of special interest is when the unknown waveform has a low signal-to-noise ratio and a low duty factor. Then, local detection becomes difficult. If one has sufficient time to observe the receiver input, however, the recurrence of the same waveform permits the possibility of "learning" or "adapting to" the waveform sent. This learning or adaptation must be done in spite of the noise and the "unknown" epoch of the waveform.

The general type of signals considered are shown in Fig. 1.1. This sketch shows possible noise-free signals that might appear at the receiver input. The particular local waveform that is sent in a given signal burst is uncertain and is one out of a finite number of local waveforms. Although the same local waveform is recurrent in each signal burst, the precise times of recurrence are uncertain. It can be seen that a wide variety of signal bursts can result, the receiver must be designed to detect any one of them.

Three basic types of recurrence-time processes are considered. They are:

1. Sporadic-Poisson process
2. Synchronous-Poisson process
3. Periodic process.

These three processes differ in that they represent three degrees of knowledge regarding the manner of recurrence of a waveform. The Sporadic-Poisson process involves the least

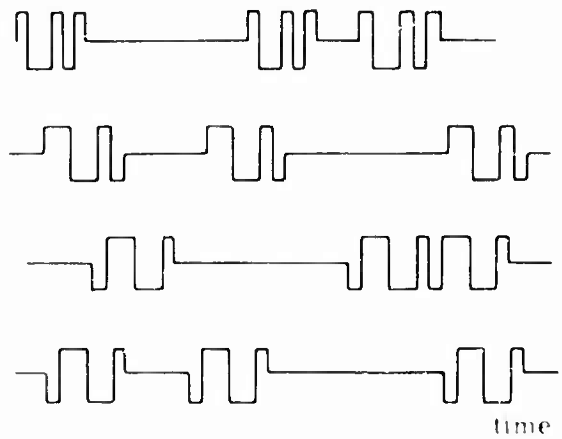

Fig. 1.1. Four "typical" bursts of signal.

amount of knowledge regarding the structure of the signal in time. The Periodic case, representing the most knowledge regarding the time recurrence, is included in this study for comparison purposes. Precise mathematical formulation of the possible signals that could occur is postponed until Chapter IV.

The basic technique of designing adaptive or sequential realizations of optimum receivers is considered in this study. As we shall see, there is no unique adaptive realization. Adaptive realizations of the optimum receiver are presented, in general block diagram form, for each of the three basic recurrence-time structures discussed above. The receiver is designed to be optimum in the sense that it makes the best decision as to presence or absence of the entire recurrence phenomenon. It is provided, sequentially in time, with two outputs; a decision output and a classification output. The detection output provides information for deciding presence or absence of the recurrence phenomenon and the classification output provides updated probabilities of the various possible fixed waveforms that could occur.

When the recurrence time process is nonperiodic, the design of the optimum receiver is complicated by receiver memory requirements. A nonsequential realization of the optimum receiver requires an exponentially growing memory. It will be shown that this difficulty can be eliminated by realizing the receiver in a sequential mode.

1.2 Background of Previous Work

The foundation of this study is the theory of signal detectability as developed by Peterson, Birdsall and Fox (Ref. 1). This theory emphasizes the central role of likelihood ratio in the receiver design. Related basic material can also be found in Helstrom (Ref. 5).

A number of authors have applied "adaptive" techniques to the problem of the detection of signals in noise (Refs. 10, 11, 15, 16, 17, 18). The problem of designing an adaptive filter for a fixed waveform whose time of arrival is unknown has been considered by Glaser (Ref. 10). In this work a statistical decision theory approach is used. Local waveform uncertainty is expressed in terms of an a priori probability density function but recurrence time uncertainty is not. The epoch is instead detected on a local basis and the assumption is made that the epoch measurement is accurate.

Jakowatz, Shuey and White (Ref. 11) have proposed an adaptive filter for detecting a recurrent fixed waveform. A simplified block diagram of their original adaptive filter is shown in Fig. 1.2. The basic operations of this filter as described by Jakowatz are: (1) comparison of a sample of the incoming waveform, x(t), with an estimate, m(t), of the unknown signal, s(t), by correlation of x(t) and m(t), (2) on the basis of the correlator output, c(t), guess whether or not a signal is contained in the current sample of x(t), and (3) at those times when a signal is guessed to be present, form a new estimate of the signal which consists of a weighted average of that sample of the input with the prior estimate.

Although basic guidelines from signal detection theory are used in the adaptive filter of Jakowatz et al, the design approach is not an optimal one as the authors indeed recognized. Two characteristic features are apparent in this adaptive filter. First, a local detection is required before any modification of the memory is made. Secondly, the receiver memory is used to remember a single waveform. This is undoubtedly an inadequate memory for the receiver to be optimum. Their adaptive filter may be, however, a practical receiver when the local waveform signal-to-noise ratio is large enough to permit good local detection. In such cases the simple implementation of a receiver with a single waveform memory may justify its suboptimum detection performance.

Several authors have considered a local detection problem in which a fixed local waveform recurs in a synchronous manner (Refs. 15 and 18). In the local detection case the problem becomes that of detecting where each of the local waveform recurrences are,

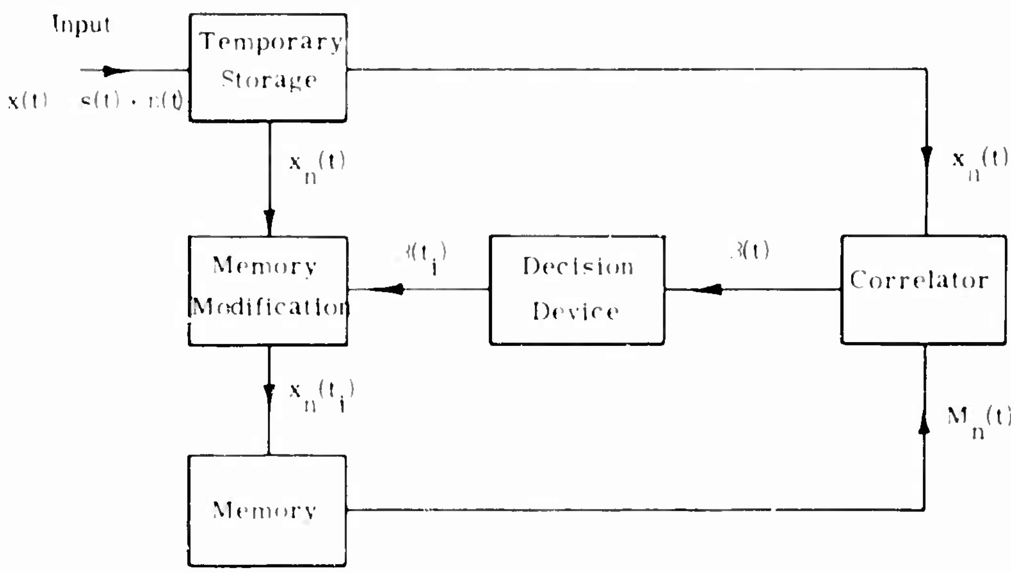

Fig. 1.2. Jakowatz, Shuey and White original adaptive filter.

using all past information. The approach is Bayesian and one of optimum receiver design. One central problem is common, however, and that is the problem of implementing an optimum receiver which requires an exponentially growing memory. As Scudder (Ref. 18) points out, the standard nonsequential realization of the optimum receiver is very complex, grows exponentially with time, and the analysis of its performance is close to impossible even using present day computers. Marcus and Swerling have recognized a similar problem of providing sufficient receiver memory in regard to a multiple-resolution-element radar problem (Ref. 12).

1.3 Method of Attack of the Problem

The Bayesian viewpoint is adhered to in this work. That is, it is assumed that some knowledge is available to the receiver designer regarding the signals and noise that will be received. The particular knowledge available must be expressible in terms of probability distribution functions.

Since the primary goal is the making of the best decision about the presence or absence of the entire recurrence phenomenon, rather than determining the location of each recurrent waveform, the problem is to decide between the two hypotheses; presence of recurrence phenomenon and noise or noise alone. If one prefers correct decisions to mistakes, Birdsall (Ref. 9) has shown that the optimum receiver is one which realizes the likelihood ratio of the observation and this fact does not depend on any specific quantity to be maximized or minimized.

Likelihood ratio plays a central role in the design of adaptive receiver realizations as it did in the design of optimum receivers in classical detection theory. The adaptive receiver realization is obtained by forming the likelihood ratio of the observation which is optimum for deciding the presence or absence of the entire recurrence phenomenon and then realizing this likelihood ratio in a sequential manner. It is interesting that receivers designed on the basis of sequentially realizing the optimum receiver often exhibit "adaptive" characteristics. The adaptive feature is, however, a result of the particular form of the realization chosen. This approach to the problem is in contrast to ones in which a block diagram of a receiver is chosen by analogy to a biological adaptation mechanism, or by extension of electronic techniques used in tracking devices.

1.4 Organization of Material

Chapters I and II provide background material for this work. Chapter II is a review of the basic signal detection theory that is relevant to the problem considered here. This chapter introduces the problem and expresses the importance of approaching optimum receiver design via likelihood ratio. In Chapter III the extension of the fixed time theory to a time varying situation is presented as well as methods of realizing the optimum receiver with an adaptive form. The inherent role that the classification problem plays in the optimum detection is also pointed out. In Chapter IV the particular types of transmitted signals considered are described in detail and defined. In Chapter V the optimum adaptive receiver design is developed in detail. Four realizations are presented for each of the three basic types of time uncertainty. This demonstrates the necessity of the adaptive receiver design for the sporadic and synchronous cases due to practical memory requirements. This is contrasted with the periodic case where no such memory problem exists. Chapter VI presents some special but interesting cases of the receivers of Chapter V.

In Chapter VII the detection performance of the optimum adaptive receiver is presented in terms of the ROC (receiver operating characteristic) for some specific cases, primarily for the Synchronous-Poisson time uncertainty. Also included in this chapter are

Monte Carlo runs which demonstrate the "adaptive" features of the adaptive realization.

In Chapter VIII conclusions to this work are presented.

CHAPTER II

REVIEW OF BASIC SIGNAL DETECTION THEORY

2.1 Classical Signal Detection Theory

Since the basis of optimum receiver design is the work of Peterson, Birdsall, and Fox (Ref. 1), it is appropriate that it be reviewed. This theory is now called classical, fixed-time detection theory. It applies to situations where the receiver input is observed over a fixed interval of time and a decision is then made concerning the presence or absence of signal during that interval. A block diagram is shown in Fig. 2.1. The transmitted signal and added noise or noise alone is presented to the receiver input. The question is whether the switch is open or closed. Classical detection theory encompasses optimum receiver design, receiver realization, and evaluation of receiver performance.

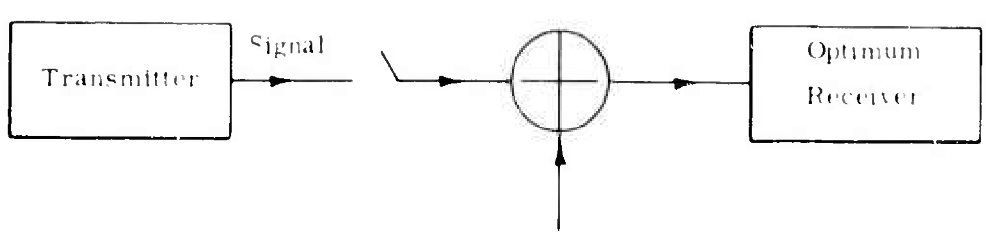

Fig. 2.1. Basic detection problem.

Optimum receiver design is approached from a decision theory viewpoint. When the input waveform to the receiver is bandlimited, it can be characterized by sample values (Ref. 1). Typically, there are 2WT independent observation samples, $(x_1, x_2, \ldots, x_{2WT})$, if W is the bandwidth over which the observations are defined and T is the total length of observation. The total observation, $(x_1, x_2, \ldots, x_{2WT})$, is considered to be made on either noise alone or signal plus noise. At the end of the observation interval, a single terminal decision is made by a device which can make two alternative decisions; conclude that signal was present during that entire observation interval or conclude that signal was not present during the entire observation interval. The time sequence in which observations and decisions are made in the fixed time theory are represented in Fig. 2.2. When the actual cause is signal plus noise, the decisions correspond to a detection and a miss, respectively. Similarly, when the actual cause is noise, there are a corresponding correct and incorrect decision. There are, therefore, two correct and two incorrect responses. There are values and costs associated with these four possible responses, and the theory prescribes the optimum receiver which makes the balance between correct and incorrect responses which optimizes some function of these values and costs.

Fig. 2.2. Observation-decision scheme for fixed time theory.

The cost of making an observation is not considered in this theory. As a result no premium is attached to making decisions rapidly. The theory of sequential analysis (Ref. 7), or deferred decision theory (Ref. 8) considers such a cost of observation. In the classical theory, the optimum receiver is one which calculates the likelihood ratio of the input observation. A decision level or threshold is then put on the likelihood ratio. When the likelihood ratio exceeds this threshold the response is "signal present" and when it falls below this threshold the response is "noise alone". The receiver design is still that of a likelihood ratio processor in deferred decision theory, but the simple output threshold is replaced with a time-varying comparison function.

Receiver realization is specification of equipment, in block diagram form, that realizes the likelihood ratio. In general there is no unique way of specifying a block diagram which realizes a mathematical equation. However, one realization may have an advantage over another in terms of equipment complexity or cost. There is no procedure at present for selecting a "best" receiver realization. Such a theory would need to incorporate an equipment cost perhaps best characterized by memory cost.

An important aspect of detection problems is evaluation of detection performance. It is often useful to evaluate the performance of suboptimum as well as optimum receivers. The evaluation of optimum receivers puts an upper bound on attainable performance. Evaluation of suboptimum receivers may reveal a receiver whose performance justifies its simpler form. In the fixed time theory, the error performance for all possible likelihood-ratio thresholds is the complete evaluation, and this is summarized by the receiver operating characteristic (ROC). This is a plot of probability of correct detection versus the probability of false alarm.

Analytical evaluation of receivers frequently becomes a difficult task. An alternative technique is an experimental approach such as simulation on a computer (Ref. 22). In the present study, a digital computer simulation of several receivers was employed (See Chapter VII).

2.2 Optimumness of Likelihood Ratio

In the formulation of the detection problem one considers the input to the receiver as being due to either of one of two causes, i.e., noise alone or a mixture of signal and noise. One of the primary conclusions that has resulted from the fixed observation theory is that the optimum receiver is one which realizes the likelihood ratio. In fact, it has been proved that the optimumness of likelihood ratio does not depend on any specific quantity to be maximized or minimized, but only on the condition that one prefers correct decisions to mistakes (Ref. 9). This is a powerful result which gives perspective to any investigation of new processing techniques since the likelihood ratio receiver puts an upper bound on attainable performance. Although the optimumness of likelihood ratio is not restricted to additive noise, most of the examples in this study will assume added white Gaussian noise.

The likelihood ratio for the fixed observation time detection problem, when the signal is known exactly, is given by

$$\ell(x_1, x_2, \ldots, x_{2WT} | s) = \frac{f(x_1, x_2, \ldots, x_{2WT} | s, SN)}{f(x_1, x_2, \ldots, x_{2WT} | N)} \tag{2.1}$$

where $f(x_1, x_2, \ldots, x_{2WT} | s, SN)$ is the probability density function of the joint observation $(x_1, x_2, \ldots, x_{2WT})$ under the condition the signal is known exactly and signal plus noise is present and $f(x_1, x_2, \ldots, x_{2WT} | N)$ is the density function of the joint observation $(x_1, x_2, \ldots, x_{2WT})$ under the condition noise alone is present. The entire signal vector, $(s_1, s_2, \ldots, s_{2WT})$ is denoted by s. As an example, in the classic case of a signal known exactly in added white Gaussian noise one may work with the logarithm of the likelihood ratio (also optimum since it is a monotone function of the likelihood ratio) yielding the familiar crosscorrelator as the optimum receiver. In this case

$$\ln f(x_1, x_2, \ldots, x_{2WT} | s) = \sum_{i=1}^{2WT} \left(x_i s_i - \frac{s_i^2}{2} \right) \qquad (2.2)$$

where s_i are the sample values of the known signal.

2.3 Composite Hypothesis Problems

In the signal-known-exactly example, the likelihood ratio gives the optimum strategy for choosing between two hypotheses (1) observation was due to noise alone, N, and (2) observation was due to signal mixed with noise, SN. For the signal-known-exactly case, both hypotheses are termed simple hypotheses. If, however, the observation under either hypothesis depends on some parameter, that hypothesis is called a composite hypothesis. An example of a composite-signal-hypothesis problem that has appeared in the literature is the problem of detecting a signal known exactly except for phase. There the parameter is the unknown phase angle, θ.

The sporadic problem which will be formulated later is a composite-signal-hypothesis problem; the parameter is the signal vector, s. The optimum receiver is then one which realizes the average likelihood ratio

$$f(x_1, x_2, \ldots, x_{2WT}) = \sum_{\substack{\text{all} \\ s \in S}} f(x_1, x_2, \ldots, x_{2WT} | s)\, p_0(s | SN) \qquad (2.3)$$

The probability $p_0(s | SN)$ is the probability a signal $s = (s_1, s_2, \ldots, s_{2WT})$ is sent under the condition that some signal-plus-noise is sent. It is based on information available prior to the observation (i.e., at time t_0). The likelihood ratio, $f(x_1, x_2, \ldots, x_{2WT} | s)$, is the likelihood ratio of the joint observation, $(x_1, x_2, \ldots, x_{2WT})$, conditional to each specific signal s that could occur. The entire ensemble of signals is denoted by S. Formally, Eq. 2.3 says that the detection output of the optimum receiver is obtained by forming the individual likelihood ratios, $\ell(x_1, x_2, \ldots, x_{2WT} | s)$, for each signal and averaging them over the *a priori* probabilities of the various signals that could occur.

As remarked earlier, the theory of signal detectability is a theory which provides for the insertion of the *a priori* knowledge. It can hardly be doubted that the designer of the receiver has some knowledge about the types of signals for which the receiver is being designed to observe. This *a priori* knowledge appears in equations for likelihood ratio in the form of the probabilities, $p_0(s|SN)$. A wide range of initial knowledge about the signal can be specified by describing the entire signal class, S, and assigning values to the probabilities, $p_0(s|SN)$. As a special case, this receiver reduces to a single crosscorrelator when only one possible signal could be sent, since $p_0(s|SN) = 1$ for that signal and zero for all others.

2.4 Memory and Signal Detectability

The classical theory of signal detectability is a full memory theory; the implicit assumption is that an unlimited amount of memory is available with which to realize the optimum receiver. The cost of providing such a full memory is an obvious practical problem in certain situations. The optimum receivers for the synchronous and sporadic recurrent waveforms present this problem. Unless special care is taken in obtaining the proper receiver realization for these cases, an impractical amount of memory may be required. It turns out that realizing such optimum receivers in a sequential form results in receivers with practical memory requirements.

Although optimum receiver design in this study will be based on the full memory theory, emphasis is placed on obtaining optimum receiver realizations with adequate but practical memory size. There is no theory yet developed on the proper utilization of receiver memory but the study of the manner in which the memory is utilized in adequate and full memory receivers should contribute to an eventual theory of the use of memory in signal detectability.

CHAPTER III

ADAPTIVE REALIZATION OF THE OPTIMUM RECEIVER

3.1 Adaptive Receiver Design Philosophy

Qualitatively, the term adaptive receiver conveys the requirements of a time-varying structure and a "learning" feature. As is evident from scanning the literature, adaptive processing schemes are not unique. The philosophical discussion of what constitutes a "true" adaptive device is not considered here.

In this chapter, a technique is developed for the design of full-memory, adaptive, optimum receivers. In the full-memory theory evolved here, the term "adaptive" or "adaptive realization" is used to label forms of optimum receivers which exhibit adaptive characteristics. Although not considered here, a different theory of adaptation would undoubtedly result if a receiver were to be designed with an inadequate memory.

Full-memory adaptive receiver design may be approached from the basic viewpoint of classical signal detection theory. The theory must center on the primary goal of making the best decisions. The mathematical operations that an adaptive receiver must make are then specified by the theory. It will be shown how the existing theory of signal detectability, because of its fundamental approach, enables the synthesis of adaptive realizations of the optimum receiver. This puts full-memory, adaptive signal processing within the framework of the theory of signal detectability. It has already been pointed out in the previous chapter that the optimum receiver under many criteria is one which realizes the likelihood ratio. There may be several different realizations, equivalent in that each processes the input to realize the same required likelihood ratio, or a monotone function of this likelihood ratio. The performance of these realizations may be equivalent; however, the different realizations may have unique advantages or disadvantages from a practical point of view. It will now be shown how the likelihood ratio can be realized in an adaptive manner.

It is convenient to consider first the adaptive receiver form for directly described signals. Direct description is the traditional description of signals in the classical theory. It becomes more convenient to use an indirect signal description in subsequent chapters. There the signal will be described indirectly in terms of a smaller ensemble of waveforms and a time structure whereby these short waveforms are assembled.

3.2 Adaptive Realization of the Optimum Receiver - Directly Described Signal Ensemble

The adaptive receiver realization which will be developed in this section operates sequentially, so that in every τ_1 seconds the observations in the past τ_1 seconds are processed. Except where otherwise noted, the notation used throughout is that X_k denotes x(t) for $0 \leq t \leq k\tau_1$ and x_k denotes x(t) for $(k-1)\tau_1 \leq t \leq k\tau_1$. In other words, a capital letter indicates the observation from the beginning of time to now (i.e., $k\tau_1$) and the lower case letter refers to the present observation which is to be processed as a unit. Sequentially in time the receiver updates whether the signal was or was not present in the entire observation, X_k, and the opinion is updated as to which signal it is if indeed signal is present.

Classical signal detection theory is a fixed-time theory. That is, much of the work in the past involved receiver design in which the processing time was chosen before building the receiver. However, in an adaptive approach the processing time remains variable. Actually, classical fixed-time theory only appears to specify a fixed observation, X_k. The theory is easily generalized to permit a receiver which operates over a variable time interval. In particular, if the optimum receiver is to be designed to work on the time interval $(0, k\tau_1)$, then the optimum receiver is one which realizes the sequence of the likelihood ratios, $\ell(X_1)$, $\ell(X_2)$, ..., $\ell(X_k)$. This receiver provides the output which is necessary for making the best decision as to presence or absence of signal from time zero to time $k\tau_1$, and does so in a running or sequential fashion. The optimumness of likelihood ratio guarantees that all available information prior to time zero, along with that available from the observation itself, has been used to make an optimum decision as to presence or absence of signal in the entire running time interval $(0, k\tau_1)$. This is called a long-term detection problem.

The sequence of likelihood ratios, $\ell(X_1)$, $\ell(X_2)$, ..., $\ell(X_k)$, could be obtained at each time $k\tau_1$ by repeated application of Eq. 2.3. This equation suggests, however, that one store all past observations as well as all the probabilities up to time $k-1$. A gross block diagram of this realization is shown in Fig. 3.1. The samples of the observation are stored in an input memory. In addition, it is necessary to make provision for storage of all the *a priori* probabilities, $p_0(s|SN)$. The storage requirements for both these purposes increase as time increases, which makes this type of realization impractical in many cases.

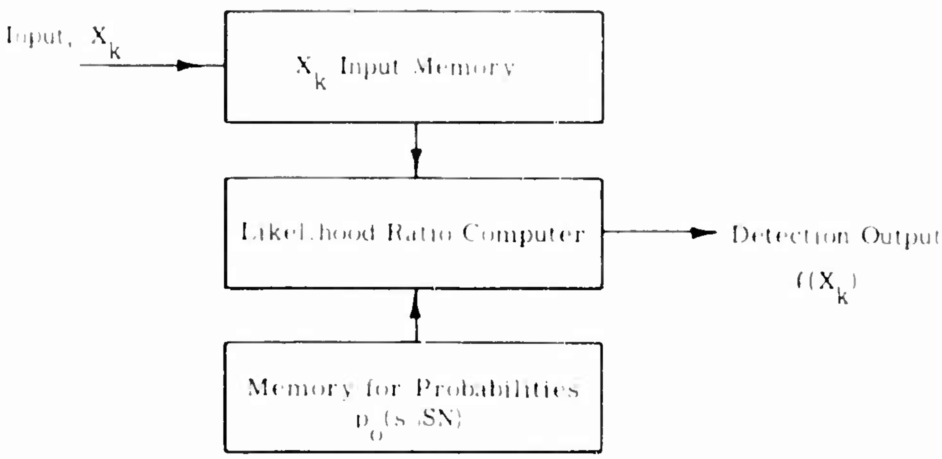

Fig. 3.1. Gross block diagram of a nonsequential receiver realization.

Another way of forming these likelihood ratios is to derive $\ell(X_k)$ from the previous one, $\ell(X_{k-1})$, together with the kth observation, and a set of updated probabilities. Several of the forms which these realizations can take will be considered later. This study is concerned primarily with this approach and its implementations. This approach is especially interesting because it leads to realizations which exhibit the features of an adaptive type of processor. First, however, these full-memory adaptive realizations are shown to be related directly to the original likelihood ratio by an equivalence transformation.

3.2.1 Sequential Realization of Likelihood Ratio - Independent Observations Conditional to SN. Many classical detection problems have dealt with the situation where either the signal transmitted was independently chosen from the signal ensemble in each unit of observation, x_i, or where only one possible known signal could be transmitted throughout the entire observation, X_k. The latter is the classic SKE (signal known exactly) case. Under certain conditions, this results in a simple recursive equation for obtaining the likelihood ratio of the observation, X_k, from the likelihood ratio of the observation X_{k-1}.

To illustrate this, consider that the likelihood ratio of the observation, X_k, is by definition

$$\ell(X_k) = \frac{f(X_k|SN)}{f(X_k|N)} \tag{3.1}$$

It is assumed that the observations are independently distributed under the background condition of noise alone. The independence of the observations under noise alone permits computation of the probability density function for a section of observation from a similar probability function for shorter sections multiplied by the probability density function for the most recent section. Thus

$$f(X_k|N) = f(X_{k-1}|N)\, f(x_k|N) \tag{3.2}$$

Since the observations are assumed independent under the condition SN, the probability density function of the observation under the condition signal plus noise can be similarly separated so that

$$f(X_k|SN) = f(X_{k-1}|SN)\, f(x_k|SN) \tag{3.3}$$

Substituting Eqs. 3.2 and 3.3 into 3.1, the likelihood ratio can be written as

$$\ell(X_k) = \left[\frac{f(X_{k-1}|SN)}{f(X_{k-1}|N)}\right]\left[\frac{f(x_k|SN)}{f(x_k|N)}\right] \tag{3.4}$$

Applying Eq. 3.1, which is the definition of likelihood ratio, Eq. 3.4 can be written as

$$\ell(X_k) = \ell(X_{k-1})\, \ell(x_k) \tag{3.5}$$

For independent observations under SN with independent noise, the likelihood ratio of the total observation, X_k, is the product of the likelihood ratios of the independent parts.[1]

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[1] For example, this is the assumption in Helstrom, Statistical Theory of Signal Detection, Chapter III, Section 4, "Sequential Testing of Hypotheses."

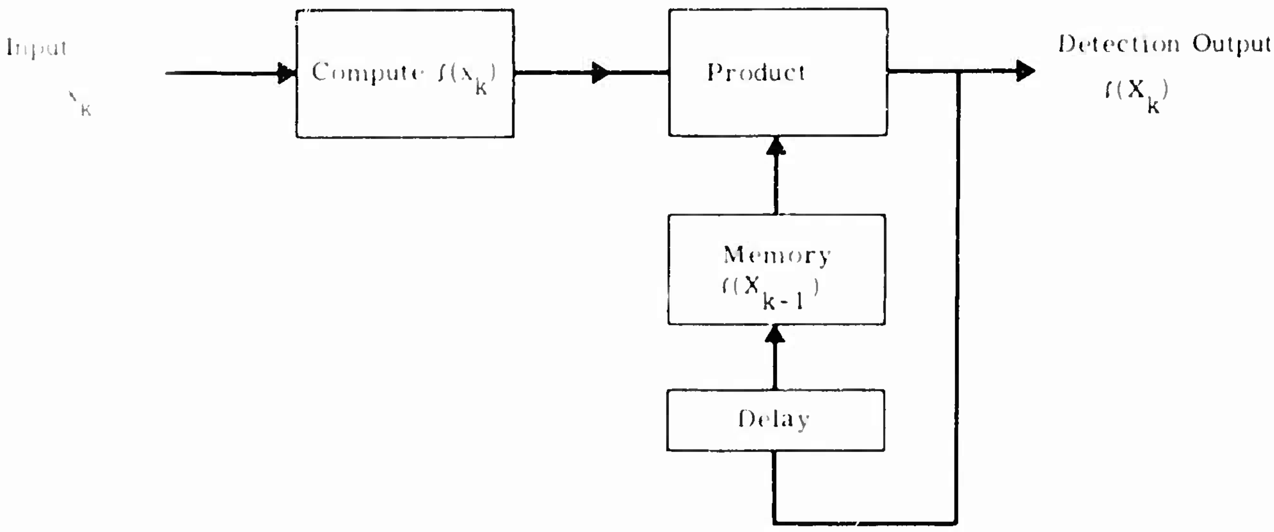

Fig. 3.2. Optimum sequential receiver, independent observations.

A simple block diagram of a receiver which realizes Eq. 3.5 is shown in Fig. 3.2. After the first unit of observation, a likelihood ratio $f(X_1)$ is computed. This is stored. Then the likelihood ratio $f(x_2)$, of the next unit of observation is computed and multiplied by $f(X_1)$ to form $f(X_2)$ which is stored. This procedure is iterated. This is a sequential form of an optimum receiver.

3.2.2 Sequential Realization of the Likelihood Ratio - Dependent Observations Conditional to SN. In this section it is shown how the likelihood ratio of the observation over an interval (0, $k-1$) can be realized, in a fashion equivalent to the above, for cases where the observations, x_i, are dependent under the hypothesis SN. This is the situation in a composite hypothesis problem. The derivation begins with the likelihood ratio, $f(X_k)$, of the entire observation, X_k, which is known to be optimum under many criteria, and transforms $f(X_k)$ into an equivalent sequential form.

Once more, we start from the likelihood ratio of the observation X_k, given by

$$f(X_k) = \frac{f(X_k \mid SN)}{f(X_k \mid N)} \tag{3.1}$$

As before, the observations, x_i, are assumed independent when noise alone is present, so Eq. 3.2 applies. In composite signal hypothesis problems, however, Eq. 3.3 does not hold and $f(X_k|SN)$ is written, instead, by definition of conditional probabilities, as

$$f(X_k|SN) = f(X_{k-1}|SN)\, f(x_k|X_{k-1}, SN) \tag{3.6}$$

Thus, $\ell(X_k)$ can be written as

$$\ell(X_k) = \left[\frac{f(X_{k-1}|SN)}{f(X_{k-1}|N)}\right]\left[\frac{f(x_k|X_{k-1}, SN)}{f(x_k|N)}\right] \tag{3.7}$$

or

$$\ell(X_k) = \ell(X_{k-1})\, \ell(x_k|X_{k-1}) \tag{3.8}$$

where we have defined

$$\ell(x_k|X_{k-1}) = \frac{f(x_k|X_{k-1}, SN)}{f(x_k|N)} \tag{3.9}$$

Note that Eq. 3.8 is similar to Eq. 3.5 except that the function $\ell(x_k|X_{k-1})$ is dependent on the entire past observations in addition to the unit observation, x_k.

Let us now determine $\ell(x_k|X_{k-1})$. To do this, let us consider in more detail the numerator, $f(x_k|X_{k-1}, SN)$, of Eq. 3.9. Solving Eq. 3.6 for $f(x_k|X_{k-1}, SN)$ one obtains

$$f(x_k|X_{k-1}, SN) = \frac{f(X_k|SN)}{f(X_{k-1}|SN)} \tag{3.10}$$

The numerator of Eq. 3.10, by definition of a composite signal hypothesis, is

$$f(X_k|SN) = \int_S f(X_k|s, SN)\, p_o(s|SN)\, ds \tag{3.11}$$

where $f(X_k|s, SN)$ is the probability density function of the observation X_k under the condition SN and where a specific signal, s, is being transmitted. At the start of the observation, specified as time t_o, the observer is uncertain as to the specific signal to be sent. This uncertainty is expressed by the probability density function, $p_o(s|SN)$. If the signal is simply added to the noise, then the observations, conditional to a specific signal, s, are independent.

Thus

$$f(X_k|s, SN) = f(X_{k-1}|s, SN)\ f(x_k|s, SN) \tag{3.12}$$

Substituting Eq. 3.12 into 3.11 results in

$$f(X_k|SN) = \int_S f(X_{k-1}|s, SN)\ f(x_k|s, SN)\ p_o(s|SN)\, ds \tag{3.13}$$

Equation 3.10 therefore becomes

$$f(x_k|X_{k-1}, SN) = \int_S \left[\frac{f(X_{k-1}|s, SN)\ p_o(s|SN)}{f(X_{k-1}|SN)}\right] f(x_k|s, SN)\, ds \tag{3.14}$$

where $f(X_{k-1}|SN)$ has been incorporated in the integrand since it is independent of the variable of integration. It is natural to define a new probability function for the signal ensemble based upon all the observations up to time $(k-1)\tau_1$. This is done by singling out the bracketed term of Eq. 3.14 and defining it as

$$p_{k-1}(s|SN) = \frac{f(X_{k-1}|s, SN)\ p_o(s|SN)}{f(X_{k-1}|SN)} \tag{3.15}$$

Substituting Eq. 3.15 into 3.14, one can write $f(x_k|X_{k-1}, SN)$ as

$$f(x_k|X_{k-1}, SN) = \int_S f(x_k|s, SN)\ p_{k-1}(s|SN)\, ds \tag{3.16}$$

which is in direct parallel to Eq. 3.11, except that the weighting probability is not the original defining density at t_o but is an up-to-date probability function based upon the observations over all the time up to the last unit of observation. The probabilities, $p_k(s|SN)$, can also be obtained from $p_{k-1}(s|SN)$ rather than $p_o(s|SN)$. We can rewrite Eq. 3.15 with the subscript indexed ahead by one,

$$p_k(s|SN) = \frac{f(X_k|s, SN)\ p_o(s|SN)}{f(X_k|SN)} \tag{3.17}$$

Solving Eq. 3.15 for $p_0(s|SN)$ and substituting this into Eq. 3.17 results in

$$p_k(s|SN) = \left[\frac{f(X_k|s,SN)}{f(X_{k-1}|s,SN)}\right]\left[\frac{f(X_{k-1}|SN)}{f(X_k|SN)}\right] p_{k-1}(s|SN) \qquad (3.18)$$

The ratio in the first bracket is conditional to the signal s and so Eq. 3.12 holds. Similarly the reciprocal of the ratio in the second bracket is $f(x_k|X_{k-1},SN)$ by Eq. 3.6. Therefore Eq. 3.18 becomes

$$p_k(s|SN) = \left[\frac{f(x_k|s,SN)}{f(x_k|X_{k-1},SN)}\right] p_{k-1}(s|SN) \qquad (3.19)$$

We have still to get the form of Eq. 3.8 for likelihood ratio. Using Eq. 3.9 for the definition of $\ell(x_k|X_{k-1})$ along with Eq. 3.16 one gets

$$\ell(x_k|X_{k-1}) = \int_S \left[\frac{f(x_k|s,SN)}{f(x_k|N)}\right] p_{k-1}(s|SN)\, ds \qquad (3.20)$$

where $f(x_k|N)$ has been brought inside the integral since it is independent of the variable of integration. Define a likelihood ratio of the unit observation, x_k, conditional to a specific signal as

$$\ell(x_k|s) = \frac{f(x_k|s,SN)}{f(x_k|N)}\,. \qquad (3.21)$$

then the conditional likelihood ratio, $\ell(x_k|X_{k-1})$ can be written as

$$\ell(x_k|X_{k-1}) = \int_S \ell(x_k|s)\, p_{k-1}(s|SN)\, ds \qquad (3.22)$$

Similarly, if numerator and denominator of Eq. 3.19 are divided by $f(x_k|N)$ and Eqs. 3.9 and 3.21 are used, the updating equation can be written as

$$p_k(s|SN) = \left[\frac{\ell(x_k|s)}{\ell(x_k|X_{k-1})}\right] p_{k-1}(s|SN) \qquad (3.23)$$

Equations 3.8, 3.22, and 3.23 form the basis of design of the sequential receiver. Table 3.1 summarizes the basic sequential receiver design equation for both dependent and independent observations under SN.

TABLE 3.1

BASIC RECEIVER DESIGN EQUATIONS

SEQUENTIAL REALIZATION OF THE LIKELIHOOD RATIO

Independent Observations Conditional to SN

Detection Output

$$f(X_k) = f(X_{k-1})f(x_k) \qquad (3.5)$$

Dependent Observations Conditional to SN

Detection Output

$$f(X_k) = f(X_{k-1})\ f(x_k|X_{k-1}) \qquad (3.8)$$

Sequential Average Likelihood Ratio

$$f(x_k|X_{k-1}) = \int_S f(x_k|s)\ p_{k-1}(s|SN)\ ds \qquad (3.22)$$

Classification Output

$$p_k(s|SN) = \frac{f(x_k|s)}{f(x_k|X_{k-1})}\ p_{k-1}(s|SN) \qquad (3.23)$$

By comparing Eqs. 3.22 and 3.23 we observe the primary earmark of adaptive operation: the feedback of results to modify the processing of subsequent observations. Thus, from $p_o(s|SN)$ and x_1 one can calculate $p_1(s|SN)$. This is used to determine the weighting on x_2 (Eq. 3.22) which in turn is used to compute $p_2(s|SN)$ and so forth. The quantities calculated are shown in Fig. 3.3.

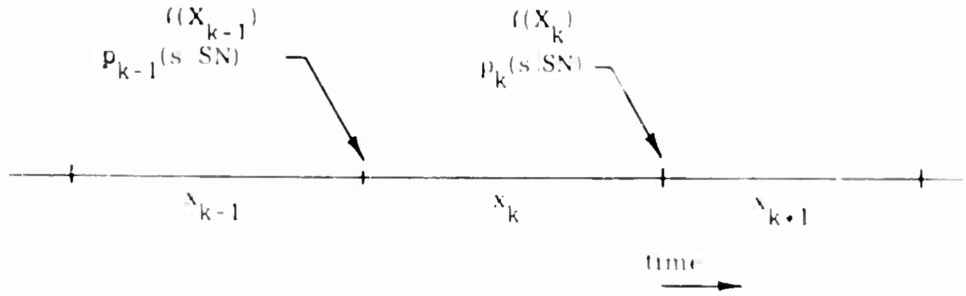

Fig. 3.3. Sequential realization representation.

It is likely that one could arrive at Eqs. 3.22 and 3.23 by proper application of Bayesian logic. The author has overtly chosen not to do this so that it is obvious that the aforementioned equations for conditional likelihood ratio and for updating knowledge are a result of simple mechanical manipulation of the formula for likelihood ratio of a complete observation, X_k.

A block diagram indicating the operation of this adaptive receiver is shown in Fig. 3.4. The likelihood ratio of the incoming observation is computed for each possible signal that could occur. These individual likelihood ratios are then weighted by up-to-date probabilities, $p_{k-1}(s|SN)$, as to which signal is being transmitted, and these products are added over all $s \epsilon S$ to obtain the conditional likelihood ratio $f(x_k|X_{k-1})$. The information regarding which signal is present, as expressed by $p_{k-1}(s)$, is then updated using the quantities $f(x_k|s)$ and $f(x_k|X_{k-1})$ which contain new information from the kth observation as to which signal is being transmitted. This forms the up-to-date probabilities, $p_k(s|SN)$, which will be used for weighting the individual likelihood ratios of the (k+1)st observation. In addition, $p_k(s|SN)$ can be displayed to provide classification information. The purpose of this section has been to show how to design optimum detection equipment which has a property normally associated with adaptive equipment: namely, the property of utilizing observations to increase knowledge and using this knowledge in interpreting subsequent observations.

We have seen how the equation for updating knowledge as to which signal was being transmitted (Eq. 3.23) gave a "learning" feature to the receiver design. This feature was absent in the realizations discussed in Section 3.2.1 since it was assumed that either (1) the signal was known exactly, in which case only the central question of its existence remains,

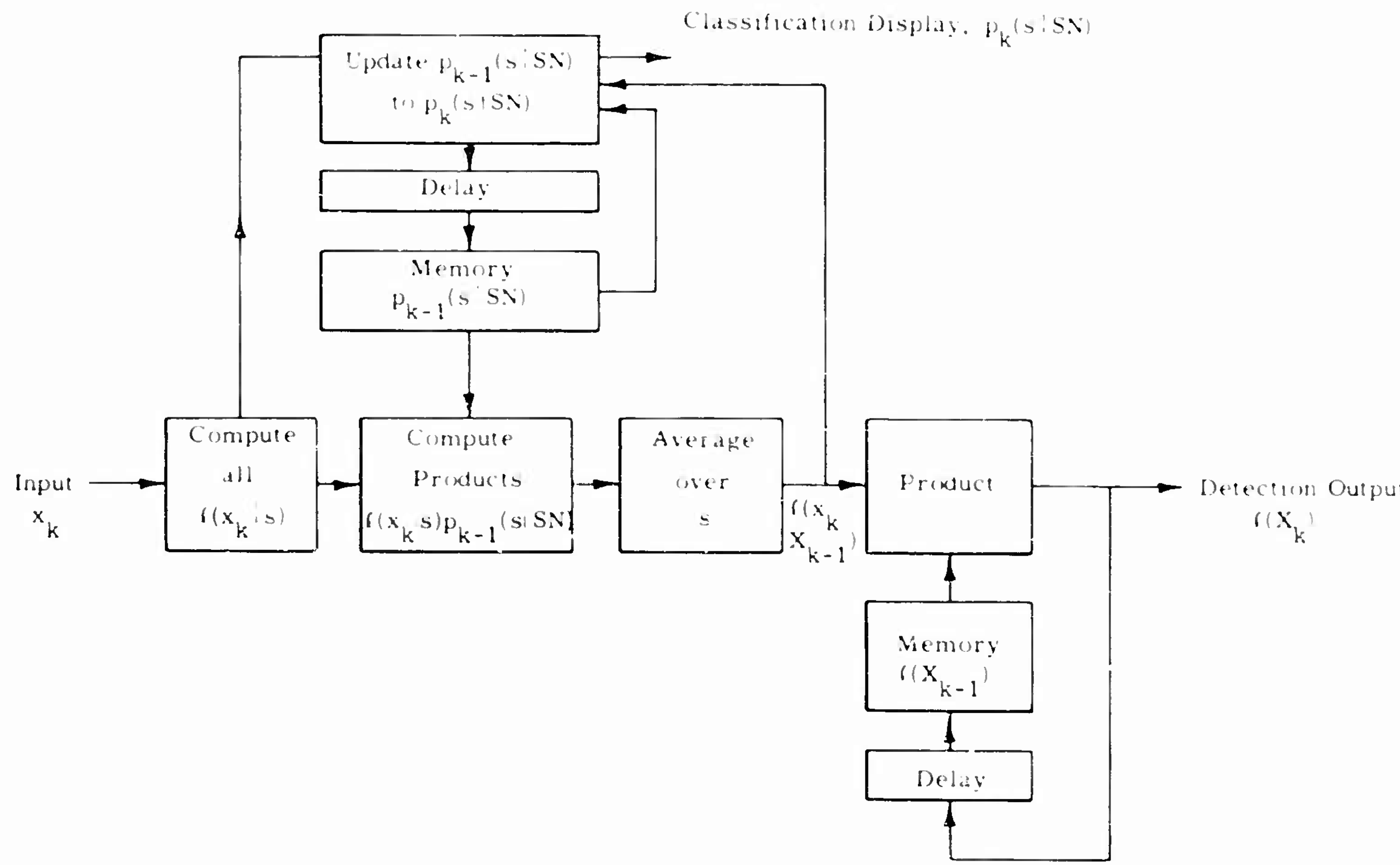

Fig. 3.4. An adaptive realization of an optimum receiver.

or (2) the signal samples in successive observation units were independent, in which case what is learned in one unit is irrelevant to observations in any other unit.

3.3 Classification of Signals in Noise

Frequently, more than just presence or absence of signal information is wanted from the receiver. In classical detection problems, information as to which signal was being transmitted was often suppressed. This resulted from the fact that the receiver was designed to answer the question of presence or absence of signal regardless of the particular signal transmitted. The realization of Fig. 3.4 displays the classification information which has always been inherent in the formation of the receiver detection output.

$p_o(s|SN)$ is the probability density function that represents our opinion, prior to any observation, as to which signal will be present. This is the classification output at time t_o. As has been shown in the previous section, updated versions of this density function, $p_k(s|SN)$, are obtained sequentially in time by the receiver and used to form the detection output, i.e., the likelihood ratio. Thus, the detection and classification outputs are obtained simultaneously and are intimately related.

CHAPTER IV

INDIRECT DESCRIPTION OF SIGNAL ENSEMBLE

In Chapter III it was shown that an optimum, full-memory, adaptive receiver design could be put within the framework of classical fixed-time theory. The basic form for an optimum, full-memory, adaptive receiver was obtained there for the case where the signal ensemble is described directly. If the design equations (Eqs. 3.8, 3.22 and 3.23) are applied directly to the case of recurrent waveforms, the resulting realizations still require a continually growing memory for storing updated probabilities, as we shall see in Chapter V.

In the next two chapters, it will be shown that optimum "adequate-memory" adaptive receiver designs will be obtained for detecting the recurrence phenomenon. An optimum "adequate-memory" receiver is one that has sufficient memory. In developing a theory for the design of an "adequate-memory" receiver, an indirect description of the signal ensemble proves useful.

4.1 Component Ensemble and Time Structure

The input voltages to the receiver, which are functions of time, are assumed to be defined for all times t in the observation interval, $0 \leq t \leq T$. They are assumed to be limited to a band of frequencies of width W. By the sampling theorem, each receiver input can be thought of as a point in a 2WT dimensional space, the coordinates of the point being the value of the function at the sample points $t = \frac{j}{2WT}$, for $1 \leq j \leq 2WT$. The notation X_k denotes a receiver input, $(x_1, x_2, \ldots, x_k)$, where $k = 2WT$ and x_j denotes the jth sample value, or coordinate.

To state the problem of detecting presence or absence of the recurrence phenomenon within the terminology of signal detection theory, it is necessary to clarify what is meant here by the word "signal." This word is often used loosely and sometimes means the noise-free emission from a transmitter, whereas at other times it refers to the noise-contaminated waveform at the receiver input. In this study a signal is the voltage waveform at the receiver input when noise is not present.

In Fig. 1.1 four "typical" segments of signals of the type of interest here were shown. A possible signal is shown in Fig. 4.1. There are intervals of no energy interrupted by occasional occurrences of the same waveform. This short waveform is called a signal component or simply a "component." A signal consists of a recurrence of the same component and the blank spaces in between.

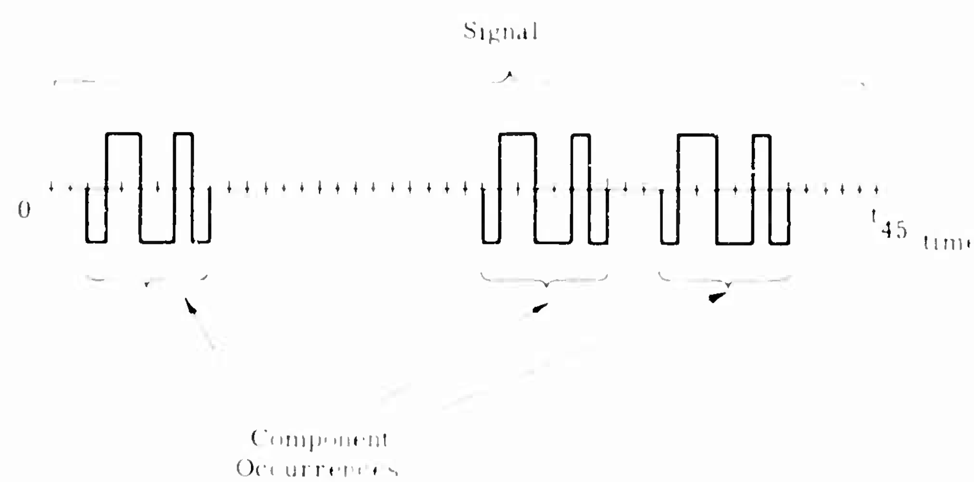

Fig. 4.1. A signal composed of components

The notation s denotes the signal, $(s_1, s_2, \ldots, s_k)$, as it would appear at the receiver input in the absence of noise where s_j denotes the jth sample value, or coordinate. C^i denotes a particular component, $(c_{i,1}, c_{i,2}, \ldots, c_{i,n_i})$, where $c_{i,j}$ denotes the jth sample of the ith component. Any value, including zero, can be assigned to these samples. By the sampling theorem, a component can be thought of as a point in a $2WT_i$ dimensional space, where T_i is the duration of the component. Since the duration, T_i, of a component can be different for each component, the number of component samples or coordinates in a component is denoted by n_i where n_i is equal to $2WT_i$ for the ith component. For example, if the component in Fig. 4.1 is labeled C^1, then $n_1 = 7$ and it is written as $C^1 = (-1, 1, 1, -1, -1, 1, -1)$ and the signal in the interval $(0, t_{45})$ is written as

$$s = (0, 0, 1, 1, 1, -1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 1,$$
$$-1, -1, 1, -1, 0, 0, 0, -1, 1, 1, -1, -1, 1, -1, 0, 0, 0, 0) \qquad (4.1)$$

In order to define the detection problem it is necessary to specify the initial signal uncertainty. This is most conveniently done by describing the signal indirectly in terms of components and their timing, thus distinguishing two types of uncertainty: uncertainty as to component character and recurrence-time uncertainty. Consider the block diagram shown in Fig. 4.2. One component out of a finite class of b components is chosen by the component generator to be characteristic of a transmission at its outset. Transmission of the characteristic component occurs only upon command by the trigger generator which introduces the recurrence-time uncertainty. Triggers may occur only at discrete times and may not occur within a component. Three basic distributions of recurrence-time intervals are considered; the Sporadic-Poisson, the Synchronous Poisson, and the Periodic. Signals of the three types are illustrated in Fig. 4.3 and defined below. The component is not restricted to a binary waveform but is shown as such for illustrative purposes. In particular, a component can have zero sample values, thus permitting signals composed of periodic pulses to be described.

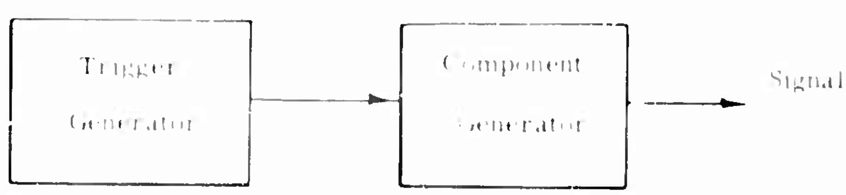

Fig. 4.2. Block diagram of signal generator.

The simplest temporal distribution of components within a signal occurs for a Periodic Time Structure. In this case a component is transmitted periodically with period T_1, where T_1 is the duration of a component. Such a signal is shown in Fig. 4.3 c. With the Sporadic-Poisson Time Structure, there is a probability of initiation of a component at each of the times $t_k = k\tau_1$ that is zero within a component and invariant at other times. This type of signal is shown in Fig. 4.3 a. For the Synchronous-Poisson Time Structure, there

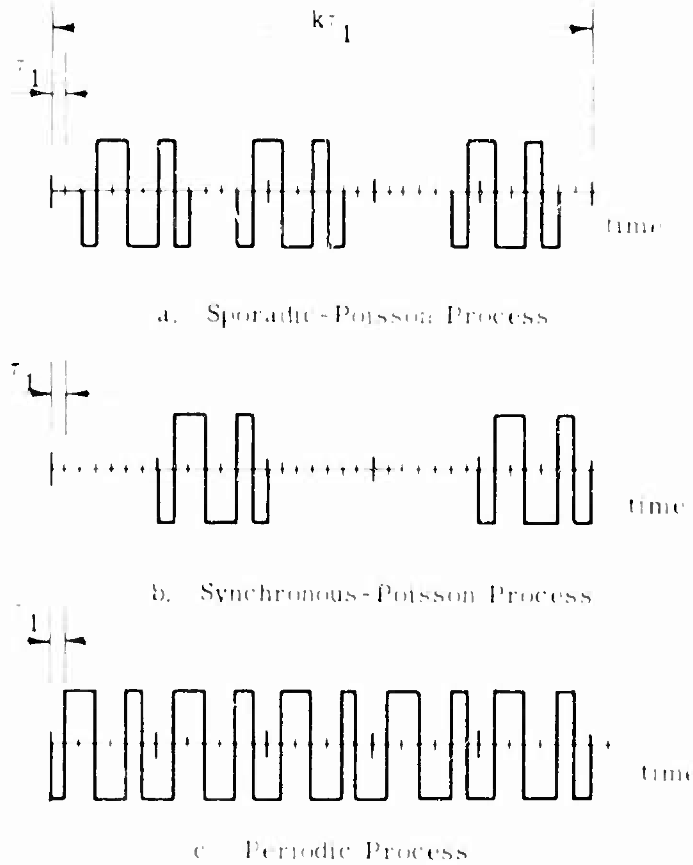

Fig. 4.3. "Typical" signals.

is a probability that a trigger will occur at times T_1 seconds apart. In the next three sections the three types of time processes will be mathematically defined.

4.2 Sporadic-Poisson Time Structure

The signal ensemble for this time structure can be indirectly described in terms of the components and time structure. The kth sample of the signal, s_k, can be defined in terms of the (k-1)st sample, s_{k-1}, and a set of transition probabilities. In other words, the signal can be defined by a one-step Markov process. In the sporadic case, a sample of the signal can be in any of the states $c_{i,1}, c_{i,2}, \ldots, c_{i,n_i}$ for $i = 1, 2, \ldots, b$, where $c_{i,j}$ corresponds to the jth position of the ith component. There are b components in the component ensemble and the ith component has n_i sample values. One other state is possible and that is where the ith component has been selected but is off. This is designated by $c_{i,0}$.

Only certain transitions from one state to another are possible. For example, if the component has seven samples and $s_{k-1} = c_{i,3}$, then $s_k = c_{i,4}$ with probability one. This is a result of the fact that once a component starts it must be completed and also no new component may start until a component is completed. The various possible transitions are visualized with the aid of Fig. 4.4. This figure shows the possible states of the (k-1)st and

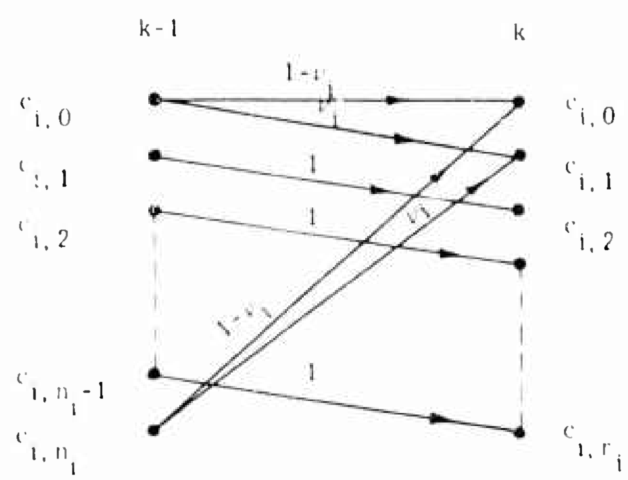

Fig. 4.4. Sporadic-Poisson process for the ith component.

kth signal samples. The arrows indicate the possible transitions. Also included on the diagram are the probabilities of the various transitions. More specifically, the properties of the Sporadic-Poisson process are defined as:

$$g(s_k \mid s_{k-1}, SN) = 1-r_i \text{ for} \begin{Bmatrix} s_k = c_{i,0}, & s_{k-1} = c_{i,0} \\ \\ s_k = c_{i,0} & s_{k-1} = c_{i,n_i} \end{Bmatrix} \tag{4.2}$$

$$g(s_k \mid s_{k-1}, SN) = r_i \quad \text{for} \begin{Bmatrix} s_k = c_{i,1}, & s_{k-1} = c_{i,0} \\ \\ s_k = c_{i,1}, & s_{k-1} = c_{i,n_i} \end{Bmatrix} \tag{4.3}$$

$$g(s_k \mid s_{k-1}, SN) = 1 \quad \text{for} \quad s_k = c_{i,j}, \quad s_{k-1} = c_{i,j-1}$$

$$\text{for} \quad j = 2, 3, \ldots, n_i \tag{4.4}$$

$$g(s_k \mid s_{k-1}, SN) = 0 \qquad \text{otherwise} \qquad (4.5)$$

The interpretation of Eq. 4.2 is that if the ith component is either off, $c_{i,0}$, or at its last component position, c_{i,n_i}, at time t_{k-1}, then it is off at time t_k with probability $1-\nu_i$. Similarly, from Eq. 4.3, if the ith component is either off or at its last component position at time t_{k-1}, then the ith component starts again at time t_k with probability ν_i. Equation 4.4 says that if the (j-1)st position of the ith component is present at time t_{k-1}, then the jth sample of the ith component is present with probability one at time t_k. Equation 4.5 says that no transitions other than the ones expressed by Eqs. 4.2 through 4.4 are possible.

4.3 Synchronous-Poisson Time Structure

The Synchronous-Poisson Time Structure is intermediate in recurrence time uncertainty between the periodic and sporadic processes. The component selected at the start of transmission is one of a finite number of b possible components. Due to the synchronous nature of the time structure, there is no detailed positional uncertainty of components as was true in the sporadic case. If a component is triggered, the time position of component samples is known exactly. This enables the component sample values to be combined into one state. $C_{i,0} = (c_{i,0}, c_{i,0}, \ldots, c_{i,0})$ is the state that results if the ith component is selected but the component is off. $C_{i,1} = (c_{i,1}, c_{i,2}, \ldots, c_{i,n_i})$ is the state that results if the ith component is selected but the component is on. The signal vector can be "blocked off" into n_i-dimensional segments, each segment being designated by S_k. For example, if the component in Fig. 4.3 b is labeled C^1, then this signal is

$$s = (S_1, S_2, S_3, S_4, S_5) = (C_{1,0}, C_{1,1}, C_{1,0}, C_{1,0}, C_{1,1}) \qquad (4.6)$$

The possible states and transition probabilities are shown in Fig. 4.5. The properties of the Synchronous-Poisson process are defined as

$$g(S_k \mid SN) = 1-\nu_i \qquad \text{for } S_k = C_{i,0} \qquad (4.7)$$

$$g(S_k \mid SN) = \nu_i \qquad \text{for } S_k = C_{i,1} \qquad (4.8)$$

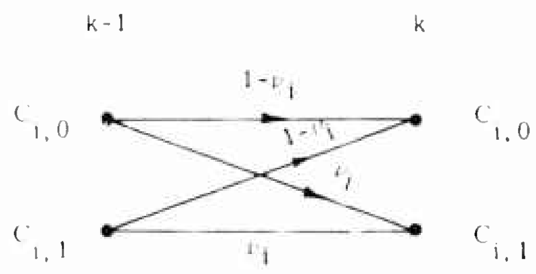

Fig. 4.5. Synchronous-Poisson process for the ith component.

The above two equations express the fact that p_i is the probability that a component will appear in the kth synchronous interval and $1-p_i$ is the probability that it will not appear in the kth interval independent of its presence or absence in any other synchronous interval.

4.4 Periodic Time Structure

The periodic process represents the least amount of time uncertainty of the three types considered. One of a finite number, b, of components is selected and recurs periodically. This recurrence process is completely deterministic. The possible states of s_k and the transition probabilities are shown in Fig. 4.6. The properties of the Periodic process are:

$$g(s_k \mid s_{k-1}, SN) = 1 \text{ for } s_k = c_{i,j},\ s_{k-1} = c_{i,j-1}; \quad j = 2, 3, \ldots, n_i \qquad (4.9)$$

$$g(s_k \mid s_{k-1}, SN) = 1 \text{ for } s_k = c_{i,1},\ s_{k-1} = c_{i,n_i} \qquad (4.10)$$

$$g(s_k \mid s_{k-1}, SN) = 0 \qquad \text{otherwise} \qquad (4.11)$$

Equation 4.9 expresses the fact that the jth component sample of the ith component occurs at time t_k if the (j-1)st component sample of the ith component is present at time t_{k-1}. The interpretation of Eq. 4.10 is that the first component sample occurs at time t_k if the last component sample occurred at time t_{k-1}. Equation 4.11 states that no transitions other than the ones defined by Eqs. 4.9 and 4.10 are possible.

In the next chapter the design of optimum adaptive receivers will be presented for each of these three time structures.

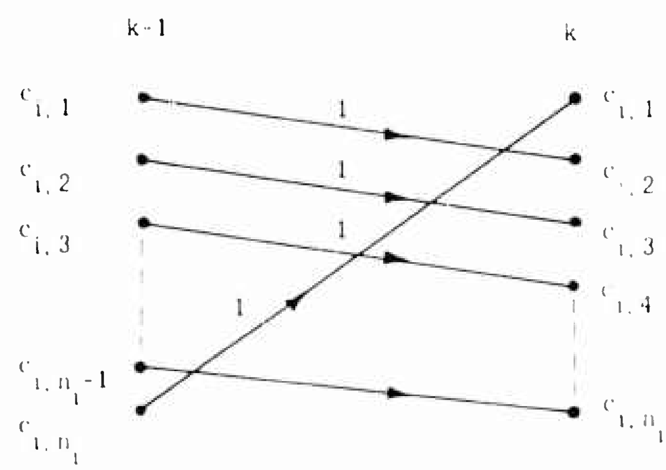

Fig. 4.6. Periodic process for ith component.

CHAPTER V

OPTIMUM ADAPTIVE RECEIVER DESIGN

In this chapter optimum adaptive receiver design is considered for a component that recurs with Sporadic-Poisson, Synchronous-Poisson, and Periodic Time Structures. A summary of the signal categories considered are shown in the chart of Fig. 5. 1.

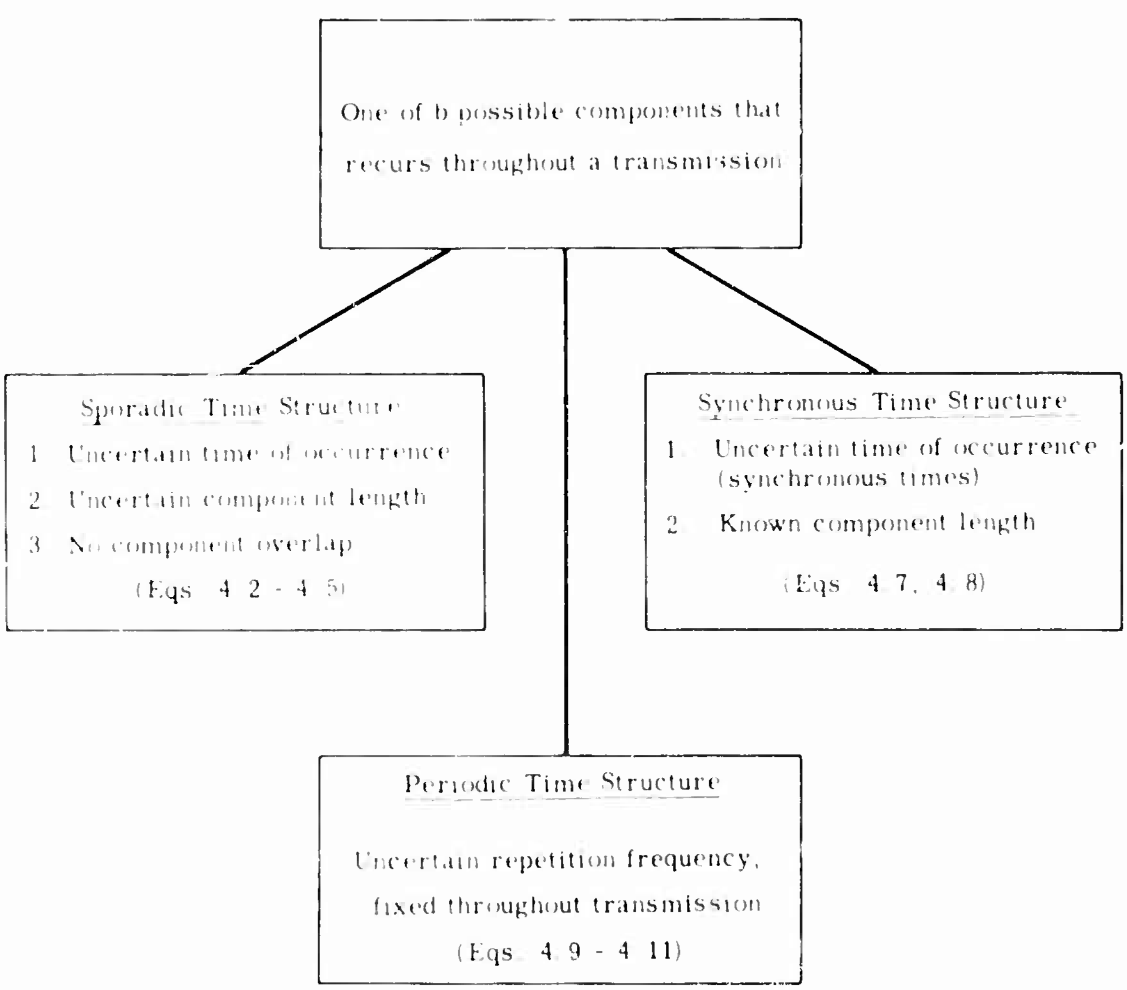

Fig. 5. 1. Summary chart: signal categories.

It is necessary to consider the adaptive realization for the Sporadic-Poisson and Synchronous-Poisson Time Structures because of practical memory requirements. This is a facet of optimum receiver design that did not exist in the periodic case. In the Periodic Time Structure, for components of duration T_1, there are the same number of signals in the signal ensemble as components in the component ensemble after observing for a time kT_1. For the Synchronous-Poisson Time Structure, however, there are 2^k times as many signals in the signal ensemble as there are components in the component ensemble after a time kT_1. In this latter case, the receiver designer is faced with an exponentially growing signal ensemble. If the receiver design is a nonsequential one, Eq. 2.3 is realized directly. This is the realization represented in Fig. 3.1 and it requires an exponentially growing memory for $p_o(s|SN)$. The nonsequential realization is therefore usually too complex to be practical.

The question arises as to whether an adaptive realization might provide a practical optimum receiver design. The optimum adaptive realization was discussed in Chapter III. The basic equations for the adaptive realizations are summarized in Table 3.1 and presented in the block diagram of Fig. 3.3. The form of the adaptive design equations in Table 3.1 is unsatisfactory since an updated probability, $p_k(s|SN)$, of each of the entire signal vectors, s, up to time t_k must still be stored, and this requires an exponentially growing memory.

In this chapter design equations are obtained of optimum receiver realizations which have a memory that remains fixed in size. The order of presentation of the receivers is from the one for the least certain time structure, the Sporadic-Poisson, to the most certain, the Periodic. Four realizations are presented for each of the three time structures. These realizations of the optimum receiver show how the detection output can be formed in many different ways. The derivation of Realization I is presented in this chapter in detail for each of the three time structures, as are the results of Realization IV. The remaining realizations are presented in Appendices A through C. Following the presentation of the receiver realizations for each time structure, the operation and use of the memory are discussed.

5.1 Optimum Adaptive Receiver Design, Sporadic-Poisson Time Structure

In this section an adaptive realization of the optimum receiver is presented for detecting signals with a Sporadic-Poisson Time Structure. One of b components is selected for transmission and <u>the same component recurs throughout a total observation</u>, X_k. The components need not have the same duration nor recur with the same duty factor. Due to the Sporadic-Poisson Time Structure, there is local positional uncertainty of a component occurrence. The probability of triggering a component at any of the times $k\tau_1$ conditional to selection of the ith component is r_i unless a component is in progress. The state diagram for the ith component has been shown in Fig. 4.4.

To review, the following basic steps are followed in the derivation of the optimum adaptive receiver realization:

1. Form the likelihood ratio, $\ell(X_k)$, of the total observation, X_k (Eq. 2.3).

2. Obtain equivalent sequential realization of the likelihood ratio, $\ell(X_k)$, in which the receiver updates information after each unit observation, x_k (Eqs. 3.8, 3.22 and 3.23).

3. Describe signal ensemble in terms of components and a time structure (Chapter IV).

In this section the properties of the Sporadic-Poisson Time Structure signal are used along with Eqs. 3.8, 3.22 and 3.23 to obtain the adaptive receivers.

5.1.1 Sporadic-Poisson Time Structure, Realization I. The derivation of this sequential realization begins with the specification of the likelihood ratio of the observation over the interval, $(0, t_k)$, which is known to be optimum. This likelihood ratio is

$$\ell(X_k) = \int_{\substack{\text{all} \\ s \in S}} \ell(X_k | s) p_o(s | SN) ds \tag{5.1}$$

In Chapter III, it was shown that this likelihood ratio could also be realized in a sequential fashion. The result was

$$\ell(X_k) = \ell(X_{k-1}) \ell(x_k | X_{k-1}) \tag{3.8}$$

where

$$f(x_k|X_{k-1}) = \int_{\substack{\text{all}\\ s\epsilon S}} f(x_k|s)p_{k-1}(s|SN)ds \tag{3.22}$$

and the information regarding which signal is present is updated by

$$p_k(s|SN) = \frac{f(x_k|s)p_{k-1}(s|SN)}{f(x_k|X_{k-1})} \tag{3.23}$$

For a finite number of possible signals, the average sequential likelihood ratio as given by Eq. 3.22 can be written as

$$f(x_k|X_{k-1}) = \sum_{s\epsilon S} f(x_k|s)p_{k-1}(s|SN) \tag{5.2}$$

where the integration has been replaced by summation. If the likelihood ratio of the kth sample of the observation depends only on the kth sample of the signal, then $f(x_k|s) = f(x_k|s_k)$. This is a condition which holds for signals in added noise. In this event, the average sequential likelihood ratio of Eq. 5.2 can be written as

$$f(x_k|X_{k-1}) = \sum_{s\epsilon S} f(x_k|s_k)p_{k-1}(s|SN) \tag{5.3}$$

This is still a summation over all the possible signal vectors that could occur during the observation X_k. One can rewrite $p_{k-1}(s|SN)$ so as to include the generator process. The vector s in sampled form is

$$p_{k-1}(s|SN) = p_{k-1}(s_1, s_2, \ldots, s_k|SN) \tag{5.4}$$

By definition of a joint probability, $p_{k-1}(s|SN)$ becomes

$$p_{k-1}(s|SN) = p_{k-1}(s_1, s_2, \ldots, s_{k-1}|SN)\, p_{k-1}(s_k|s_1, s_2, \ldots, s_{k-1}, SN) \tag{5.5}$$

Now, $p_{k-1}(s_k|s_1, s_2, \ldots, s_{k-1}, SN)$ is the probability, before taking the kth observation, of the kth sample of the transmitted signal under the condition that signal and noise are present and one has exact knowledge of the k-1 samples of the transmitted signal. This probability is not a function of the observation but only of the previous samples of the signal. However, the state of the kth sample of the signal depends only on the state of s_{k-1}. In other words,

$$p_{k-1}(s_k | s_1, s_2, \ldots, s_{k-1}, SN) = g(s_k | s_{k-1}, SN) \tag{5.6}$$

since the state of s_k is independent of the states of $s_1, s_2, \ldots, s_{k-2}$. Therefore Eq. 5.3 can be written as

$$f(x_k | X_{k-1}) = \sum_{s \in S} f(x_k | s_k)\, p_{k-1}(s_1, s_2, \ldots, s_{k-1} | SN)\, g(s_k | s_{k-1}, SN) \tag{5.7}$$

Now, the summation is over all the vectors, s, in the total space S of signals that could possibly occur. Expanding this summation to sum over one dimension at a time of the signal vector for each of the k possible samples gives

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} \sum_{s_1 = c_{i,0}}^{c_{i,n_i}} \sum_{s_2 = c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_k = c_{i,0}}^{c_{i,n_i}} f(x_k | s_k) g(s_k | s_{k-1}, SN) p_{k-1}(s_1, s_2, \ldots, s_{k-1} | SN) \tag{5.8}$$

Since the sums are finite, the order of summation may be interchanged in any desired fashion. Thus, Eq. 5.8 can be written as

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} \left[\sum_{s_k = c_{i,0}}^{c_{i,n_i}} \sum_{s_{k-1} = c_{i,0}}^{c_{i,n_i}} \sum_{s_1 = c_{i,0}}^{c_{i,n_i}} \sum_{s_2 = c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2} = c_{i,0}}^{c_{i,n_i}} f(x_k | s_k) g(s_k | s_{k-1}, SN) p_{k-1}(s_1, s_2, \ldots, s_{k-1} | SN) \right] \tag{5.9}$$

Now $f(x_k | s_k)$ depends only on the summation over s_k, and $g(s_k | s_{k-1}, SN)$ depends only on the summation over s_k and s_{k-1}. Factoring these terms out gives

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} \sum_{s_k = c_{i,0}}^{c_{i,n_i}} f(x_k | s_k) \sum_{s_{k-1} = c_{i,0}}^{c_{i,n_i}} g(s_k | s_{k-1}) \sum_{s_1 = c_{i,0}}^{c_{i,n_i}} \sum_{s_2 = c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2} = c_{i,0}}^{c_{i,n_i}} p_{k-1}(s_1, \ldots, s_{k-1} | SN)$$

The term in brackets is a joint probability of k-2 variables summed over the first k-2 variables. This, by definition of marginal probabilities, can be written as

$$\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2}=c_{i,0}}^{c_{i,n_i}} p_{k-1}(s_1, s_2, \ldots, s_{k-1}|SN) = p_{k-1}(s_{k-1}|SN) \tag{5.11}$$

This permits us to write Eq. 5.10 as

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)\, p_{k-1}(s_{k-1}|SN) \tag{5.12}$$

Many of the $g(s_k|s_{k-1}, SN)$ terms of Eq. 5.12 may be zero, depending upon the generator process. From Chapter IV the properties of the generator process for the Sporadic-Poisson process are:

$$g(s_k|s_{k-1}, SN) = 1-r_i \quad \text{for} \begin{Bmatrix} s_k = c_{i,0}, & s_{k-1} = c_{i,0} \\ \vdots & \vdots \\ s_k = c_{i,0}, & s_{k-1} = c_{i,n_i} \end{Bmatrix} \tag{4.2}$$

$$g(s_k|s_{k-1}, SN) = r_i \quad \text{for} \begin{Bmatrix} s_k = c_{i,1}, & s_{k-1} = c_{i,0} \\ \vdots & \vdots \\ s_k = c_{i,1}, & s_{k-1} = c_{i,n_i} \end{Bmatrix} \tag{4.3}$$

$$g(s_k|s_{k-1}, SN) = 1 \quad \text{for} \quad s_k = c_{i,j}, \quad s_{k-1} = c_{i,j-1}, \quad j = 2, 3, \ldots, n_i \tag{4.4}$$

$$g(s_k|s_{k-1}, SN) = 0 \qquad \text{otherwise} \tag{4.5}$$

Substituting these generator properties into Eq. 5.12 gives

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} \Bigg\{ (1-\nu_i) \Big[p_{k-1}(s_{k-1}=c_{i,0} | SN) + p_{k-1}(s_{k-1}=c_{i,n_i} | SN) \Big] f(x_k | s_k=c_{i,0})$$

$$+ \nu_i f(x_k | s_k=c_{i,1}) \Big[p_{k-1}(s_{k-1}=c_{i,0} | SN) + p_{k-1}(s_{k-1}=c_{i,n_i} | SN) \Big]$$

$$+ \sum_{j=2}^{n_i} f(x_k | s_k=c_{i,j}) p_{k-1}(s_{k-1}=c_{i,j-1} | SN) \Bigg\} \tag{5.13}$$

Now, $c_{i,0}$ has the value zero and for a known signal in added white Gaussian noise,

$$f(x_k | s_k) = e^{\frac{1}{N}\left[x_k s_k - \frac{s_k^2}{2}\right]} \tag{5.14}$$

where N is the noise power in the bandwidth W. Therefore

$$f(x_k | s_k = c_{i,0}) = f(x_k | s_k = 0) = 1 \tag{5.15}$$

For convenience, let us use the notation

$$b_{i,j}(k) = p_k(s_k = c_{i,j} | SN) \tag{5.16}$$

The interpretation of the probability, $b_{i,j}(k)$ is that it is the probability that the signal sample at time t_k is the jth sample of the ith component under the condition that signal and noise are present and that the previous k observations have been seen. Using the $b_{i,j}(k)$ notation along with the definition of $f(x_k | s_k = c_{i,0})$ for signals in added white Gaussian noise as given by Eq. 5.15, one obtains

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} \Bigg\{ (1-\nu_i)\Big[b_{i,0}(k-1) + b_{i,n_i}(k-1)\Big] + \nu_i f(x_k | s_k = c_{i,1}) \Big[b_{i,0}(k-1) + b_{i,n_i}(k-1)\Big]$$

$$+ \sum_{j=2}^{n_i} f(x_k | s_k = c_{i,j})\, b_{i,j-1}(k-1) \Bigg\} \tag{5.17}$$

Factoring out $b_{i,0}(k-1) + b_{i,n_i}(k-1)$ permits Eq. 5.17 to be written as

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \left\{ \left[b_{i,0}(k-1) + b_{i,n_i}(k-1) \right] \left[(1-r_i) + r_i f(x_k|s_k = c_{i,1}) \right] \right.$$

$$\left. + \sum_{j=2}^{n_i} f(x_k|s_k = c_{i,j})\, b_{i,j-1}(k-1) \right\} \qquad (5.18)$$

which is the expression for the average sequential likelihood ratio for the Sporadic-Poisson generator process.

In this realization the probabilities $b_{i,j}(k)$ must be updated as each unit observation, x_k, is taken. In Chapter III the general equation for updating information was shown to be

$$p_k(s|SN) = \frac{p_{k-1}(s|SN)\, f(x_k|s)}{f(x_k|X_{k-1})} \qquad (3.23)$$

We now need to put this in the form of the $b_{i,j}(k)$ probabilities as in Eq. 5.18.

Note that the denominator, $f(x_k|X_{k-1})$, is a normalizing factor given by Eq. 5.18. Also, as before, let $f(x_k|s) = f(x_k|s_k)$. As remarked earlier, this assumption holds for problems of signals in added noise. Expanding the signal vector, s, in terms of its samples permits Eq. 3.23 to be written as

$$p_k(s_1, s_2, \ldots, s_k|SN) = \frac{p_{k-1}(s_1, s_2, \ldots, s_k|SN)\, f(x_k|s_k)}{f(x_k|X_{k-1})} \qquad (5.19)$$

Let us now sum both sides of Eq. 5.19 over the first k-1 samples of the ith component.

$$\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} p_k(s_1, s_2, \ldots, s_k|SN) =$$

$$\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} \frac{p_{k-1}(s_1, s_2, \ldots, s_k|SN)\, f(x_k|s_k)}{f(x_k|X_{k-1})} \qquad (5.20)$$

The sequential average likelihood ratio, $\ell(x_k|X_{k-1})$, is independent of the summation over the k-1 sample values of the signal so that the denominator may be removed from the summations. $\ell(x_k|s_k)$ is also independent of the summations and can be factored out. This means that Eq. 5.20 can be written

$$\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} p_k(s_1, s_2, \ldots, s_k|SN) =$$

$$\frac{\ell(x_k|s_k)}{\ell(x_k|X_{k-1})} \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} p_{k-1}(s_1, s_2, \ldots, s_k|SN) \tag{5.21}$$

The left hand side of Eq. 5.21 is by definition of marginal probabilities

$$\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} p_k(s_1, s_2, \ldots, s_k|SN) = p_k(s_k|SN) \tag{5.22}$$

Therefore Eq. 5.21 becomes

$$p_k(s_k|SN) = \frac{\ell(x_k|s_k) \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} p_{k-1}(s_1, s_2, \ldots, s_k|SN)}{\ell(x_k|X_{k-1})} \tag{5.23}$$

As before, by definition of a joint probability, we can write

$$p_{k-1}(s_1, s_2, \ldots, s_k|SN) = p_{k-1}(s_1, s_2, \ldots, s_{k-1}|SN)\, p_{k-1}(s_k|s_1, s_2, \ldots, s_{k-1}, SN) \tag{5.24}$$

Now considering generator processes, which can be expressed as a function $g(s_k|s_{k-1}, SN)$, we can write Eq. 5.24 as

$$p_{k-1}(s_1, s_2, \ldots, s_k|SN) = p_{k-1}(s_1, s_2, \ldots, s_{k-1}|SN)\, g(s_k|s_{k-1}, SN) \tag{5.25}$$

Inserting this expression into Eq. 5.23 results in

$$p_k(s_k | SN) = \frac{f(x_k | s_k) \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} p_{k-1}(s_1, s_2, \ldots, s_{k-1} | SN) g(s_k | s_{k-1}, SN)}{f(x_k | X_{k-1})} \tag{5.26}$$

Since this equation involves finite summations, the order of summation can be interchanged. Summing with respect to s_{k-1} first and factoring out $g(s_k | s_{k-1}, SN)$ from the summation over the first k-2 samples of the signal, since it is independent of that summation, results in

$$p_k(s_k | SN) = \frac{f(x_k | s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k | s_{k-1}, SN) \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2}=c_{i,0}}^{c_{i,n_i}} p_{k-1}(s_1, s_2, \ldots, s_{k-1} | SN)}{f(x_k | X_{k-1})} \tag{5.27}$$

Once more by definition of marginal probabilities, the summation in the numerator over the first k-2 samples can be written as $p_{k-1}(s_{k-1} | SN)$. Therefore Eq. 5.27 can be written as

$$p_k(s_k | SN) = \frac{f(x_k | s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k | s_{k-1}, SN) p_{k-1}(s_{k-1} | SN)}{f(x_k | X_{k-1})} \tag{5.28}$$

Insertion of the generator process, $g(s_k | s_{k-1}, SN)$, for the Sporadic-Poisson process results in a reduction of terms under the summation since only certain transitions are permitted. These properties were previously defined by Eqs. 4.2 through 4.5. Equation 5.28 then takes on three basic forms:

$$p_k(s_k = c_{i,0} | SN) = \frac{(1-\gamma_i)\left[p_{k-1}(s_{k-1} = c_{i,0} | SN) + p_{k-1}(s_{k-1} = c_{i,n_i} | SN)\right]}{f(x_k | X_{k-1})} \tag{5.29}$$

$$p_k(s_k = c_{i,1} \mid SN) = \frac{\nu_i \left[p_{k-1}(s_{k-1} = c_{i,0} \mid SN) + p_{k-1}(s_{k-1} = c_{i,n_i} \mid SN) \right] f(x_k \mid s_k = c_{i,1})}{f(x_k \mid X_{k-1})} \tag{5.30}$$

$$p_k(s_k = c_{i,j} \mid SN) = \frac{p_{k-1}(s_k = c_{i,j-1} \mid SN)\, f(x_k \mid s_k = c_{i,j})}{f(x_k \mid X_{k-1})} \tag{5.31}$$

for $j = 2, 3, \ldots, n_i$

If we use the $b_{i,j}(k)$ notation as defined by Eq. 5.16, Eqs. 5.29, 5.30 and 5.31 become

$$b_{i,0}(k) = \frac{(1-\nu_i)\left[b_{i,0}(k-1) + b_{i,n_i}(k-1) \right]}{f(x_k \mid X_{k-1})} \tag{5.32}$$

$$b_{i,1}(k) = \frac{\nu_i \left[b_{i,0}(k-1) + b_{i,n_i}(k-1) \right] f(x_k \mid s_k = c_{i,1})}{f(x_k \mid X_{k-1})} \tag{5.33}$$

$$b_{i,j}(k) = \frac{b_{i,j-1}(k-1)\, f(x_k \mid s_k = c_{i,j})}{f(x_k \mid X_{k-1})} \tag{5.34}$$

for $j = 2, 3, \ldots, n_i$

5.1.2 Operation of the Adaptive Receiver. The basic equations of Realization I for the Sporadic-Poisson Time Structure are summarized in Table 5.1. These equations can be interpreted by considering a simplified, illustrative example. Suppose there are two possible components, C^1 and C^2, each with three possible sample values; i.e., $b = 2$ and $n_1 = n_2 = 3$. Table 5.2 summarizes the receiver design equations for the example.

TABLE 5.1

BASIC RECEIVER DESIGN EQUATIONS, SPORADIC-POISSON TIME STRUCTURE REALIZATION I

Optimum Detection Output

$$f(X_k) = f(X_{k-1})\, f(x_k | X_{k-1}) \tag{3.8}$$

Sequential Average Likelihood Ratio

$$f(x_k | X_{k-1}) = \sum_{i=c}^{b} \Bigg\{ \Big[b_{i,0}(k-1) + b_{i,n_i}(k-1) \Big] \Big[1 - \epsilon_i + \epsilon_i f(x_k | s_k, c_{i,1}) \Big]$$

$$+ \sum_{j=2}^{n_i} b_{i,j-1}(k-1) f(x_k | s_k, c_{i,j}) \Bigg\} \tag{5.18}$$

Classification - Component Identification and Position

$$b_{i,0}(k) = \frac{(1-\epsilon_i) \Big[b_{i,0}(k-1) + b_{i,n_i}(k-1) \Big]}{f(x_k | X_{k-1})} \tag{5.32}$$

$$b_{i,1}(k) = \frac{\epsilon_i \Big[b_{i,0}(k-1) + b_{i,n_i}(k-1) \Big] f(x_k | s_k, c_{i,1})}{f(x_k | X_{k-1})} \tag{5.33}$$

$$b_{i,j}(k) = \frac{b_{i,j-1}(k-1) f(x_k | s_k, c_{i,j})}{f(x_k | X_{k-1})} \tag{5.34}$$

for $j = 2, 3, \ldots, n_i$

Classification - Component Identification

$$p_k(C^i | SN) = \sum_{j=0}^{n_i} b_{i,j}(k) \tag{5.35}$$

TABLE 5.2

ILLUSTRATIVE EXAMPLE OF THE BASIC EQUATIONS
SPORADIC-POISSON TIME STRUCTURE
REALIZATION I

Optimum Detection Output

$$f(X_k) = f(X_{k-1})\, f(x_k \mid X_{k-1}) \tag{3.8}$$

Sequential Average Likelihood Ratio

$$\begin{aligned} f(x_k \mid X_{k-1}) = {} & \left[b_{1,0}(k-1) + b_{1,3}(k-1)\right](1-p_1) - \left[b_{1,0}(k-1) + b_{1,3}(k-1)\right] p_1 f(x_k \mid s_k = c_{1,1}) \\ & + b_{1,1}(k-1) f(x_k \mid s_k = c_{1,2}) + b_{1,2}(k-1) f(x_k \mid s_k = c_{1,3}) \\ & + \left[b_{2,0}(k-1) + b_{2,3}(k-1)\right](1-p_2) + \left[b_{2,0}(k-1) + b_{2,3}(k-1)\right] p_2 f(x_k \mid s_k = c_{2,1}) \\ & + b_{2,1}(k-1) f(x_k \mid s_k = c_{2,2}) + b_{2,2}(k-1) f(x_k \mid s_k = c_{2,3}) \end{aligned} \tag{5.36}$$

Classification - Component Identification and Position

$$b_{1,0}(k) = \frac{\left[b_{1,0}(k-1) + b_{1,3}(k-1)\right](1-p_1)}{f(x_k \mid X_{k-1})} \tag{5.37}$$

$$b_{1,1}(k) = \frac{\left[b_{1,0}(k-1) + b_{1,3}(k-1)\right] p_1 f(x_k \mid s_k = c_{1,1})}{f(x_k \mid X_{k-1})} \tag{5.38}$$

$$b_{1,2}(k) = \frac{b_{1,1}(k-1) f(x_k \mid s_k = c_{1,2})}{f(x_k \mid X_{k-1})} \tag{5.39}$$

$$b_{1,3}(k) = \frac{b_{1,2}(k-1) f(x_k \mid s_k = c_{1,3})}{f(x_k \mid X_{k-1})} \tag{5.40}$$

$$b_{2,0}(k) = \frac{\left[b_{2,0}(k-1) + b_{2,3}(k-1)\right](1-r_2)}{f(x_k | X_{k-1})} \tag{5.41}$$

$$b_{2,1}(k) = \frac{\left[b_{2,0}(k-1) + b_{2,3}(k-1)\right] r_2 f(x_k | s_k = c_{2,1})}{f(x_k | X_{k-1})} \tag{5.42}$$

$$b_{2,2}(k) = \frac{b_{2,1}(k-1) f(x_k | s_k = c_{2,2})}{f(x_k | X_{k-1})} \tag{5.43}$$

$$b_{2,3}(k) = \frac{b_{2,2}(k-1) f(x_k | s_k = c_{2,3})}{f(x_k | X_{k-1})} \tag{5.44}$$

Classification - Component Identification

$$p_k(C^1 | SN) = b_{1,0}(k) + b_{1,1}(k) + b_{1,2}(k) + b_{1,3}(k) \tag{5.45}$$

$$p_k(C^2 | SN) = b_{2,0}(k) + b_{2,1}(k) + b_{2,2}(k) + b_{2,3}(k) \tag{5.46}$$

Each term in the sum of Eq. 5.36 and each numerator of Eqs. 5.37 through 5.44 is the product of three basic factors. One factor is a probability or combination of probabilities of the $b_{i,j}(k-1)$ type. These probabilities contain all past information relevant to the optimum detection. The second factor is a probability associated with the generator process which is r_1, r_2, $1-r_1$, $1-r_2$, or 1. The third factor is a likelihood ratio term, $f(x_k | s_k = c_{i,j})$. This is the factor which extracts the proper new information from the unit observation, x_k. In addition, a normalizing factor, $f(x_k | X_{k-1})$, appears in the denominator of Eqs. 5.37 through 5.44. In Fig. 5.2, the sequential quantities that are calculated are represented on a time axis. The $b_{i,j}(k-1)$ terms relate to the time just prior to the kth observation. The probability associated with the generator process is combined with the $b_{i,j}(k)$ terms to obtain an a priori probability about what will occur during the observation x_k. These are combined with a $f(x_k | s_k = c_{i,j})$ term to get $f(X_k)$ and a new set of $b_{i,j}(k)$ terms.

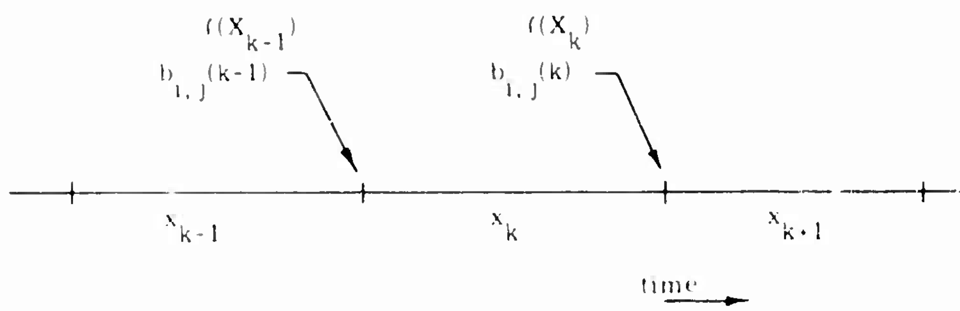

Fig. 5.2. Sequential quantities, Sporadic-Poisson receiver, Realization I.

In Fig. 5.3 a state diagram is shown for our illustrative example. Using this figure as a reminder of the various possible component positions let us interpret each term in the sum of Eq. 5.36. The term $b_{1,0}(k-1) \cdot b_{1,3}(k-1)$ is the probability after k-1 observations

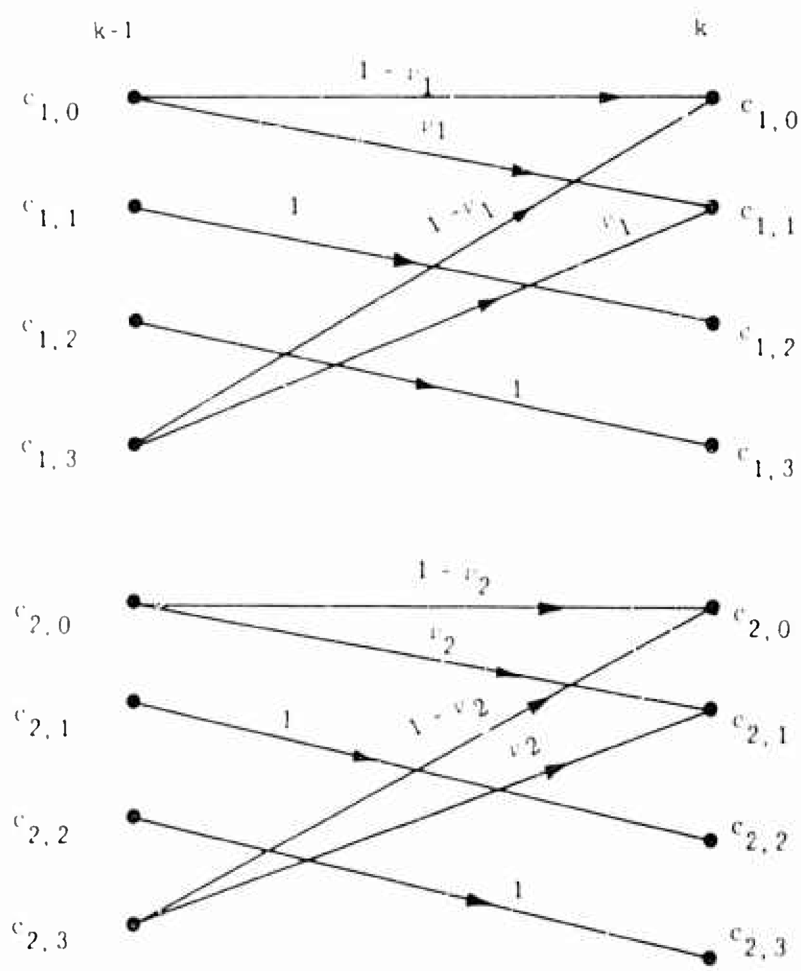

Fig. 5.3. Sporadic-Poisson process, illustrative example.

that component C^1 has been selected but is either off, $c_{1,0}$, or at its last component sample, $c_{1,3}$, at time t_{k-1}. This is multiplied by the probability, $1-c_1$, that the component will not start at time t_k under the condition component C^1 has been selected. This is then multiplied by the likelihood ratio of the kth observation if the component C^1 had been selected but is off at time t_k. This likelihood ratio has the value one.

The interpretation of the second term in the sum of Eq. 5.36 is that $\left[b_{1,0}(k-1) + b_{1,3}(k-1)\right]c_1$ is the probability after k-1 observations that component C^1 has been selected and will start at time t_k. This probability is multiplied by the likelihood ratio of the kth observation if the first sample of the component C^1, which is $c_{1,1}$, is present at time t_k.

In the third term of Eq. 5.36, $b_{1,1}(k-1)$ is the probability after k-1 observations that the first sample of component C^1, denoted by $c_{1,1}$, was present at time t_{k-1}. Also hidden is a factor of one which is the probability that the second sample of component C^1, $c_{1,2}$, occurs at time t_k if the first sample of component C^1, $c_{1,1}$, occurred at time t_{k-1}. This is then multiplied by the likelihood ratio $\ell(x_k|s_k = c_{1,2})$, of the observation, x_k. This is the likelihood ratio of s_k had the second sample, $c_{1,2}$, of component C^1 occurred at time t_k.

The interpretation of the fourth term of Eq. 5.36 is analagous to that of the third term. The fifth through eighth terms refer to component C^2, and their interpretation is analagous to that of the first four terms. The reader will notice that terms of Eq. 5.36 appear individually in the numerators of Eqs. 5.37 through 5.44. The denominator, $\ell(x_k|X_{k-1})$, is a normalizing factor. All eight probabilities $b_{i,j}(k)$ could be displayed as a classification output, but it is more likely that the only information wanted is which of the two components is presented. The component identification output, which can be displayed is given by Eqs. 5.45 and 5.46 in Table 5.2.

In Fig. 5.4 a block diagram of Realization I is shown for the general case. This receiver operates sequentially in time, extracting and updating information after each sample of the observation, x_k. Two outputs are provided sequentially in time. One is the logarithm of the likelihood ratio from time zero to time t_k, which is $\ln \ell(X_k)$, the detection output. This output is used to decide presence or absence of a recurrence phenomenon in the interval $(0, t_k)$. The other output is the classification output, $p_k(C^i|SN)$. This output provides information, in the form of updated probabilities, as to which component has been recurrent from time zero to time t_k.

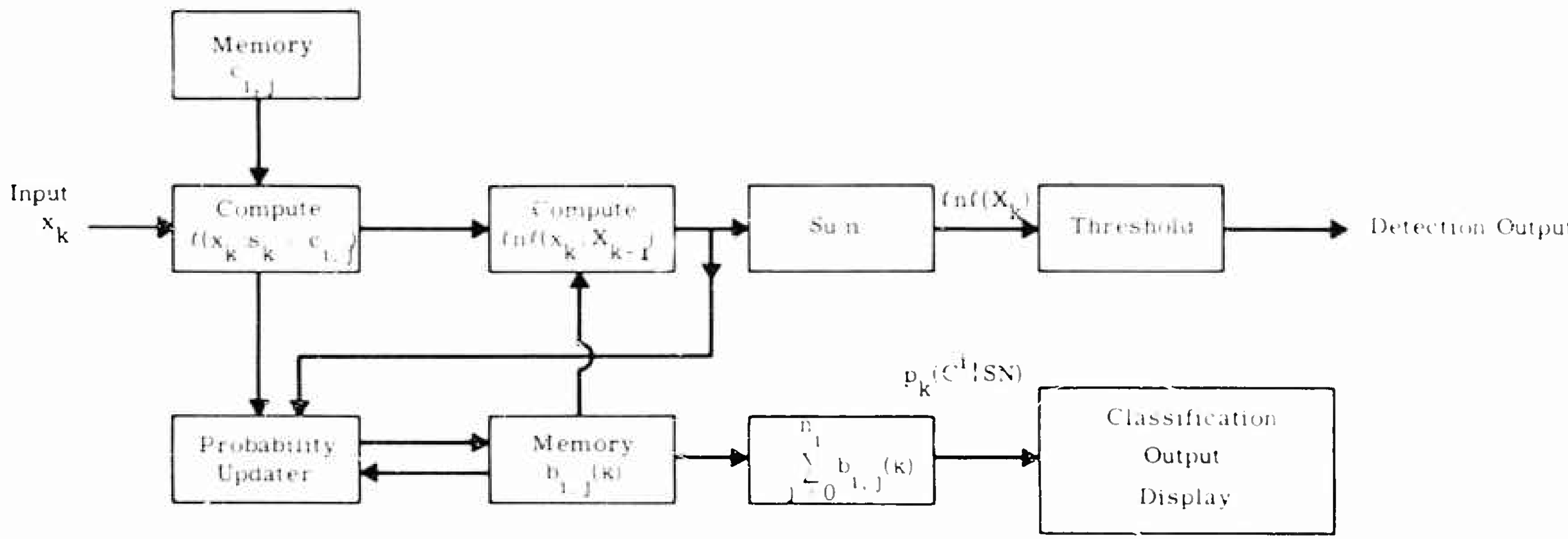

Fig. 5.4. Adaptive receiver realization, Sporadic-Poisson Time Structure, Realization I.

The summer stores a number which represents the logarithm of the likelihood ratio of the observation X_{k-1}. When the next observation, x_k, is made, the possible component waveform samples, $c_{i,j}$, stored in fixed memory are used to form the likelihood ratio of the kth observation for each of the possible components and component positions. These individual likelihood ratios are weighted by the updated probabilities of the various components and component positions which are stored in the temporary memory and which are the result of combining initial knowledge and information from the observation, X_{k-1}. This weighting is performed in the box labeled $\ell n(x_k | X_{k-1})$. The output of this box is added to $\ell n \ell(X_{k-1})$, which is already in the summer to form $\ell n \ell(X_k)$, the detection output over $(0, t_k)$. This output is compared with a threshold to provide a yes-no decision.

Simultaneously, information from the observation, x_k, as provided at the output of the average likelihood ratio box is combined with classification information from the observations, X_{k-1}, in the probability updater. The probability updater performs the operations specified by Eqs. 5.32 through 5.34. The updated probabilities, $b_{i,j}(k)$, replace $b_{i,j}(k-1)$ in the temporary memory, and the receiver is ready to accept the k+1 observation. A classification output could be taken directly from the temporary memory. It is more likely that a display of the updated probabilities, $p_k(C^i | SN)$, Eq. 5.35 in Table 5.1, is wanted and this can be obtained by summing $b_{i,j}(k)$ over all j.

5.1.3 Other Receiver Realizations and the Use of Memory. In Section 5.1.1, a realization of the optimum receiver for a sporadic-recurrent component is presented. Although the receiver realization discussed here was obtained by formal manipulations of a likelihood ratio equation, its nature is intuitively satisfying. It uses each observation, x_k, to "learn" as much as possible which component is present. This information is stored in the form of the $b_{i,j}(k)$ matrix in the temporary memory. This knowledge is kept current by combining knowledge of the generator process, the information contained in all previous observations, and information obtained in the kth observation. As time progresses, this receiver "adapts" to the particular component waveshape that is recurrent, and it "adapts" locally to position within a component.

In this realization $b_{i,j}(k-1)$ must be kept up to date as each observation is taken. These probabilities express up-to-date knowledge on which component is present as well as component positional information. Although the receiver takes into account all the possible time patterns of components, its memory need not store all of these patterns. Realization I has a temporary memory which is continually updated and which has a finite size of $\sum_{i=1}^{b} n_i$ words. This finite memory is a primary practical feature of this realization.

Realization I is not unique. In Appendix A, three other realizations of the optimum receiver are presented. Realization II (see Appendix A.1) is similar to Realization I except for the fact that information about component identification and component position are updated separately. Therefore, Realization II requires a finite-size temporary memory of $\sum_{i=1}^{b} n_i$ words for component positional information and b words for component identifcation information.

Another receiver realization is Realization III (see Appendix A.2). This is a receiver which has a channel for each of the b possible components. Each channel calculates the likelihood ratio of the observation conditional to presence of the ith component and the channel outputs are then weighted by the a priori probabilities, $p_0(C^i|SN)$, of the selection of each of the components, and then summed. This receiver looks "less adaptive" since $p_0(C^i|SN)$ is not explicitly updated at each step in time. $\sum_{i=1}^{b} n_i$ words of temporary memory are needed to store component identification information and positional information and b words to store $\ell(X_k|C^i)$ terms.

An important practical realization is Realization IV (see Appendix A.3). It is a b-channel receiver that appears to require the least number of computations of the four realization presented. The basic design equations are summarized in Table 5.3 and a block diagram is shown in Fig. 5.5. By comparing Table 5.3 with Table 5.1 one can see the simplification in computations of Realization IV. In this realization a quantity $Q_{i,j}(k) = \ell(X_k)b_{i,j}(k)$, instead of $b_{i,j}(k)$, is stored in temporary memory for each possible component and component position. A finite-size memory of $\sum_{i=1}^{b} n_i$ words is needed to store the $Q_{i,j}(k)$ terms. The updating equations for the $Q_{i,j}(k)$ terms are, however, simpler than those required for the $b_{i,j}(k)$ terms in Realization I. Moreover, in Realization IV, the likelihood ratio is calculated by simple addition of the $Q_{i,j}(k)$ terms and the classification output is obtained almost as simply.

TABLE 5.3

BASIC RECEIVER DESIGN EQUATIONS, SPORADIC-POISSON TIME STRUCTURE
REALIZATION IV

Optimum Detection Output

$$f(X_k) = \sum_{i=1}^{b} \sum_{j=0}^{n_i} Q_{i,j}(k) \tag{A.38}$$

Information Updating

$$Q_{i,0}(k) = \left[Q_{i,0}(k-1) + Q_{i,n_i}(k-1)\right](1-\epsilon_i) \tag{A.35}$$

$$Q_{i,1}(k) = \left[Q_{i,0}(k-1) + Q_{i,n_i}(k-1)\right]\epsilon_i f(x_k | s_k = c_{i,1}) \tag{A.36}$$

$$Q_{i,j}(k) = Q_{i,j-1}(k-1) f(x_k | s_k = c_{i,j}) \tag{A.37}$$

for $j = 2, 3, \ldots, n_i$

Classification - Component Identification and Position

$$b_{i,j}(k) = \frac{Q_{i,j}(k)}{f(X_k)} \tag{A.45}$$

Classification - Component Identification

$$p_k(C^i | SN) = \sum_{j=0}^{n_i} b_{i,j}(k) = \frac{\sum_{j=0}^{n_i} Q_{i,j}(k)}{f(X_k)} \tag{A.46}$$

Figures 5.6 and 5.7 show a more detailed block diagram of Realization IV for signals in added white Gaussian noise. Figure 5.6 shows one channel of the first portion of the receiver which computes $f(X_k)p_k(C^i|SN)$. Each channels only "looks" for the ith component, taking into account all possible time patterns of that component. Figure 5.7 shows how each

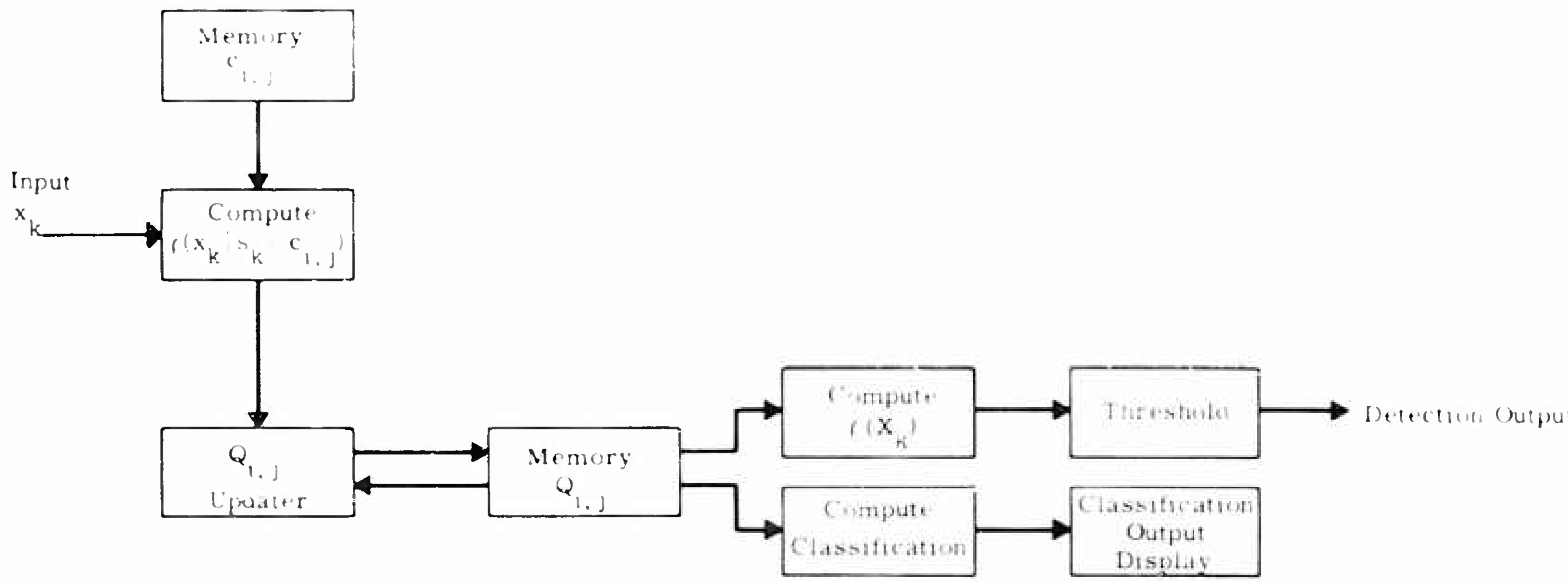

Fig. 5.5. Adaptive receiver realization, Sporadic-Poisson Time Structure, Realization IV.

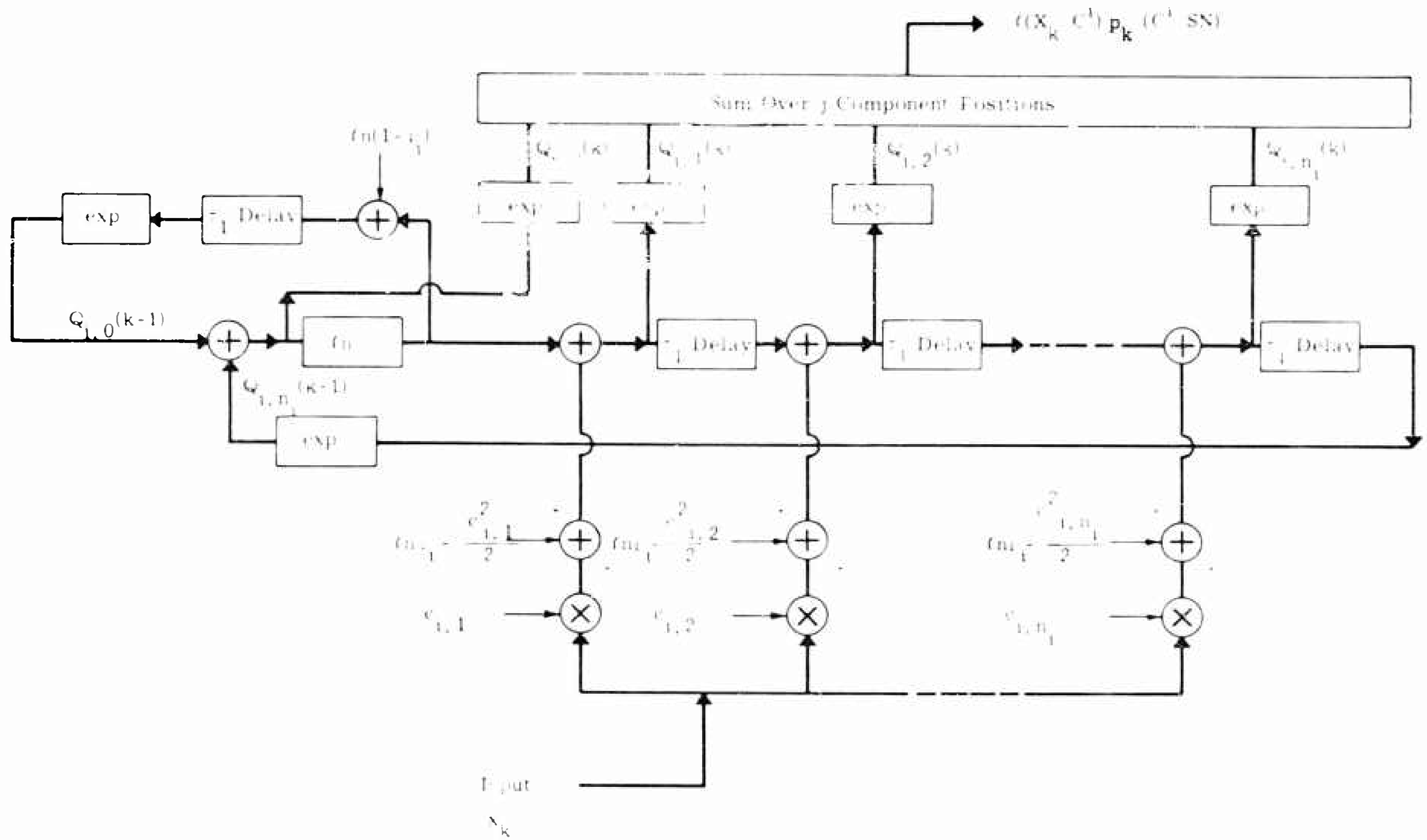

Fig. 5.6. Adaptive receiver realization, Sporadic-Poisson Time Structure, Realization IVa, front end.

of these channel outputs are combined to form the detection and classification outputs. This realization uses the logarithm of Eqs. A.35 through A.37. New information obtained in the unit observation, x_k, is combined with information obtained from previous observations in the series of adders shown to the left of the τ_1 delays where τ_1 is the smallest possible time shift on a component. The unit observation, x_k, is first processed to determine how likely it arose from the various possible positions of the ith component. This processing consists of correlating x_k with each of the possible positions that a component could be in and adding bias terms, $(m_i - c_{i,j})^2/2$. These outputs are then applied to a series of adders, each separated by a τ_1 delay. The outputs of these adders are the logarithms of the $Q_{i,j}(k)$ terms which contain all the information about the likelihood of the jth position of the ith component being present during the kth observation. These terms are summed over j giving the likelihood that the ith component has been recurrent. As one can see from Eq. A.37, the $Q_{i,j}(k)$ term is obtained at time t_k, and it is calculated from a similar quantity, $Q_{i,j}(k-1)$, at time t_{k-1}. The τ_1 delays provide the memory delay for this computation. The "loop" on the far left in Fig. 5.6 calculates $Q_{i,0}(k) + Q_{i,n_i}(k)$ along with its logarithm which is used to make the computations specified by Eqs. A.35 and A.36.

The output, $\ell(X_k|C^i)p_k(C^i|SN)$, becomes one of the inputs to the remainder of the receiver shown in Fig. 5.7. The detection output, $\ell n \ell(X_k)$, is obtained by summing the terms, $\ell(X_k|C^i)p_k(C^i|SN)$ of the b channels. The classification outputs are obtained by taking logarithms of $\ell(X_k|C^i)p_k(C^i|SN)$ for each channel and subtracting the logarithm of the detection output, $\ell n \ell(X_k)$.

Figure 5.8 shows another version of one of the input channels, which could be used in place of the realization shown in Fig. 5.6. It is quite similar except it implements Eqs. A.35 through A.37 directly, rather than the logarithm of these equations. As a result some of the adders must be replaced by multipliers.

The important feature in common to all four realizations is the fact that the size of the temporary memory remains fixed and "slides" in time. This is of practical importance not only for receiver design but also for receiver evaluation. A nonsequential realization would have required a growing memory. Such a realization is impractical to build. A receiver must be designed before it can be evaluated, the sequential or adaptive realizations provide simpler expressions to work with.

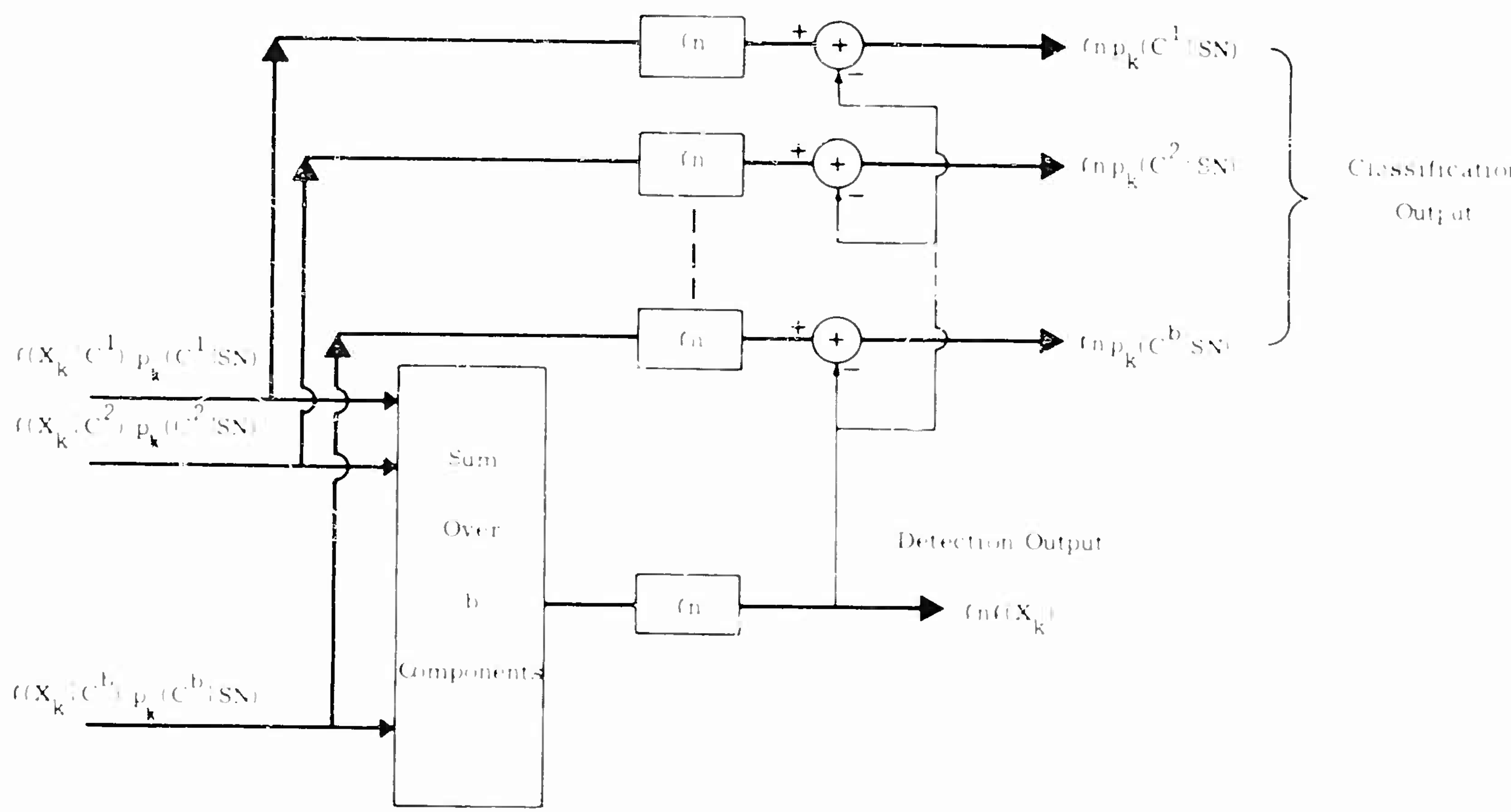

Fig. 5.7. Block diagram of formation of detection and classification outputs, Sporadic-Poisson Time Structure, Realization IV.

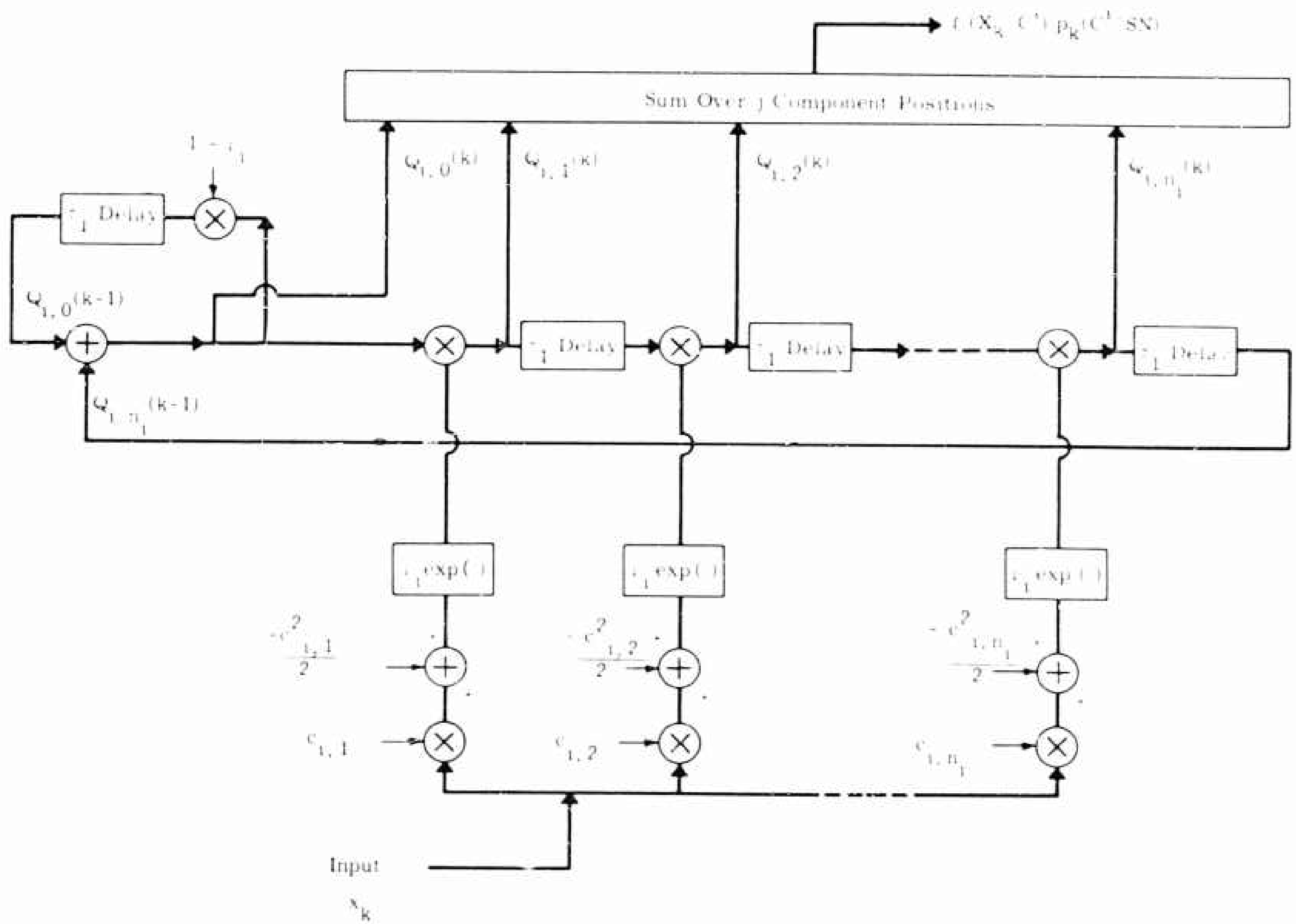

Fig. 5.8. Adaptive receiver realization, Sporadic-Poisson Time Structure, Realization IVb, front end.

5.2 Optimum Adaptive Receiver Design, Synchronous-Poisson Time Structure

In this section an adaptive realization of the optimum receiver is presented for detecting signals with a Synchronous-Poisson Time Structure. This time structure provides interesting cases in that the amount of time uncertainty is between the periodic and sporadic time processes. The uncertainty is in the exact component waveform transmitted and in the component recurrence times associated with the Synchronous-Poisson Time Structure. One of the b components is selected for transmission and the same component recurs throughout a total observation, X_k. Primarily for convenience and simplicity, it is assumed that all components in the finite ensemble are of common duration and the possible starting times of a component are known.

Due to the synchronous nature of the time structure, there is no positional uncertainty of components. Therefore, the component samples can be combined into one state. Thus, $C_{i,0} = (c_{i,0}, c_{i,0}, \ldots, c_{i,0})$ represents absence and $C_{i,1} = (c_{i,1}, c_{i,2}, \ldots, c_{i,n_i})$ represents presence of the ith component. The probability of triggering a component, conditional to selection of the ith component, is ϵ_i. The state diagram for the ith component has been shown in Fig. 4.5.

The basic steps in the development of the receiver realization for detecting signals with the Synchronous-Poisson Time Structure began with steps 1-3, given on page 35. In this section the properties of the signal for the Synchronous-Poisson Time structure are combined with Eqs. 3.8, 3.22 and 3.23 to obtain the adaptive receiver.

5.2.1 Synchronous-Poisson Time Structure, Realization I. In the Synchronous-Poisson case, component position is known exactly, but whether a component is present or not is uncertain. Therefore the receiver can operate sequentially in time blocks equal to a component duration. In this section x_k is an n_i-dimensional observation having the duration of a component and S_k is an n_i-dimensional segment of the signal, $s = S_1, S_2, \ldots, S_k$, which is either the ith recurrence phenomenon with the component on, $C_{i,1}$, or the ith recurrence phenomenon with the component off, $C_{i,0}$. With this change in the notation, the sequential average likelihood ratio analagous to Eq. 5.2 is

$$f(x_k \mid X_{k-1}) = \sum_{s \in S} f(x_k \mid s)\, p_{k-1}(s \mid SN) \qquad (5.47)$$

The signal ensemble space, S, can be partitioned into b disjoint subspaces, S_i. Each S_i subspace contains all those signals that might result from the ith component alone. This is a result of the restriction that a given component, C^i, is selected and fixed at the beginning of each long transmission. Thus, Eq. 5.47 can be written as

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{s \in S_i} f(x_k|s)p_{k-1}(s|C^i, SN)p_{k-1}(C^i|SN) \tag{5.48}$$

Expanding the vector, s, into sample form, Eq. 5.48 becomes

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_k=C_{i,0}}^{C_{i,1}} f(x_k|S_1, S_2, \ldots, S_k)$$

$$p_{k-1}(S_1, S_2, \ldots, S_k|C^i, SN)p_{k-1}(C^i|SN) \tag{5.49}$$

Since s is the receiver input if there were no noise,

$$f(x_k|S_1, S_2, \ldots, S_k) = f(x_k|S_k) \tag{5.50}$$

Due to the independence of the signal recurrence (see page 30) we write

$$p_{k-1}(S_1, S_2, \ldots, S_k|C^i, SN) = p_{k-1}(S_1, S_2, \ldots, S_{k-1}|C^i, SN)p_{k-1}(S_k|C^i, SN) \tag{5.51}$$

Substitution of Eqs. 5.50 and 5.51 into Eq. 5.49 results in

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_k=C_{i,0}}^{C_{i,1}} \Big[f(x_k|S_k)p_{k-1}(S_1, S_2, \ldots, S_{k-1}|C^i, SN)$$

$$p_{k-1}(S_k|C^i, SN)p_{k-1}(C^i|SN) \Big] \tag{5.52}$$

Since we are dealing with finite sums, the order of summation can be interchanged. Reordering the summations and factoring, Eq. 5.52 can be put in the form

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN) \sum_{S_k=C_{i,0}}^{C_{i,1}} \Bigg\{ f(x_k|S_k)p_{k-1}(S_k|C^i,SN)$$

$$\left[\sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} p_{k-1}(S_1,S_2,\ldots,S_{k-1}|C^i,SN) \right] \Bigg\} \qquad (5.53)$$

where the term in brackets is equal to one. Therefore

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN) \Big[f(x_k|S_k=C_{i,0})p_{k-1}(S_k=C_{i,0}|C^i,SN)$$

$$+ f(x_k|S_k=C_{i,1})p_{k-1}(S_k=C_{i,1}|C^i,SN) \Big] \qquad (5.54)$$

By definition of the signal generator process considered here, (Eqs. 4.7 and 4.8), $p_{k-1}(S_k=C_{i,0}|C^i,SN) = 1-\epsilon_i$ and $p_{k-1}(S_k=C_{i,1}|C^i,SN) = \epsilon_i$. Also for zero energy signals in added noise, $f(x_k|S_k=C_{i,0}) = 1$. We can then put Eq. 5.54 in its final form.[1]

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN) \Big[1-\epsilon_i + \epsilon_i f(x_k|S_k=C_{i,1}) \Big] \qquad (5.55)$$

It is also necessary to obtain equations that update which component is being transmitted. The updating equation is

$$p_k(s|SN) = \frac{f(x_k|s)p_{k-1}(s|SN)}{f(x_k|X_{k-1})} \qquad (3.23)$$

Using the definition of conditional probabilities, this equation can be written as

---

[1] An expression similar to Eq. 5.55 with $b = 1$ (in which case $p_{k-1}(C^1|SN) = 1$) arose from a Synchronous-Poisson trigger process in the paper, "A Sequential Test for Radar Detection of Multiple Targets," W. B. Kendall and I. S. Reed, IRE Trans. on Information Theory, Vol. IT-9, January, 1963.

$$p_k(s|C^i,SN)p_k(C^i|SN) = \frac{f(x_k|s)p_{k-1}(s|C^i,SN)p_{k-1}(C^i|SN)}{f(x_k|X_{k-1})} \tag{5.56}$$

Writing s in sample form and summing both sides of Eq. 5.56 over all possible signals for the ith component gives

$$p_k(C^i|SN)\left[\sum_{S_1=C_{i,0}}^{C_{i,1}}\sum_{S_2=C_{i,0}}^{C_{i,1}}\cdots\sum_{S_k=C_{i,0}}^{C_{i,1}} p_k(S_1,S_2,\ldots,S_k|C^i,SN)\right]$$

$$= p_{k-1}(C^i|SN)\frac{\sum_{S_k=C_{i,0}}^{C_{i,1}} f(x_k|S_k)p_{k-1}(S_k|C^i,SN)\sum_{S_1=C_{i,0}}^{C_{i,1}}\sum_{S_2=C_{i,0}}^{C_{i,1}}\cdots\sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} p_{k-1}(S_1,S_2,\ldots,S_{k-1}|C^i,SN)}{f(x_k|X_{k-1})} \tag{5.57}$$

The bracketed terms on each side of Eq. 5.57 are equal to one so that we have

$$p_k(C^i|SN) = p_{k-1}(C^i|SN)\frac{\left[f(x_k|S_k=C_{i,0})p_{k-1}(S_k=C_{i,0}|C^i,SN) + f(x_k|S_k=C_{i,1})p_{k-1}(S_k=C_{i,1}|C^i,SN)\right]}{f(x_k|X_{k-1})} \tag{5.58}$$

The terms in brackets in the numerator of Eq. 5.58 become, as before, $1-r_i + r_i f(x_k|S_k=C_{i,1})$. The updating equation for component information is

$$p_k(C^i|SN) = \frac{\left[1-r_i + r_i f(x_k|S_k=C_{i,1})\right]p_{k-1}(C^i|SN)}{f(x_k|X_{k-1})} \tag{5.59}$$

Equations 5.55 and 5.59 are the basic equations for the adaptive realization. Defining the component conditional sequential likelihood ratio as

$$f(x_k|X_{k-1},C^i) = 1-r_i + r_i f(x_k|S_k=C_{i,1}) \tag{5.60}$$

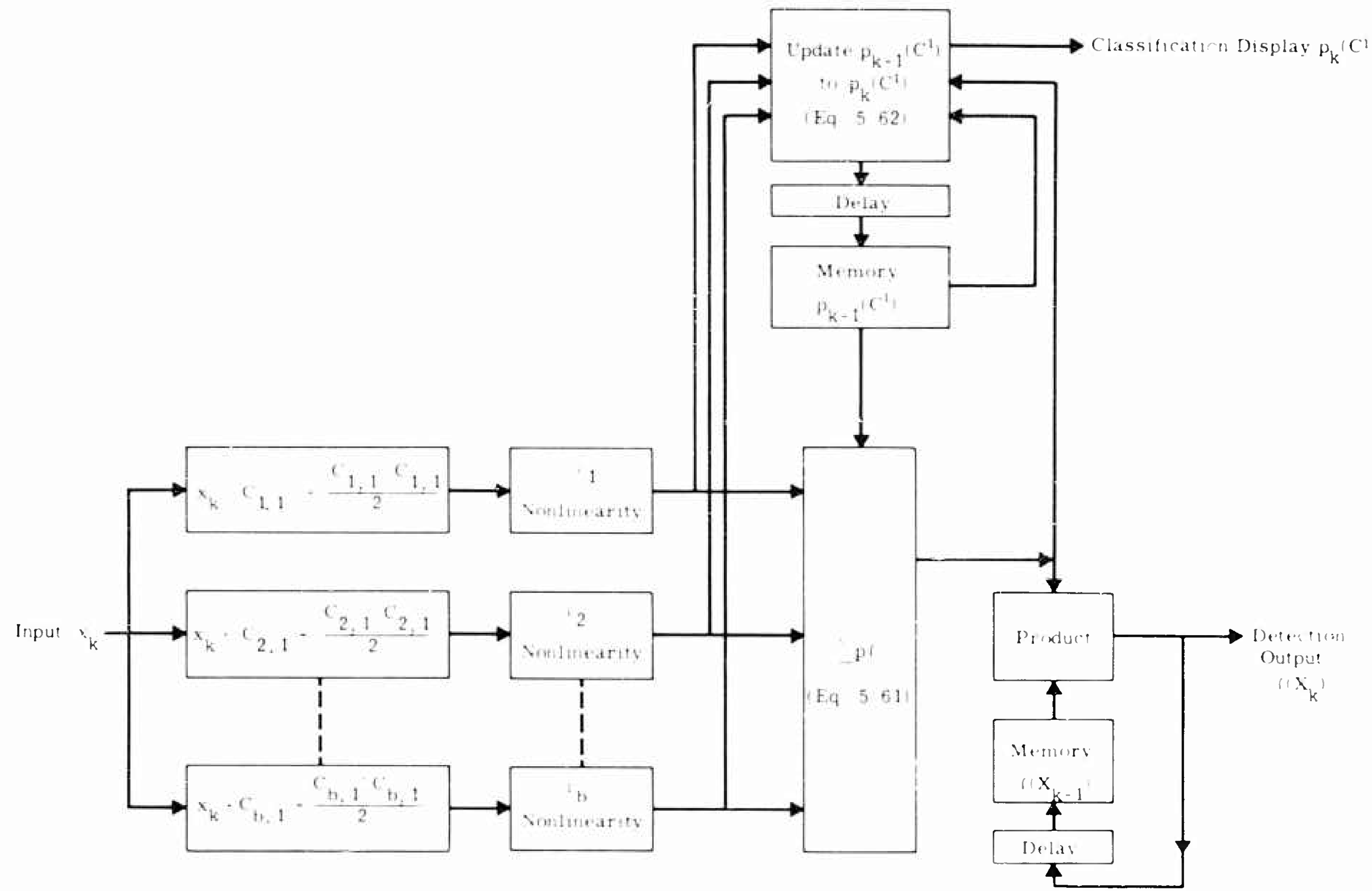

Fig. 5.9. Adaptive receiver realization, Synchronous-Poisson Time Structure Realization I.

one can put Eq. 5.55 into the form:

$$\ell(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN)\,\ell(x_k|X_{k-1}, C^i) \tag{5.61}$$

and Eq. 5.59 can be put in the form

$$p_k(C^i|SN) = \frac{p_{k-1}(C^i|SN)\,\ell(x_k|X_{k-1}, C^i)}{\ell(x_k|X_{k-1})} \tag{5.62}$$

The basic receiver design equations for this realization are summarized in Table 5.4.

TABLE 5.4

BASIC RECEIVER DESIGN EQUATIONS: SYNCHRONOUS-POISSON TIME STRUCTURE
(COMMON COMPONENT DURATION)
REALIZATION I

Optimum Detection Output

$$L(X_k) = L(X_{k-1})\,\ell(x_k|X_{k-1}) \tag{3.8}$$

Sequential Average Likelihood Ratio

$$\ell(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN)\,\ell(x_k|X_{k-1}, C^i) \tag{5.61}$$

Component Conditional Sequential Likelihood Ratio

$$\ell(x_k|X_{k-1}, C^i) = 1 - r_i + r_i\,\ell(x_k|S_k C_{i,1}) \tag{5.60}$$

Classification - Component Identification

$$p_k(C^i|SN) = \frac{p_{k-1}(C^i|SN)\,\ell(x_k|X_{k-1}, C^i)}{\ell(x_k|X_{k-1})} \tag{5.62}$$

5.2.2 Operation of the Adaptive Receiver. Figure 5.9 is a block diagram of Realization I for components in added white Gaussian noise. In this case

$$\ell(x_k \mid S_k, C_{i,1}) = e^{\left[x_k \cdot C_{i,1} - \frac{C_{i,1} \cdot C_{i,1}}{2}\right]} \tag{5.63}$$

where N, the noise power in the receiver bandwidth, is one. The receiver input, x_k, is correlated with each possible component that could occur, and the bias $C_{i,1} \cdot C_{i,1}/2$ subtracted. These outputs are then passed through a nonlinearity.

$$\ln \ell(x_k \mid X_{k-1}, C^i) = \ln \left\{ 1 - \nu_i + \nu_i e^{\left[x_k \cdot C_{i,1} - \frac{C_{i,1} \cdot C_{i,1}}{2}\right]} \right\} \tag{5.64}$$

This nonlinearity depends on the trigger probability, ν_i, which is also the duty factor in the Synchronous-Poisson Time Structure. This nonlinearity is called the "ν nonlinearity." One could write this equation in words as $\ell(x_k \mid X_{k-1}) = (1-\nu_i)$ (likelihood ratio of x_k given the selection of the ith component but no component occurrence) + ν_i (likelihood ratio of x_k given the selection of the ith component and component occurrence). Thus, the likelihood ratio of the observation, x_k, is computed as if a component occurred and this is "watered down" because of the recurrence uncertainty. Figure 5.10 shows a plot of the ν nonlinearity for several values of ν.

The outputs of these nonlinearities, $\ell(x_k \mid X_{k-1}, C^i)$ are then weighted by updated knowledge, $p_{k-1}(C^i \mid SN)$, as to which component is being sent. This forms the sequential average likelihood ratio, $\ell(x_k \mid X_{k-1})$, which is combined with $\ell(X_{k-1})$ to provide the detection output, $\ell(X_k)$.

The receiver also updates $p_{k-1}(C^i \mid SN)$, the component information, to $p_k(C^i \mid SN)$ and stores these updated probabilities in preparation for the next observation. These probabilities can be read out to form a classification output.

5.2.3 Other Receiver Realizations and the Use of Memory. In Section 5.2.1, a realization of the optimum receiver for a synchronous-recurrent component is presented. Realization I has a temporary memory which is continually updated and which has a finite

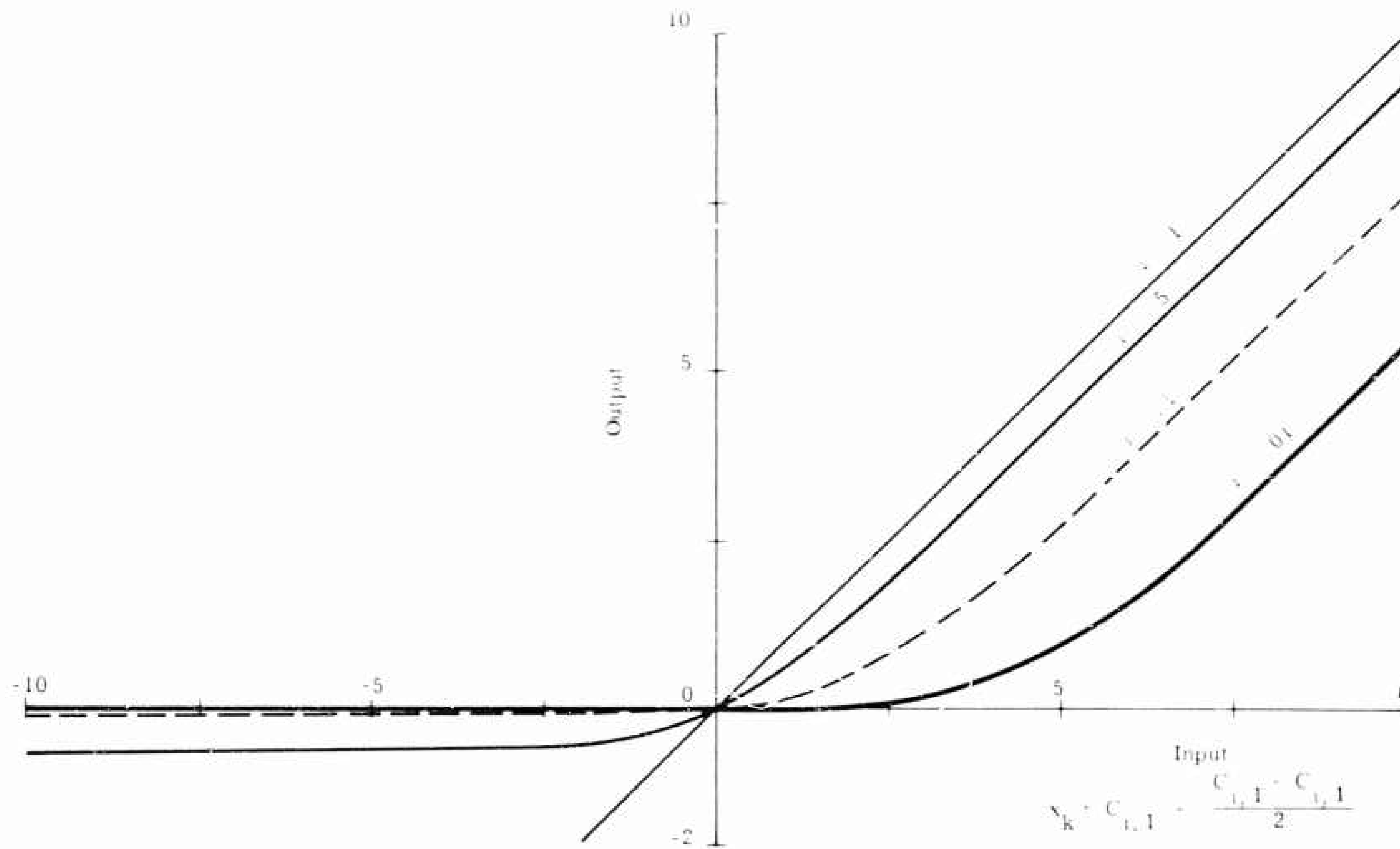

Fig. 5.10. Nonlinearity

size of b words. Although the receiver takes into account all the possible time patterns of components, its memory need not store all of these patterns. This finite memory is a primary practical feature of this realization.

Two other realizations of the optimum receiver are derived in Appendix B. Realization III (see Appendix B.1) is a b-channel receiver. The likelihood ratio of the observation X_k conditional to presence of each of the b components is computed sequentially in separate channels and the outputs are weighted by the a priori probabilities, $p_0(C^i | SN)$, of the selection of each of the components and then summed. In this receiver b words of memory are needed to store the likelihood ratios, $\ell(X_k | C^i)$.

Realization IV is an important practical receiver since it appears to be the simplest (see Appendix B.2). The basic design equations are summarized in Table 5.5 and a block diagram is shown in Fig. 5.11. This is a b-channel receiver. The ith channel correlates

TABLE 5.5

BASIC RECEIVER DESIGN EQUATIONS, SYNCHRONOUS-POISSON TIME STRUCTURE
REALIZATION IV

Optimum Detection Output

$$\ell(X_k) \quad \sum_{i=1}^{b} Q_i(k) \qquad (B.19)$$

Information Updating

$$Q_i(k) \quad Q_i(k-1)\left[1 - \quad_i \quad \cdot \quad_i \; \ell(X_k \quad S_k \quad C_{i,1} \;)\right] \qquad (B.17)$$

Classification - Component Identification

$$p_k(C^i | SN) \quad \frac{Q_i(k)}{\ell(X_k)} \qquad (B.20)$$

the input with the ith component and subtracts a bias term $C^i \cdot C^i / 2$. This quantity is then fed into a ... nonlinearity to form $Q_i(k)$. The $Q_i(k)$ terms are stored and accumulated for each of the components by means of the channel adders and the T_1 delays. These terms are exponentiated, summed, and the logarithm formed to ... in the detection output. The

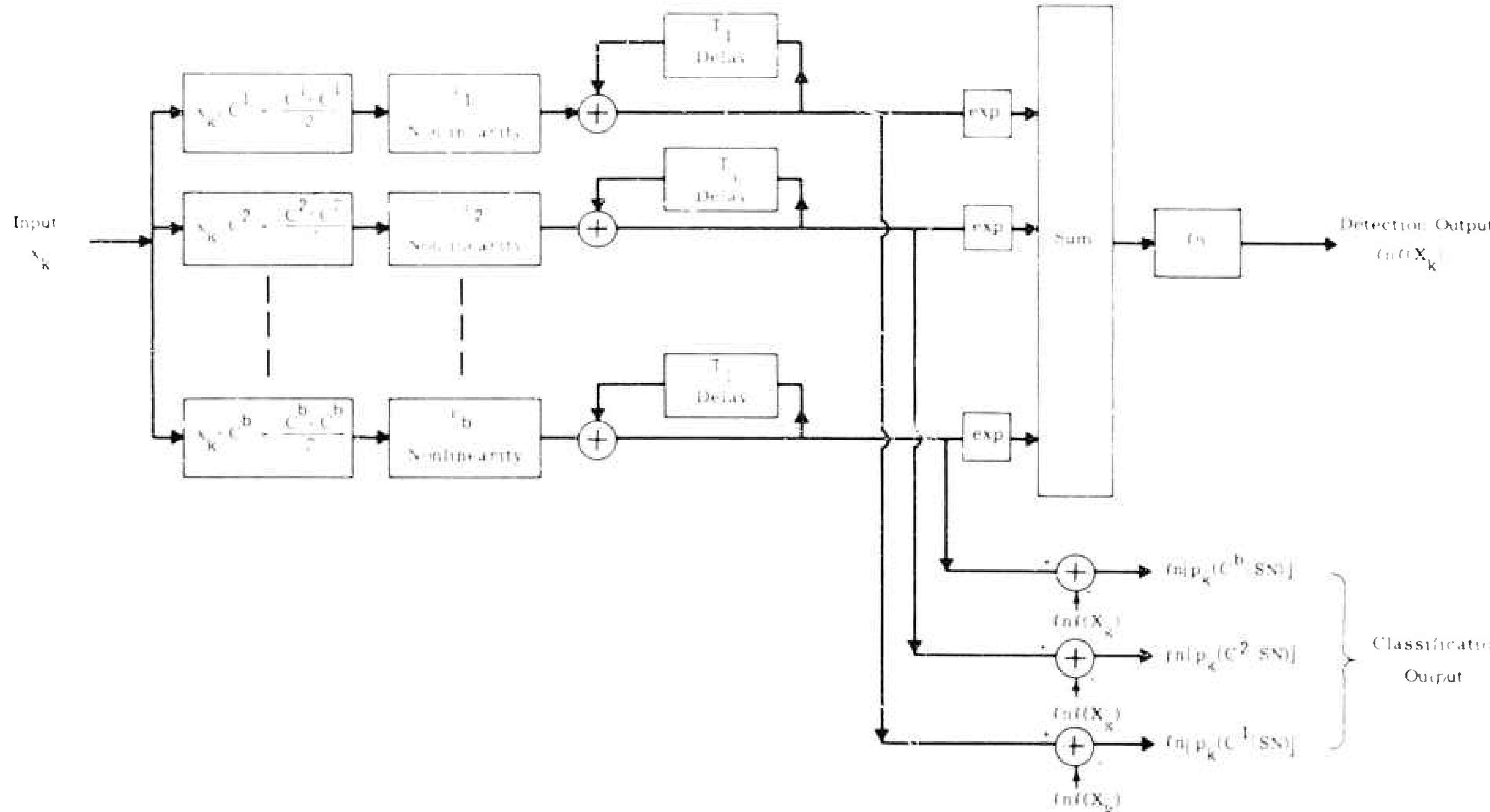

Fig. 5.11. Adaptive receiver realization, Synchronous-Poisson Time Structure, Realization IV.

classification output is obtained by subtracting $\ln f(X_k)$ from the output of each of the recirculating delays. By comparing Table 5.5 with Table 5.4 one can see the simplification in computation of Realization IV. In this realization b words of memory are required to store $Q_1(k) = f(X_k)p_k(C^1 | SN)$ in temporary memory.

The receivers presented in Section 5.2 are different realizations of the optimum receiver. The particular realization of the optimum receiver chosen determines whether the receiver "looks" adaptive. Realization I has an adaptive feature in that component information is updated. On the other hand, Realizations III and IV have a separate channel for each possible component and the "learning" of which component is being sent is not an obvious feature.

A problem in receiver design that has emerged when dealing with time uncertainty and nonperiodic components is the problem of receiver complexity or memory. Since it is uncertain whether a component will start or not, the receiver designer is presented with an exponentially growing number of time patterns or signals. In the Synchronous-Poisson Time Structure, the ensemble of possible signals grows like $b2^k$ where k is the index on time and b is the number of components in the component ensemble. The implementation or simulation of such a receiver designed on the basis of this growing ensemble can rapidly become impractical. On the other hand, the adaptive or sequential realizations presented have been designed by describing useful signal ensembles indirectly in terms of components. The result is a receiver design which utilizes a fixed size memory. The important reason for wanting sequential or adaptive realizations is not their adaptive-looking nature, but the fact that this is a way of realizing the optimum receiver with a fixed size memory.

5.3 Optimum Adaptive Receiver Design, Periodic Time Structure, Unknown Repetition Frequency.

In this section an adaptive realization of the optimum receiver is presented for detecting signals with a Periodic Time Structure. This is the most certain of the three time structures considered and it differs from the sporadic and synchronous cases in that it is learnable. One of b components is selected for transmission and the same component recurs periodically throughout a total observation, X_k. The repetition frequency and start of the period are initially unknown but fixed throughout a transmission.

The development of the receiver in this section is similar to the development of the receiver designed for the Sporadic-Poisson Time Structure (see Section 5.1). The first three steps are given on page 35. In this section the properties of the signal for the Periodic Time Structure are combined with Eqs. 3.8, 3.22, and 3.23 to obtain the adaptive receiver.

5.3.1 Periodic Time Structure, Unknown Repetition Frequency, Realization I. This realization follows the development of the receiver for the Sporadic-Poisson Time Structure given in Section 5.1.1 up to Eq. 5.12. That equation for the sequential average likelihood ratio was:

$$\ell(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} \ell(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1},SN)\, p_{k-1}(s_{k-1}|SN) \qquad (5.12)$$

Recall that the signal properties are defined in terms of the generator process, $g(s_k|s_{k-1},SN)$, by Eqs. 4.9, 4.10, and 4.11. The possible states of a signal sample, s_k, are the possible component samples, $c_{i,j}$, for $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, n_i$. The number of samples, n_i, of a component can in general be variable so that b possible components can be defined to represent b possible repetition frequencies. $c_{i,0}$ is not an allowed state in the periodic case since some portion of a component is always present. If the development in Section 5.1.1 up through Eq. 5.12 is modified for the periodic case by summing over the allowed states, $c_{i,1}, c_{i,2}, \ldots, c_{i,n_i}$, an analagous equation becomes:

$$\ell(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{s_k=c_{i,1}}^{c_{i,n_i}} \ell(x_k|s_k) \sum_{s_{k-1}=c_{i,1}}^{c_{i,n_i}} g(s_k|s_{k-1},SN)\, p_{k-1}(s_{k-1}|SN) \qquad (5.65)$$

The properties of the generator process for the Periodic Time Structure are:

$$g(s_k|s_{k-1},SN) = 1 \quad \text{for} \quad s_k = c_{i,j} \qquad s_{k-1} = c_{i,j-1} \quad \text{for} \quad j = 2, 3, \ldots, n_i \qquad (4.9)$$

$$g(s_k|s_{k-1},SN) = 1 \quad \text{for} \quad s_k = c_{i,1} \qquad s_{k-1} = c_{i,n_i} \qquad (4.10)$$

$$g(s_k|s_{k-1},SN) = 0 \qquad \text{otherwise} \qquad (4.11)$$

Using these properties and the notation of Eq. 5.16, $b_{i,j}(k) = p_k(s_k = c_{i,j}|SN)$, one can write Eq. 5.65 as

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \left[b_{i,n_i}(k-1) f(x_k|s_k = c_{i,1}) + \sum_{j=2}^{n_i} b_{i,j-1}(k-1) f(x_k|s_k = c_{i,j}) \right] \tag{5.66}$$

This is the equation for the sequential average likelihood ratio.

The equations that update component identification and positional information are obtained by following steps similar to those that lead to Eqs. 5.32, 5.33 and 5.34 for the Sporadic-Poisson Time Structure. In the periodic case the sums are only over the states $c_{i,1}, c_{i,2}, \ldots, c_{i,n_i}$. Thus Eq. 5.28 becomes

$$p_k(s_k|SN) = \frac{f(x_k|s_k) \sum_{s_{k-1}=c_{i,1}}^{c_{i,n_i}} p_{k-1}(s_{k-1}|SN)\, g(s_k|s_{k-1}, SN)}{f(x_k|X_{k-1})} \tag{5.67}$$

Substituting the properties of the generator process (Eqs. 4.9, 4.10, and 4.11) into Eq. 5.67 and using the notation $b_{i,j}(k) = p_k(s_k = c_{i,j}|SN)$, one can write the component updating equations as

$$b_{i,1}(k) = \frac{b_{i,n_i}(k-1) f(x_k|s_k = c_{i,1})}{f(x_k|X_{k-1})} \tag{5.68}$$

$$b_{i,j}(k) = \frac{b_{i,j-1}(k-1) f(x_k|s_k = c_{i,j})}{f(x_k|X_{k-1})} \quad \text{for } j = 2, 3, \ldots, n_i \tag{5.69}$$

where component identification information is obtained by forming

$$p_k(C^i|SN) = \sum_{j=1}^{n_i} b_{i,j}(k) \tag{5.70}$$

The design equations for this realization are summarized in Table 5.6.

TABLE 5.6

BASIC RECEIVER DESIGN EQUATIONS, PERIODIC TIME STRUCTURE
UNKNOWN REPETITION FREQUENCY
REALIZATION I

Optimum Detection Output

$$f(X_k) = f(X_{k-1})\, f(x_k | X_{k-1}) \tag{3.8}$$

Sequential Average Likelihood Ratio

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} \left[b_{i,n_i}(k-1)\, f(x_k | s_k = c_{i,1}) + \sum_{j=2}^{n_i} b_{i,j-1}(k-1)\, f(x_k | s_k = c_{i,j}) \right] \tag{5.66}$$

Classification - Component Identification and Position

$$b_{i,1}(k) = \frac{b_{i,n_i}(k-1)\, f(x_k | s_k = c_{i,1})}{f(x_k | X_{k-1})} \tag{5.68}$$

$$b_{i,j}(k) = \frac{b_{i,j-1}(k-1)\, f(x_k | s_k = c_{i,j})}{f(x_k | X_{k-1})} \tag{5.69}$$

for $j = 2, 3, \ldots, n_i$

Classification - Component Identification

$$p_k(C^i | SN) = \sum_{j=1}^{n_i} b_{i,j}(k) \tag{5.70}$$

5.3.2 Operation of the Adaptive Receiver. In Realization I, the Periodic Time Structure, the optimum receiver stores information obtained from the past observations, X_{k-1} and initial knowledge of the situation, in the form of probabilities, $b_{i,j}(k-1)$ (see Table 5.6). Since the interpretation of the terms $b_{i,j}(k-1)$ and $f(x_k | s_k = c_{i,j})$ is similar to that given for the Sporadic-Poisson Time Structure in Section 5.1.2 it will not be repeated. Note that in the periodic receiver there is no state, $c_{i,0}$. This is reflected in the absence of $b_{i,0}(k-1)$, $1-\epsilon_i$ and ϵ_i terms in the receiver operations.

5.3.3 Other Receiver Realizations and the Use of Memory. In Realization I, component identification and positional information are stored in a temporary memory as the probabilities $b_{i,j-1}(k-1)$. These probabilities are updated after each unit observation, x_k. $\sum_{i=1}^{b} n_i$ words of memory are needed to store these probabilities.

In Appendix C, three other realizations of the optimum receiver are presented. In Realization II (Appendix C.1) component identification and positional information are updated separately. $\sum_{i=1}^{b} n_i$ words are required in a temporary memory to store component positional information and b words to store component identification information.

In Realization III, there is a channel for each of the b possible components. Each channel computes the likelihood ratio conditional to presence of the ith component and the channel outputs are then weighted by the a priori probabilities of each of the possible components that could occur. $\sum_{i=1}^{b} n_i$ words are needed to store component identification and positional information and b words to store the $\ell(X_k|C^i)$ terms.

Realization IV (Appendix C.3) is the simplest of the four realizations. These receiver design equations are summarized in Table 5.7 and a block diagram is shown in Fig. 5.12.

TABLE 5.7

BASIC RECEIVER DESIGN EQUATIONS, PERIODIC TIME STRUCTURE
UNKNOWN REPETITION FREQUENCY
REALIZATION IV

Optimum Detection Output

$$\ell(X_k) = \sum_{i=1}^{b} \sum_{j=1}^{n_i} Q_{i,j}(k) \qquad (C.13)$$

Information Updating

$$Q_{i,1}(k) = Q_{i,n_i}(k-1)\, f(x_k|s_k, c_{i,1}) \qquad (C.11)$$

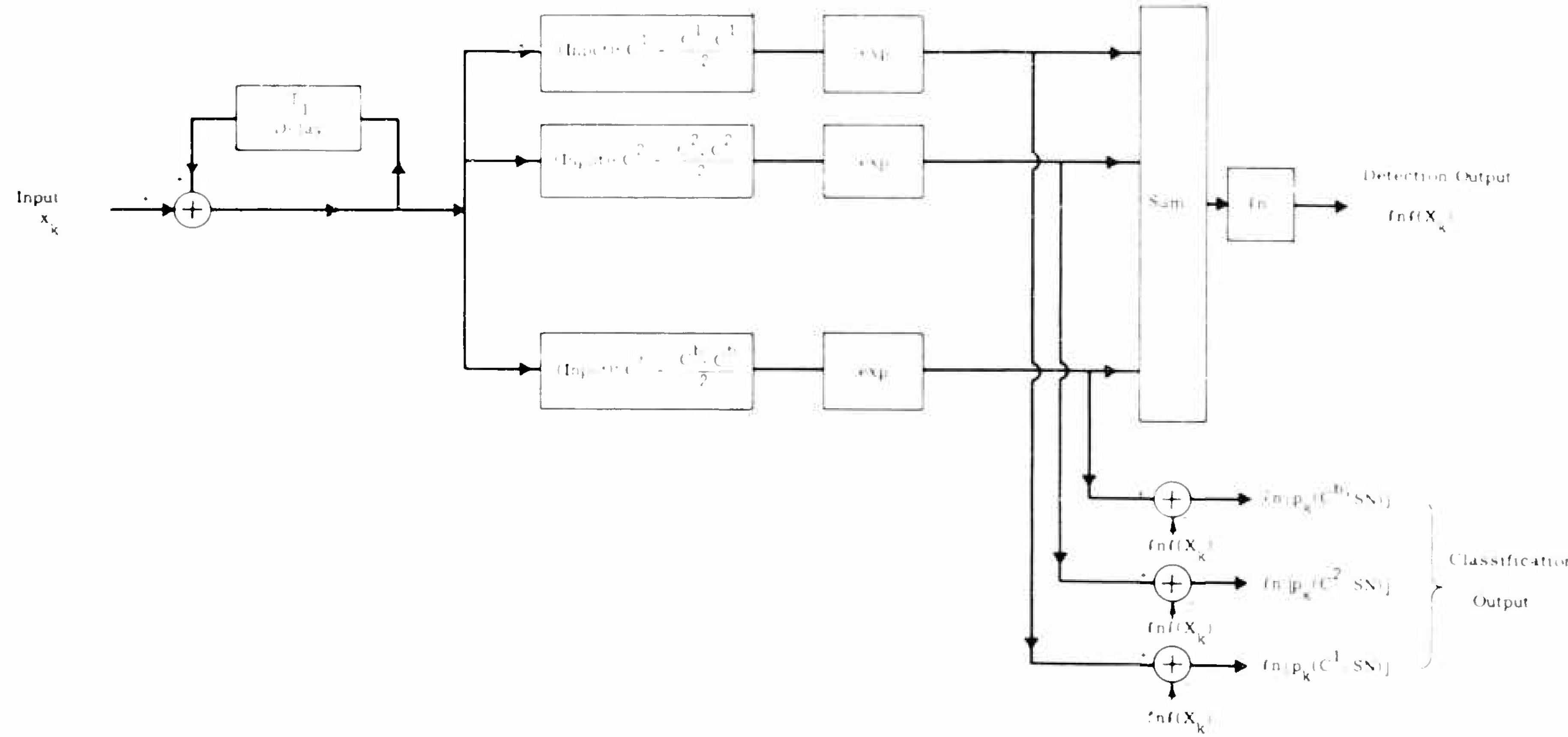

Fig. 5. 12. Adaptive receiver realization, Periodic Time Structure, Realization IV

$$Q_{i,j}(k) = Q_{i,j-1}(k-1) f(x_k \mid s_k = c_{i,j}) \qquad \text{(C.12)}$$

for $j = 2, 3, \ldots, n_i$

Classification - Component Identification and Position

$$b_{i,j}(k) = \frac{Q_{i,j}(k)}{f(X_k)} \qquad \text{(A.44)}$$

Classification - Component Identification

$$p_k(C^i \mid SN) = \frac{\sum_{j=1}^{n_i} Q_{i,j}(k)}{f(X_k)} \qquad \text{(A.45)}$$

Here the receiver updates the quantities $Q_{i,j}(k) = f(X_k) b_{i,j}(k)$ directly using Eqs. C.11 and C.12. From the $Q_{i,j}(k)$ terms the likelihood ratio can be calculated by simple addition. $\sum_{i=1}^{b} n_i$ words are required to store the $Q_{i,j}(k)$ terms just as in Realization I. However, by comparing Tables 5.6 and 5.7, it can be seen that the operations performed by Realization IV are much simpler.

In all four receiver realizations for the Periodic Time Structure, the receiver memory is finite. This result is not surprising here since this signal ensemble does not grow with time.

5.4 Optimum Adaptive Receiver Design, Periodic Time Structure, Known Repetition Frequency

In this section an adaptive realization of the optimum receiver is presented for detecting signals with a Periodic Time Structure in which the repetition frequency and the start of the period are known. This is the usual classical periodic case. One of b components is selected for transmission and the same component recurs periodically throughout a total observation, X_k. In this case the observations can be processed in blocks of time equal to a component duration. The notation used is the same as that used in the Synchronous-Poisson Time Structure. In other words, x_k is an n_1-dimensional observation having the duration of a component and S_k is an n_1-dimensional segment of the signal. The optimum receiver is the same as that which would result if r_1 were set equal to one in Eq. B.17, Table 5.5. The equations for the receiver design are presented in Table 5.8.

TABLE 5.8

BASIC RECEIVER DESIGN EQUATIONS, PERIODIC TIME STRUCTURE
KNOWN REPETITION FREQUENCY, KNOWN START OF PERIOD
REALIZATION IV

Optimum Detection Output

$$f(X_k) = \sum_{i=1}^{b} Q_i(k) \qquad \text{(B.14)}$$

Information Updating

$$Q_i(k) = Q_i(k-1)\, f(x_k \mid S_k = C^i) \qquad (5.71)$$

Classification - Component Identification

$$p_k(C^i \mid SN) = \frac{Q_i(k)}{f(X_k)} \qquad \text{(B.20)}$$

Let us consider this receiver in more detail for the case of added white Gaussian noise. In that case, for the noise power, N, equal to one,

$$f(x_k \mid S_k = C^i) = e^{\left[x_k \cdot C^i - \frac{C^i \cdot C^i}{2}\right]} \qquad (5.72)$$

and so

$$Q_i(k) = Q_i(k-1)\, e^{\left[x_k \cdot C^i - \frac{C^i \cdot C^i}{2}\right]} \qquad (5.73)$$

But by repeated application of Eq. 5.73, $Q_i(k)$ can be written as

$$Q_1(k) = Q_1(0)\, e^{\left[-k \frac{C^1 \cdot C^1}{2} + C^1 \cdot \sum_{j=1}^{k} x_j\right]} \tag{5.74}$$

and

$$G(X_k) = \sum_{i=0}^{b} Q_i(0)\, e^{\left[-k \frac{C^i \cdot C^i}{2} + C^i \cdot \sum_{j=1}^{k} x_j\right]} \tag{5.75}$$

From this equation one can see that the observations themselves can first be added in synchronous intervals and this sum correlated with each of the possible components.

When the component is known exactly, a monotone function of the likelihood ratio, which is also optimum, is simply

$$M(X_k) = C^1 \cdot \sum_{j=1}^{k} x_j \tag{5.76}$$

In this case, the observations themselves may be simply accumulated and the sum correlated with the known component, C^1.

5.5 Comparison of Receivers for Synchronous-Poisson and Periodic Time Structures

It is interesting to compare the optimum receivers for the Synchronous-Poisson and Periodic Time Structures when the repetition frequency is known. First, consider the case of component known exactly (CKE) in added white Gaussian noise. A block diagram of the optimum receiver for the Periodic Time Structure, obtained by setting b = 1 in the equations of Table 5.8, is shown in Fig. 5.13a. A realization of the optimum receiver for the Synchronous-Poisson Time Structure, obtained by setting b = 1 in Table 5.5, is shown in Fig. 5.13b.

In the periodic case, the adder and T_1 delay recirculate the input waveshapes, $x_1, x_2, \ldots, x_k$. Recall that in this periodic case, x_i represents an input observation of $2WT_1$ samples. After the observation x_k, the receiver has formed $x_1 + x_2 + \ldots + x_k$ and this average waveshape is correlated with the component.

The optimum receiver (see Fig. 5.13b) for the Synchronous-Poisson Time Structure does not simply add the input waveshape in synchronous intervals. Instead, a more abstract quantity, a likelihood ratio, is recirculated. The observation, x_k, is first correlated with the known component waveshape, a bias term is subtracted, and this quantity is then fed into a nonlinearity, which is a function of the duty factor. The synchronous sum of such nonlinear functions of the input signal and noise waveshapes are stored.

Next, let us compare the optimum receivers for the Synchronous-Poisson and Periodic Time Structures when the component is known statistically (CKS). A block diagram of the receiver for the Periodic Time Structure was shown in Fig. 5.12 and the receiver for the Synchronous-Poisson Time Structure was shown in Fig. 5.11. The receiver for the periodically recurrent component is simpler in two respects, the number of recirculating delays and the nonlinearities. In the periodic case, the observations are recirculated by means of a single adder and a T_1 delay. These outputs are fed into b parallel channels where they are correlated with each of the possible components that could occur, exponentiated, and summed in a final summer. In the receiver for the synchronous case, however, a likelihood ratio, rather than an input signal plus noise waveshape, is circulated. The input observation is correlated with each of the possible components, fed into a nonlinearity which depends on the duty factor, , and then stored and recirculated. These outputs are then summed to form the detection output. Thus, the receiver for the periodic case is much simpler since the input waveshape can be recirculated with a single adder and delay.

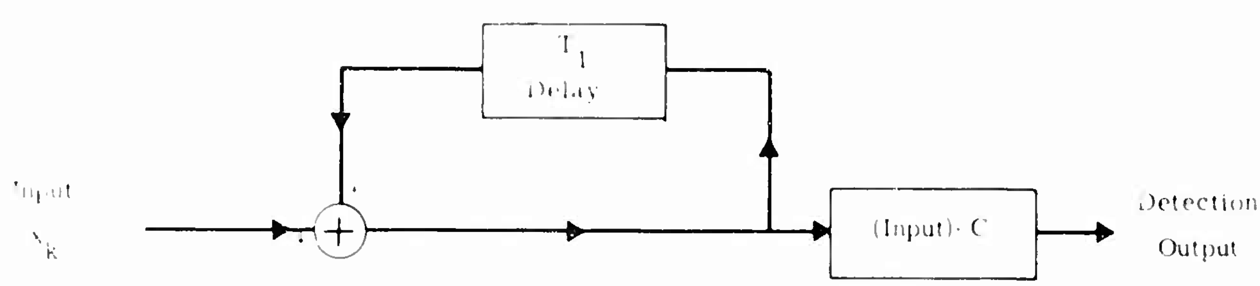

Fig. 5.13.a. Optimum receiver, CKE, Periodic Time Structure

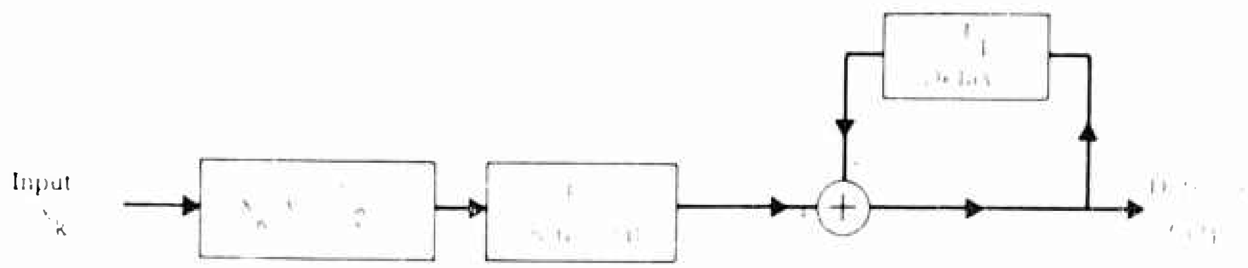

Fig. 5.13b Optimum receiver, CKF Synchronous-Poisson Time Structure

5.6 Summary of Chapter V

In this chapter optimum receiver designs have been developed to detect a recurrence phenomenon in noise. The receivers are time varying in the sense that they are capable of processing an increasingly longer observation and are capable of making a corresponding optimum decision as to presence or absence of the recurrence phenomenon in that observation. There is uncertainty in which component, out of a finite class of components, will be selected and there is uncertainty in the recurrence times of a component. Three basic types of recurrence-time uncertainty are considered, Sporadic-Poisson, Synchronous-Poisson, and Periodic.

The approach used in designing the receiver has been to solve an over-all optimization problem and then to realize this optimum receiver in a sequential manner that works on a component basis. In other words, since the primary goal is detecting the recurrence phenomenon during the observation X_k, the likelihood ratio of the observation, X_k, is formed and put into an equivalent form in which component information is updated. Since the receiver development starts with the likelihood ratio of the observation X_k, it is optimum. We are assured that the operation of the receiver on a local component basis is correct since the receiver design is a result of transforming this likelihood ratio into an equivalent form. A contrasting method of attack would be to formulate the detection problem on the basis of detecting a single component and to then combine these results in a manner that would result in an optimum decision over the entire observation.

Providing the required amount of receiver memory is a basic difficulty which emerges in the design of the optimum receiver for nonperiodic components. There are as many signals in the signal ensemble as there are components in the component ensemble multiplied by the number of possible component time patterns. This is a fixed-size ensemble for all time for components recurring with a Periodic Time Structure. However, for the Sporadic-Poisson and Synchronous-Poisson Time Structures, the signal ensemble grows with time. If the receivers are designed using classical terminology, they become too complex. To obtain receivers with a fixed-size memory or receiver structure, the signal ensemble is described i directly in terms of components and the time structure. Other time structures besides the three considered could be studied.

We have seen how the optimum receiver can be put into different forms. Different aspects of the receiver operations are explicitly displayed by the particular realization chosen. It is an interesting sidelight that sequential realizations, such as Realizations I and II, often appear to work in an adaptive manner. These realizations display an explicit updating of component information, giving them a "learning" feature. On the other hand, in Realizations III and IV, it is not so obvious that the receiver is learning the component selected since the receiver does not explicitly work with component quantities. In any of the realizations, classification information can be obtained regardless of whether the receiver appears to use it or not.

The quantities stored in the receiver memory depend on the time structure of the signal and the particular realization chosen. For the Sporadic-Poisson Time Structure, component identification and local component positional information are stored and updated. In Realization I this information is combined in the $b_{i,j}(k)$ matrix. $\sum_{i=1}^{b} n_i$ words are required in a temporary memory to store this information. In Realization II, component identification, $p_k(C^i|SN)$, and local component positional information, $b'_{i,j}(k)$, are stored in separate temporary memories. $\sum_{i=1}^{b} n_i$ words of memory are needed for $b'_{i,j}(k)$ and b words for $p_k(C^i|SN)$ terms. In Realization III, $\sum_{i=1}^{b} n_i$ words of temporary memory are needed to store component identification and positional information and b words to store $f(X_k|C^i)$ terms. In Realization IV, $\sum_{i=1}^{b}$ words of memory are needed to store $Q_{i,j}(k)$ terms.

For the Synchronous-Poisson Time Structure, only component identification information must be stored since there is no uncertainty about component position. 'n Realization I this information is stored as $p_k(C^1 \; SN)$ and b words of temporary memory are required. In Realizations III and IV, b branches are needed for each of the possible components.

For the Periodic Time Structure, unknown repetition frequency, component identification and local positional information are stored in a temporary memory. In Realization I this information is combined in the $b_{i,j}(k)$ matrix and $\sum_{i=1}^{b} n_i$ words are needed to store this information. In Realization II, component identification and positional information have been separated so that $\sum_{i=1}^{b} n_i$ words are needed to store positional information and b words for identification information. In Realization III, $\sum_{i=1}^{b} n_i$ words are used to store component identification and positional information and b words to store $(X_k \; C^1)$ terms. In Realization IV, $\sum_{i=1}^{b} n_i$ words are used to store the $Q_{i,j}(k)$ terms. When the repetition frequency is known as well as the start of the period, only b words of component identification information must be stored.

The fixed-size memory or receiver structure of the adaptive realizations presented in this chapter is important for two reasons. First, it is a necessary realization in terms of providing a practical receiver implementation. The second interest is in regard to optimum receiver performance. In order to examine the effects of time uncertainty on detectability for the optimum receiver it is first necessary to design this receiver. The adaptive realization provides a receiver that is more manageable, in many cases, and can therefore be evaluated analytically or by simulation techniques with a digital computer. The much simpler adaptive realizations enable us to study how time uncertainty affects the performance of the optimum receiver. This is an area of study that begins in Chapter VII.

CHAPTER VI

OPTIMUM RECEIVER DESIGN — SPECIAL CASES

In Chapter V the design of optimum receivers was carried through in rather general terms. In this chapter several miscellaneous cases of receiver design will be considered. (The reader who is interested in receiver performance and the effect of time uncertainty can go to Chapter VII without loss of continuity.)

6.1 Finite Class of Periodic Equal Amplitude Pulses, Known Exactly Except for Repetition Frequency.

Consider the problem of optimum detection of a periodic pulse sequence when the pulse waveshape is known exactly but the repetition frequency can be one of a finite number of values. This class of signals can be thought of as a finite class of periodically recurrent components where each component has n_i sample values. Each component is then of the form $C^1 = (c_{1,1}, 0, 0, 0, \ldots, 0)$ in which the number of component samples is equal to n_1. The various possible repetition frequencies are specified by stating the class of n_i values.

Any of the basic four realizations could of course be considered, but Realization IV is the simplest and we will consider it. From Table 5.7 the information updating equations are given by

$$Q_{1,1}(k) = Q_{1,n_1}(k-1) f((x_k - s_k c_{1,1})) \qquad (C.11)$$

$$Q_{1,j}(k) = Q_{1,j-1}(k-1) f((x_k - s_k c_{1,j})) \qquad (C.12)$$

$$\text{for } j = 2, 3, \ldots, n_1$$

But for signals in added white Gaussian noise

$$f(x_k \mid s_k = c_{i,j}) = 1 \text{ for } j = 2, 3, \ldots, n_i \tag{6.1}$$

so Eq. C.12 becomes

$$Q_{i,j}(k) = Q_{i,j-1}(k-1) \text{ for } j = 2, 3, \ldots, n_i \tag{6.2}$$

The detection output, the likelihood ratio of the observation X_k, is given by

$$\ell(X_k) = \sum_{i=1}^{b} \sum_{j=1}^{n_i} Q_{i,j}(k) \tag{C.13}$$

which can be written using Eq. C.11 and 6.2 as

$$\ell(X_k) = \sum_{i=1}^{b} \left[f(x_k \mid s_k = c_{i,1})\, Q_{i,n_i}(k-1) + \sum_{j=2}^{n_i} Q_{i,j-1}(k-1) \right] \tag{6.3}$$

If all pulses are of equal amplitude "a", then $c_{i,1} = a$ for all i and Eq. 6.3 can be written as

$$\ell(X_k) = f(x_k \mid s_k = a) \sum_{i=1}^{b} Q_{i,n_i}(k-1) + \sum_{i=1}^{b} \sum_{j=2}^{n_i} Q_{i,j-1}(k-1) \tag{6.4}$$

But

$$\sum_{j=2}^{n_i} Q_{i,j-1}(k-1) = 1 - Q_{i,n_i}(k-1) \tag{6.5}$$

So Eq. 6.4 can be written

$$\ell(X_k) = b + \left[f(x_k \mid s_k = a) - 1 \right] \sum_{i=1}^{b} Q_{i,n_i}(k-1) \tag{6.6}$$

From Eq. 6.6 one can see that the optimum receiver forms the likelihood ratio of the unit observation, x_k, given a pulse is present and subtracts from this the value one. This is multiplied by the sum $\sum_{i=1}^{b} Q_{i,n_i}(k-1)$ which is $\ell(X_{k-1}) \sum_{i=1}^{b} p_{k-1}(s_{k-1} = c_{i,n_i})$. So $\sum_{i=1}^{b} Q_{i,n_i}$ has the interpretation of being the likelihood ratio of the observation X_{k-1} times the probability after taking the (k-1)st observation that the(k-1)st observation is the last sample value just prior to a pulse occurrence. Even though $\ell(X_k)$ requires only the sum, $\sum_{i=1}^{b} Q_{i,n_i}(k-1)$, both component identification and positional information must be updated to keep the sum up to date.

6.2 Unknown Duty Factor

The possibility of a component recurring with a duty factor which is one of b possible duty factors has already been incorporated into the receiver design equations since ν_i can have a different value for each possible component. For example, if the component is known exactly (CKE) and the time structure is Synchronous-Poisson, Realization IV becomes a single crosscorrelator that correlates the unit observation, x_k, with the component C and subtracts the bias term $C \cdot C/2$. This is then fed into b parallel ν nonlinearities and these outputs are summed. The block diagram of such a realization is shown in Fig. 6.1.

6.3 Overlapping Recurrent Component Versus Nonoverlapping Recurrent Component, Example

In previous sections, all signals considered have been assumed to be composed of nonoverlapping recurrent components. Overlapping component examples can be formulated in a similar manner. The receiver design, however, rapidly increases in complexity since overlapping means many more states are now possible.

To illustrate an overlapping component case, consider a two-sample sporadically recurrent component. Since there are only two samples to the component, there is only one possible overlap position. This overlap situation is defined as the $c_{i,z}$ state. The state diagram for this case is presented in Fig. 6.2. The updating equations follow the same general patterns as before. The results for Realization IV are

$$Q_{i,0}(k) = \left[Q_{i,0}(k-1) + Q_{i,2}(k-1)\right](1-\nu_i) \tag{6.7}$$

$$Q_{i,1}(k) = \left[Q_{i,0}(k-1) + Q_{i,2}(k-1)\right]\nu_i f(x_k \mid s_k = c_{i,1}) \tag{6.8}$$

$$Q_{i,2}(k) = \left[Q_{i,1}(k-1) + Q_{i,z}(k-1)\right](1-\nu_i) f(x_k \mid s_k = c_{i,2}) \tag{6.9}$$

$$Q_{i,z}(k) = \left[Q_{i,1}(k-1) + Q_{i,z}(k-1)\right]\nu_i f(x_k \mid s_k = c_{i,z}) \tag{6.10}$$

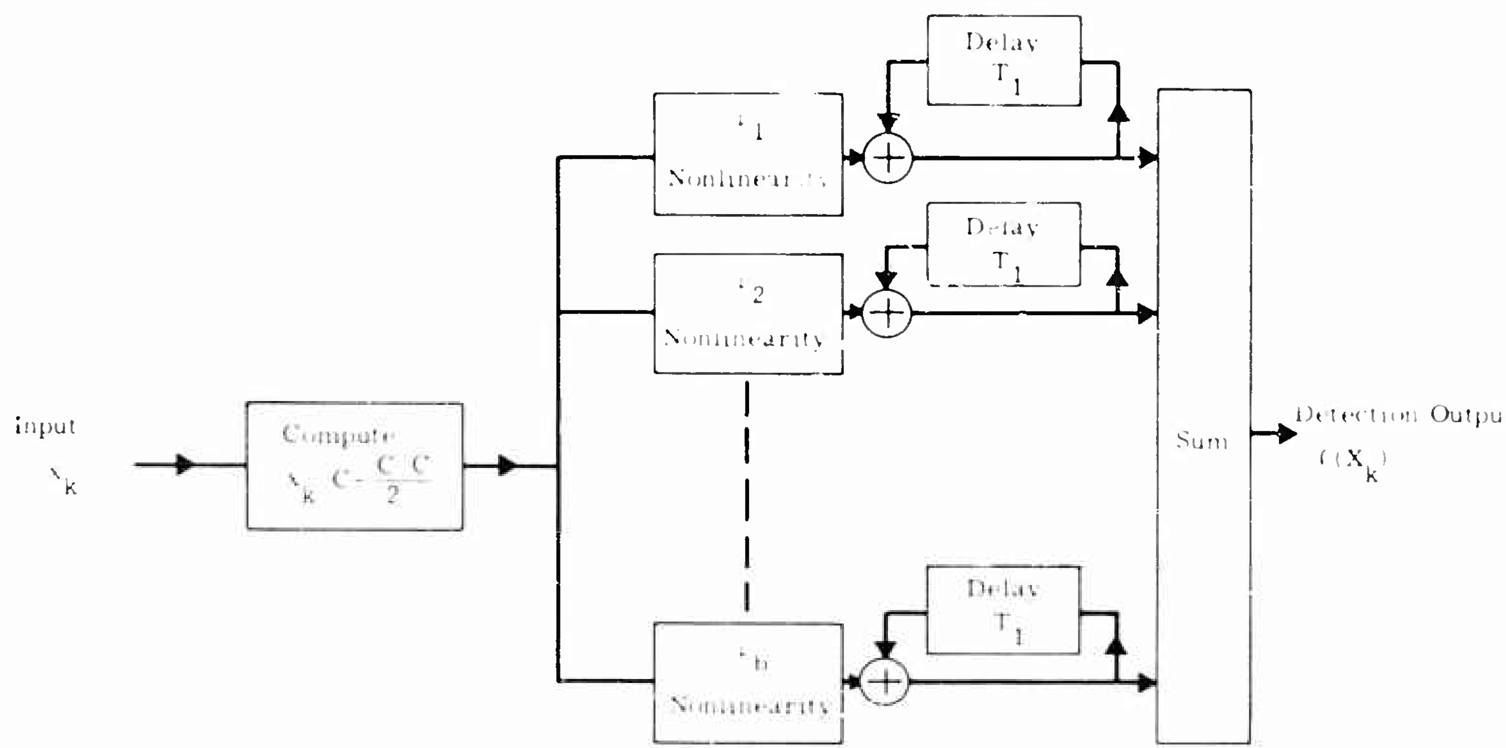

Fig. 6.1. Adaptive receiver realization, CKE, Synchronous-Poisson Time Structure, unknown duty factor.

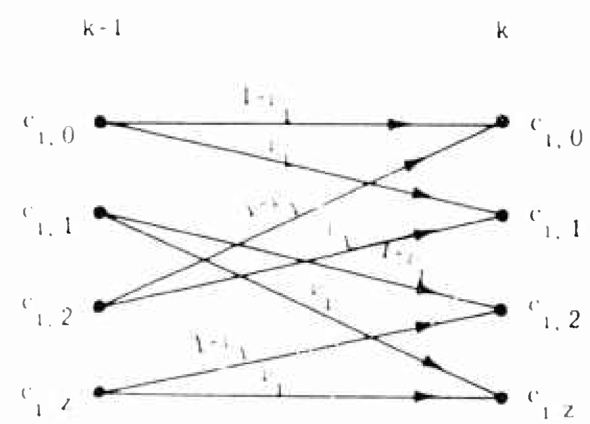

Fig. 6.2. Sporadic-Poisson process for ith component, overlapping components.

and the over-all likelihood ratio, $\ell(X_k)$, becomes

$$\ell(X_k) = \sum_{i=1}^{b} \left\{ \left[Q_{i,0}(k-1) + Q_{i,2}(k-1)\right]\left[(1-r_i) + r_i \ell(x_k \mid s_k = c_{i,1})\right] \right.$$

$$\left. + \left[Q_{i,1}(k-1) + Q_{i,z}(k-1)\right]\left[(1-r_i)\ell(x_k \mid s_k = c_{i,2}) + r_i \ell(x_k \mid s_k = c_{i,z})\right] \right\} \tag{6.11}$$

The state diagram for two nonoverlapping components is shown in Fig. 6.3. Now, the updating equations for Realization IV become

$$Q_{i,0}(k) = \left[Q_{i,0}(k-1) + Q_{i,2}(k-1)\right](1-r_i) \tag{6.12}$$

$$Q_{i,1}(k) = \left[Q_{i,0}(k-1) + Q_{i,2}(k-1)\right] r_i \ell(x_k \mid s_k = c_{i,1}) \tag{6.13}$$

$$Q_{i,2}(k) = Q_{i,1}(k-1)\ell(x_k \mid s_k = c_{i,2}) \tag{6.14}$$

and the detection output is

$$\ell(X_k) = \sum_{i=1}^{b} \left\{ \left[Q_{i,0}(k-1) + Q_{i,2}(k-1)\right] \left[1-r_i + r_i \ell(x_k \mid s_k = c_{i,1})\right] + Q_{i,1}(k-1)\ell(x_k \mid s_k = c_{i,2}) \right\} \tag{6.15}$$

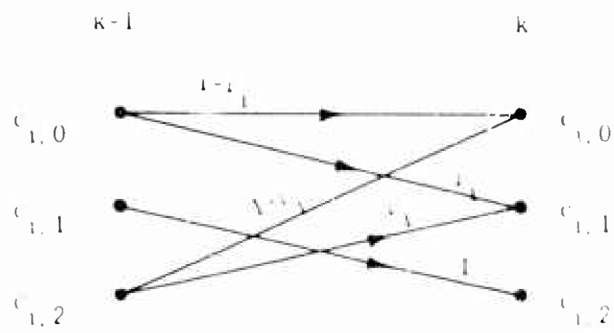

Fig. 6.3. Sporadic-Poisson process for ith component, nonoverlapping components.

Comparing Eqs. 6.11 and 6.15 it is apparent that if component overlap is possible the receiver complexity increases. It is important to note, however, that the receiver can still be realized with a fixed size memory when component overlap is possible.

CHAPTER VII

PERFORMANCE OF THE OPTIMUM ADAPTIVE RECEIVER

Chapters V and VI considered several optimum receiver designs. Although these receivers are optimal for signal detection, the detection performance remains to be investigated. Detection performance, which may be summarized by a receiver operating characteristic (ROC), depends upon waveform uncertainties and noise of the particular problem. The receiver designs in Chapter V are rather general. In this chapter the performance of the optimum receiver is evaluated for several specific signals. Emphasis will be placed on evaluation of some special, useful examples of the Synchronous-Poisson and Periodic Time Structures. The evaluation of an optimum receiver for the detection of a signal with a Synchronous-Poisson Time Structure is new work. The evaluation of the receiver for a Periodic Time Structure signal is taken from the literature and is included for comparison purposes (Ref. 1).

We are interested in the following items:

1. The operation of an adaptive receiver realization.
2. Detection performance of the optimum adaptive receiver for some special cases.
3. Effect of component uncertainty on detectability.
4. Effect of component recurrence time uncertainty on detectability.
5. Comparison between the optimum adaptive receiver and the simple energy detector.
6. Comparison of the performance of other suboptimum receivers with the optimum receiver.

Before considering these items, let us briefly review the basic techniques of receiver evaluation.

7.1 Review of Receiver Evaluation

The detection performance of a receiver performance may be summarized on a receiver operating characteristic (ROC). The ROC is a graphical means of portraying the quality of detection in a situation involving signal, noise and a receiver (Refs. 1 and 19). When noise is present, the detection process is always accompanied by the possibility of making errors. In the basic decision problem there are two types of errors, false alarms and misses, and two types of correct decisions, correct detection and correct rejection. A false alarm is the result of responding "signal present" when the noise was actually the cause and a miss is the result of responding "signal absent" when signal was indeed present. A correct detection is the result of responding "signal present" when signal was actually present, and a correct rejection is the result of responding "signal absent" when signal was indeed absent. In a detection problem there are probabilities associated with each of these types of errors and correct decisions. The notation used for these probabilities is

$P(A\|N)$	probability of false alarm
$P(B\|SN)$	probability of a miss
$P(A\|SN)$	probability of a correct detection
$P(B\|N)$	probability of a correct rejection

where

A	is the response "signal present"
B	is the response "signal absent"
SN	the hypothesis "signal mixed with noise"
N	the hypothesis "noise alone"

The probabilities of errors and correct decisions are not independent since

$$P(A|SN) + P(B|SN) = 1 \tag{7.1}$$

and

$$P(A|N) + P(B|N) = 1 \tag{7.2}$$

Therefore all of the available information can be conveyed by a plot of the relationship between the probability of detection, P(A|SN), and the probability of false alarm, P(A|N). Such a plot is made by determining the probability of detection versus the probability of false alarm for all possible threshold settings on the receiver output.

An ROC is called "normal" if it can be parameterized by the normal probability distribution as follows:

$$P(A|SN) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(u+d')^2}{2}} du \qquad (7\text{-}3)$$

when

$$P(A|N) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \qquad (7\text{-}4)$$

If we use the notation

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}} du \qquad (7\text{-}5)$$

then it becomes convenient to described the normal ROC as

$$P(A|SN) = \Phi(x + d'), \text{ when } P(A|N) = \Phi(x) \qquad (7\text{-}6)$$

Therefore, when the ROC is normal we can characterize the entire curve by the parameter, d'. It is frequently convenient to plot the ROC on double probability paper which linearizes the normal ROC curves.

It is interesting that for signals in added white Gaussian noise, the two extremes of knowledge regarding the signal results in a normal ROC. If a signal of energy E is known exactly and the noise power per cycle per second is N_0, then the ROC for the optimum receiver is normal and the parameter d' has the value

$$d' = \sqrt{\frac{2E}{N_0}} \qquad (7\text{-}7)$$

This case is a valuable reference case since it is an upper bound on possible detection performance. It is sometimes convenient to plot the ROC as a function of the parameter $d = (d')^2$. The ROC's for the signal known exactly case for several values of the parameters $d = (d')^2$ are plotted in Fig. 7.1.

When the parameters of the signal waveform are very uncertain and distributed over wide ranges we are at the other extreme of knowledge regarding the signal. The normal ROC is frequently a limiting curve in such situations. For example, when the signal itself is a sample of white Gaussian noise of T seconds duration in a bandwidth W cycles per second wide, and the signal-to-noise ratio is sufficiently small and WT is sufficiently large, then d' is approximated by (Ref. 1)

$$d' = \sqrt{WT}\left(\frac{S}{N}\right)_{in} \tag{7.6}$$

where $\left(\frac{S}{N}\right)_{in}$ is the input signal-to-noise ratio.

In general, in order to evaluate an optimum receiver, we need the distribution of the likelihood ratio, or a monotone function of it, under both signal and noise and noise alone. These density functions may be as shown in Fig. 7.2. For a given threshold setting, the striped area under the $f_{s(t)+n(t)}$ curve is equal to $P(A|SN)$ and the cross-hatched area under the $f_{n(t)}$ curve is equal to $P(A|N)$. An ROC is obtained by plotting $P(A|SN)$ versus $P(A|N)$ for all possible threshold settings. In practice, there is often considerable difficulty in expressing analytically the probability density functions of the likelihood ratio under signal and noise and noise alone. Although the appropriate integrals can be specified, their evaluation frequently becomes difficult.

An alternative technique for receiver evaluation is one in which a digital computer is used as an experimental tool. This is Monte Carlo technique. The receiver operations are simulated on the digital computer and the signal mixed with noise and noise density functions are sampled. Even though this technique is an approximate one, it is quite useful. However, the usefulness of this method is limited by the number of trials that can be run feasibly on the computer.

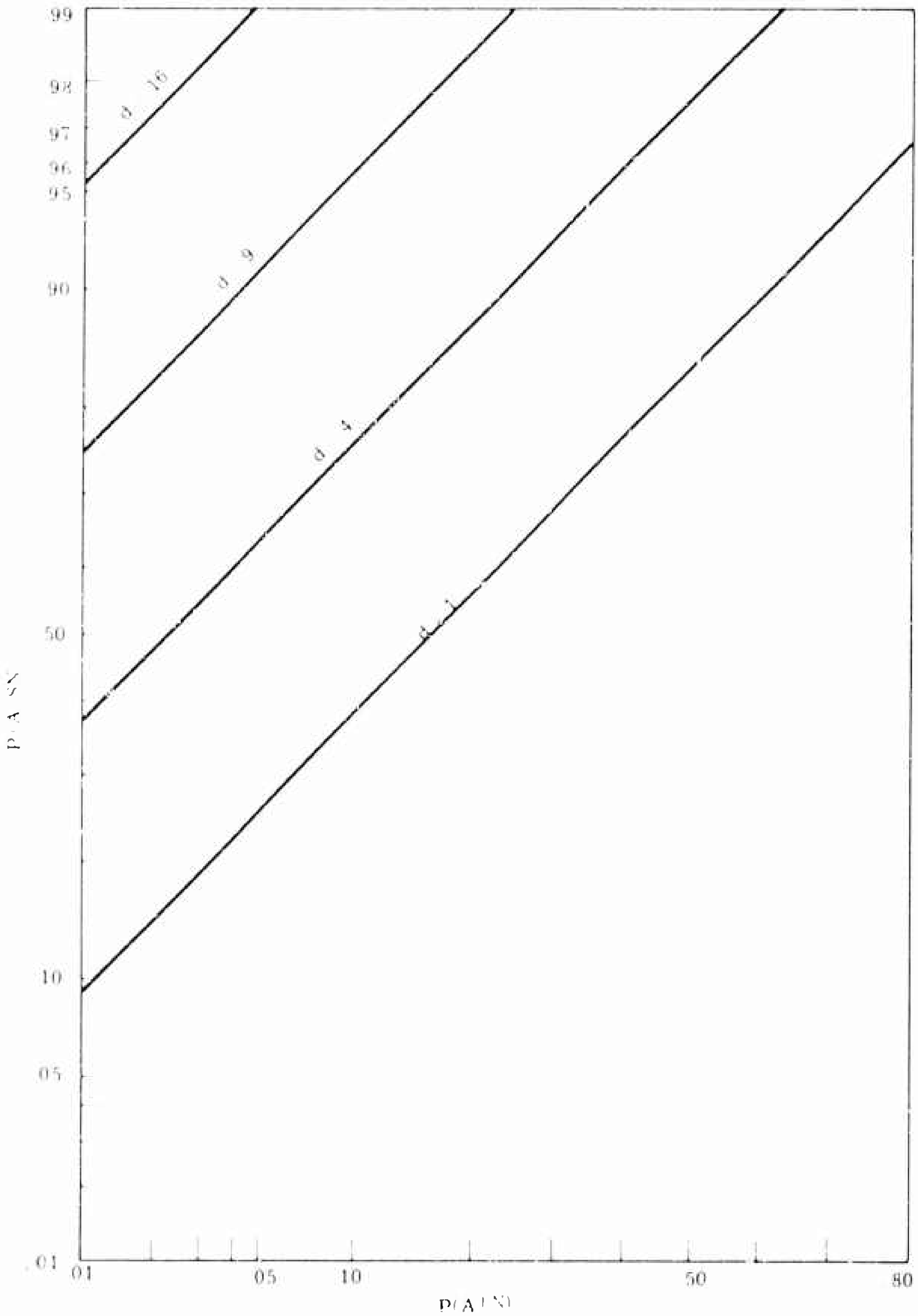

Fig. 7.1. ROC for signal known exactly in added white Gaussian noise.

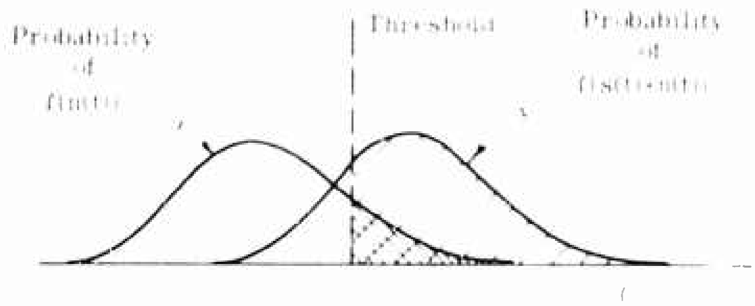

Fig. 7.2 Probability density function of likelihood ratio under noise and signal plus noise

In this chapter, the evaluation of the receiver is done analytically where possible and supplemented by results from the Monte Carlo method. Before obtaining the ROC for several cases, we wish to display the operations of an adaptive receiver realization.

7.2 Simulation and Operations of an Adaptive Receiver Realization

Although the design of the optimum adaptive receiver realization has been developed for a number of cases in previous chapters, the question remains as to how it operates. Recall that the approach to adaptive receiver design has been an optimal one. The adaptive design is a result of realizing the optimum receiver in a sequential manner. Although the detection performance of various realizations of an optimum receiver are equivalent, the receiver form and the operations performed by these receivers may appear quite different. In this section we consider simulating an adaptive receiver on a digital computer for a rather specific case to observe how the detection and classification outputs grow or decay with time.

An optimum adaptive receiver realization was simulated for a signal having a component ensemble of eight orthogonal components and a Synchronous Poisson Time Structure. Both a detection output, $\ln \ell(X_k)$ and a classification output, $p_k(C^1|SN)$, were printed out sequentially in time. The receiver simulated was that of Realization IV, whose operations are summarized in Table 5.5. Of course, $\ln \ell(X_k)$ and $p_k(C^1|SN)$ are available from any of the four realizations. The adaptive receiver simulation was programmed for an IBM 7090 digital computer. Because of the large amount of output data, the digital results were converted to analog form in a digital analog converter and the results printed out on a pen recorder.

The component ensemble chosen consists of eight equal energy orthogonal components. That is

$$\int_0^{T_1} C^i(t)\,C^j(t)\,dt = E_c\,\delta_{ij} \tag{7.9}$$

where

E_c is the component energy into 1 ohm

$C^i(t)$ is the ith component waveform

T_1 is the component duration

δ_{ij} is the Dirac delta-function

The noise alone and signal plus noise density functions were approximated by a representative set of 59 discrete probabilities. Details concerning the computer simulation techniques are contained in Appendix D.

One hundred runs were made. The triggering probability, was 1 and $\frac{2E_c}{N_0} = 1$. A priori each of the eight possible components was assumed equally likely. Figures 7.3 through 7.13 show the result of these runs. Twelve functions of time are displayed simultaneously for each run. The total duration of each of the runs plotted is 1000 times a component duration. Let us define each of these functions. The function labeled "signal energy" is the total accumulated signal energy from the start of transmission. Full scale corresponds to 64 occurrences of the component. After 64 occurrences the pen is reset to zero, i.e., the signal energy is plotted mod 64.

The third column labeled [illegible] is the sequential detection output when signal plus noise is present. The scaling of the detection output is from −9 to +9. This output is the value of the logarithm of the likelihood ratio at each time, t_k, for the particular set of observations obtained up to that time.

The fourth through eleventh columns are classification outputs. The scaling is from zero to a full scale of one. Each of the columns, labeled $P(C^i|SN)$ for $i = 1, 2, \ldots, 8$ is an updated probability of presence of the ith component conditional to the fact that signal plus noise (i.e., the recurrence phenomenon plus noise) is present.

Fig. 7.3 Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure signal and noise, runs 0 - 8.

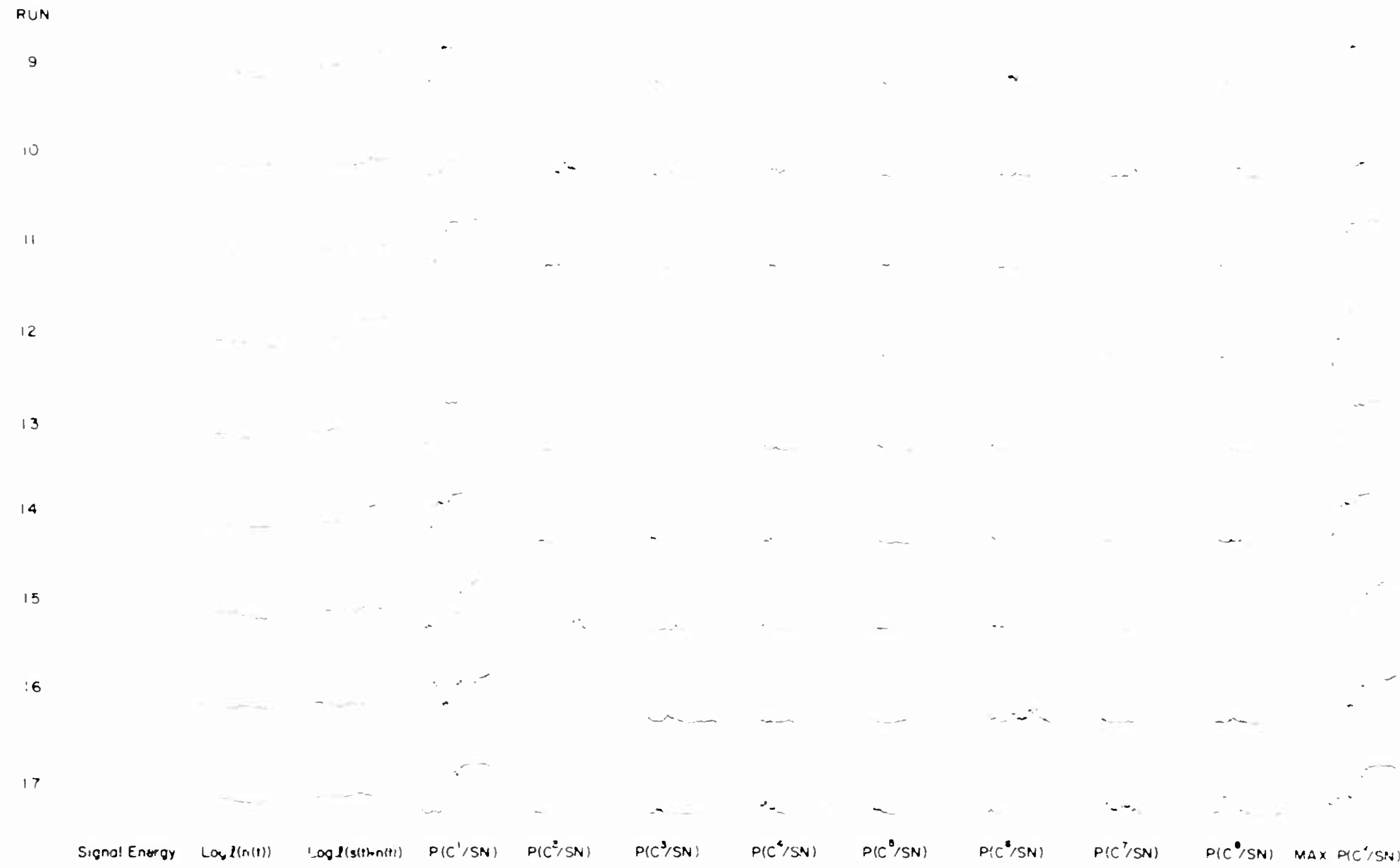

Fig. 7.4. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 9 - 17.

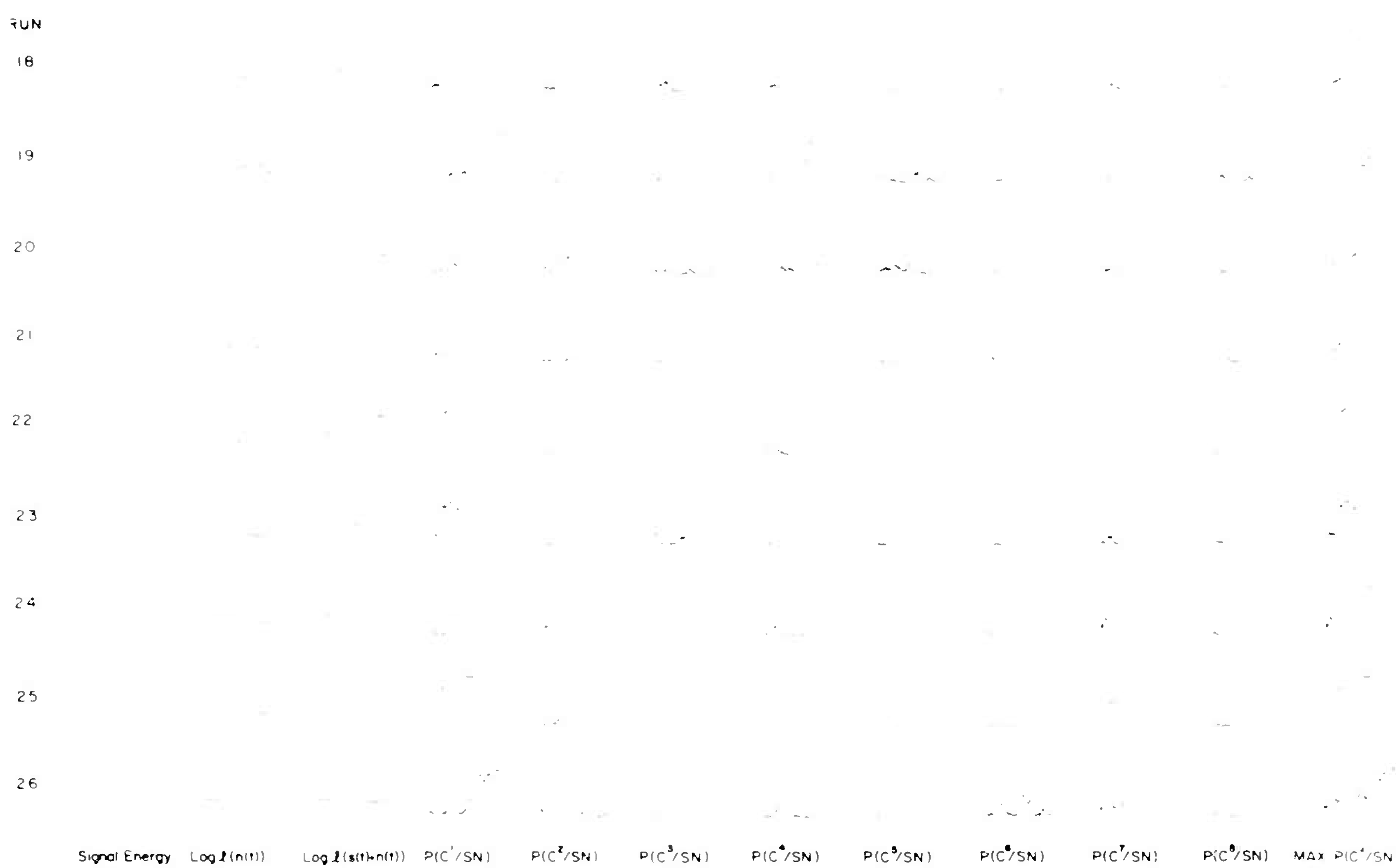

Fig. 7.5. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 18 – 26.

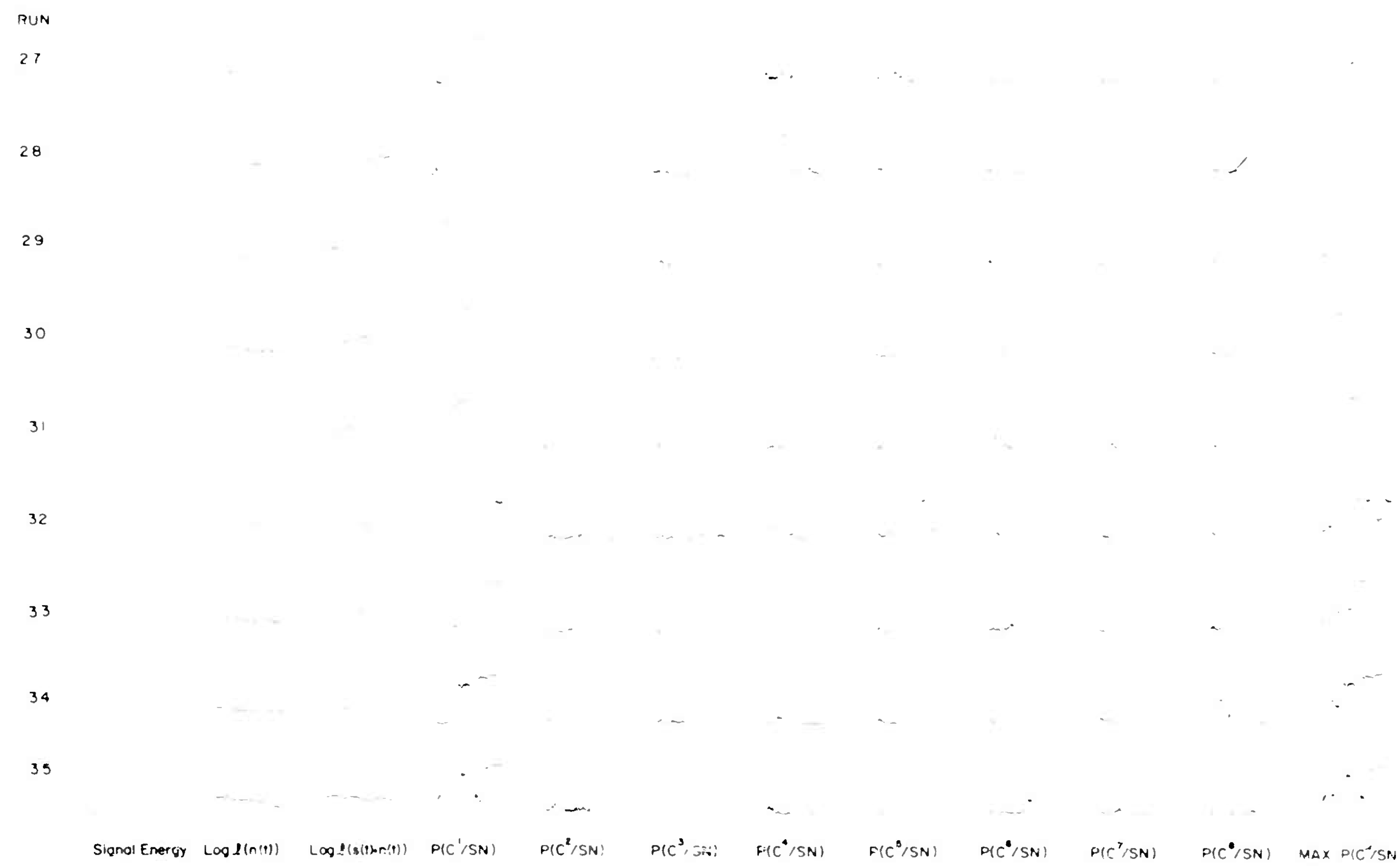

Fig. 7.6. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 27 - 35.

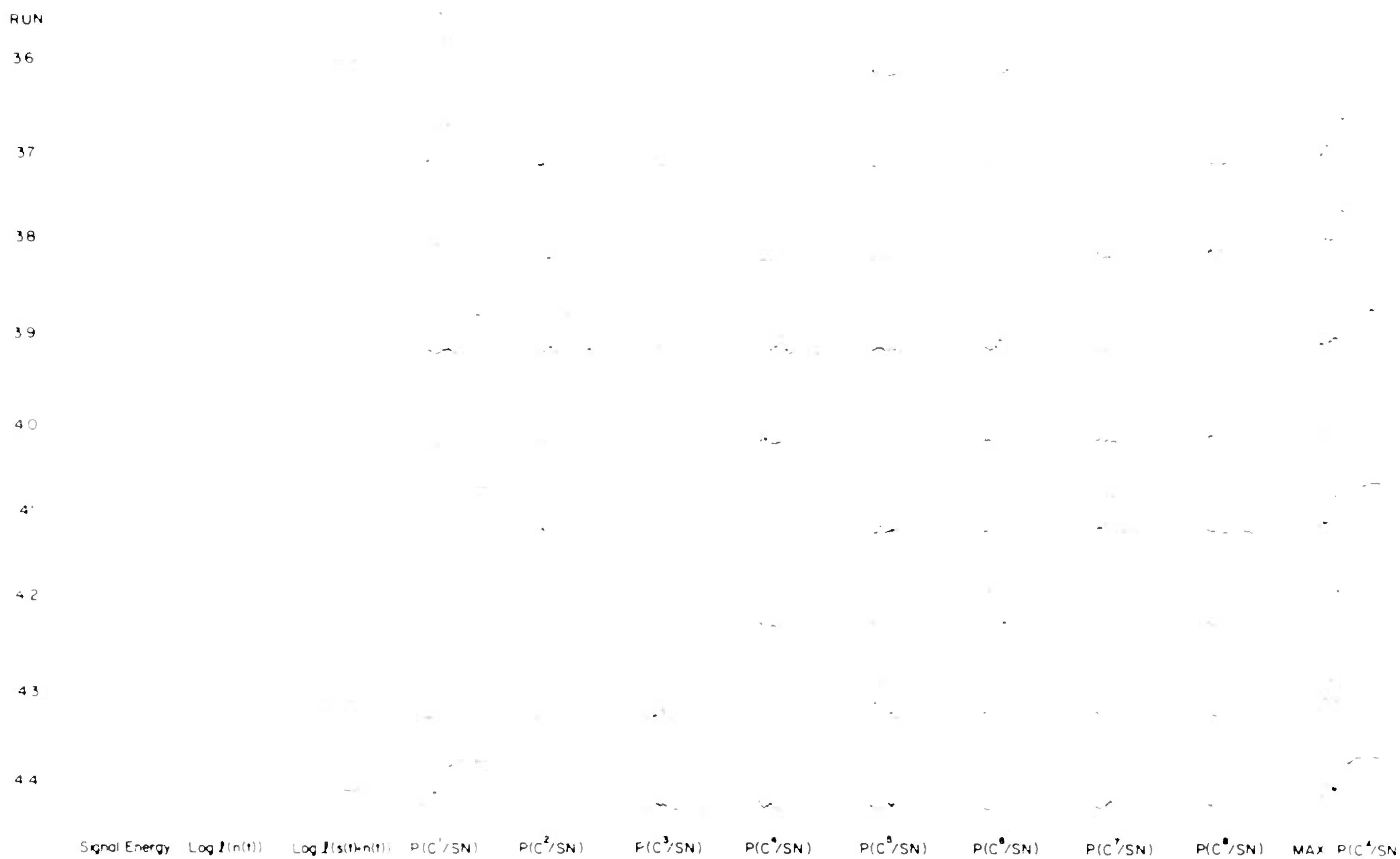

Fig. 7-7 Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 36 - 44.

Fig. 7.8 Digital computer simulation of adaptive receiver, one of eight orthogonal components. Synchronous-Poisson Time Structure, signal and noise, runs 45 - 53.

Fig. 7.9 Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure signal and noise, runs 54 - 62.

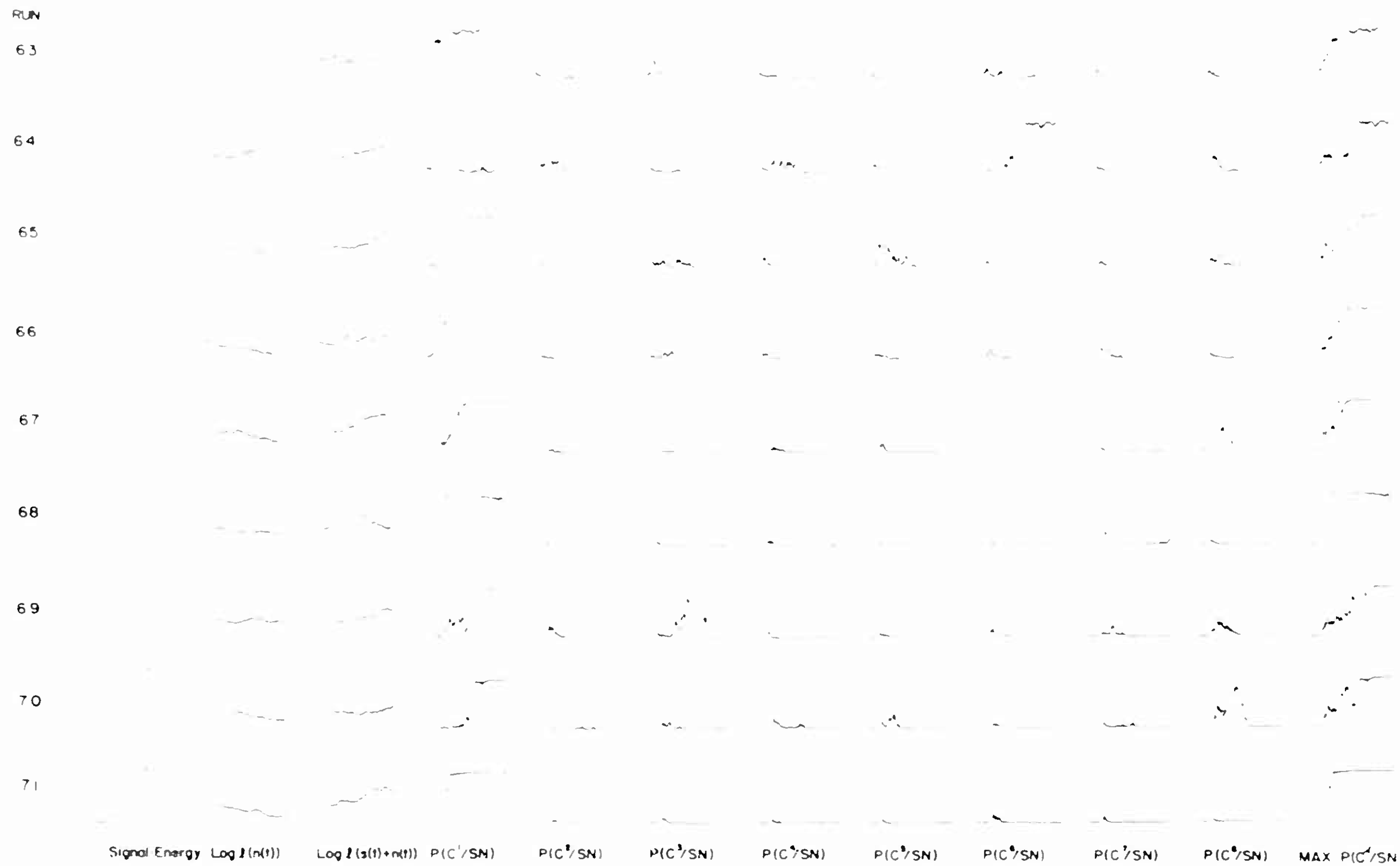

Fig. 7.10 Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 63-71.

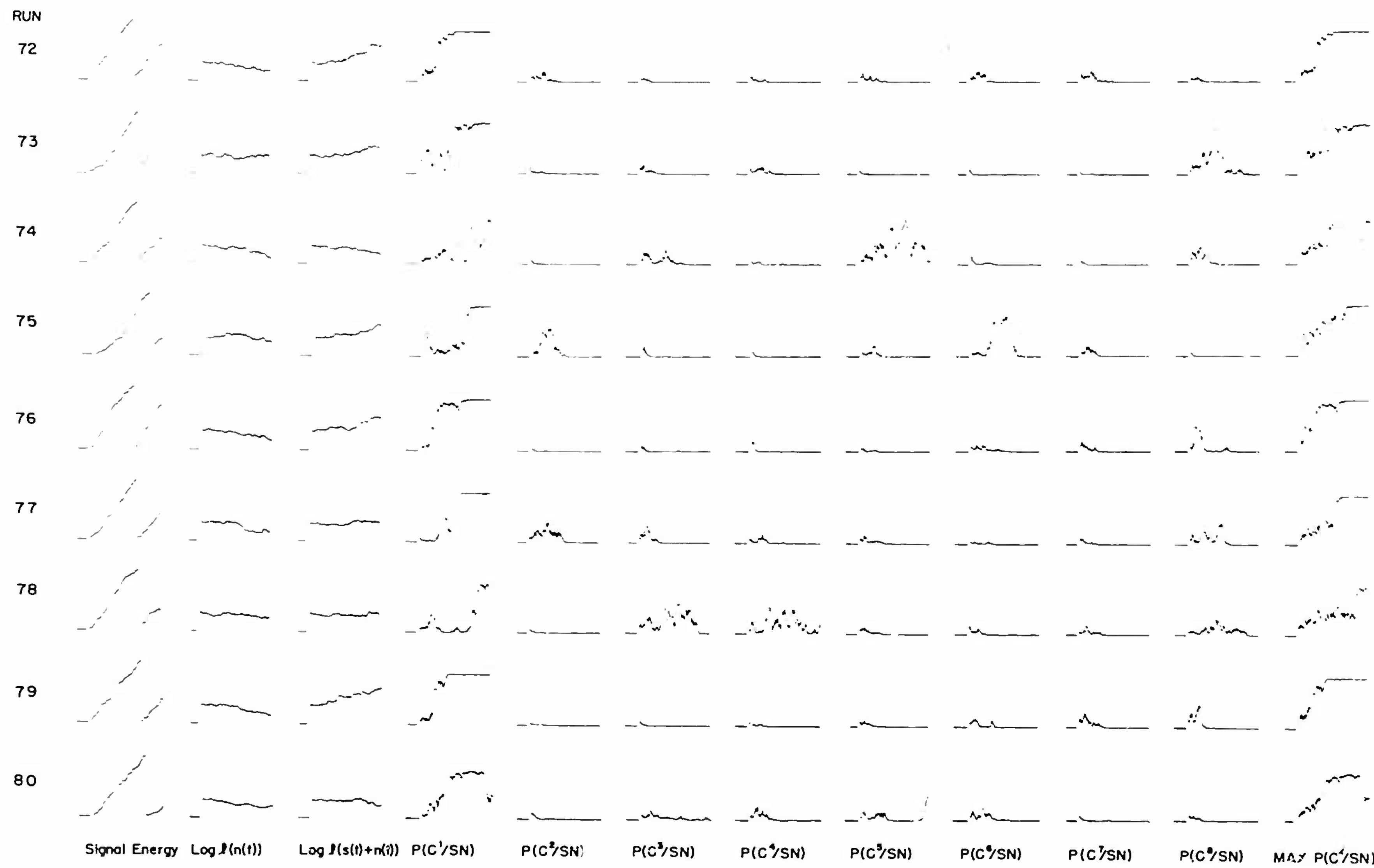

Fig. 7. 11. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 72 - 80.

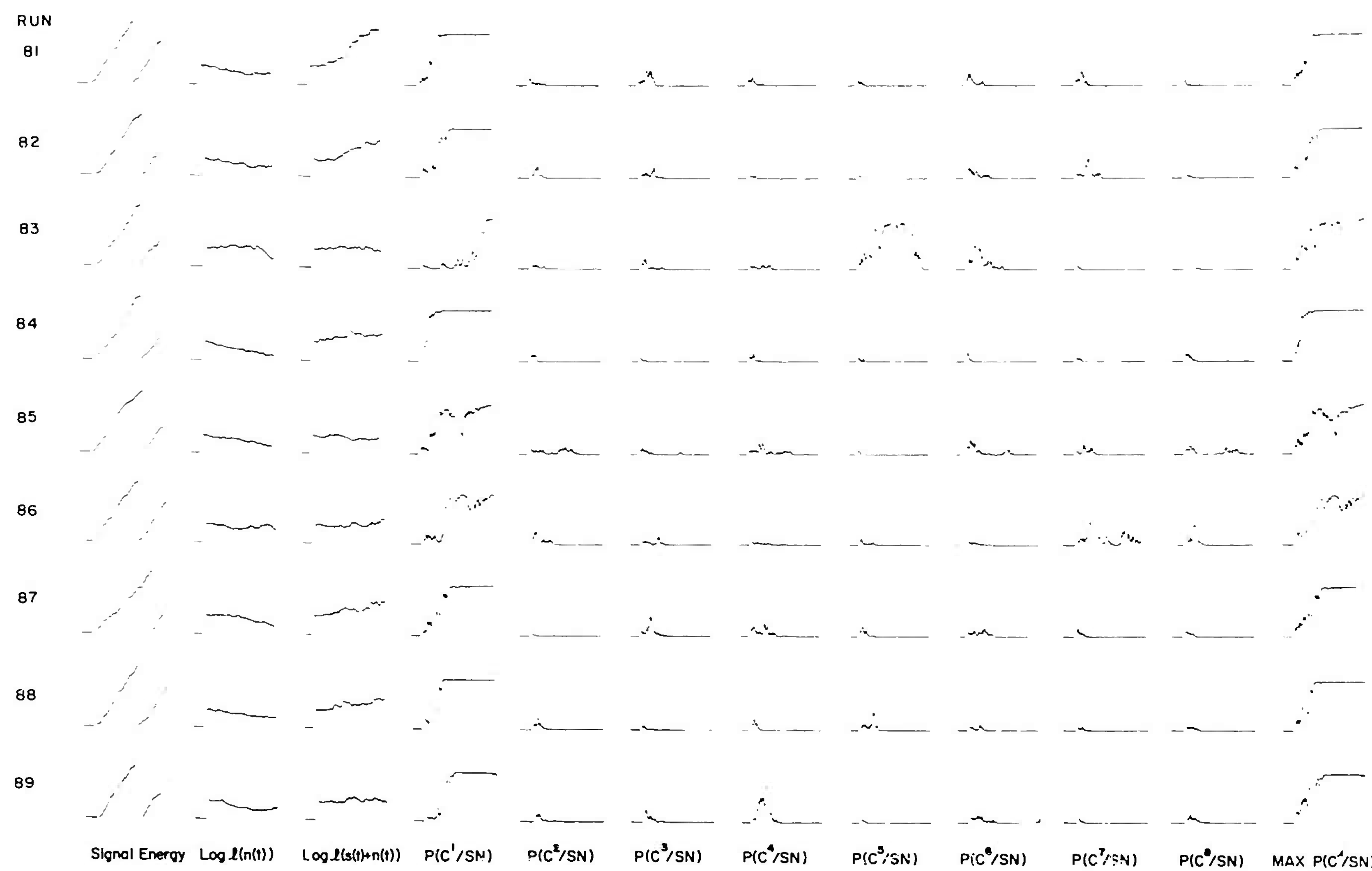

Fig. 7. 12. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 81 - 89.

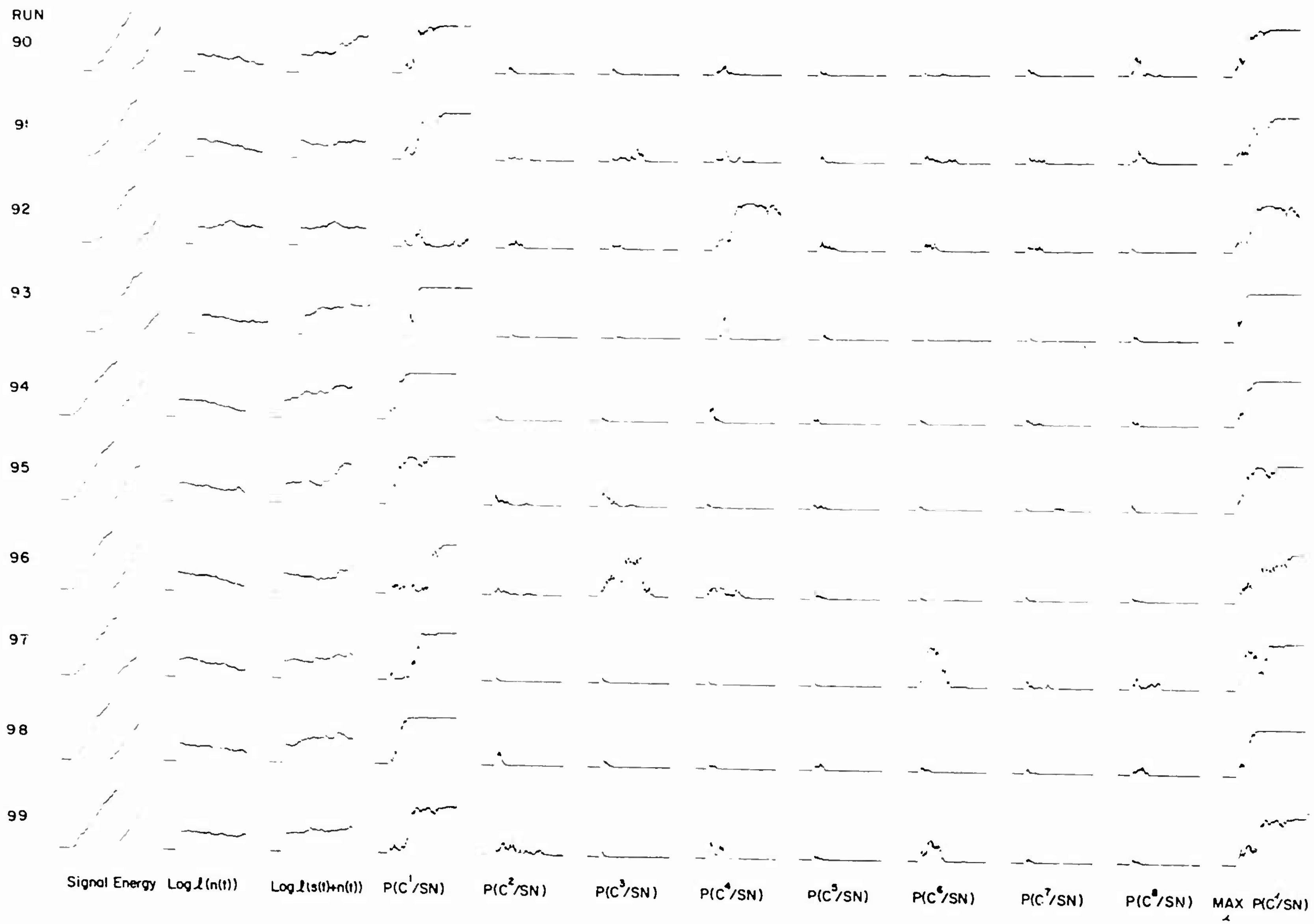

Fig. 7. 13. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, signal and noise, runs 90 - 99.

The final column, labeled $\max_i P(C^i|SN)$, is a function constructed from the classification outputs. It is obtained by picking the largest of the eight classification outputs at each instant in time. The scaling goes from zero to one. The second column, labeled Log $f(n(t))$, is the sequential detection output when noise alone is present to the receiver input. (The initial negative pulse in each of the traces of detection and classification outputs should be ignored.)

In runs 0 through 99, component C^1 was the actual component transmitted and it was recurrent in all of the runs. A wide variety of detection and classification responses result. The reader can obtain an idea of the number of component arrivals by looking at the signal energy plotted as a function of time in the first column.

Since each of the eight components was assumed equally likely at the start of a transmission, the probabilities, $P(C^i|SN)$, were each initially set at 1/8. In a majority of runs, the classification output, $P(C^i|SN)$, rises abruptly after a sufficient number of components have recurred. Due to the noise and the fluctuations in total signal energy from run to run, the time of rapid build-up of $P(C^i|SN)$ varies. For instance, in runs 0, 3 and 71 the abrupt changes occur early whereas in a run such as 70 there is a considerable delay before the receiver "learns" which component is being transmitted. On the other hand, there are runs where no abrupt rise in the classification output, $P(C^i|SN)$, occurs even though C^1 is being transmitted. Such cases are shown in runs 20, 26, 35, 59, 68 and 74. There are, in fact, a few runs in which the receiver has "learned" the wrong component. This has happened in runs 64 and 92.

To see how the detection and classification outputs respond to noise alone, an additional set of 27 runs were made. These runs are shown in Figs. 7.14 through 7.16 as runs 100 through 126. The labeling and scaling in these runs is the same as in the first 100 runs except an additional quantity, labeled "Selected i", is plotted in the last column. This is a plot of the component whose probability, $P(C^i|SN)$, is a maximum at each instant of time. The scaling on the "Selected i" column is quantized in unit steps from zero to eight. In the noise alone runs of Figs. 7.14 through 7.16 the detection output, in general, drifts downward. In general, the classification outputs, $P(C^i|SN)$, give no consistent indication of any particular component. There are occasions, such as runs 115 and 122, where the receiver "learns a component even though noise alone is present. The fact that the

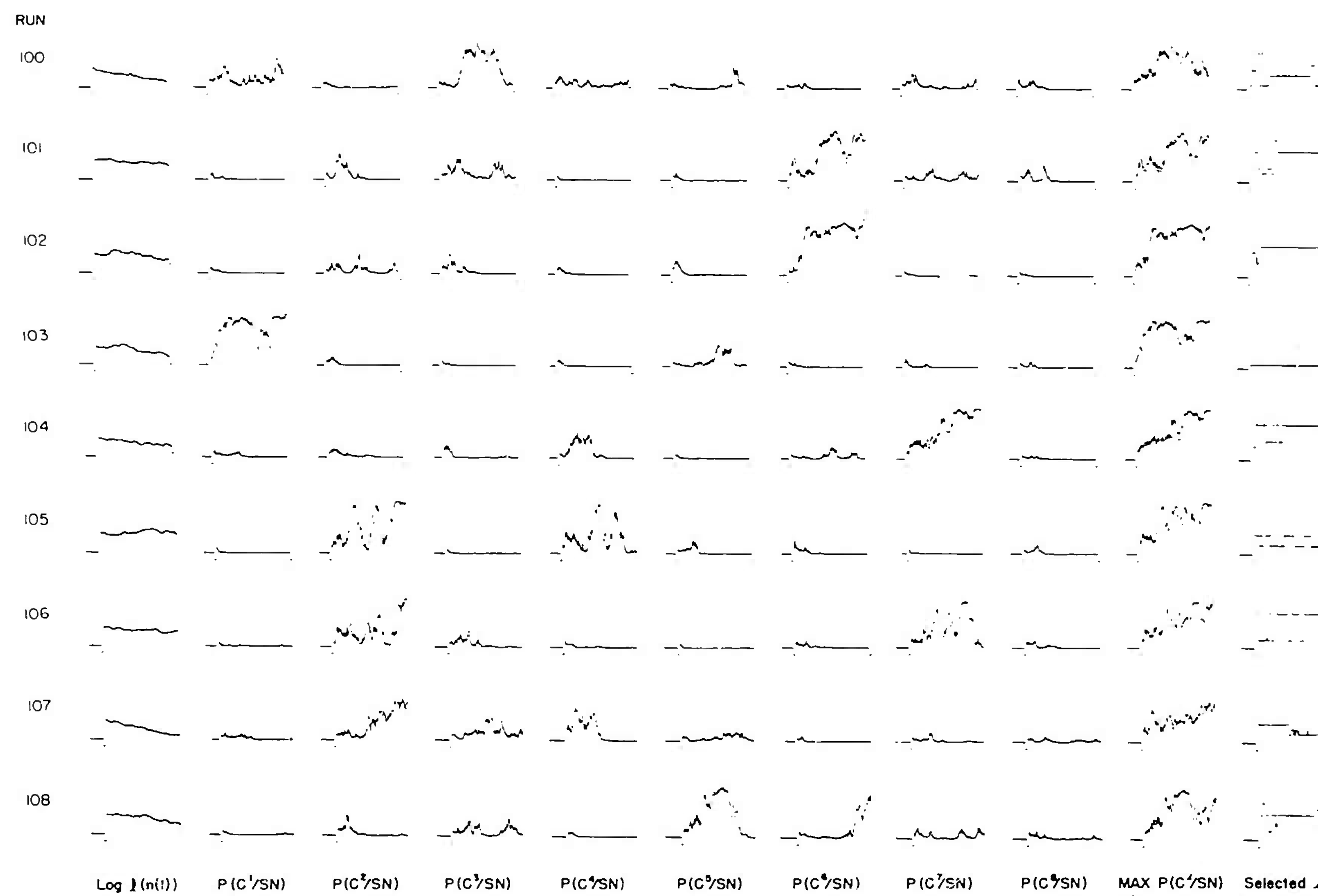

Fig. 7.14 Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, noise alone, runs 100 - 108.

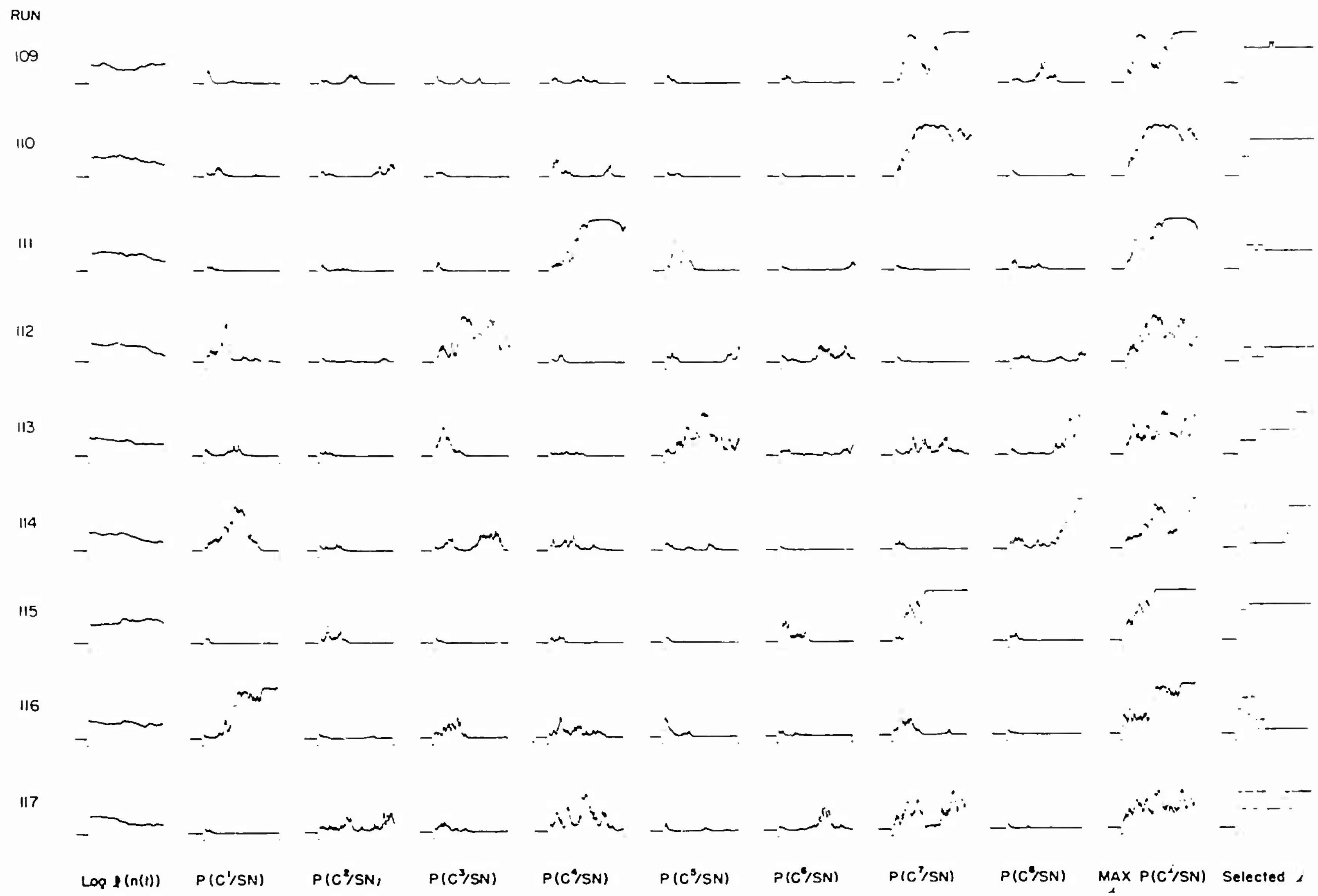

Fig. 7.15 Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, noise alone, runs 109 - 117.

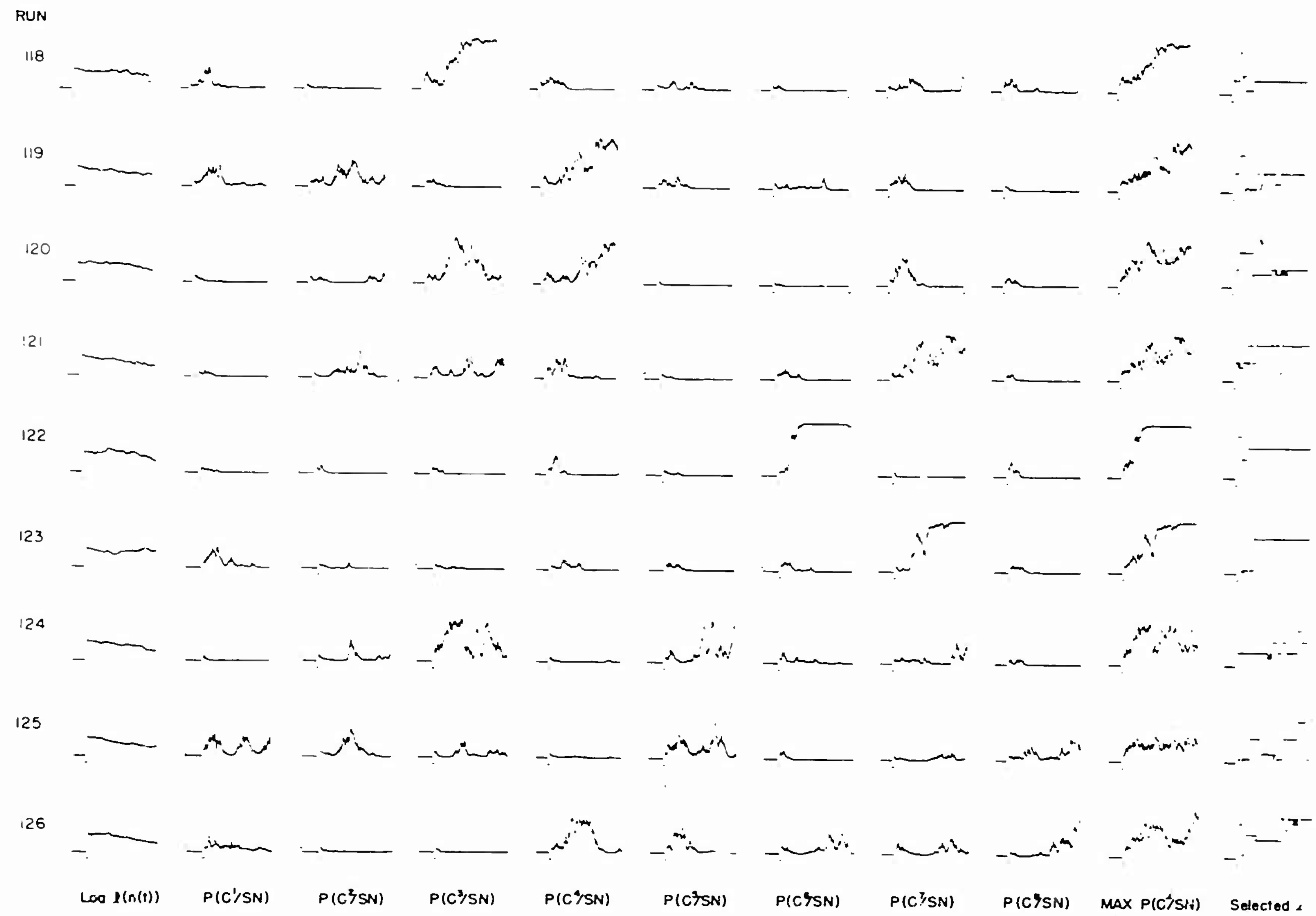

Fig. 7. 16. Digital computer simulation of adaptive receiver, one of eight orthogonal components, Synchronous-Poisson Time Structure, noise alone, runs 118 - 126.

adaptive realization occasionally indicates the "wrong" component is of course not a fault of the adaptive realization but a reflection of the statistical nature of the noise and signal uncertainties.

The 127 individual runs have been displayed in Figs. 7. 3 through 7. 16 in order to observe how an optimum receiver which has been realized by an adaptive realization operates. Recall that an important motivation for the development of an optimum adaptive receiver is the complex nature of the nonsequential realization (see Chapter V). In the problem simulated here, there are eight possible components in the component ensemble and 2^{1000} possible time patterns of component occurrences. In terms of a formal nonsequential receiver realization, this would require the storage of a priori probabilities for $8 \cdot 2^{1000}$ thousand-dimensional signal vectors along with $8 \cdot 2^{1000}$ multipications of each of these probabilities by the likelihood ratio conditional to each of the time patterns. Such a receiver realization is much too complex to be simulated even on modern digital computers. Such a realization also appears "nonadaptive." On the other hand, by going to the adaptive mode, the optimum receiver has been simulated by storing eight probabilities, $P(C^i | SN)$, and continuously updating them as the observations come in. Although the primary reason for operating the optimum receiver in the adaptive mode was to greatly reduce receiver complexity, the resultant adaptive realization displays "learning" features which are hidden in the nonsequential mode of operation.

Although it is interesting to look at each of the runs of the adaptive receiver, the variety of receiver outputs is too great to tell just how well the receiver is performing. In order to evaluate the adaptive receiver properly we need to obtain the ROC (receiver operating characteristic). This can be done by properly using the data from all 100 runs to obtain the approximate ROC at several points. From these ROC's we can obtain a meaningful estimate of the way the detectability builds in time. This will be deferred to Section 7. 3. 2. 3.

7. 3 Receiver Performance

In Section 7. 1 the basic problem of receiver evaluation in terms of the ROC was reviewed. In Section 7. 2 individual operating runs of an optimum adaptive receiver are displayed. The ROC for a number of cases will be obtained in this section.

It is necessary to determine the probability density function of the likelihood ratio or a monotone function of it under both hypotheses in order to obtain the ROC for the optimum receiver. Two approaches can be used to obtain these density functions -- an analytical approach and an experimental approach. While the analytical approach can lead to "exact" answers, considerable difficulties in performing the necessary integrations frequently result. The experimental approach referred to is a Monte Carlo technique using the digital computer as an experimental tool. In this approach it is necessary to represent the input noise and signal plus noise density function by a discrete set of probabilities. It is also necessary to make a sufficient number of runs in order to obtain confidence in the results. The total number of runs, however, is limited by the cost of computing time.

The receiver evaluation is separated into three parts according to the time structure; the Periodic, Synchronous-Poisson, and Sporadic-Poisson Time Structures. The simplest time structure is the periodic structure. This type of time structure is characteristic of many active detection and ranging systems working in a stable medium; the detection of such signals has been well understood for a number of years. It is included here so comparisons can be readily made. The next order of complexity in time structure is the Synchronous-Poisson Time Structure. It is like the periodic case in that if a component occurs, it starts only at synchronous times. That is, it starts only at integral multiples of a component duration. If it were always triggered a periodic signal would be generated. However, it is only triggered some small percentage of the time. The third order of time structure complexity is the Sporadic-Poisson Time Structure. In this time structure a component can start at times other than multiples of a component duration.

The component uncertainty is represented by a component ensemble consisting of equal energy orthogonal components. There are M components of common duration T_1 The minimum bandwidth must be $\frac{M}{2T_1}$ so that $2WT_1$ is at least M.

In this chapter a number of experimental ROC's have been obtained for the Synchronous-Poisson Time Structure. These will be compared with the known performance for the Periodic Time Structure. Since uncertainty increases in going from the Periodic to the Synchronous-Poisson to the Sporadic-Poisson Time Structure, and since performance necessarily drops as uncertainty increases, the results of the Synchronous-Poisson case can be used as an upper bound on the detection performance for the Sporadic-Poisson case.

We are especially interested in cases where the duty factor is low and the input signal-to-noise ratio is sufficiently small so that a receiver could not make a good decision on the basis of a single component occurrence.

In the classical theory, the SKE (signal known exactly) is an important reference case. For this case there is no uncertainty regarding the signal. In the recurrent component problems we will use the CKE (component known exactly) for various time structures as a basic reference case. Since component uncertainty creates a more difficult receiver evaluation problem, it is useful to have the detection performance of the CKE case as an upper bound.

7.3.1 Receiver Performance - Periodic Time Structure (Known Period, Known Start).

7.3.1.1 CKE (Component Known Exactly). When a known component recurs in time, and the period and starting time of the component are known, the signal is known exactly. This is then an SKE (signal known exactly) case and the detectability is

$$d = k\frac{2E_c}{N_o} \qquad (7.10)$$

where

- k is the number of components observed
- E_c is the energy of a single component of duration T_1
- N_o is the noise power per unit bandwidth

Since k is a measure of time, this equation shows that d increases linearly with time.

7.3.1.2 CKS (component known statistically), One of M Orthogonal Components. This is a case in which there is uncertainty about the component, but the period and start of the component are known. This is one of M orthogonal signals, and one can use the results in the literature to obtain the detectability, d (Ref. 1). At time t_k the total signal energy is $k E_c$ and the detectability is

$$d = \ln\left[1 - \frac{1}{M} + \frac{1}{M}\exp\left(k\frac{2E_c}{N_o}\right)\right] \qquad (7.11)$$

where M is the number of orthogonal components in the component ensemble. By varying M an idea of the effect of component uncertainty on detectability can be obtained.

7.3.1.3 Performance of the Energy Detector. A very simple detector which is not optimum is the power or energy detector. It is interesting to compare the optimum receiver to it, in order to see how much performance is gained by the more complex optimum receiver. For small signal-to-noise ratios, the performance of the energy detector is (see Appendix E)

$$d = \frac{k}{4M}\left(\frac{2E_c}{N_o}\right)^2 \tag{7.12}$$

7.3.1.4 Effect of Component Uncertainty. The effect of component uncertainty is shown in Fig. 7.17. Using Eqs. 7.10 and 7.11, the detectability is plotted as a function of time for no component uncertainty, (CKE), and two degrees of uncertainty, (CKS, one of eight and one of 100 orthogonal components). There is no uncertainty in the time structure since this is a periodic case in which the period and start of the component is known. The detectability for the CKE case rises linearly with time. There is a threshold effect for the two CKS cases plotted. The slope of the detectability curves for CKS approaches the slope of the CKE curve after the receiver has obtained sufficient evidence that a particular component has been transmitted. The effect of component uncertainty is a rather mild function of M in that the vertical displacement of the CKS from the CKE curve is $\ln M$ for large processing times. If one compares the detectability of the CKE and CKS case, the ratio eventually approaches unity. In either case, the CKE curve provides a useful upper bound on detectability.

7.3.1.5 Performance of a Receiver That Does Not Utilize Repeatability of a Component. The optimum receiver for M orthogonal components, whose performance is given by Eq. 7.11, would look as though it "learned" which component is being sent if it had been realized with an adaptive realization. Let us now consider the performance of a suboptimum receiver that is optimum for a component duration but which does not utilize what it has "learned" about the component to process subsequent information. In other words, at the start of each occurrence of the periodic component the receiver anticipates one of M orthogonal components and it can use no component information obtain from the previous observations. The detectability for each interval T_1 (a component

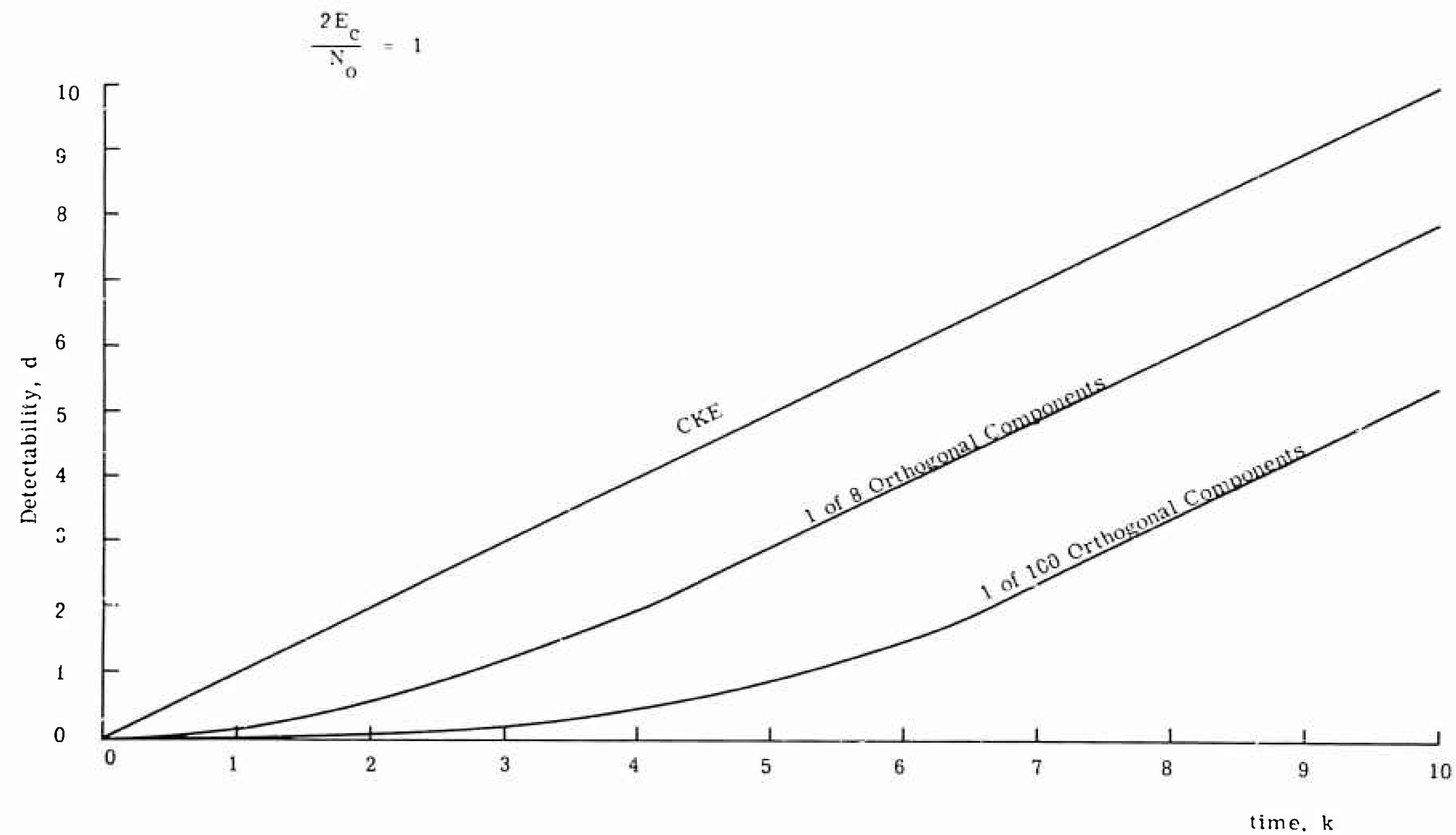

Fig. 7.17. Effect of component uncertainty on detectability, Periodic Time Structure

duration) is then

$$\ln\left[1 - \frac{1}{M} + \frac{1}{M}\exp\left(\frac{2E_c}{N_o}\right)\right]$$

and the detectability at time t_k is given by

$$d = k \ln\left[1 - \frac{1}{M} + \frac{1}{M}\exp\left(\frac{2E_c}{N_o}\right)\right] \qquad (7.13)$$

The performance equation of this suboptimum receiver differs from that of the optimum receiver in the argument of the exponential. In Eq. 7.13 the argument of the exponential is simply $\frac{2E_c}{N_o}$, a quantity associated with a time interval equal to a component duration. The variable k, representing time, is outside of the logarithm and so the detectability eventually rises linearly with time although locally there are exponential segments. In the case of the optimum receiver, the time variable, k, appears in the exponential of Eq. 7. 1 which gives rise to the knee in the detectability curve for the optimum receiver. A comparison of the detectability of the optimum receiver, which makes use of what has been "learned" about the component sent, with the suboptimum receiver, which does not, is shown in Fig. 7.18.

7.3.1.6 Comparison of the Optimum Receiver with the Energy Detector. In Fig. 7.19 the performance of the optimum receiver for one of eight orthognal components is compared with the performance of the energy detector. Once past the "threshold," the detectability of the optimum receiver increases rapidly over the energy detector.

7.3.2 Receiver Performance, Synchronous-Poisson Time Structure, (Common Component Duration). In the previous section receiver performance was obtained for detecting a component generated by a periodic triggering process. In this section, the Synchronous-Poisson triggering process is considered. That is, the probability that a component will occur in a synchronous interval is ν, the probability that it will not occur is $1-\nu$, and the occurrences are independent from one interval to another. In the Synchronous-Poisson Time Structure ν is also the duty factor and the average signal energy in t_k seconds is $\nu k E_c$.

Most of the ROC curves presented in this section were obtained by using the digital computer as an experimental tool. This is an approximate but useful technique. The

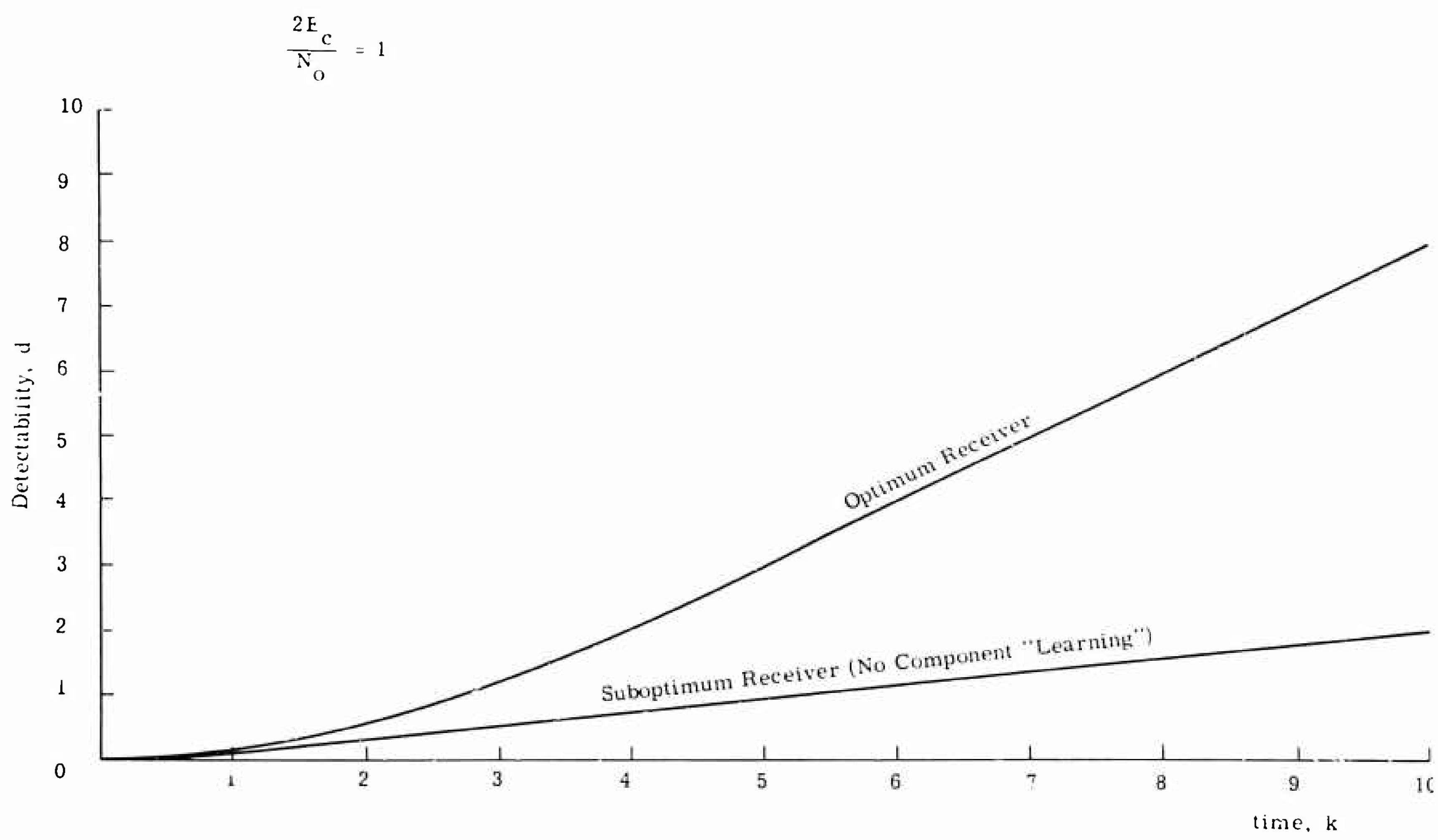

Fig. 7. 18. Comparison of optimum and a suboptimum receiver that ignores component identification information, Periodic Time Structure.

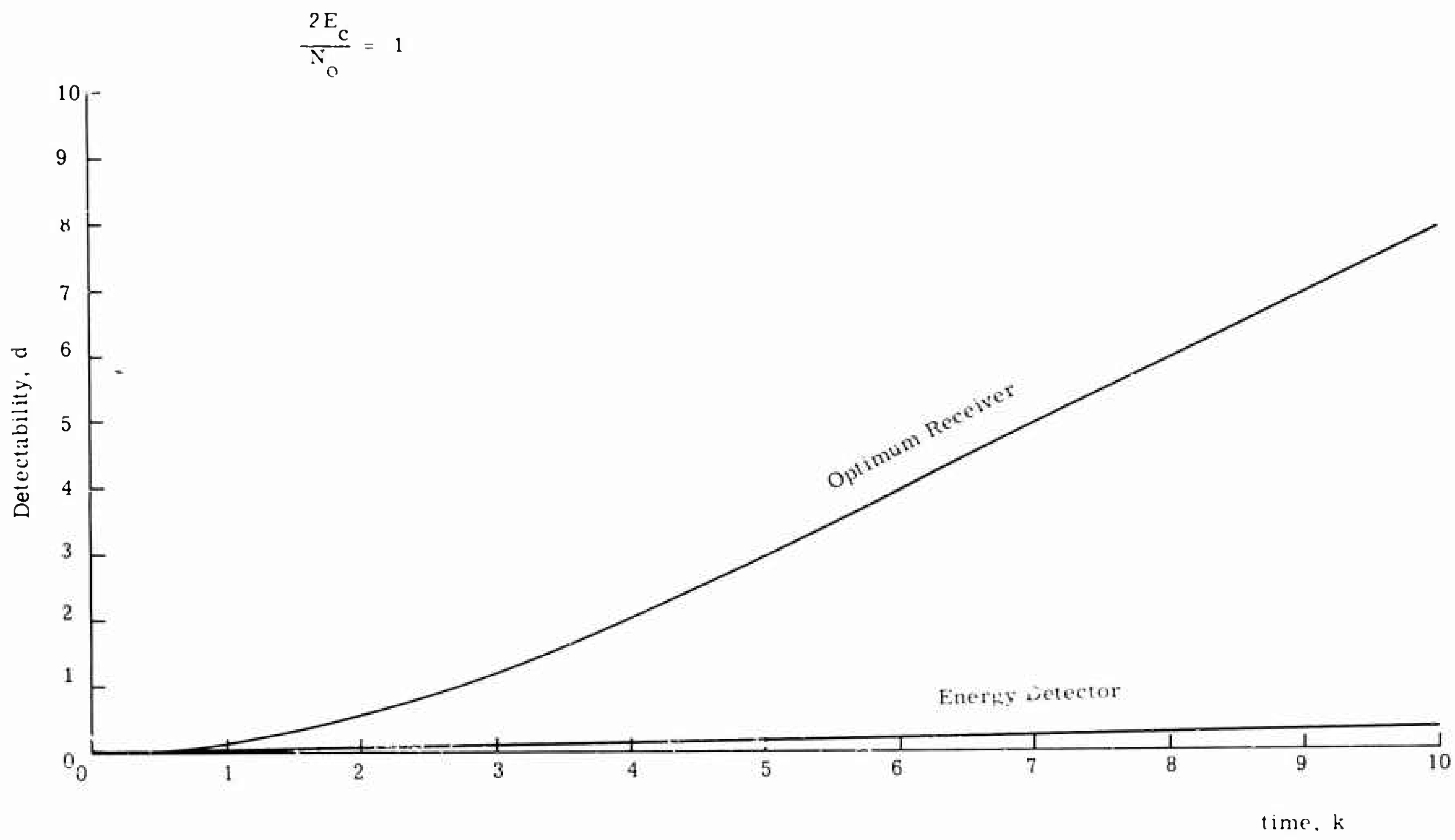

Fig. 7.19. Comparison of optimum receiver and energy detector performance, CKS (one of eight orthogonal components) Periodic Time Structure.

performance of the optimum receiver for CKE will occupy much of this section. Comparing this performance with the performance where the occurrence times of the component are known exactly shows the effect of the Synchronous-Poisson Time Structure on detectability. Also, the CKE case puts an upper bound on performance when there is component uncertainty.

7.3.2.1 CKE (Component Known Exactly). When the component is known exactly and the time structure is Synchronous-Poisson, the optimum receiver cross-correlates the input observation with the component waveform and subtracts a bias. This is fed into a "ν nonlinearity" and its output integrated. A block diagram of this receiver was shown in Fig. 5.12b. The performance of the optimum receiver for the case of a component known exactly was experimentally determined on the digital computer for values of ν (duty factor) and $\frac{2E_c}{N_o}$ shown in Table 7.1. For each set of parameters, 500 simulation runs

TABLE 7.1

VALUES OF PARAMETERS RUN
CKE, SYNCHRONOUS-POISSON TIME STRUCTURE

ν	$\frac{2E_c}{N_o}$
.0125	1
.0707	2
.0707	4
.1	4
.1	.02
.1	1
.1	1.3
.1	2
.1	4
.1414	2
.1414	4
.2	4

were made on the digital computer to determine the ROC. The duration of a run depended on the particular set of parameters. For example, for $\nu = .1$ and $\frac{2E_c}{N_o} = 1$, a run lasted 1000 times a component duration. The probability distribution of the optimum receiver output (the likelihood ratio) was obtained for several points in time under both hyptheses, SN and N. Time is indexed by k, the number of synchronous intervals. From the probability distributions of the receiver output the ROC was obtained. A normal approximation to the data points was made. Further details on the computer simulation are discussed in Appendix D.

The ROC's for the parameters listed in Table 7. 1 are presented in Figs. 7. 20 through 7. 30. For a given set of parameters, ν and $\frac{2E_c}{N_o}$, one can see from the ROC's how detectability builds in time as k increases. It is easier to show this effect if we read the ROC along the negative diagonal (i. e., read the ROC where the probability of each of the two possible types of errors are equal) and plot this detectability, d, as a function of time. This has been done in Figs. 7. 31 through 7. 33 for the parameters listed in Table 7. 1, except for $\nu = .0125$ and $\frac{2E_c}{N_o} = 1$. From these curves one can see that detectability is nearly a linear function of time. In Fig. 7. 31 detectability, d, is plotted as a function of time for $\nu = .1$ and $\frac{2E_c}{N_o} = .02$, 1, 1.3, 2, and 4. As $\frac{2E_c}{N_o}$ increases, the slope of the curves increase, as one would expect. In Figs. 7. 32 and 7. 33 detectability is plotted versus time with the duty factor, ν, as a parameter. These curves are also nearly linear and increase in slope as the duty factor increases. Using this data, the effect of the Synchronous-Poisson Time Structure and component uncertainty on detectability will be investigated in subsequent sections.

7. 3. 2. 2 CKS (One of Eight Orthogonal Components). ROC curves are plotted in Fig. 7. 34 for the case of one of eight orthogonal components for $\nu = .1$, $\frac{2E_c}{N_o} = 1$, and k = 100, 250, 500, 750, and 1000. The data for these ROC curves was obtained from the receiver simulation displayed in Section 7. 2. These ROC's were obtained from 100 runs rather than the 500 runs for the ROC's of the CKE case.

7. 3. 2. 3 Effect of Component Uncertainty. A preliminary idea of the effect of component uncertainty on receiver performance for the Synchronous-Poisson Time Structure is obtained by comparing the CKE and CKS curves for $\nu = .1$, $\frac{2E_c}{N_o} = 1$. This comparison is made in Fig. 7. 35 in which the detectability is plotted as a function of time. The CKS curve exhibits a threshold effect. After approximately k = 100, the detectability

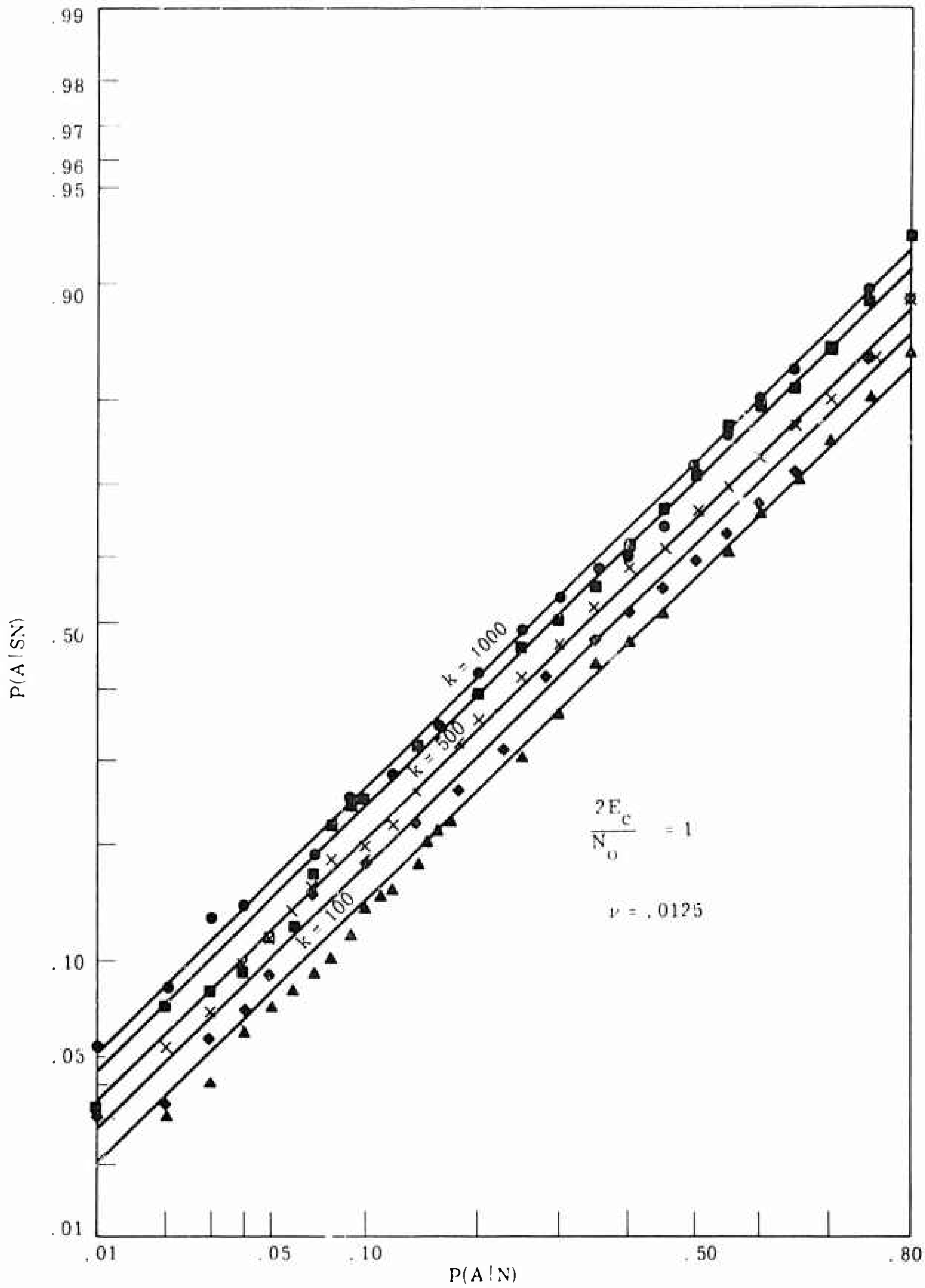

Fig. 7.20. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 1$, $\nu = .0125$.

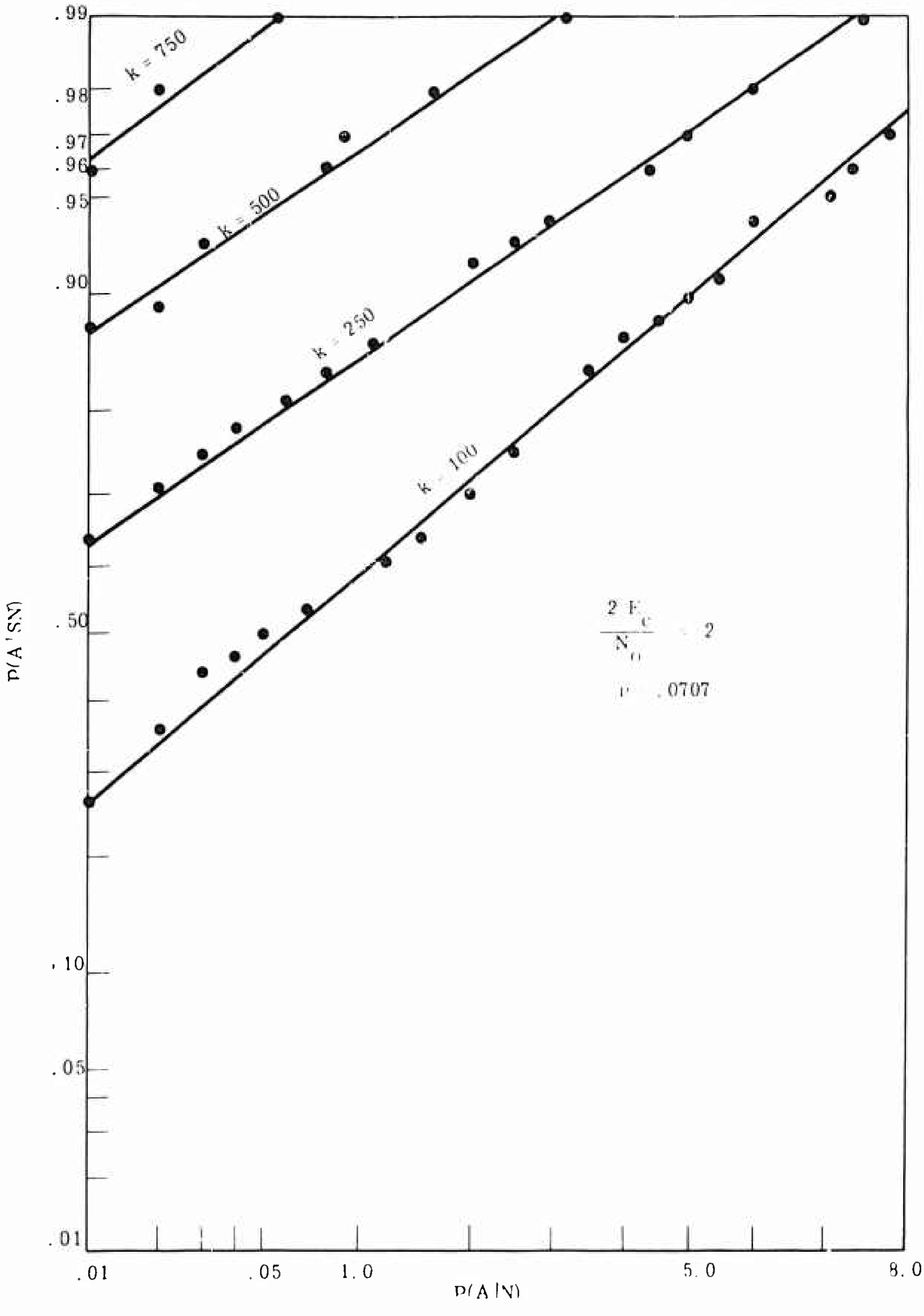

Fig. 7.21. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 2$, $\nu = .0707$.

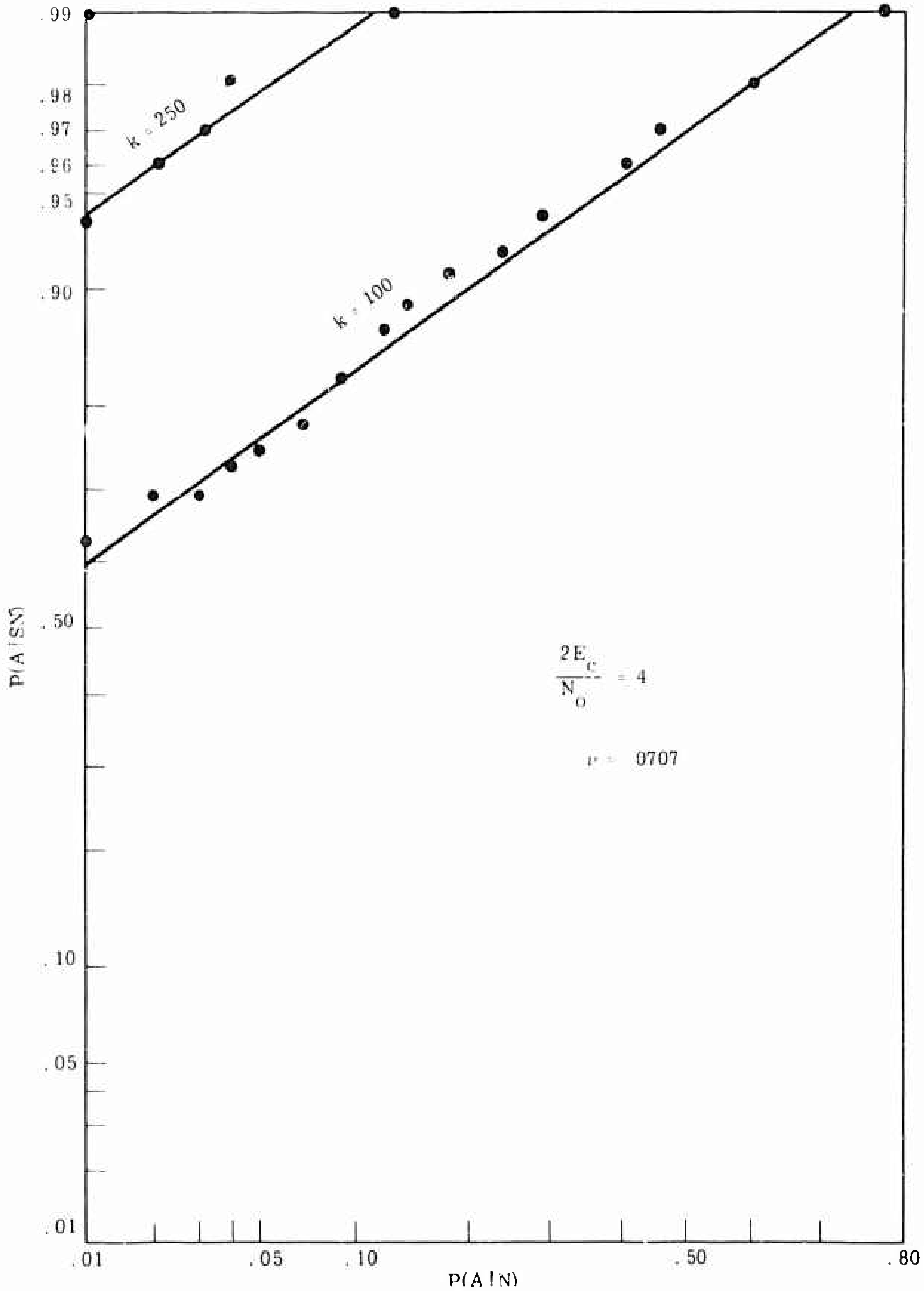

Fig. 7.22. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 4$, $\nu = .0707$.

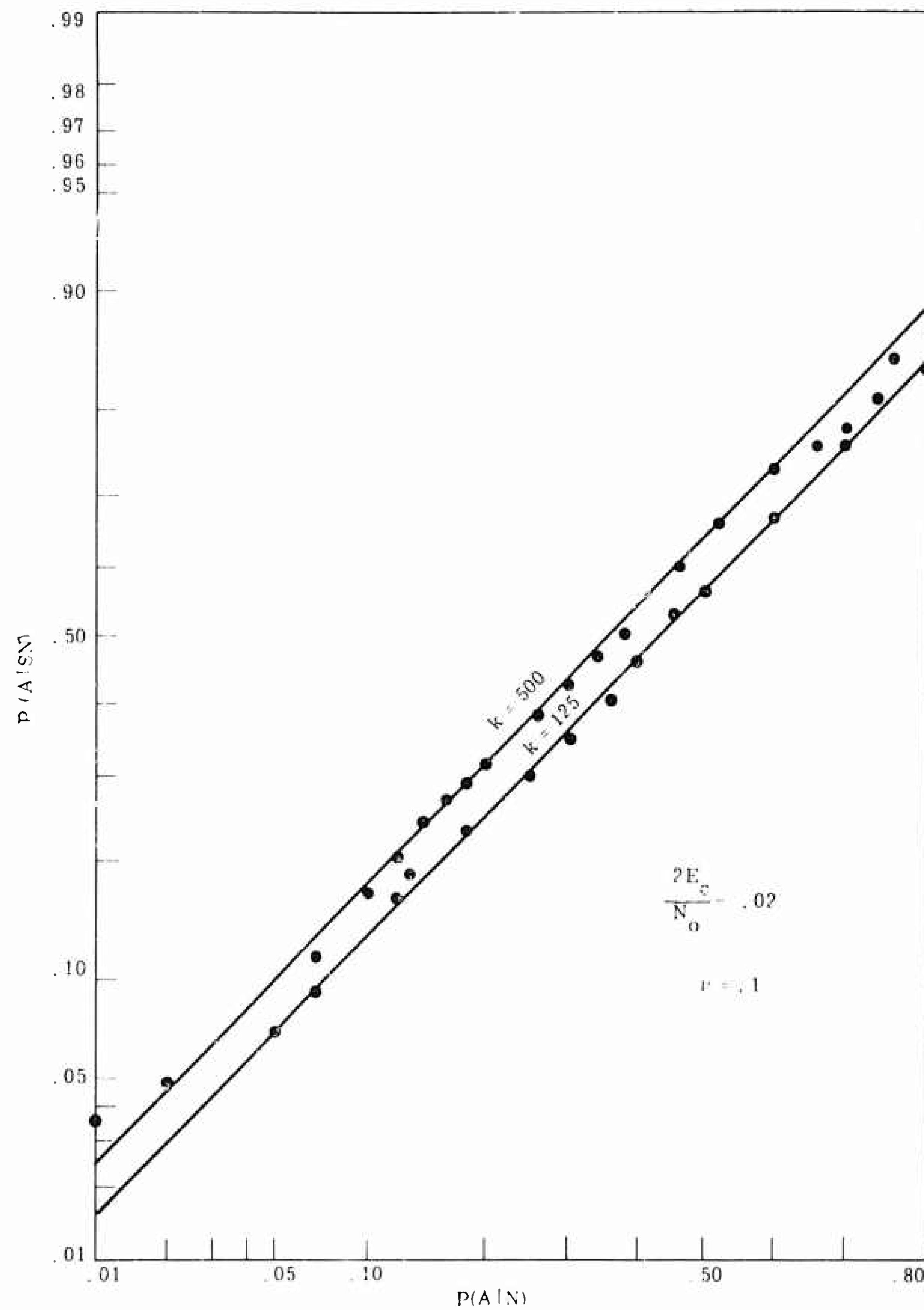

Fig. 7.23. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = .02$, $\nu = .1$.

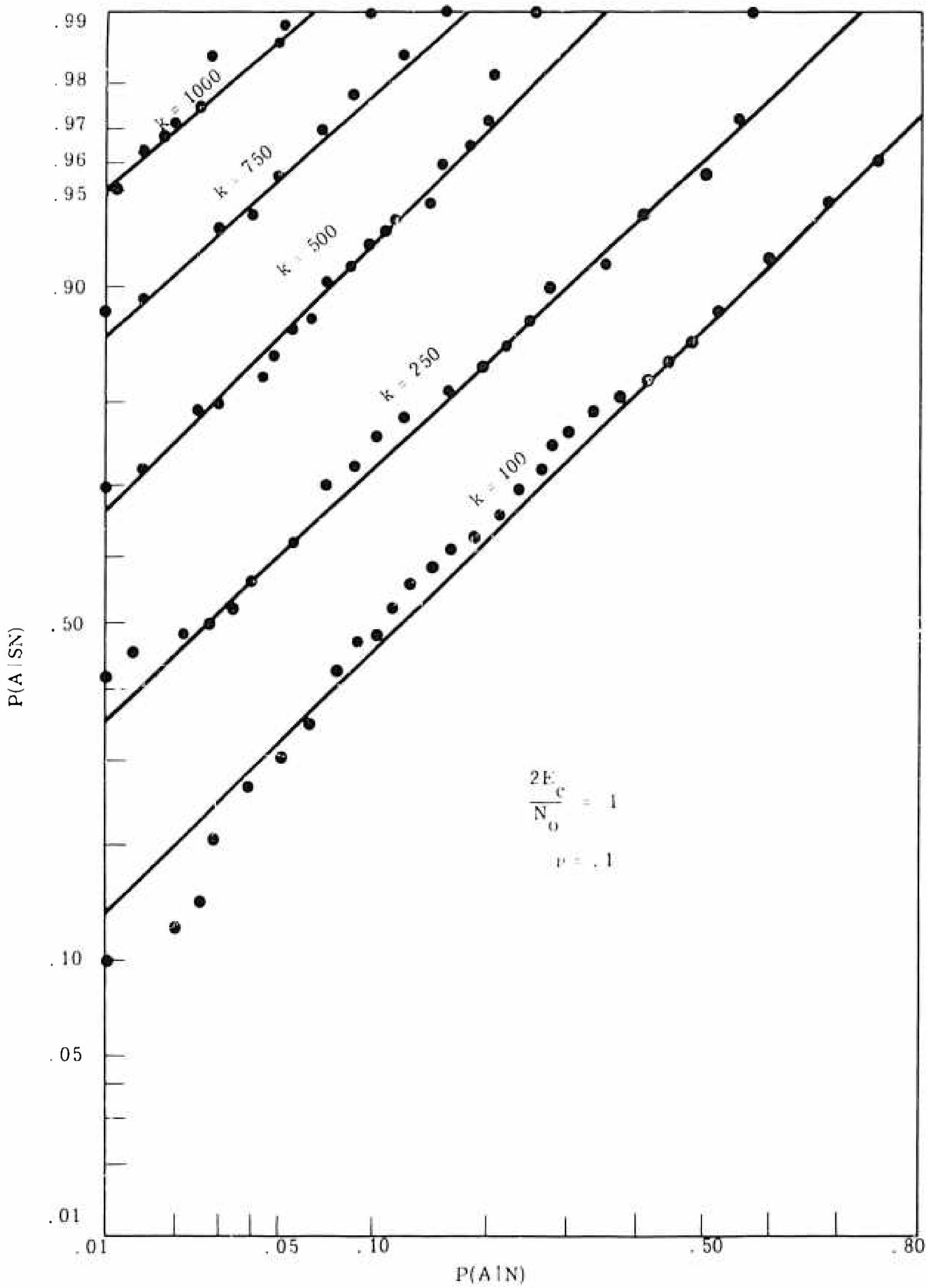

Fig. 7.24. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 1$, $\nu = .1$.

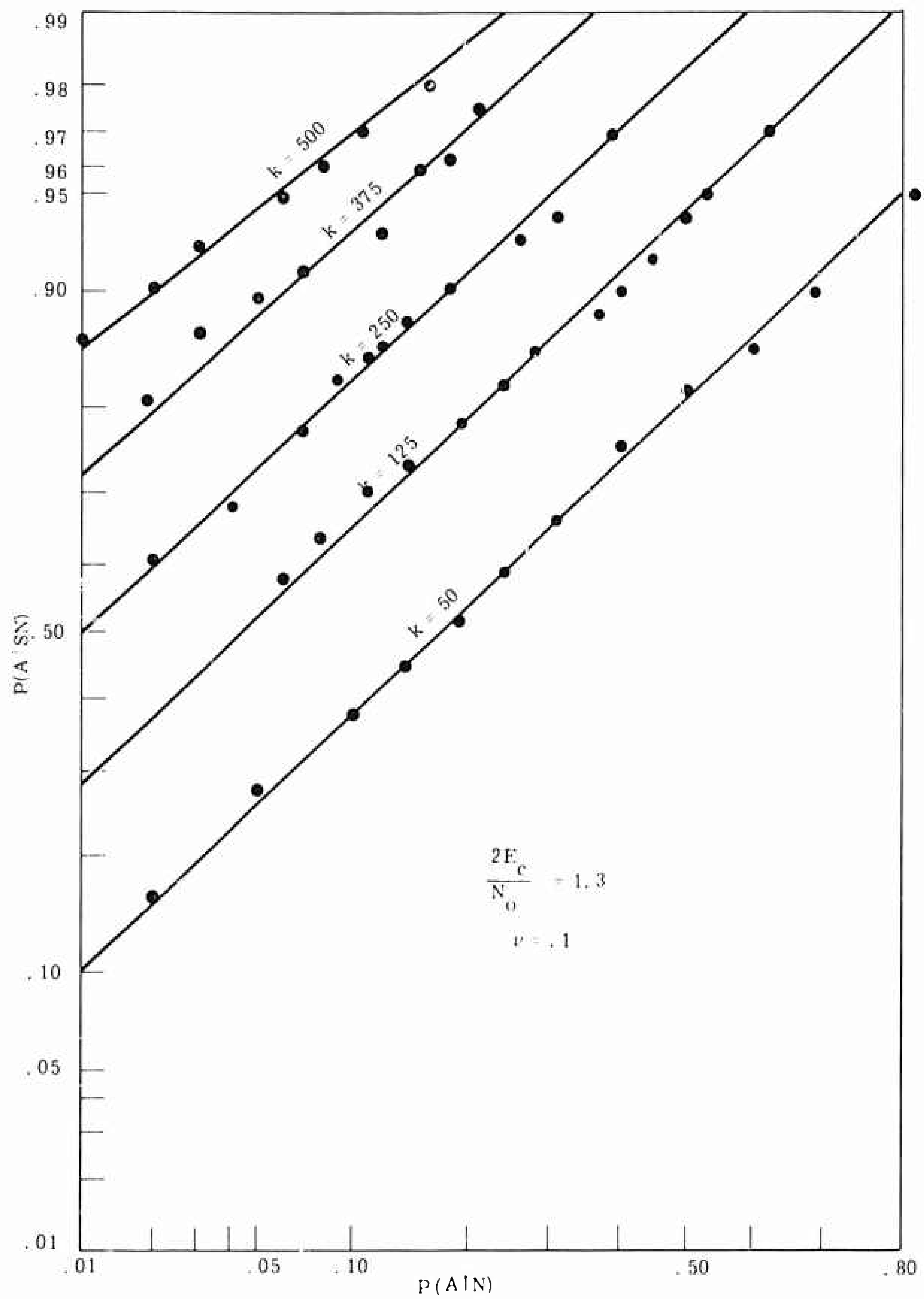

Fig. 7.25. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 1.3$, $\nu = .1$.

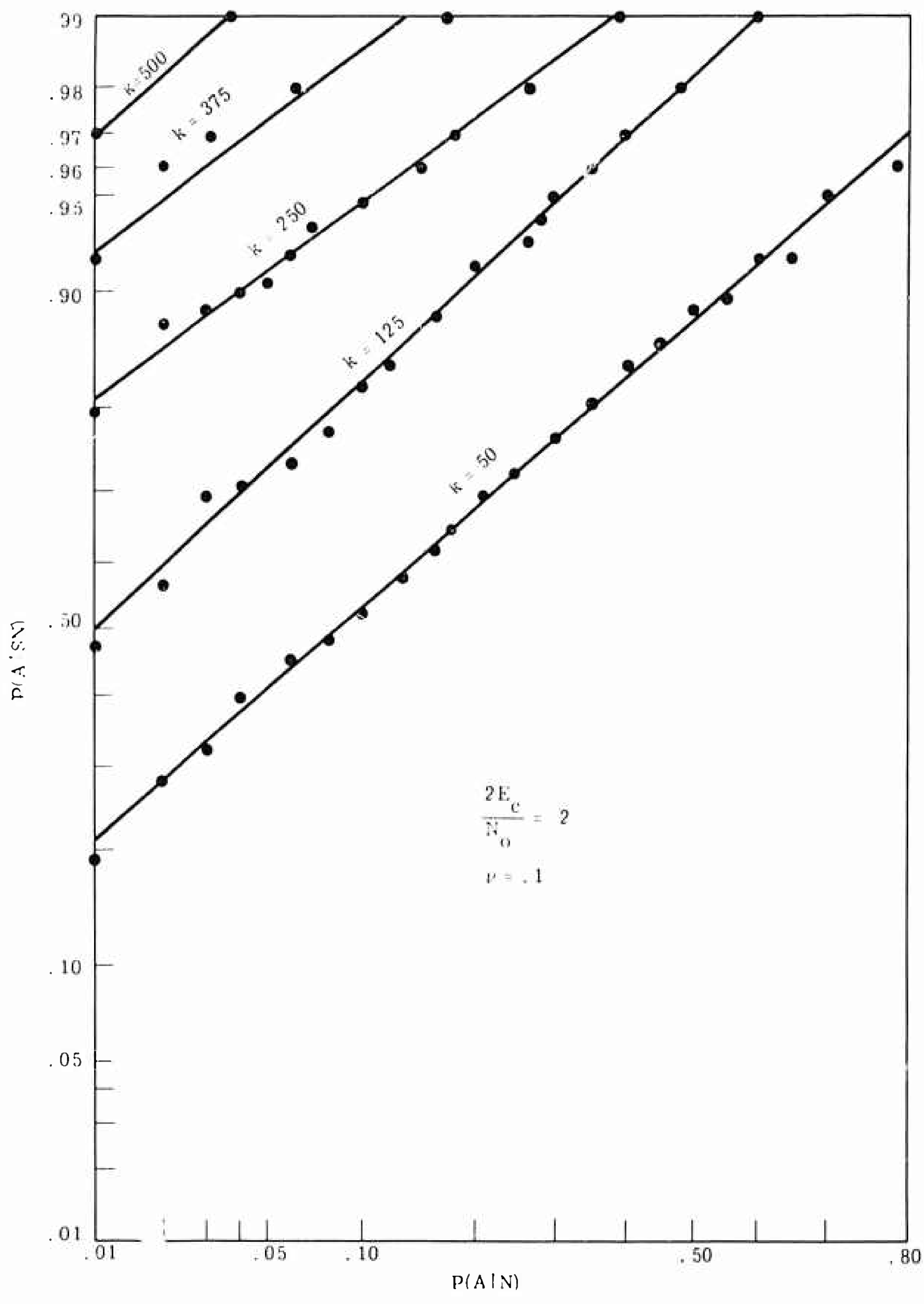

Fig. 7.26. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 2,\ \nu = .1.$

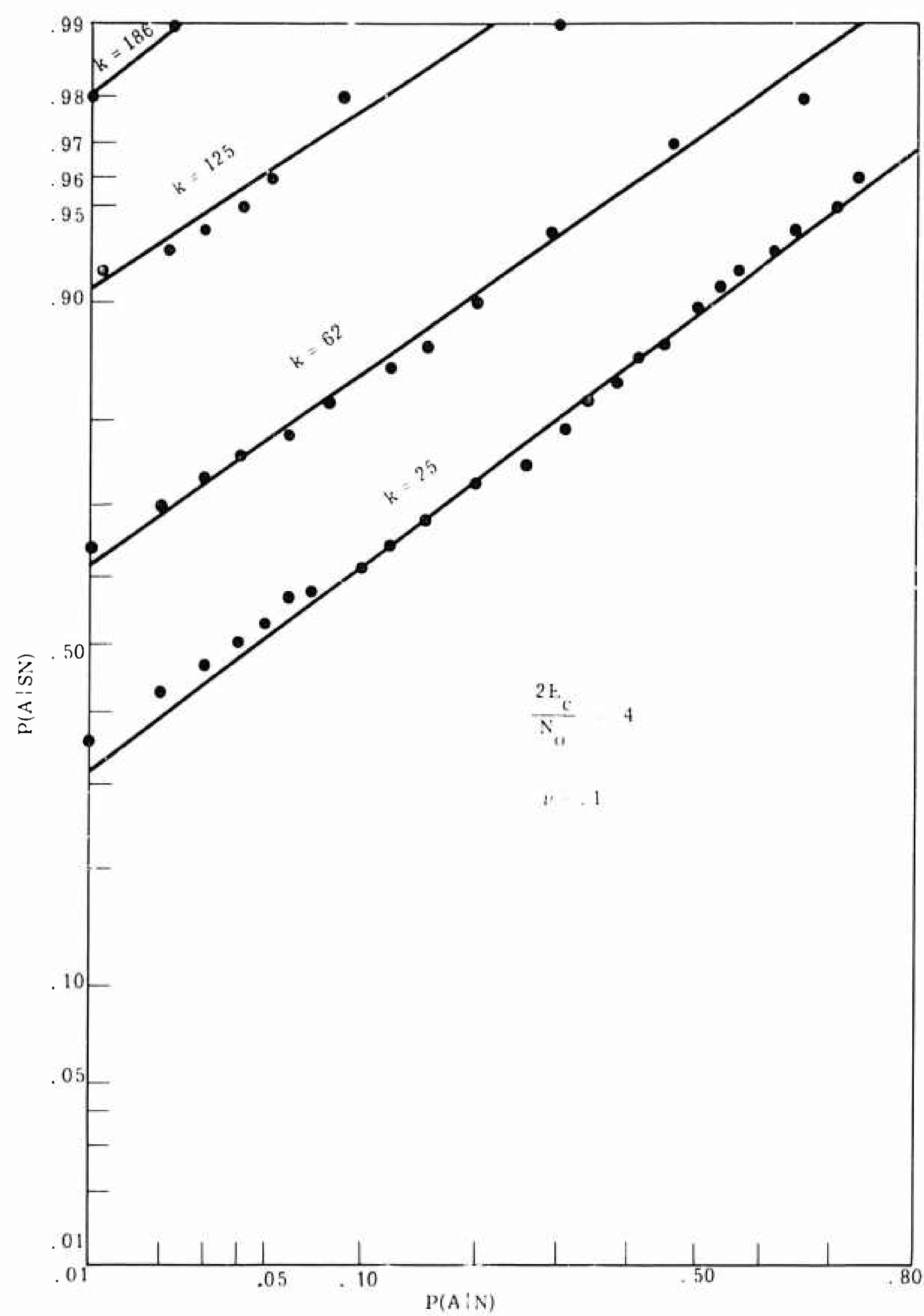

Fig. 7.27. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 4$, $\nu = .1$.

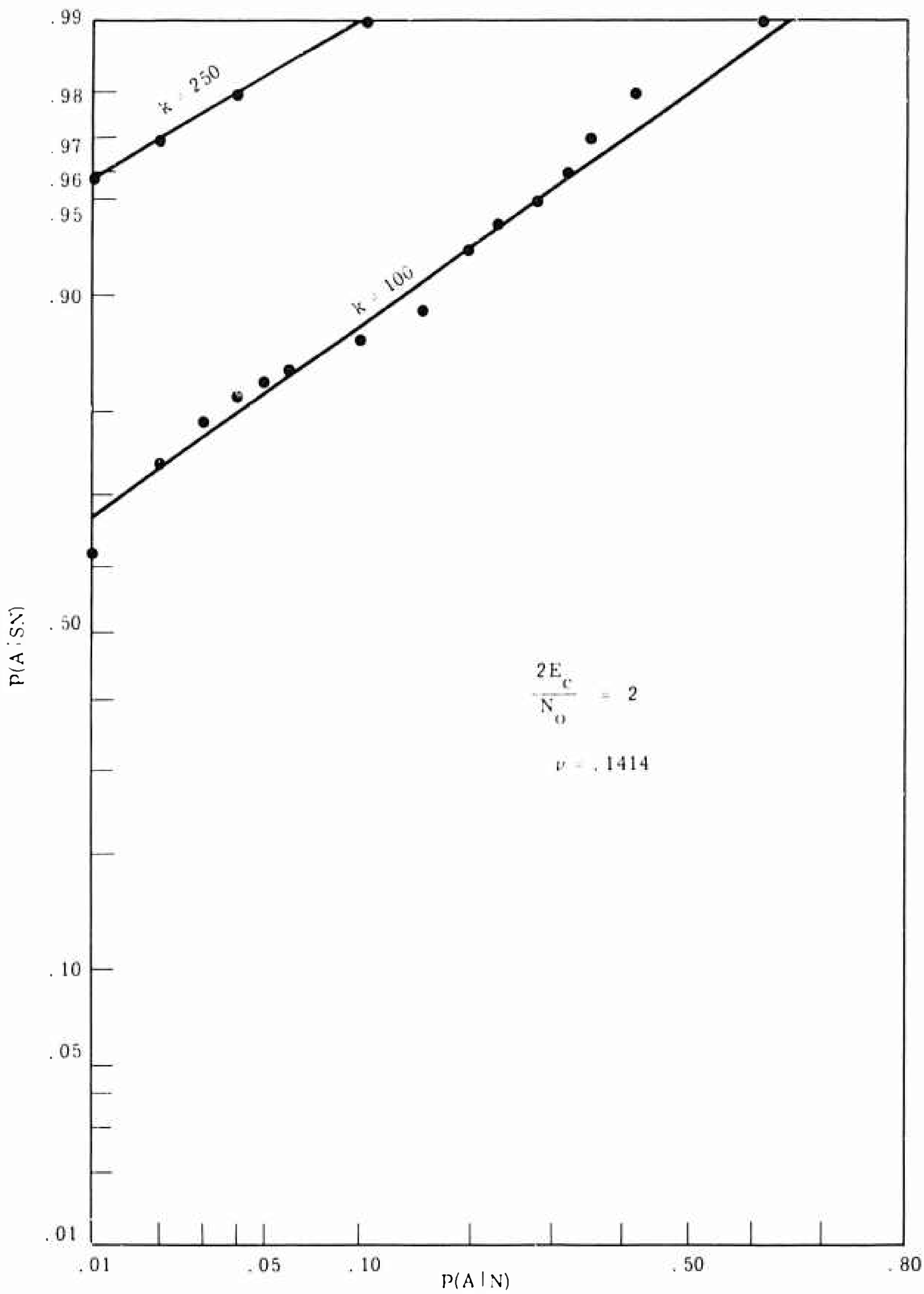

Fig. 7.28. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 2$, $\nu = .1414$.

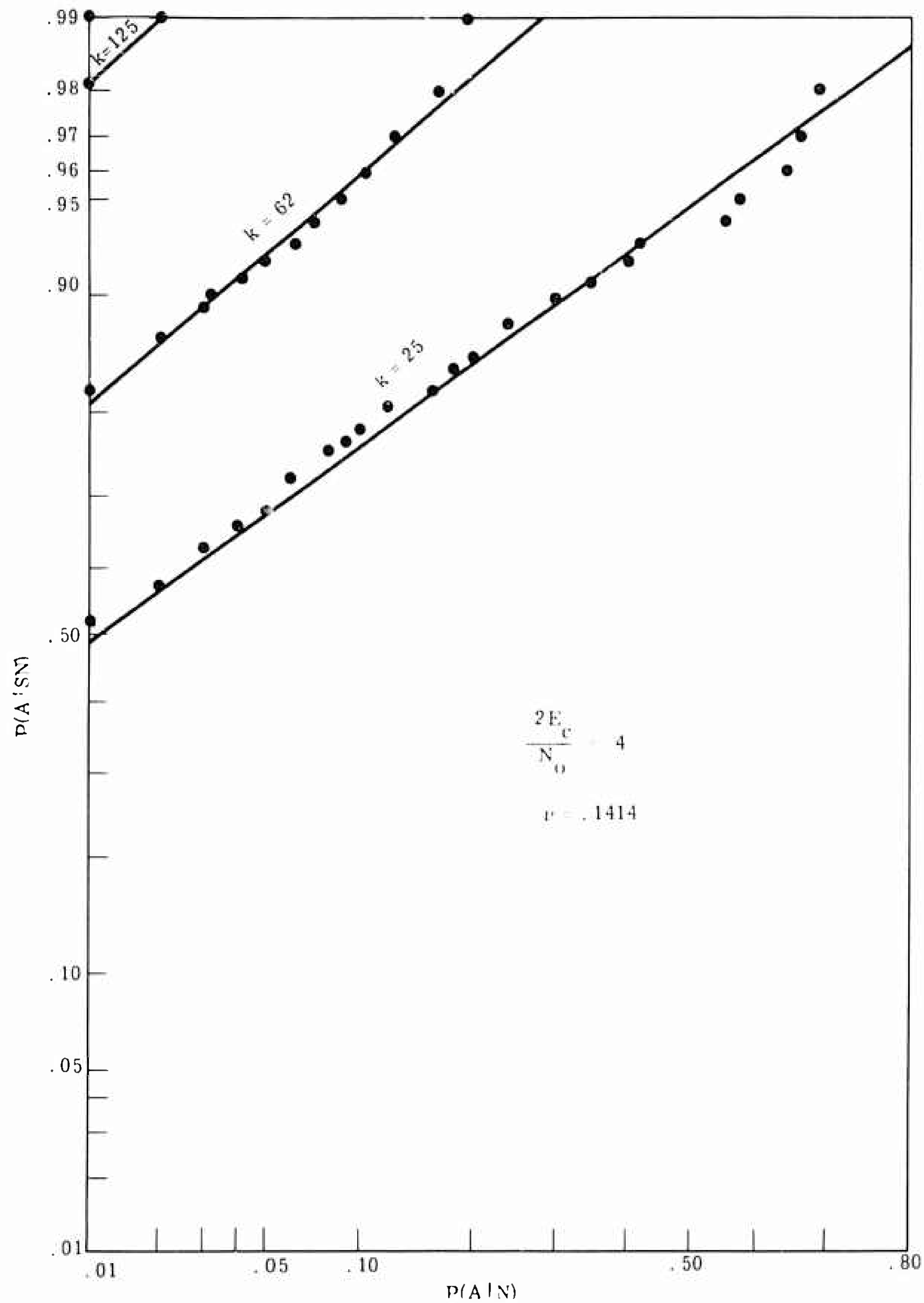

Fig. 7.29. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 4$, $\nu = .1414$.

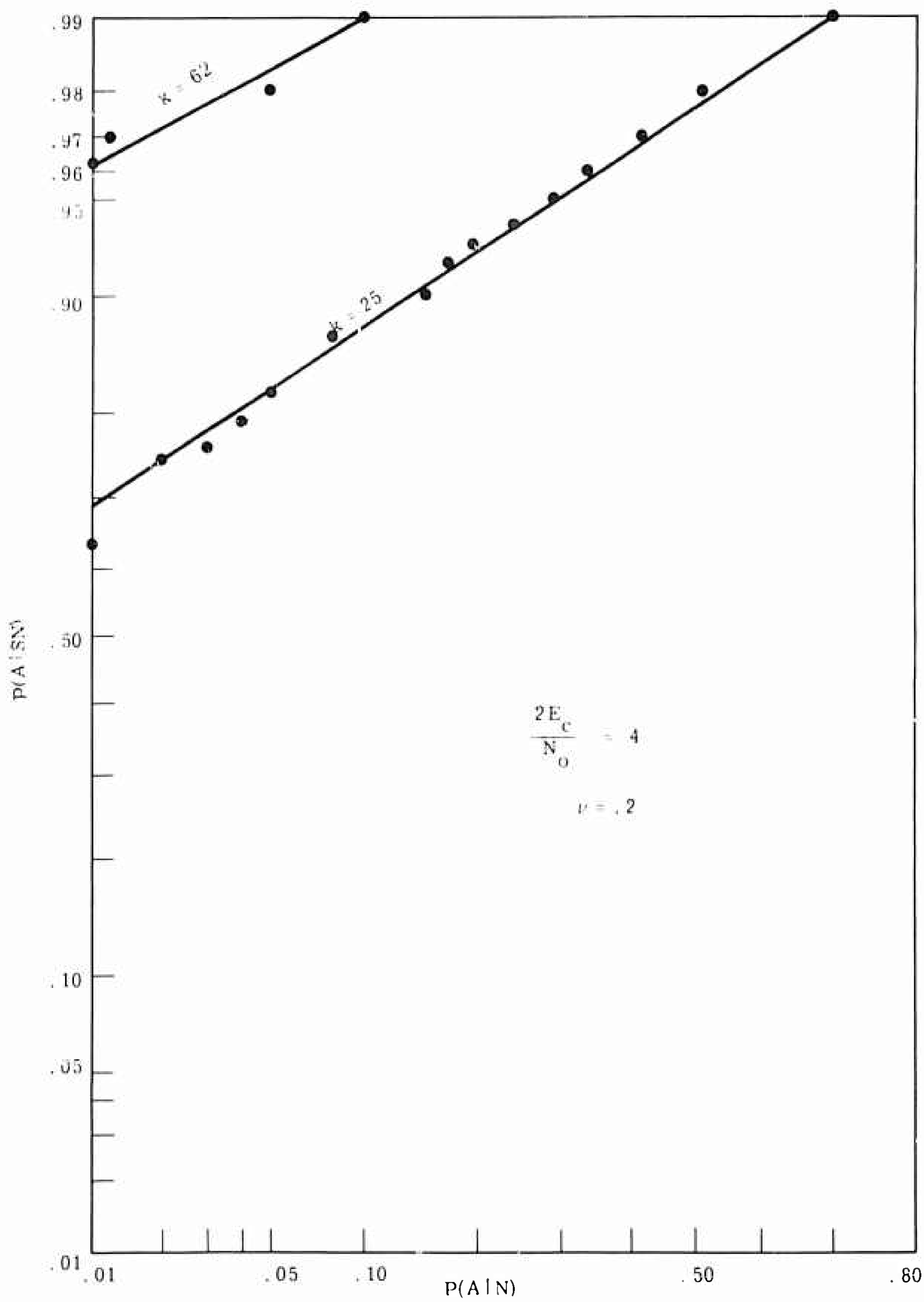

Fig. 7.30. ROC for optimum receiver, CKE, Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 4$, $\nu = .2$.

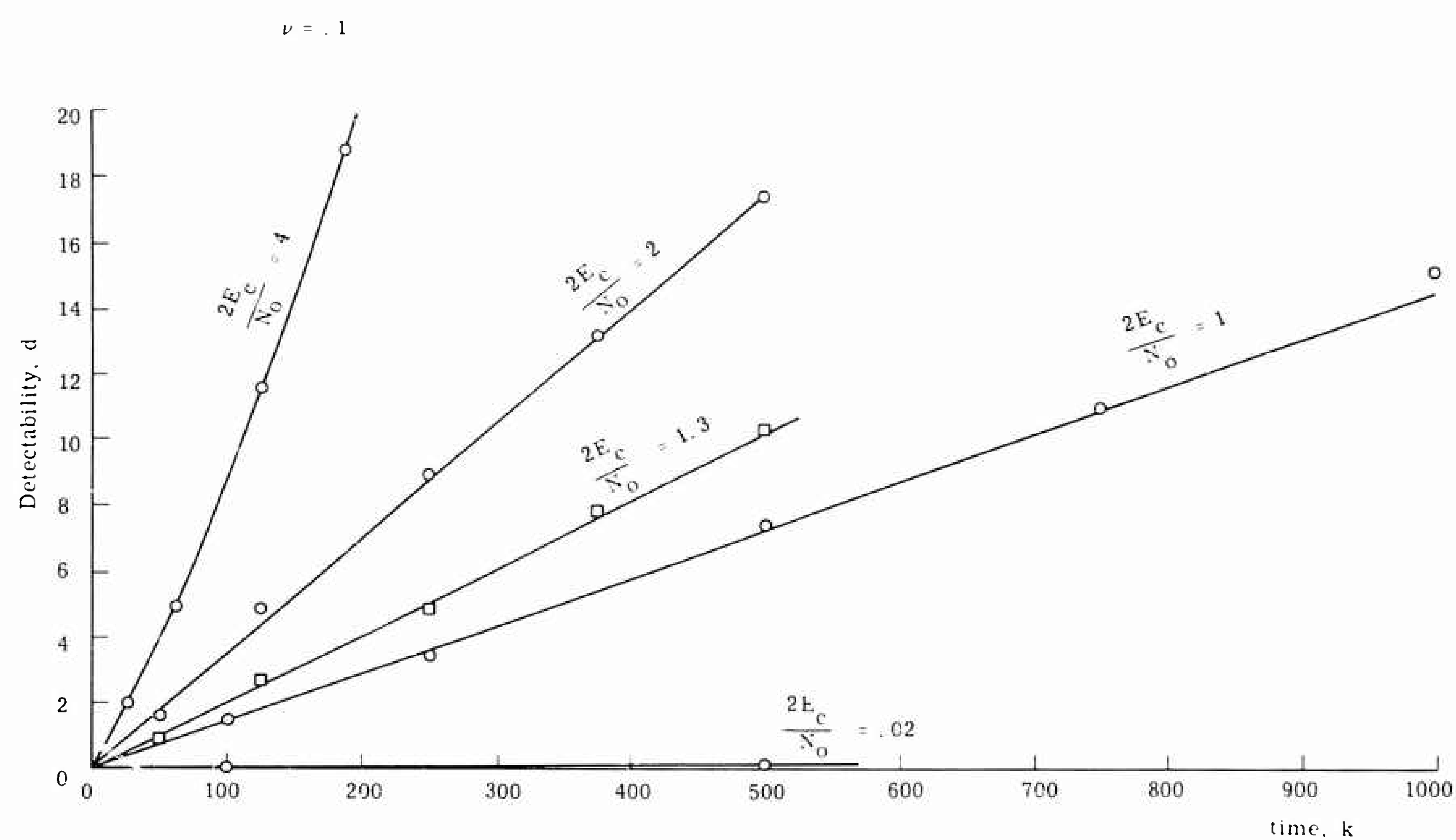

Fig. 7.31. Detectability vs. time, optimum adaptive receiver, Synchronous-Poisson Time Structure, $\nu = 1$, $\frac{2E_c}{N_o} = .02, 1, 1.3, 2, 4.$

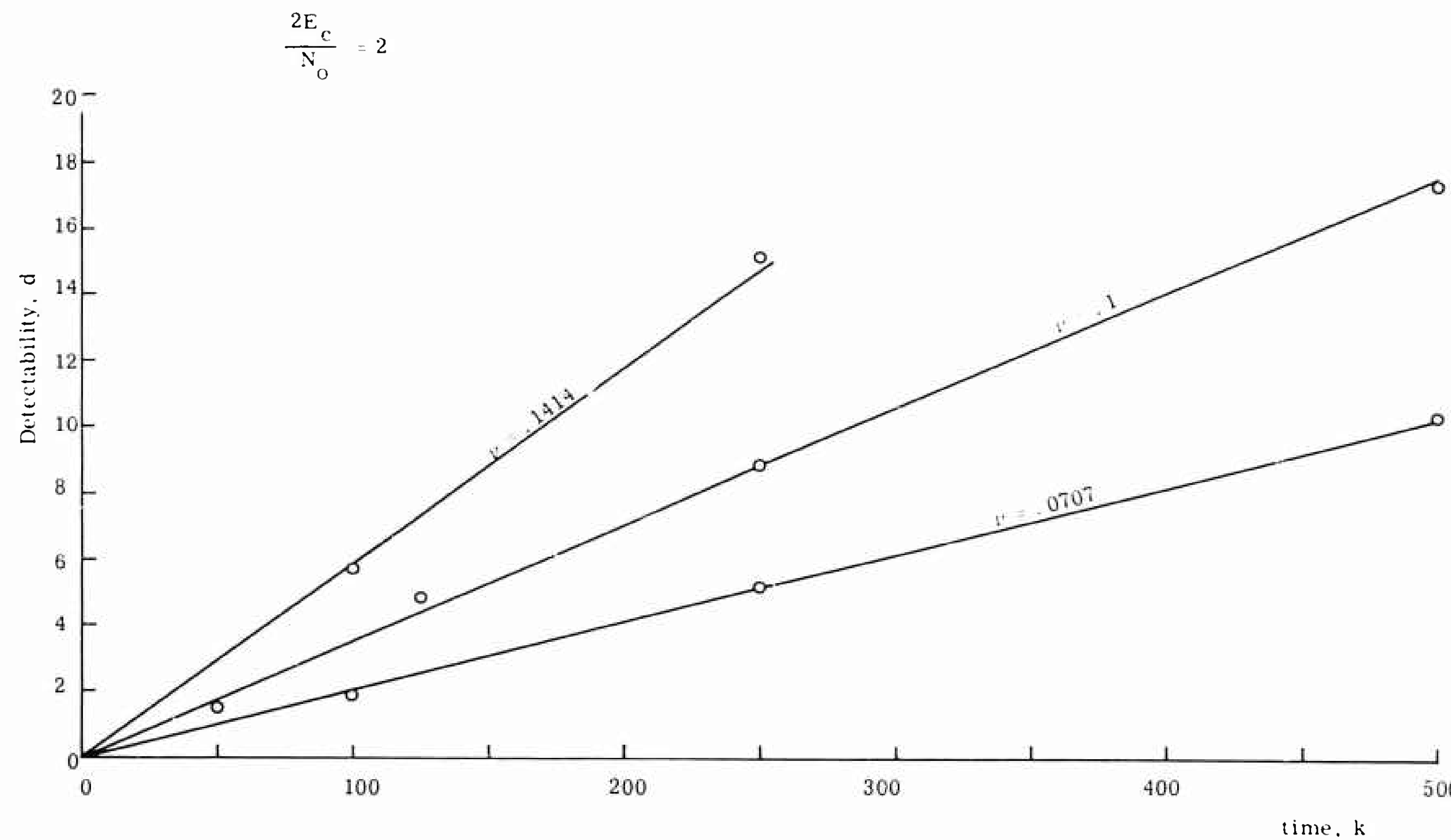

Fig. 7.32. Detectability vs. time, optimum adaptive receiver, Synchronous-Poisson Time Structure, $\nu = .0707, .1, .1414, \frac{2E_c}{N_o} = 2.$

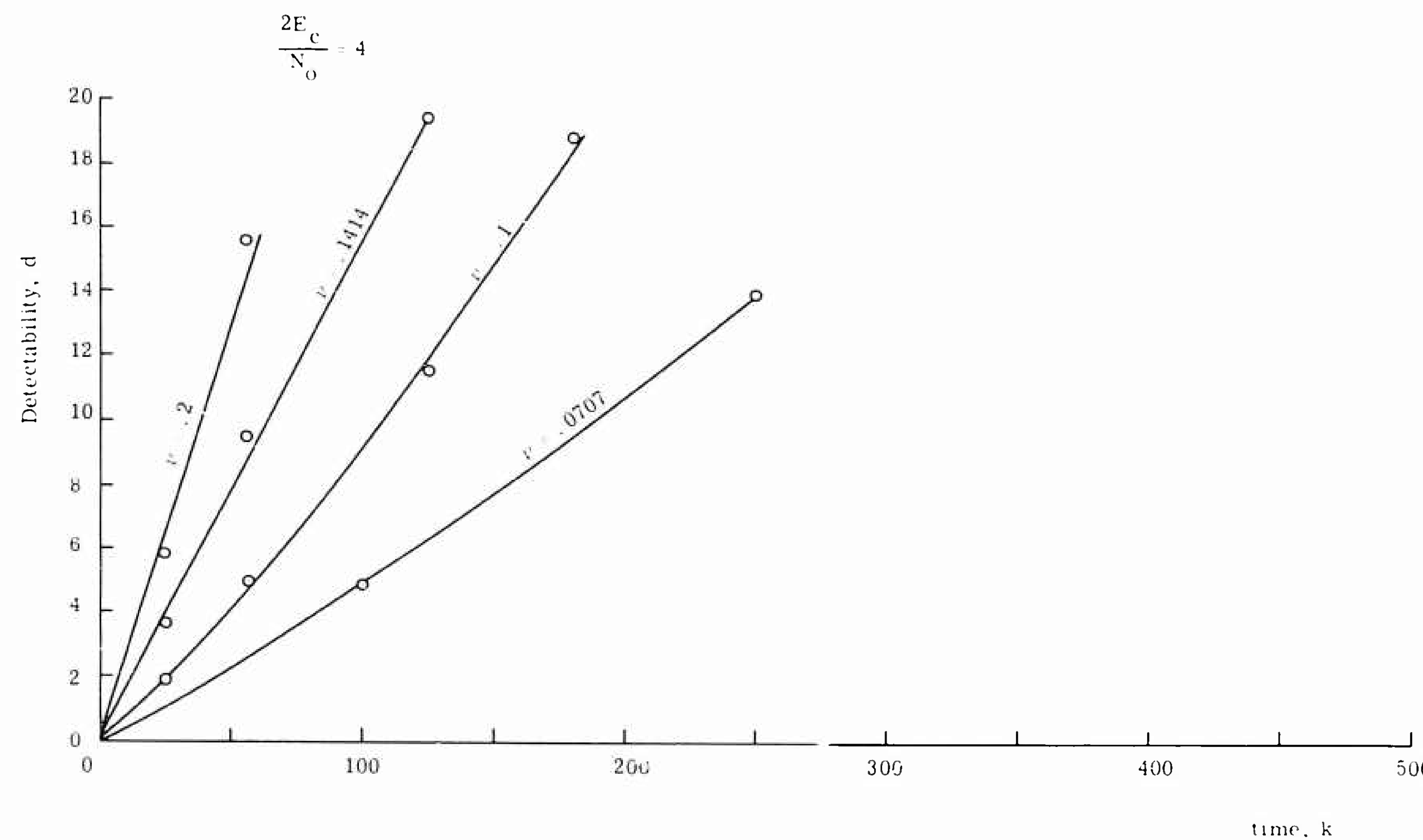

Fig. 7.33. Detectability vs. time, optimum adaptive receiver Synchronous-Poisson Time Structure, $\nu = .0707, .1, .1414, .2, \frac{2E_c}{N_o} = 2.$

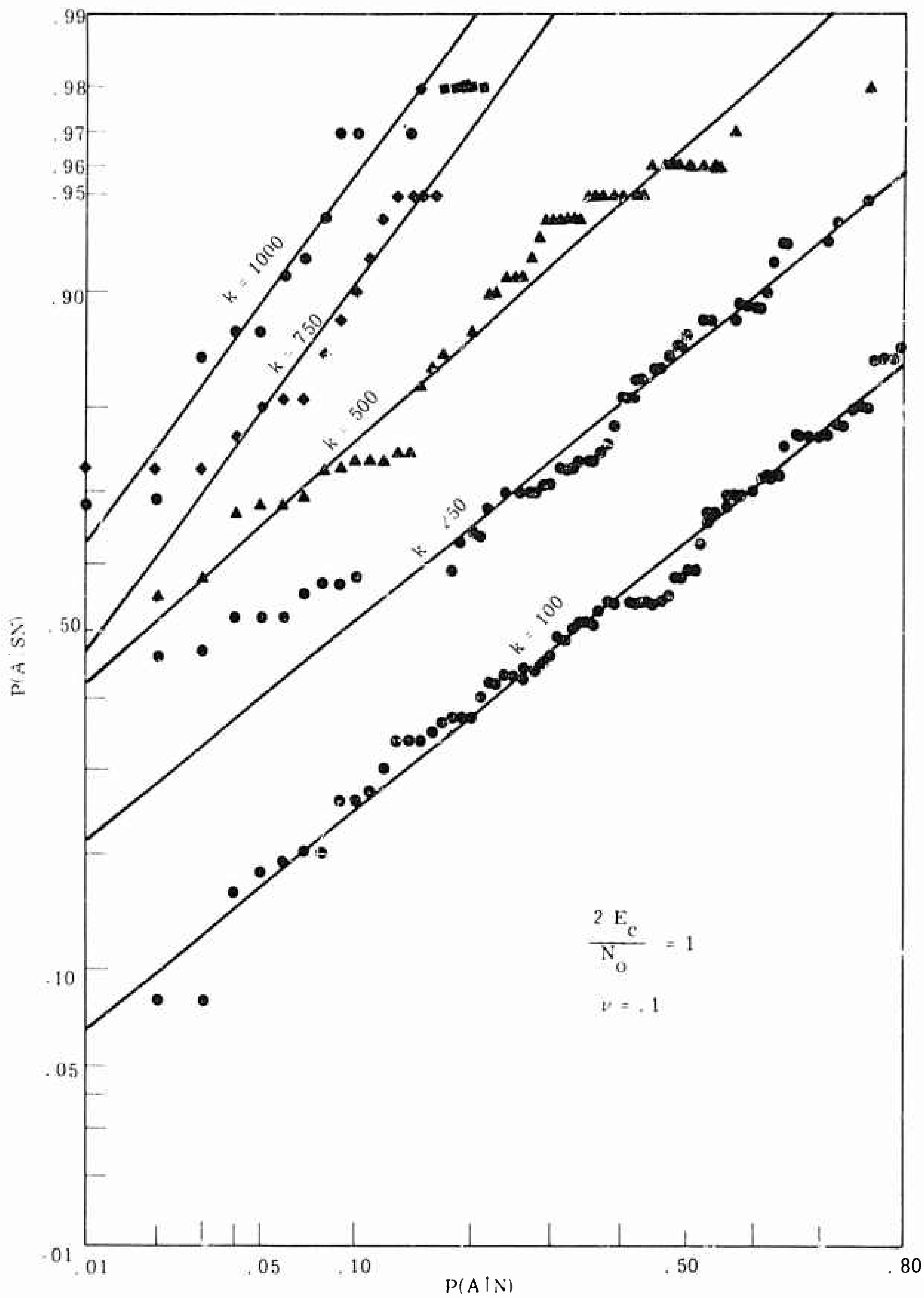

Fig. 7.34. ROC for optimum receiver, CKS (one of eight orthogonal components), Synchronous-Poisson Time Structure, $\frac{2E_c}{N_o} = 1$, $\nu = .1$.

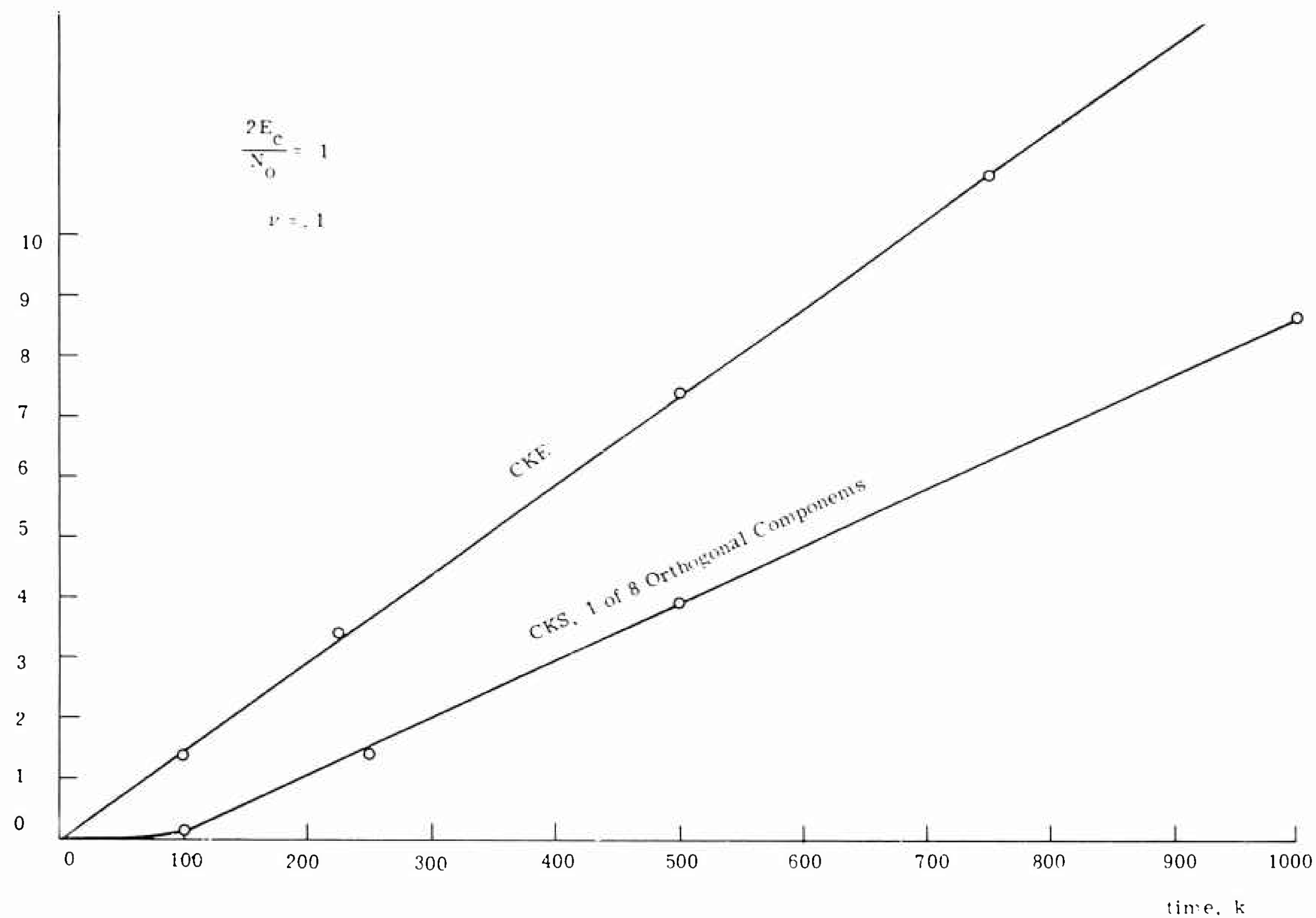

Fig. 7.35. Effect of component uncertainty on detectability, Synchronous-Poisson Time Structure.

rises almost linearly. The slow rise in d, below k = 100, occurs during the time when the receiver is "learning" which component is being sent. The effect of component uncertainty on receiver performance appears to be relatively small once a sufficient processing time has elapsed. There is undoubtedly a tradeoff between component uncertainty and the time it takes for detectability to reach a constant slope. These effects are similar to those observed in the periodic case shown in Fig. 7. 17. The tilt in the ROC curves for high k values and the fact that the slope of the CKS curve in Fig. 7. 35 does not quite approach the slope for the CKE curve may be due to the smaller number of runs (100) used for the CKS data as compared to the 500 runs used for the CKE curves. This analysis is only the start of a study of the effect of component uncertainty on detectability.

7.3.2.4 Effect of the Synchronous-Poisson Time Structure on Detectability. The CKE is an important case. The performance of the optimum receiver, when the component is known exactly, puts an upper bound on attainable performance when there is initial component uncertainty. In other words, the performance of the optimum adaptive receiver designed for a relatively known component can never exceed the performance of the optimum receiver designed for a component known exactly even after the adaptive receiver has "learned" which component is being sent. Even then, the receiver is still faced with uncertain component arrival times. We now wish to investigate the effect of the uncertainty in component arrival times on the detection performance of the optimum receiver. To do this we will need to know the performance of an optimum receiver had the arrival times been known exactly.

When the component and arrival times are known exactly, the optimum receiver is one which gates on only when a component is known to occur, and at those times crosscorrelates the input observation with the component waveform and subtracts a bias term proportional to the component energy. These outputs are then integrated to form the detection output. Although the signal is known exactly in any given transmission, the number of components that occur in an interval, $(0, t_k)$ varies from one transmission to the next. In fact, the number of components that occur is described by the binomial distribution. The detection performance is then a performance averaged over the various number of components that could occur.

This case was simulated on the digital computer for $\nu = .1$ and $\frac{2E_c}{N_o} = 1, 2, 4,$ and the resulting ROC's are plotted in Figs. 7.36 through 7.38. The detectability, d, is read off these curves along the negative diagonal and plotted as a function of time. This is shown in Fig. 7.39 for $\nu = .1$, and $\frac{2E_c}{N_o} = 1, 2,$ and 4.

The average number of components that occur in the interval $(0, t_k)$ is νk. If the actual number of components that occurred on each transmission were equal to the average number, the detectability would be given simply by

$$d = \nu k \frac{2E_c}{N_o} \tag{7.14}$$

This analytical approximation is plotted in Fig. 7.39 along with the experimental curve. Both the analytical equation and the curve that results from the experimental runs are approximations to the true curve. The agreement between the two approximations is best for $\frac{2E_c}{N_o} = 1$. The simplicity of Eq. 7.14 makes it a useful rule of thumb equation for the performance of a receiver which knows the component arrival times exactly.

The analytical equation for detectability for the CKE, known arrival times, case is compared with the detectability for the CKE, Synchronous-Poisson Time Structure in Fig. 7.40. These performance curves are shown for $\nu = .1$ and $\frac{2E_c}{N_o} = 1, 2, 4$. The difference in the detection performance is due to the uncertain component arrival times. This shows that even when the component is known exactly, a fairly high price must be paid in the detectability by even the optimum receiver when the recurrence times of the component are this uncertain. For example, when $\frac{2E_c}{N_o} = 1$, for the same detectability, signal processing time must be increased 6.85 times that required if the component recurrence times are known exactly. For $\frac{2E_c}{N_o} = 2$, it is 5.7 times longer and for $\frac{2E_c}{N_o} = 4$ it is about 3.4 times longer. Thus, it is at low component signal-to-noise ratios where component recurrence time uncertainties affect detectability the greatest.

Figure 7.41 shows the same comparison for $\frac{2E_c}{N_o} = 2$, and $\nu = .0707$, .1, and .1414. For $\nu = .0707$, an increase in processing time of about 6.4 times longer is required, in order to attain the same detectability, than would be required if the component recurrence times were known exactly. For $\nu = .1$, it is 5.67 and for $\nu = .1414$ it is 4.74. These curves indicate that component recurrence time uncertainty affects detectability the most at low duty factors.

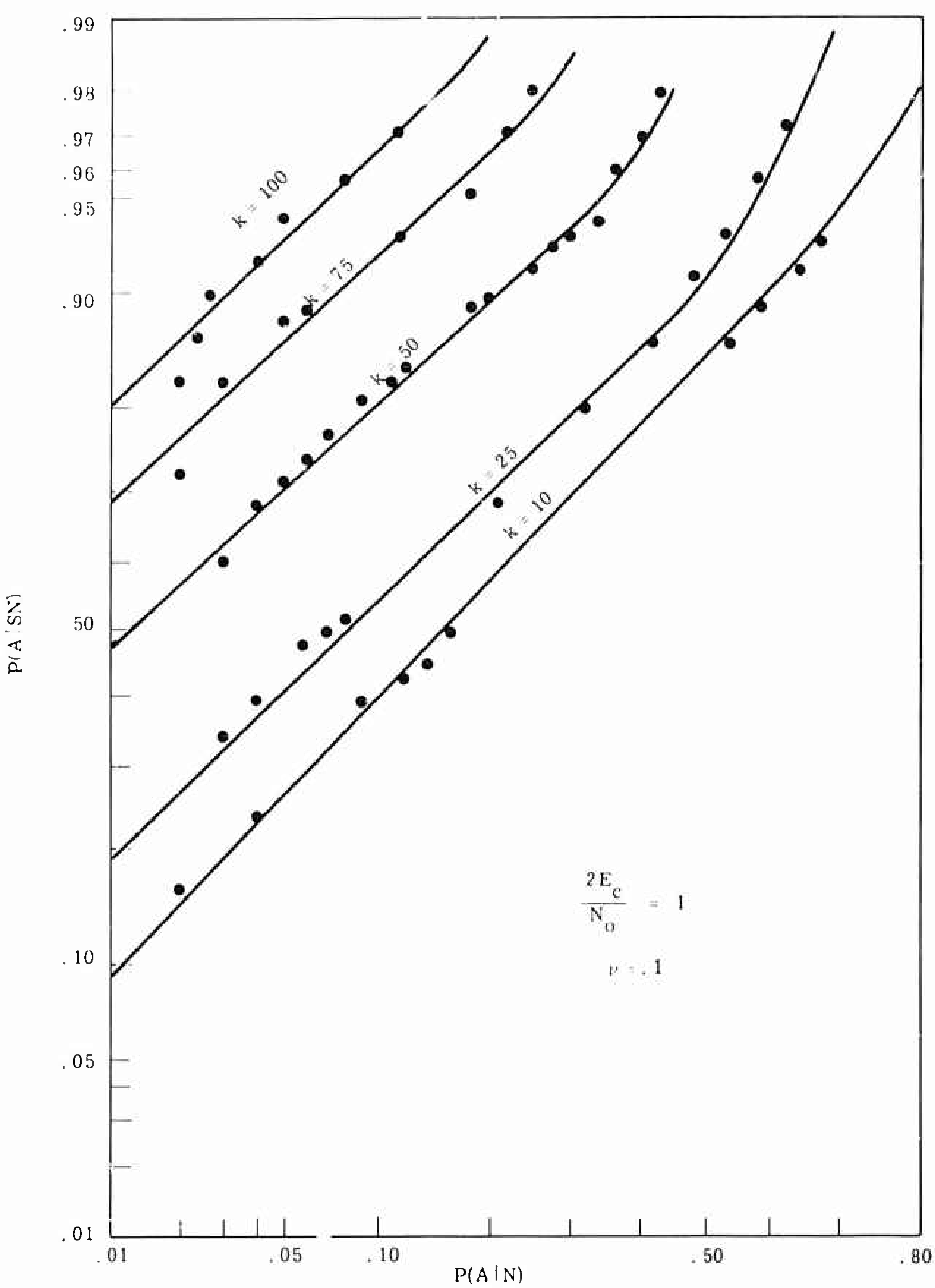

Fig. 7.36 ROC for optimum receiver, CKE, arrival times known exactly, $\frac{2E_c}{N_o} = 1$, $\nu = .1$.

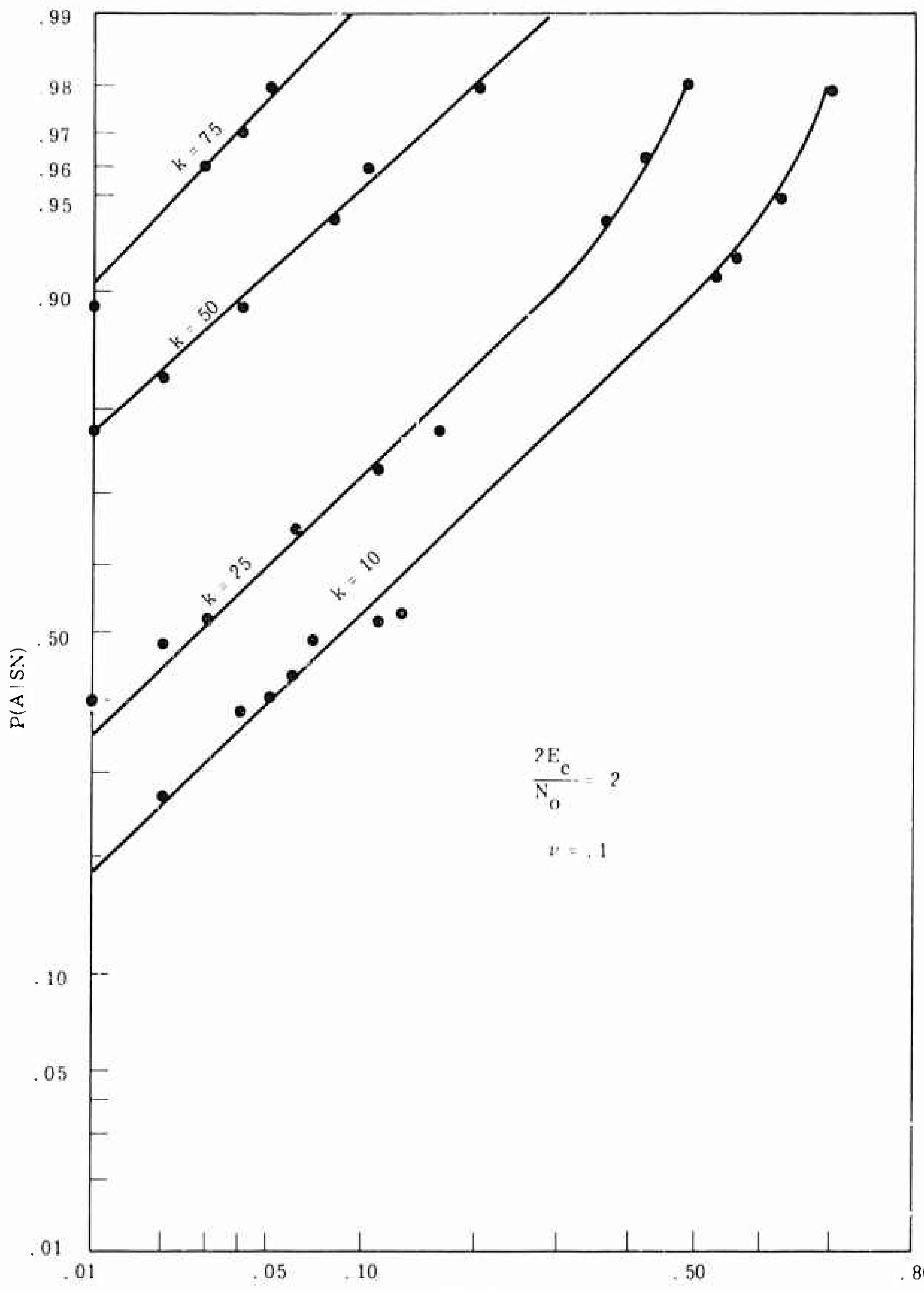

Fig. 7.37. ROC for optimum receiver, CKE, arrival times known exactly, $\frac{2E_c}{N_o} = 2$, $\nu = .1$.

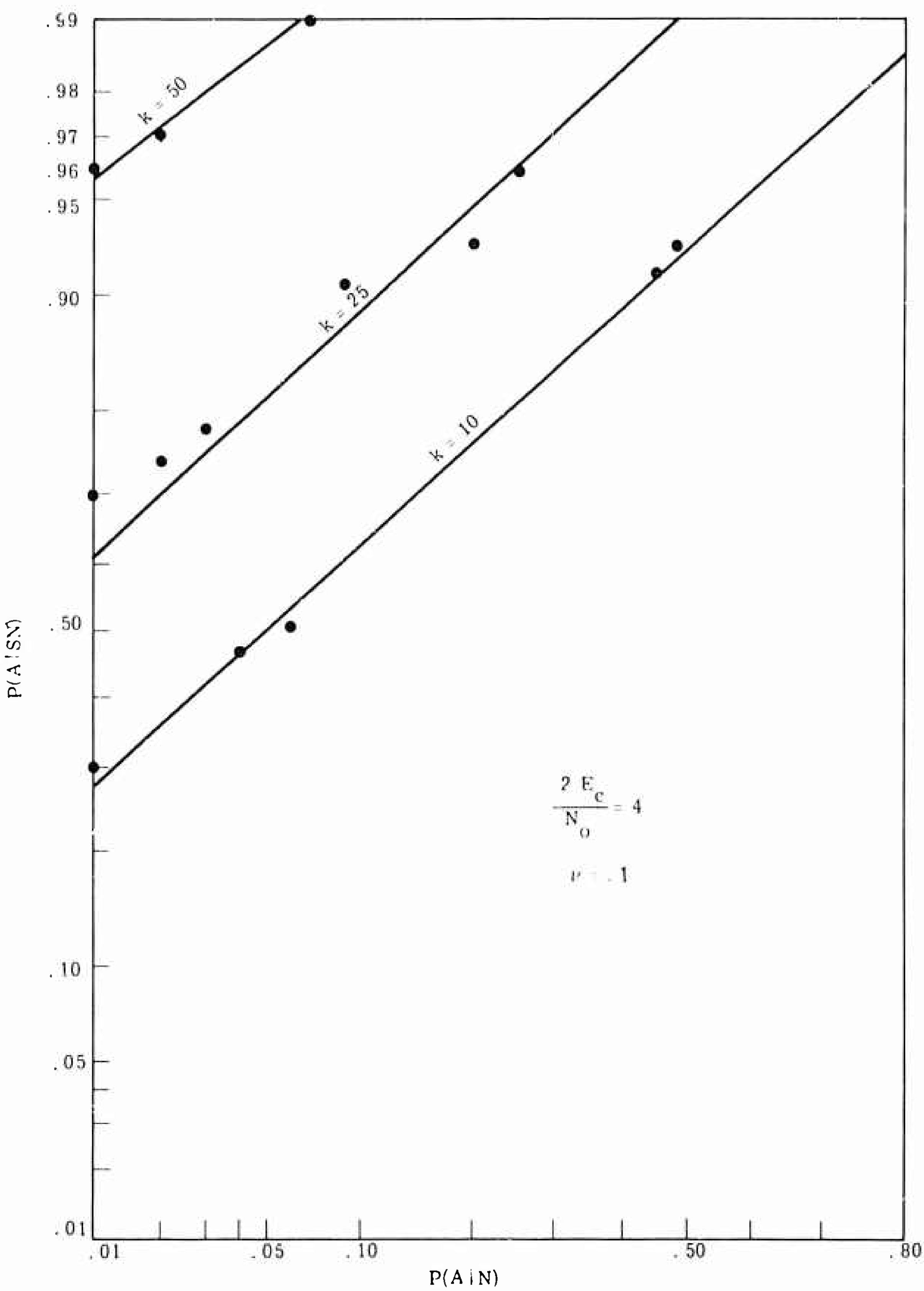

Fig. 7.38. ROC for optimum receiver, CKE, arrival times known exactly, $\frac{2E_c}{N_o} = 4$, $\nu = .1$.

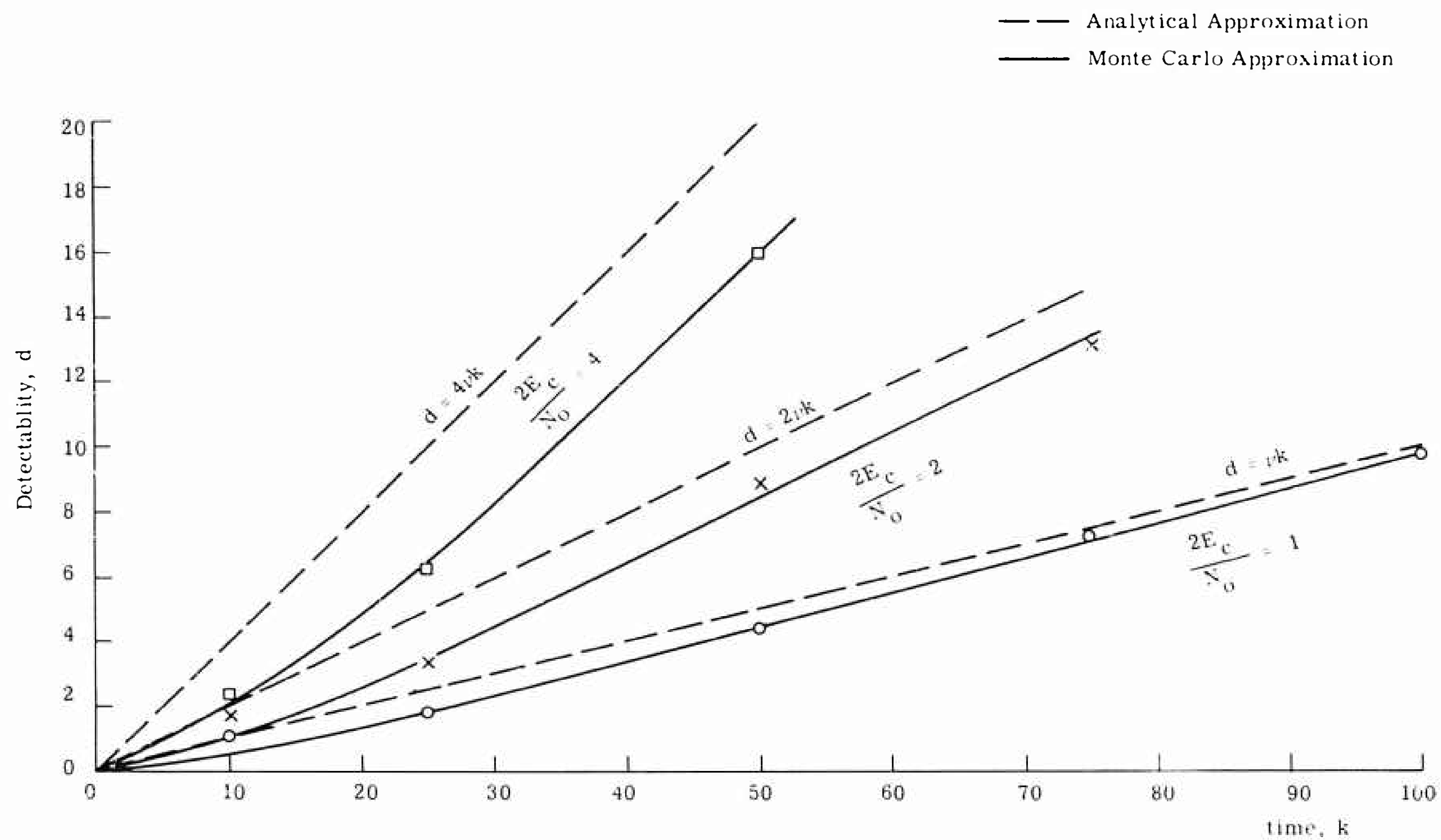

Fig. 7.39. Comparison of analytical approximation and Monte Carlo data for CKE, arrival times known exactly.

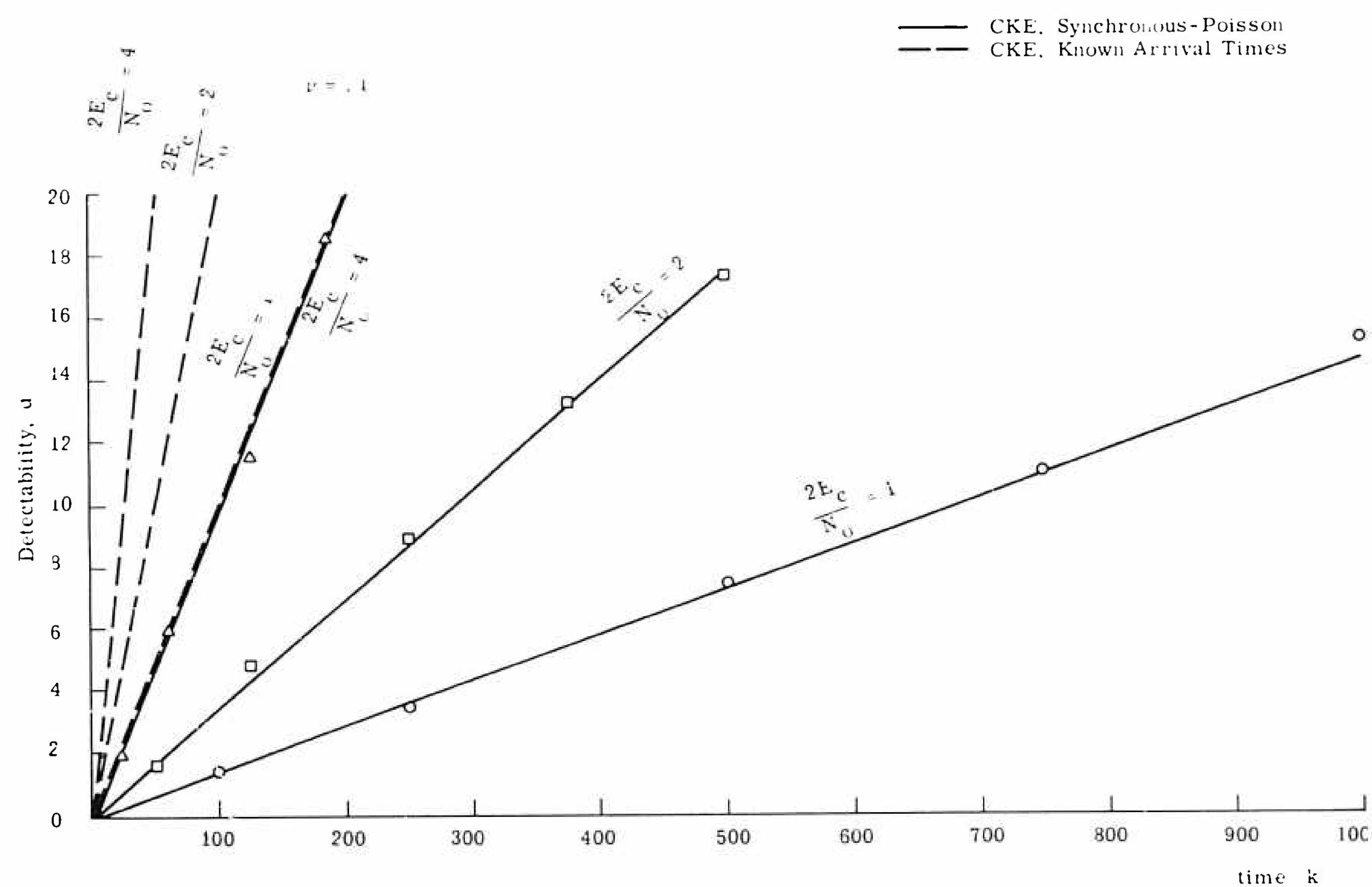

Fig. 7.40. Comparison of detectability of CKE, Synchronous-Poisson Time Structure and CKE, arrival times known exactly, $\nu = .1$, $\frac{2E_c}{N_o} = 1, 2, 4.$

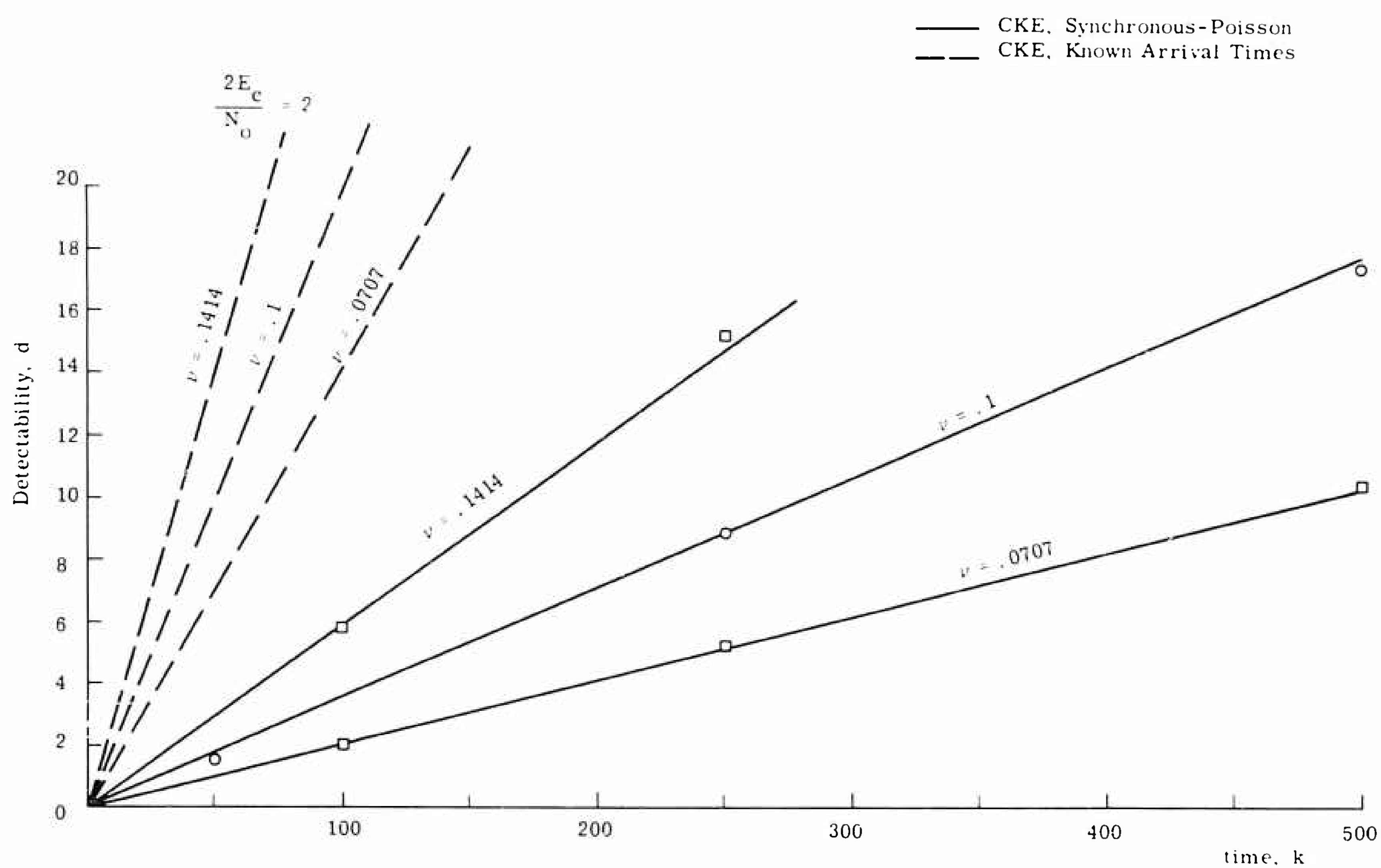

Fig. 7.41. Comparison of detectability of CKE, Synchronous-Poisson Time Structure, and CKE, arrival times known exactly, $\frac{2E_c}{N_o} = 2$, $\nu = .0707, .1, .1414$.

In general, it can be seen that the effect of Synchronous-Poisson time uncertainty on detectability is substantial for the range of values of duty factor, ν, and $\frac{2E_c}{N_o}$ considered here. Component recurrence time uncertainty, of the Synchronous-Poisson Time Structure, degrades performance the most at low component signal-to-noise ratios and low duty factors.

7.3.2.5 Comparison of the Performance of the Optimum Receiver with the Energy Detector (One of Eight Orthogonal Components). The optimum adaptive receiver has already been discussed for the case of one of eight orthogonal components (see Section 7.2). It uses a temporary memory for storing probabilities of each of the eight components and continually updates these probabilities with new information obtained in subsequent observations. On the other hand, the energy detector has one square-law nonlinearity followed by an integrator. The energy detector also has no classification capability. It is interesting to see how the detection performance of such a limited memory receiver compares with the performance of the optimum receiver. The performance of the energy detector for the Synchronous-Poisson time uncertainty signals has been derived in Appendix E for one of M orthogonal components, and for small $\frac{2E_c}{N_o}$; it is

$$d = \frac{\nu^2 k}{4M} \left[\frac{2E_c}{N_o} \right]^2 \tag{7.15}$$

The performance of the energy detector is degraded by a factor of ν^2 which is the duty factor squared, and by the component uncertainty, expressed by M. Figure 7.42 is a comparison of the detectability of the optimum receiver and the energy detector. After about k = 100, the detectability of the optimum receiver increases rapidly over the energy detector. This shows the value of the optimal use of the receiver memory.

7.3.2.6 Effect of the "ν Nonlinearity" On Receiver Performance. When the design of the optimum receiver for the Synchronous-Poisson Time Structure was compared with the optimum receiver for the Periodic Time Structure, many striking similarities were found (see Section 5.4). In fact, the primary difference was the presence of a "ν nonlinearity" in the receiver designed for the Synchronous-Poisson Time Structure (see Fig. 5.13b). In the optimum receiver for the Periodic Time Structure, $x_i \cdot C - \frac{C \cdot C}{2}$

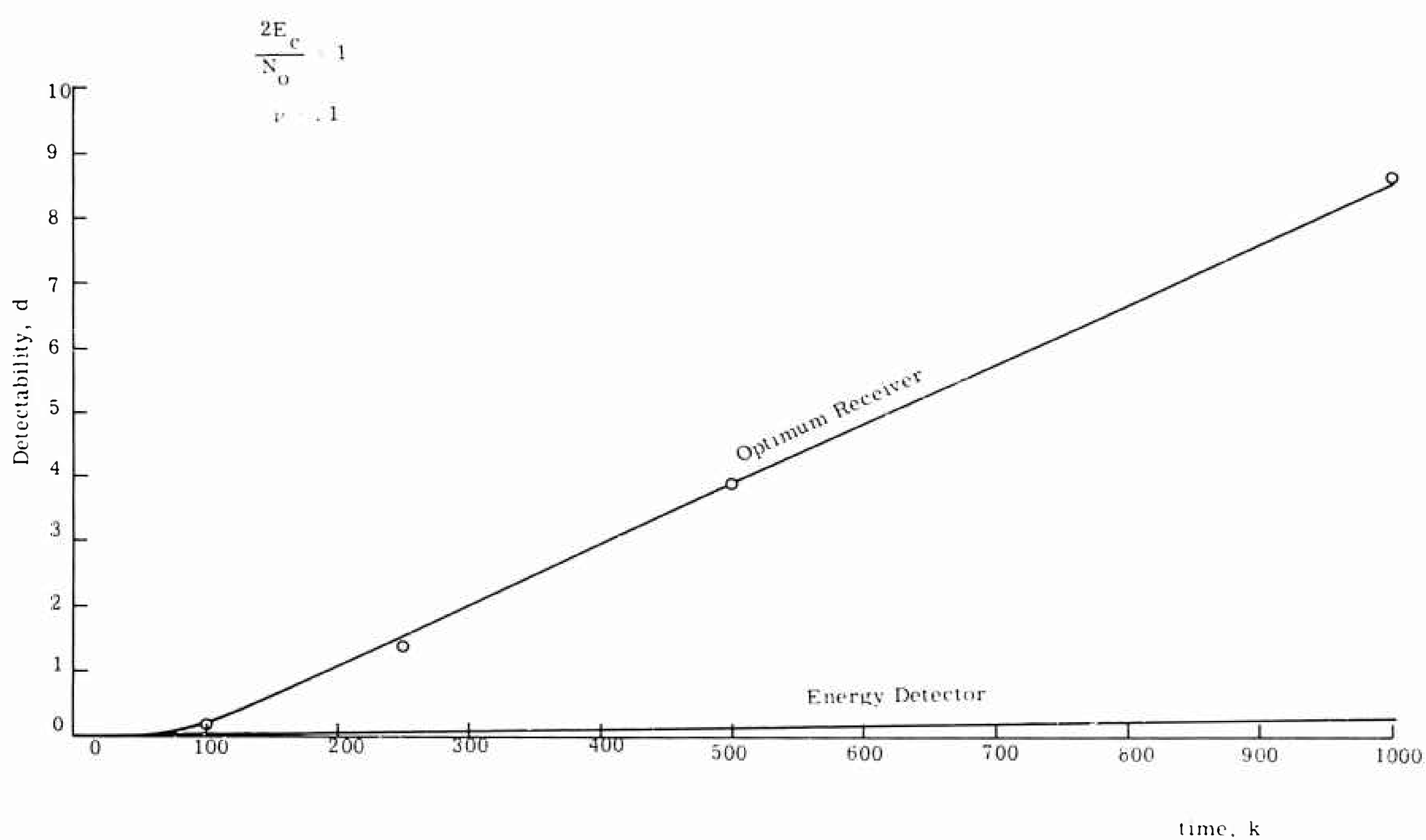

Fig. 7.42. Comparison of optimum receiver and energy detector performance. CKS, (one of eight orthogonal components).

was circulated through the delay forming

$$\sum_{i=1}^{k} x_i \cdot C - \frac{C \cdot C}{2} \quad .$$

Since this can be written as

$$\ell(X_k) = \sum_{i=1}^{k} x_i \cdot C - \frac{C \cdot C}{2} = -\frac{k\,C \cdot C}{2} + C \cdot \sum_{i=1}^{k} x_i \tag{7.16}$$

one can see that for the periodic case the observations themselves could be simply added in synchronous intervals and this sum, $\sum_{i=1}^{k} x_i$, correlated with the component C. In the optimum receiver for the Synchronous-Poisson Time Structure, however, the observations must first be correlated with the component, passed through a ν nonlinearity and then summed. A natural question arises as to how important this nonlinearity is. Since the input to the nonlinearity is a random variable, more than just the shape of the nonlinearity must be examined. In this section the effect on detectability of the ν nonlinearity will be studied by evaluating the detection performances of two receivers. These two receivers are: (1) the optimum receiver for CKE, Synchronous-Poisson Time Structure, (Fig. 5.13b) and (2) a suboptimum receiver for the CKE, Synchronous-Poisson Time Structure (Fig. 5.13a). The first receiver is optimum and includes the ν nonlinearity and the second receiver (which is suboptimum for the synchronous case but happens to be optimum for the periodic case) does not have a nonlinearity. The optimum receiver has already been evaluated using Monte Carlo techniques. The suboptimum receiver has been evaluated analytically and the derivation of this result is presented in Appendix F. The performance of this suboptimum receiver is

$$d = \nu^2 k \frac{2E_c}{N_o} \tag{7.17}$$

The performance of this receiver is affected by ν squared. One ν accounts for the fact that signal energy is reduced by ν and the other ν accounts for the uncertainty in recurrence times of components.

In Figs. 7.43 through 7.45 the performance of the optimum and suboptimum receivers are compared. The performance curves for the optimum receiver are obtained from Monte

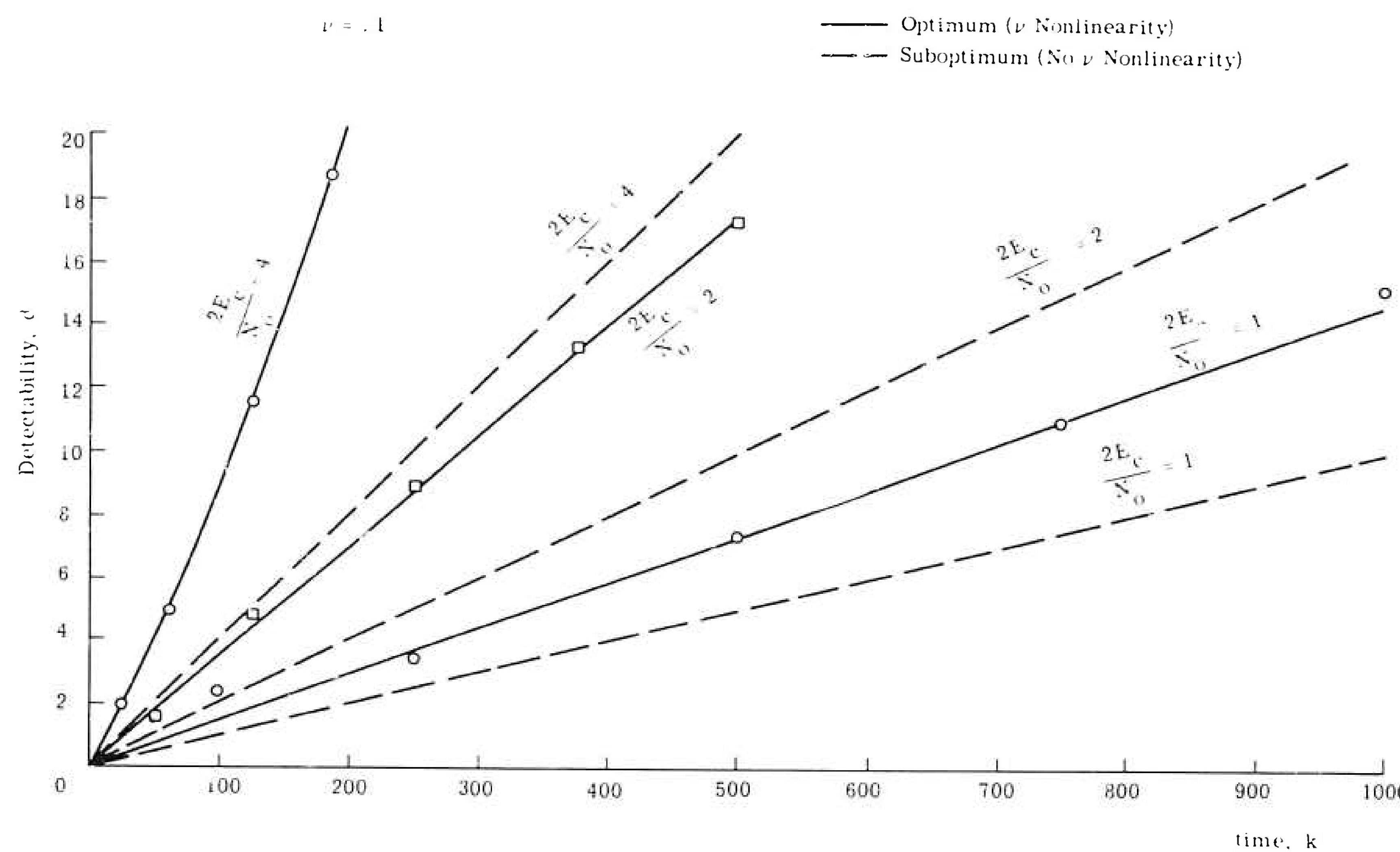

Fig. 7.43. Comparison of detectability of an optimum and suboptimum receiver, CKE, Synchronous-Poisson Time Structure, $\nu = .1$, $\frac{2E_c}{N_o} = 1, 2, 4$.

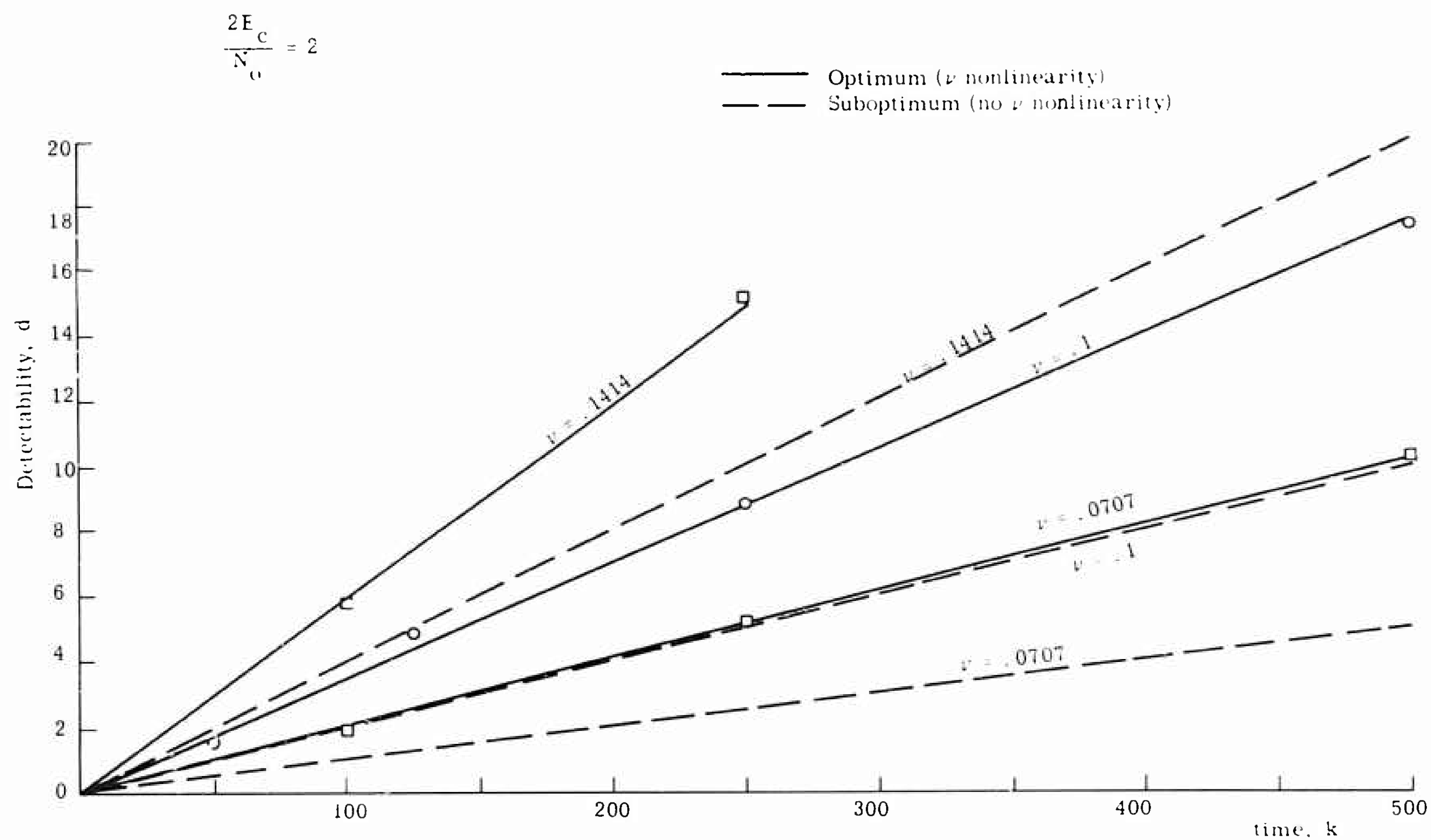

Fig. 7.44. Comparison of detectability of an optimum and suboptimum receiver, CKE, Synchronous-Poisson Time Structure, ν = .0707, .1, .1414,

$$\frac{2E_c}{N_o} = 2.$$

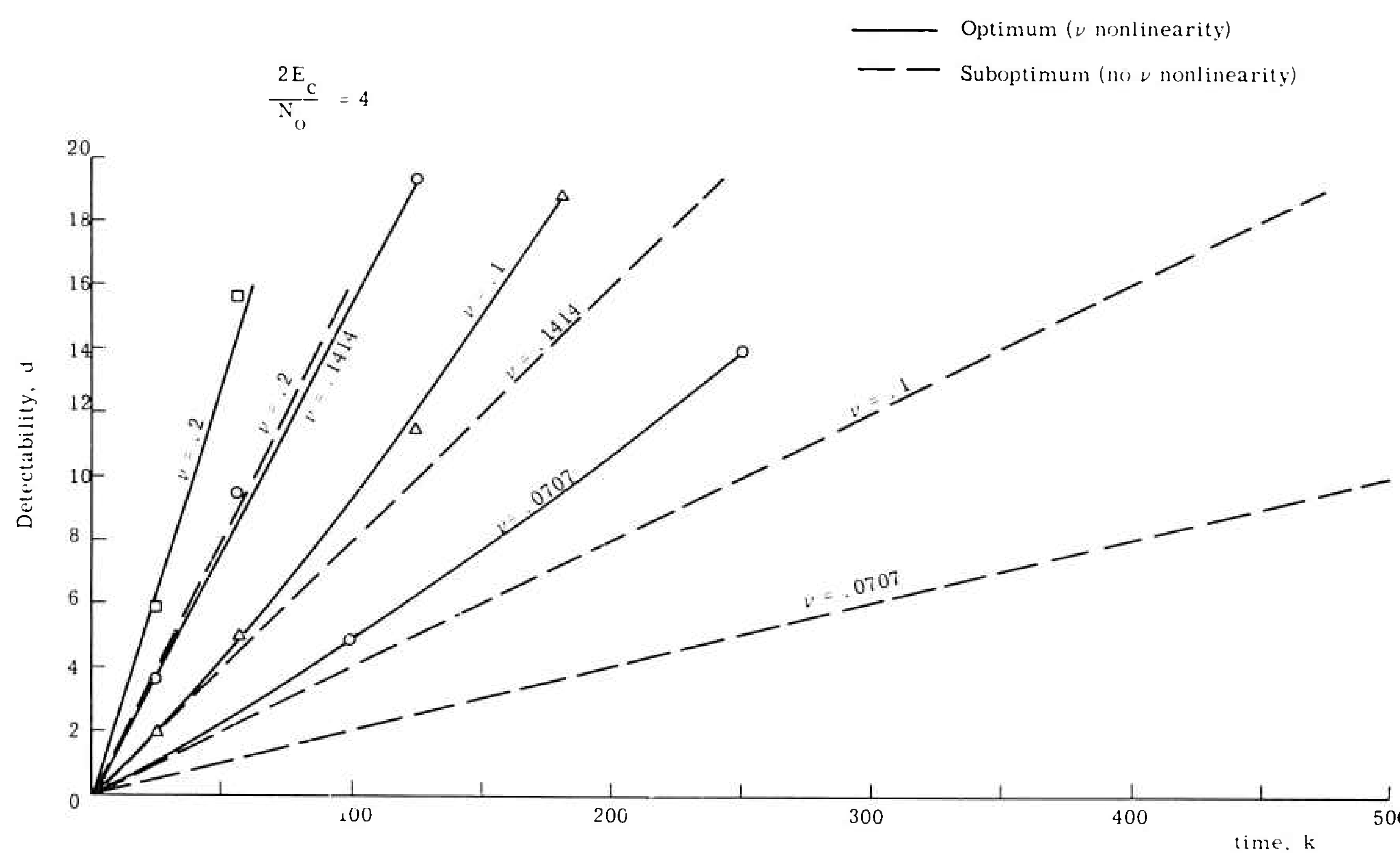

Fig. 7.45. Comparison of detectability of an optimum and suboptimum receiver, CKE, Synchronous-Poisson Time Structure, $\nu = .0707, .1, .1414, .2$

$$\frac{2E_c}{N_o} = 4.$$

Carlo runs (see Figs. 7.20 through 7.30), and the performance of the suboptimum receiver is given by Eq. 7.17. Let us discuss each of these three figures. In Fig. 7.43 detectability vs. time is plotted for the two receivers for $\nu = .1$, $\frac{2E_c}{N_o} = 1, 2,$ and 4. The increased processing time necessary for the suboptimum receiver to reach the same level of detection performance as the optimum receiver can be determined by comparing the performance of the two receivers at a constant d. The ratio of optimum receiver processing time to suboptimum receiver processing time required to reach the same detectability is:

$\frac{2E_c}{N_o}$	Ratio of optimum to suboptimum receiver processing time.
1	1.46
2	1.76
4	2.25

This data shows that the importance of the ν nonlinearity increases as $\frac{2E_c}{N_o}$ increases.

In Fig. 7.44 the performance of the optimum and suboptimum receivers are plotted for $\frac{2E_c}{N_o} = 2$ and $\nu = .0707$, .1, and .1414. In Fig. 7.45 similar data is presented for $\frac{2E_c}{N_o} = 4$ and $\nu = .0707$, .1, .1414, and .2. The ratio of processing times required by the suboptimum and optimum receivers to reach the same detectability is plotted in Figs. 7.46 and 7.47. The ratio of processing times for $\nu = 1$ is one since then the two receivers are identical. It is difficult to obtain data for very low values of ν because of the longer runs required on the digital computer. Figures 7.46 and 7.47 show that the importance of the ν nonlinearity increases as the duty factor, ν, decreases.

In conclusion, the importance of the ν nonlinearity in the optimum receiver increases as $\frac{2E_c}{N_o}$ increases and the duty factor decreases.

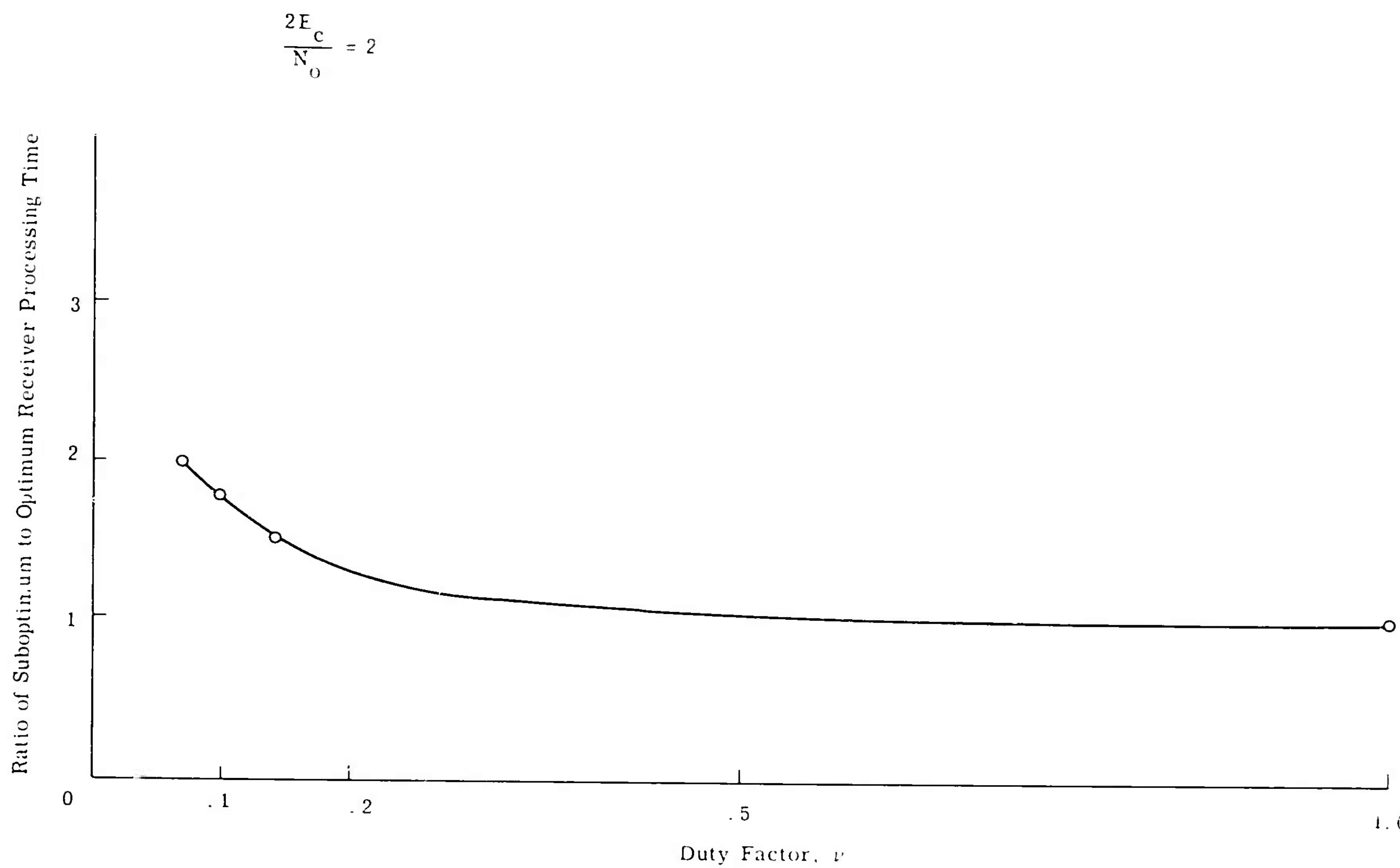

Fig. 7.46. Ratio of increased processing time required by suboptimum receiver vs. duty factor, $\frac{2E_c}{N_o} = 2$.

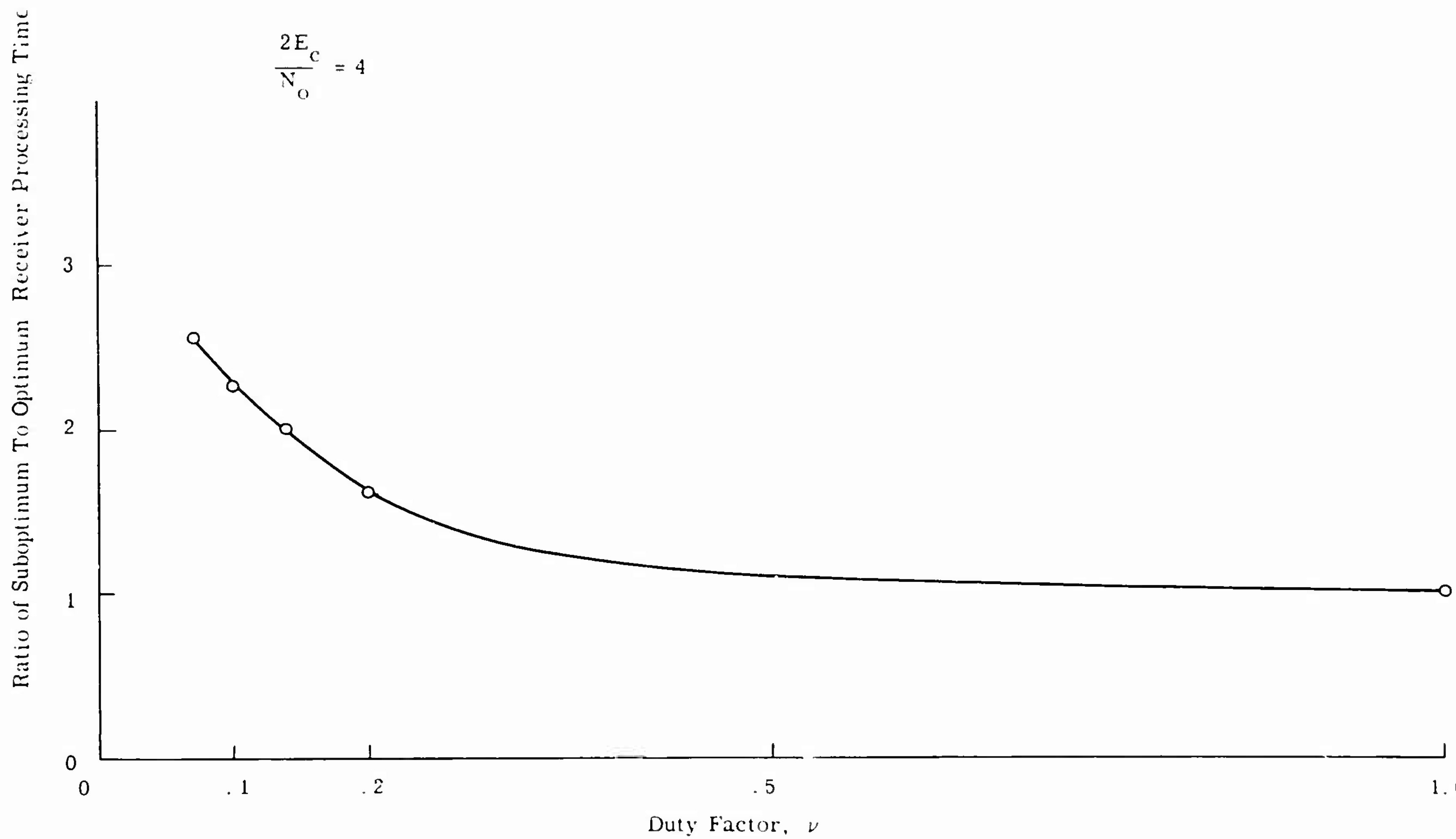

Fig. 7.47. Ratio of increased processing time required by suboptimum receiver vs. duty factor, $\frac{2E_c}{N_o} = 4$.

7.4 Summary

In this chapter an adaptive receiver realization was simulated on a digital computer for one of eight orthogonal components, Synchronous-Poisson Time Structure. Individual runs of the detection and classification outputs as a function of time were displayed to illustrate how such a receiver operates. However, it is difficult to judge receiver performance from the individual runs.

The performance of the optimum receiver for a CKE, Synchronous-Poisson Time Structure was evaluated experimentally on the digital computer for various values of the parameters ν (duty factor), $\frac{2E_c}{N_o}$, and k (time). The detectability builds up in time in a nearly linear fashion. The signal detectability is reduced when the arrival times are uncertain. For example, for $\frac{2E_c}{N_o} = 1$ and a duty factor of 10%, the processing time required to reach a specified detectability is 6.85 times longer than would be required if the component recurrence times were known exactly. This extra required processing time increases as component signal-to-noise ratio and duty factor decrease.

By comparing the detectability for one of eight orthogonal components with the CKE case, it was found that component uncertainty affects detectability in a rather mild manner after sufficient processing time has elapsed. This effect is similar to that which occurs when the component recurs periodically. The performance of the optimum receiver for one of eight orthogonal components was compared with the energy detector to show the value of the optimal use of the receiver memory.

The importance of the ν nonlinearity in the optimum receiver for the CKE, Synchronous-Poisson Time Structure was investigated by evaluating a suboptimum receiver which does not contain the ν nonlinearity. It was found that the importance of the ν nonlinearity increases as $\frac{2E_c}{N_o}$ increases and the duty factor decreases.

CHAPTER VIII

SUMMARY

8.1 Conclusions

An exciting new area of research is the application of adaptive processing techniques to the problem of detecting signals in noise. Adaptive techniques have been considered by several researchers in regard to detecting an unknown, but fixed, waveform that recurs randomly in time. For such a detection situation it seems quite natural to postulate an adaptive device to "learn" this waveform in order to aid the detection process. However, a basic contribution of this study has been to show how the theory of signal detectability can be extended to include techniques of optimum receiver design for problems of this type.

Most past work in detection theory considers signals whose time structure is periodic. In the usual radar problem, the time structure is basically periodic of known repetition frequency but unknown start of the period and parameters such as amplitude are assumed unknown. A significant difference in this study is in the consideration of the detection of signals in noise in which the time structure is nonperiodic.

A rather general problem is considered to which adaptive techniques have been applied by others. A fixed waveform, called a component, is initially uncertain but learnable. One of b components is selected prior to the start of transmission and the same component recurs at uncertain times which are unlearnable. The receiver must be capable of detecting such a recurrence phenomenon in noise. This problem is formulated as an over-all optimization problem in detection theory rather than as a problem in which an adaptive receiver is postulated. Detection theory provides a mathematical model in which initial knowledge about component and recurrence time uncertainties are expressed in terms of a priori probabilities. The component is uncertain in the sense that one of a finite number of b components is selected for transmission. The recurrence-time uncertainties studied are of three basic types: Periodic, Synchronous-Poisson, and a Sporadic-Poisson

Time Structure. The basic philosophy is to design an optimum receiver which makes the best decision as to presence or absence of the recurrence phenomenon in the entire observation, X_k, and to realize this optimum receiver with an equivalent adaptive realization.

From detection theory it is known that the optimum receiver forms the likelihood ratio, $\ell(X_k)$, of that observation. If the receiver is to run in time, it must keep forming the likelihood ratio of the entire observation as k increases. In Chapter III it was shown how this optimum receiver could be realized in an alternate equivalent form. This is a form in which the likelihood ratio of the observation, X_k, is realized in a sequential manner. The operations performed by the sequential and nonsequential receivers appear quite different although the receivers are equivalent for detection purposes. The sequential receiver is called an adaptive realization because of the explicit manner in which it updates knowledge of the situation. A classification output, which is a set of updated probabilities, can be conveniently made available.

A basic difficulty in receiver design emerges when considering signals with a nonperiodic time structure which does not appear in the classical periodic cases. This is the problem of providing sufficient receiver memory to store probabilities of signals in an ensemble that grows rapidly in time. To design a practical optimum receiver, it was found necessary to develop an indirect description of the signal ensemble. In an indirect description, the signal ensemble is described in terms of a component ensemble and a time structure. Optimum adaptive receivers were designed for the Sporadic-Poisson, Synchronous-Poisson, and Periodic Time Structures using this technique and the sequential realization.

The proper use and updating of the contents of the temporary memory for the adaptive realizations are specified by the design procedure. When the time structure is periodic, the starting times of the component are known, and the possible components are of common duration, the temporary memory of the optimum receiver stores and sums the input waveshape to the receiver. If the period is unknown or if the time structure is Synchronous-Poisson, then the temporary memory stores a more abstract quantity such as an updated probability of each possible period or component. Finally, in the Sporadic-Poisson Time Structure, the adaptive realization stores and continually updates component identification and local component positional information in its temporary memory. The updating procedures have been formalized and presented.

An adaptive receiver realization was simulated on the digital computer, and its operation was displayed for a number of runs. These displays show the "adaptive" nature of this type of realization. However, it was found difficult to judge receiver performance from any single run. Instead, performance is presented in terms of the ROC (receiver operating characteristic).

This study also contributes to the understanding of the effect of time uncertainty on detectability. Since the adaptive realization provides a receiver of manageable form, it becomes feasible to evaluate its performance. Evaluation of the performance of the optimum receiver for a particular time structure then sets an upper bound on the performance of any other receiver in that same environment. The effect of Synchronous-Poisson Time Structure uncertainty on detectability for the case of a component known exactly (CKE) was investigated. This is an important first case. Even when the component is initially uncertain and has been "learned" the performance of the optimum receiver for that case cannot exceed the CKE case. The performance of the optimum receiver for the CKE, Synchronous-Poisson Time Structure, was presented in terms of the ROC for various values of the average duty factor, $\frac{2E_c}{N_o}$, and time. The detectability, d, builds almost linearly in time.

The price in performance that must be paid by even the optimum receiver, because component arrival times are not known exactly, was investigated. For example, for a $\frac{2E_c}{N_o} = 1$ and an average duty factor of 10%, the receiver processing time required to reach the same detectability is about 6.85 times longer than would be required if component recurrence times were known exactly. This extra processing time required is a decreasing function of $\frac{2E_c}{N_o}$ and a decreasing function of average duty factor. These results show that even when a component is known exactly, the effect on detectability can be substantial.

The results of comparing the detectability for one of eight orthogonal components with the CKE, Synchronous-Poisson Time Structure, suggest that component uncertainty affects detectability in a rather mild manner after a sufficient amount of time has elapsed. This situation is similar to that of the periodic case. The performance of the optimum receiver for one of eight orthogonal components was compared with the energy detector to show its superior performance.

The importance of storing and updating probability or likelihood ratio terms in the temporary memory for the optimum receiver, CKE, Synchronous-Poisson Time Structure, was investigated by comparing its performance with a receiver which stores and modifies input waveshape. From these results it was found that storing probabilities or likelihood ratios in the temporary memory rather than input waveshape became more important as $\frac{2E_c}{N_o}$ increased and the average duty factor decreased. For example, at $\frac{2E_c}{N_o} = 2$ and an average duty factor of .0707, about twice as much processing time is required by the receiver that circulates input waveshape to obtain the same detectability as the optimum receiver.

8.2 Future Work

There are a number of directions in which future work can go. First, the effect of time uncertainty on detectability has just begun. Although the optimum receiver has been designed, its actual performance in terms of the ROC remains to be determined. The effect of the Sporadic-Poisson Time Structure and component uncertainties on detectability remains to be investigated.

The problem of optimum receiver design for an infinite component ensemble with a learnable parameter and a Synchronous-Poisson or Sporadic-Poisson Time Structure is an area of investigation. At present, such a problem could be attacked in an approximate manner by representing such a component ensemble as finite and using the receiver design techniques presented in this study.

Another area of investigation is the design of optimum receivers for learnable time structures which are initially uncertain. In this study the design of the optimum receiver for detecting a recurrence phenomenon of unknown period was presented and this could be extended to more complicated time structures.

Three basic time structures have been considered: the Periodic, Synchronous-Poisson, and Sporadic-Poisson. The extension of the same optimum receiver design approach could be considered for many other types of time uncertainty.

APPENDIX A

OPTIMUM ADAPTIVE RECEIVER REALIZATIONS SPORADIC-POISSON TIME STRUCTURE

A. 1 Realization II

The optimum adaptive receiver realized in Section 5. 1. 1 is not unique A slight modification of Realization I results if one writes the joint probability, $b_{i,j}(k)$ as

$$b_{i,j}(k) = b'_{i,j}(k)\, p_k(C^i | SN) \qquad \text{(A. 1)}$$

where $b'_{i,j}(k)$ is the probability of the jth component sample under the condition that the ith component and SN are present and that k observations have been taken. Writing $b_{i,j}(k)$ in this manner emphasizes the classification output, $p_k(C^i | SN)$. Substituting Eq. A. 1 into Eq. 5. 18 for the sequential average likelihood ratio results in

$$\ell(x_k | X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i | SN) \Bigg\{ \Big[b'_{i,0}(k-1) + b'_{i,n_i}(k-1) \Big] \Big[(1-\nu_i) + \nu_i \ell(x_k | s_k = c_{i,1}) \Big] + \sum_{j=2}^{b} b'_{i,j-1}(k-1) \ell(x_k | s_k = c_{i,j}) \Bigg\} \qquad \text{(A. 2)}$$

where $p_k(C^i | SN)$ has been factored out. We can therefore define a conditional sequential likelihood ratio, $\ell(x_k | X_{k-1}, C^i)$

$$\ell(x_k | X_{k-1}, C^i) = \Big[b'_{i,0}(k-1) + b'_{i,n_i}(k-1) \Big] \Big[(1-\nu_i) + \nu_i \ell(x_k | s_k = c_{i,1}) \Big] + \sum_{j=2}^{n_i} b'_{i,j-1}(k-1) \ell(x_k | s_k = c_{i,j}) \qquad \text{(A. 3)}$$

The preceding equation gives the likelihood ratio of the observation x_k assuming the ith component has been recurrent. Substituting Eq. A. 3 into Eq. A. 2 results in the average sequential likelihood ratio, $f(x_k | X_{k-1})$, becoming

$$f(x_k | X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i | SN) f(x_k | X_{k-1}, C^i) \tag{A. 4}$$

The updating of component information still requires equations similar to Eq. 5. 32 through 5. 34. If one makes the substitution

$$b_{i,j}(k-1) = b'_{i,j}(k-1) p_{k-1}(C^i | SN) \tag{A. 5}$$

and factors out $p_{k-1}(C^i | SN)$, then Eqs. 5. 32 through 5. 34 become

$$b'_{i,0}(k) p_k(C^i | SN) = \frac{(1-\nu_i)\left[b'_{i,0}(k-1) + b_{i,n_i}(k-1)\right] p_{k-1}(C^i | SN)}{f(x_k | X_{k-1})} \tag{A. 6}$$

$$b'_{i,1}(k) p_k(C^i | SN) = \frac{\nu_i\left[b'_{i,0}(k-1) + b_{i,n_i}(k-1)\right] p_{k-1}(C^i | SN) f(x_k | s_k = c_{i,1})}{f(x_k | X_{k-1})} \tag{A. 7}$$

$$b'_{i,j}(k) p_k(C^i | SN) = \frac{b'_{i,j-1}(k-1) p_{k-1}(C^i | SN) f(x_k | s_k = c_{i,j})}{f(x_k | X_{k-1})} \tag{A. 8}$$

$$\text{for } j = 2, 3, \ldots, n_i$$

Instead of updating products of the form $b'_{i,j-1}(k-1) p_{k-1}(C^i | SN)$, $b'_{i,j-1}(k-1)$ and $p_{k-1}(C^i | SN)$ can be updated separately by noticing that

$$b'_{i,0}(k) = \frac{(1-\nu_i)\left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right]}{f(x_k | X_{k-1}, C^i)} \tag{A. 9}$$

$$b'_{i,1}(k) = \frac{\nu_i\left[b'_{i,j-1}(k-1) + b'_{i,n_i}(k-1)\right] f(x_k \mid s_k = c_{i,1})}{f(x_k \mid X_{k-1}, C^i)} \tag{A. 10}$$

$$b'_{i,j}(k) = \frac{b'_{i,j-1}(k-1) f(x_k \mid s_k = c_{i,j})}{f(x_k \mid X_{k-1}, C^i)} \tag{A. 11}$$

and

$$p_k(C^i \mid SN) = \frac{p_{k-1}(C^i \mid SN) f(x_k \mid X_{k-1}, C^i)}{f(x_k \mid X_{k-1})} \tag{A. 12}$$

The updating of component information can be implemented by updating of the probability of the ith component, $p_k(C^i \mid SN)$, and updating the probability of the jth component sample given the ith component, $b'_{i,j}(k)$. This then gives an alternative realization of the optimum adaptive receiver. The design equations for Realization II are summarized in Table A. 1.

TABLE A. 1

BASIC RECEIVER DESIGN EQUATIONS, SPORADIC-POISSON TIME STRUCTURE

REALIZATION II

Optimum Detection Output

$$f(X_k) = f(X_{k-1}) f(x_k \mid X_{k-1}) \tag{3. 8}$$

Sequential Average Likelihood Ratio

$$f(x_k \mid X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i \mid SN) f(x_k \mid X_{k-1}, C^i) \tag{A. 4}$$

Component Conditional Sequential Likelihood Ratio

$$f(x_k | X_{k-1}, C^i) = \left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right]\left[1-\nu_i + \nu_i f(x_k | s_k = c_{i,1})\right]$$

$$+ \sum_{j=2}^{n_i} b'_{i,j-1}(k-1) f(x_k | s_k = c_{i,j}) \qquad \text{(A. 3)}$$

Classification - Component Position

$$b'_{i,0}(k) = \frac{(1-\nu_i)\left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right]}{f(x_k | X_{k-1}, C^i)} \qquad \text{(A. 9)}$$

$$b'_{i,1}(k) = \frac{\nu_i\left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right] f(x_k | s_k = c_{i,1})}{f(x_k | X_{k-1}, C^i)} \qquad \text{(A. 10)}$$

$$b'_{i,j}(k) = \frac{b'_{i,j-1}(k-1) f(x_k | s_k = c_{i,j})}{f(x_k | X_{k-1}, C^i)} \qquad \text{(A. 11)}$$

for $j = 2, 3, \ldots, n_i$

Classification - Component Identification

$$p_k(C^i | SN) = \frac{p_{k-1}(C^i | SN) f(x_k | X_{k-1}, C^i)}{f(x_k | X_{k-1})} \qquad \text{(A. 12)}$$

A block diagram of the realization is illustrated in Fig. A. 1. This realization is basically the same as Realization I in Section 5. 1. 1. In Realization II, however, the information regarding which component is present is kept in a temporary memory separate from the updated component positional information. This means that Realization II requires a greater amount of temporary memory. On the other hand, Realization I requires a summer, $\sum_{j=0}^{n_i} b_{i,j}(m)$ to calculate the component classification information.

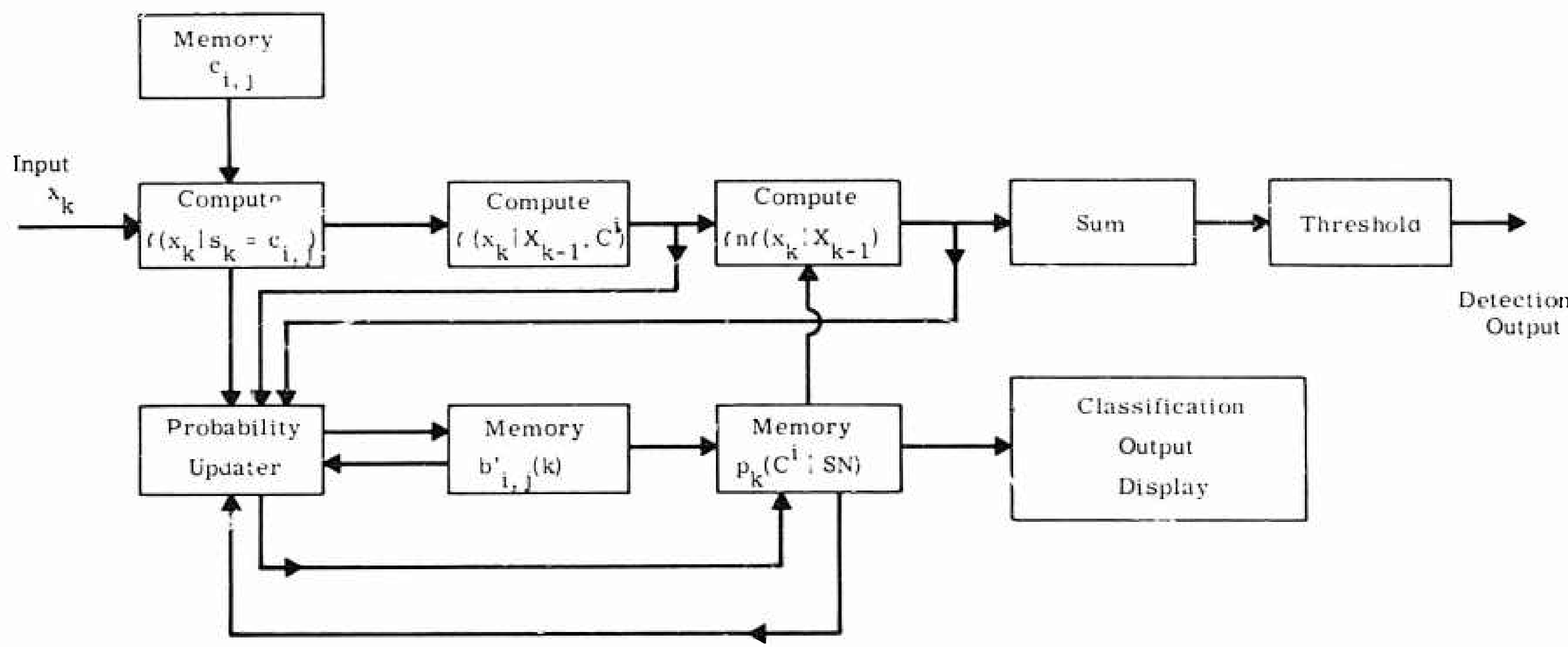

Fig. A.1. Adaptive receiver realization, Sporadic-Poisson Time Structure, Realization II.

A. 2 Realization III

There is a third adaptive receiver realization which is a "b" channel receiver in which each channel "looks" for one of the components. For a finite number of signals in the signal ensemble the integral in Eq. 5. 1 becomes a sum so that the likelihood ratio of the observation X_k is

$$\ell(X_k) = \sum_{s \epsilon S} \ell(X_k | s) p_o(s | SN) \qquad (A. 13)$$

Now the signal space, S, can be partitioned into b disjoint subspaces, S_i. Each S_i subspace contains all those signals that might result from the ith component alone. This is a result of the restriction that a given component, C^i, is selected and fixed at the beginning of each long transmission. Thus Eq. A. 13 can be written as

$$\ell(X_k) = \sum_{i=1}^{b} \sum_{s \epsilon S_i} \ell(X_k | s) p_o(s | C^i, SN) p_o(C^i | SN) \qquad (A. 14)$$

where by definition of a joint probability, $p_o(s | SN)$ has been written as

$$p_o(s | SN) = p_o(s | C^i, SN) p_o(C^i | SN) \qquad (A. 15)$$

First, the summation in Eq. A. 14 is carried out over each subspace, S_i. Since $p_o(C^i | SN)$ is a factor for each sum over S_i, Eq. A. 14 can be written as

$$\ell(X_k) = \sum_{i=1}^{b} p_o(C^i | SN) \sum_{s \epsilon S_i} \ell(X_k | s) p_o(s | C^i, SN) \qquad (A. 16)$$

or

$$\ell(X_k | C^i) = \sum_{i=1}^{b} \ell(X_k | C^i) p_o(C^i | SN) \qquad (A. 17)$$

Now the sum over the space S_i is the likelihood ratio, $\ell(X_k | C^i)$ of the observation X_k under the condition the ith component is present. In other words,

$$\ell(X_k | C^i) = \sum_{s \epsilon S_i} \ell(X_k | s) p_o(s | C^i, SN) \qquad (A. 18)$$

This likelihood ratio, $\ell(X_k|C^i)$, can in turn be realized in a sequential fashion for each of the b possible components. So

$$\ell(X_k|C^i) = \ell(X_{k-1}|C^i)\,\ell(x_k|X_{k-1}, C^i) \tag{A. 19}$$

where $\ell(x_k|X_{k-1}, C^i)$ is given by Eq. A. 13 along with Eqs. A. 9, A. 10 and A. 11. The classification output for component identification is given by

$$p_k(C^i|SN) = \frac{p_o(C^i|SN)\,\ell(X_k|C^i)}{\ell(X_k)} \tag{A. 20}$$

Table A. 2 summarizes the design equations. This realization has a channel for each of the b possible components. A gross block diagram of one channel of the receiver, Realization III, is shown in Fig. A. 2. In this realization each of the branches calculates the likelihood ratio of the entire observation, X_k, under the condition the ith component is being sent. The outputs of each channel are then weighted by the <u>a priori</u> probabilities, $p_o(C^i|SN)$, of the selection of each of the components and these are summed to form the likelihood ratio, $\ell(X_k)$. This realization looks "less adaptive" since $p_o(C^i|SN)$ is not explicitly updated at each step in time. It differs from Realization II in that it has a separate channel for each of the possible components.

TABLE A. 2

BASIC RECEIVER DESIGN EQUATIONS, SPORADIC-POISSON TIME STRUCTURE

REALIZATION III

<u>Optimum Detection Output</u>

$$\ell(X_k) = \sum_{i=1}^{b} \ell(X_k|C^i)\,p_o(C^i|SN) \tag{A. 18}$$

Component Conditional Likelihood Ratio

$$f(X_k | C^i) = f(X_{k-1} | C^i) f(x_k | X_{k-1}, C^i) \tag{A. 19}$$

Component Conditional Sequential Likelihood Ratio

$$f(x_k | X_{k-1}, C^i) = \left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right]\left[1 - \nu_i + \nu_i f(x_k | s_k = c_{i,1})\right] + \sum_{j=2}^{n_i} b'_{i,j-1}(k-1) f(x_k | s_k = c_{i,j}) \tag{A. 3}$$

Classification - Component Position

$$b'_{i,j}(k) = \frac{(1-\nu_i)\left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right]}{f(x_k | X_{k-1}, C^i)} \tag{A. 9}$$

$$b'_{i,1}(k) = \frac{\nu_i\left[b'_{i,0}(k-1) + b'_{i,n_i}(k-1)\right] f(x_k | s_k = c_{i,j})}{f(x_k | X_{k-1}, C^i)} \tag{A. 10}$$

$$b'_{i,j}(k) = \frac{b'_{i,j-1}(k-1) f(x_k | s_k = c_{i,j})}{f(x_k | X_{k-1}, C^i)} \tag{A. 11}$$

Classification - Component Identification

$$p_k(C^i | SN) = \frac{p_o(C^i | SN) f(X_k | C^i)}{f(X_k)} \tag{A. 20}$$

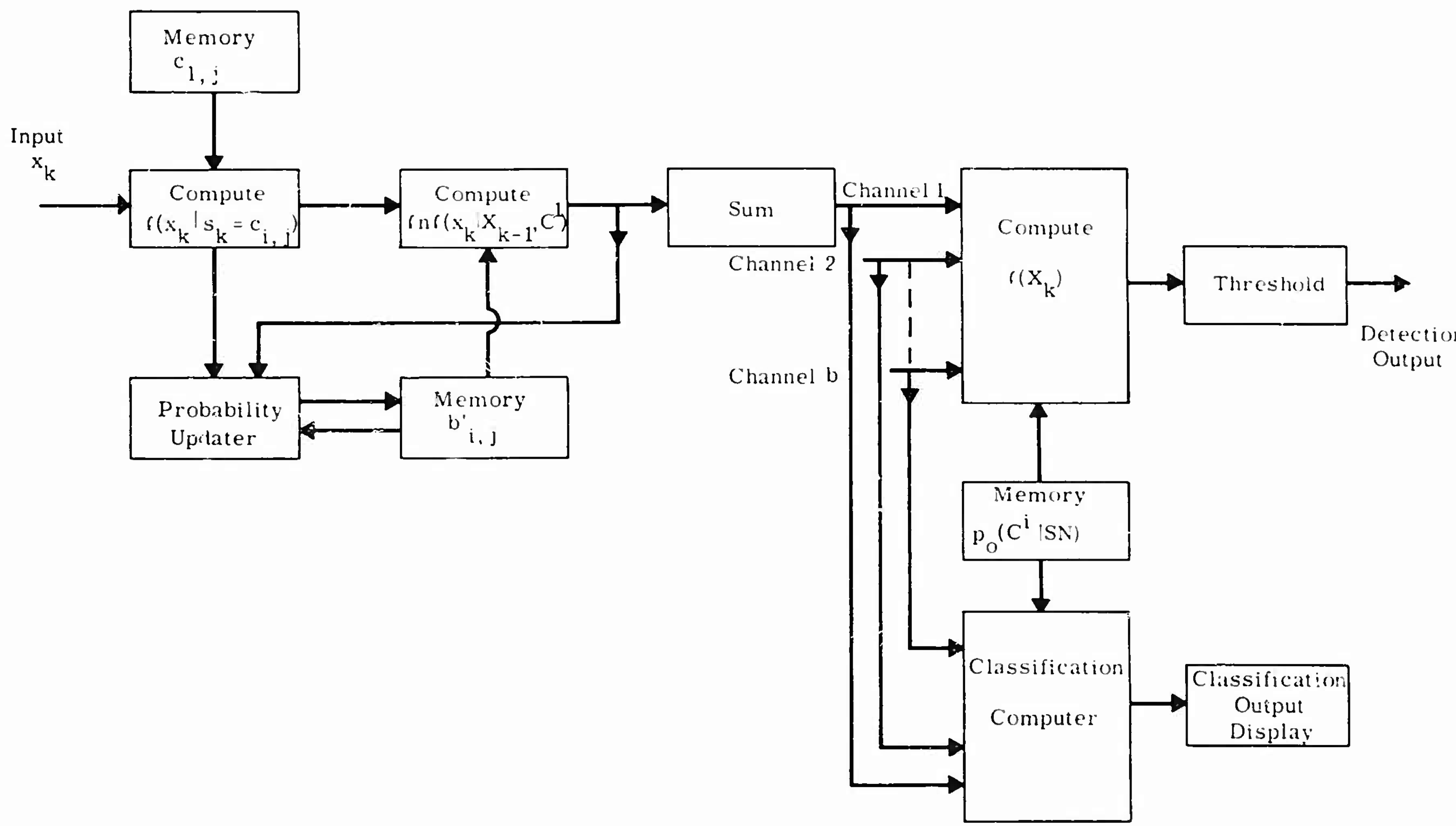

Fig. A.2. Adaptive receiver realization, Sporadic-Poisson Time Structure, Realization III.

A.3 Realization IV

We now come to one of the most interesting and useful realizations. The receiver realizations discussed previously are adaptive and operate in an intuitively appealing manner. A simpler sequential realization is possible. Let us begin the derivation of this realization by considering the likelihood ratio of the observation X_{k-1}. This likelihood ratio is

$$\ell(X_{k-1}) = \sum_{i=1}^{b} \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} \ell(X_{k-1}|s_1, s_2, \ldots, s_{k-1}) p_o(s_1, s_2, \ldots, s_{k-1}|SN) \tag{A.21}$$

Since these are finite sums, the order of summation may be interchanged. Leaving the summations with respect to i and s_{k-1} until last, one can write

$$\ell(X_{k-1}) = \sum_{i=1}^{b} \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} \left[\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2}=c_{i,0}}^{c_{i,n_i}} \ell(X_{k-1}|s_1, s_2, \ldots, s_{k-1}) p_o(s_1, s_2, \ldots, s_{k-1}|SN) \right] \tag{A.22}$$

Denoting the quantity in brackets by

$$Q_{s_{k-1}}(k-1) \triangleq \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2}=c_{i,0}}^{c_{i,n_i}} \ell(X_{k-1}|s_1, s_2, \ldots, s_{k-1}) p_o(s_1, s_2, \ldots, s_{k-1}|SN)$$

gives

$$\ell(X_{k-1}) = \sum_{i=1}^{b} \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} Q_{s_{k-1}}(k-1) \tag{A.24}$$

Now, the likelihood ratio of the observation X_k is defined to be

$$\ell(X_k) = \sum_{i=1}^{b} \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_k=c_{i,0}}^{c_{i,n_i}} \ell(X_{k-1}|s_1, s_2, \ldots, s_{k-1}) \ell(x_k|s_k) p_o(s_1, s_2, \ldots, s_k|SN) \tag{A.25}$$

and by definition of a joint probability

$$p_o(s_1, s_2, \ldots, s_k | SN) = p_o(s_1, s_2, \ldots, s_{k-1} | SN) p_o(s_k | s_1, s_2, \ldots, s_{k-1}, SN) \qquad (A.26)$$

For the generator processes under consideration

$$p_o(s_k | s_1, s_2, \ldots, s_{k-1}, SN) = g(s_k | s_{k-1}, SN) \qquad (A.27)$$

the state of the signal sample, s_k, depends only on the state of the previous sample, s_{k-1}. Substituting Eq. A.27 into A.26 results in

$$p_o(s_1, s_2, \ldots, s_k | SN) = p_o(s_1, s_2, \ldots, s_{k-1} | SN)\, g(s_k | s_{k-1}, SN) \qquad (A.28)$$

and substituting Eq. A.28 into A.25 gives

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_k=c_{i,0}}^{c_{i,n_i}} \left[f(X_{k-1} | s_1, s_2, \ldots, s_{k-1}) f(x_k | s_k) \right]$$

$$\left[g(s_k | s_{k-1}, SN) p_o(s_1, s_2, \ldots, s_{k-1} | SN) \right] \qquad (A.29)$$

One can select the order of summation so that the sum over s_k and s_{k-1} follows the sum over $s_1, s_2, \ldots, s_{k-2}$. Also, $g(s_k | s_{k-1}, SN)$ can be factored out of the summation over the first k-2 sums and $f(x_k | s_k)$ factored out of the sum over the first k-1 sums. Equation A.29 can be written then as

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k | s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} \Bigg\{ g(s_k | s_{k-1}, SN)$$

$$\left[\sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_{k-2}=c_{i,0}}^{c_{i,n_i}} f(X_{k-1} | s_1, s_2, \ldots, s_{k-1}) p_o(s_1, s_2, \ldots, s_{k-1}, SN) \right] \Bigg\}$$

(A.30)

Notice that the multiple sum in brackets is what we have defined in Eq. A. 23 as $Q_{s_{k-1}}(k-1)$. Making this substitution, Eq. A. 30 becomes

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)\, Q_{s_{k-1}}(k-1). \qquad (A.31)$$

Now in a manner completely analagous to that which resulted in Eq. A. 24, one can write the likelihood ratio of the observation X_k as

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} Q_{s_k}(k) \qquad (A.32)$$

Therefore Eq. A. 31 can be written as

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} Q_{s_k}(k) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)\, Q_{s_{k-1}}(k-1)$$

$$(A.33)$$

By inspecting both sides of Eq. A. 33 it can be seen that the general equation for updating $Q_{s_k}(k)$ in terms of $Q_{s_k}(k-1)$ is given by

$$Q_{s_k}(k) = f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)\, Q_{s_{k-1}}(k-1) \qquad (A.34)$$

We now want the updating equation for each state that s_k can be in. Let us use the notation $Q_{s_k=c_{i,j}}(k) = Q_{i,j}(k)$. By definition of the Sporadic-Poisson generator process as given by Eqs. 4. 2 through 4. 5, many of the state transitions in the formal sum of Eq. A. 34 are zero. For $s_k = c_{i,0}$, the only nonzero values of $g(s_k|s_{k-1}, SN)$ are $g(s_k = c_{i,0}|s_{k-1}=c_{i,0}, SN) = 1-\nu_i$ and $g(s_k = c_{i,0}|c_{i,n_i}, SN) = 1-\nu_i$. Since $f(x_k|s_k = c_{i,0}) = 1$ for signals in added white Gaussian noise, Eq. A. 34 becomes for $s_k = c_{i,0}$

$$Q_{i,0}(k) = \left[Q_{i,0}(k-1) + Q_{i,n_i}(k-1)\right](1-\nu_i) \qquad (A.35)$$

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The signal can only reach state $s_k = c_{i,1}$ from $s_{k-1} = c_{i,0}$ or $s_{k-1} = c_{i,n_i}$, and the probabilities associated with these transitions are $g(s_k = c_{i,1} | s_{k-1} = c_{i,0}, SN) = \nu_i$ and $g(s_k = c_{i,1} | s_{k-1} = c_{i,n_i}, SN) = \nu_i$. So for $s_k = c_{i,1}$, Eq. A. 34 can be written as

$$Q_{i,1}(k) = \left[Q_{i,0}(k-1) + Q_{i,n_i}(k-1)\right] \nu_i f(x_k | s_k = c_{i,1}) \tag{A. 36}$$

For any of the other states, $c_{i,j}$, where j is not equal to 0 or 1, the state $s_k = c_{i,j}$ can only be reached from $s_{k-1} = c_{i,j-1}$. The probability of this transition is one. In this case Eq. A. 34 becomes

$$Q_{i,j}(k) = Q_{i,j-1}(k-1) f(x_k | s_k = c_{i,j}) \tag{A. 37}$$

Now, the likelihood ratio of the observation X_k was given by Eq. A. 32 as

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_k = c_{i,0}}^{c_{i,n_i}} Q_{s_k}(k) \tag{A. 32}$$

Using the notation $Q_{s_k}(k)\big|_{= c_{i,j}} = Q_{i,j}(k)$, Eq. A. 32 can be written as

$$f(X_k) = \sum_{i=1}^{b} \sum_{j=0}^{n_i} Q_{i,j}(k) \tag{A. 38}$$

Equations A. 35 through A. 38 are the basic equations of this realization. This receiver is much simpler than the previous ones. If one compares the updating equations for the $Q_{i,j}(k)$ matrix with that for the $b_{i,j}(k)$ matrix of Realization I in Section 5. 1. 1, one sees that they are quite similar except that the updating of $b_{i,j}(k)$ is more involved. In Realization I, each $b_{i,j}(k)$ term had to be multiplied by a sequential likelihood ratio, $f(x_k | s_k = c_{i,j})$, to obtain the likelihood ratio, $f(X_k)$, of the observation X_k. In Realization IV, however, the likelihood ratio at time t_k is simply given by Eq. A. 34. In order to see how a classification output is obtained from the $Q_{i,j}(k)$ terms let us first look at another interpretation of $Q_{i,j}(k)$. We know from Eq. 3. 8 that

$$f(X_k) = f(X_{k-1}) f(x_k | X_{k-1}) \tag{3. 8}$$

By definition

$$f(x_k|X_{k-1}) \triangleq \sum_{i=1}^{b} \sum_{s_1=c_{i,0}}^{c_{i,n_i}} \sum_{s_2=c_{i,0}}^{c_{i,n_i}} \cdots \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k)p_{k-1}(s_1, s_2, \ldots, s_k|SN) \qquad (A.39)$$

For generator processes which can be expressed as a function, $g(s_k|s_{k-1}, SN)$, one obtains, as before, for the average sequential likelihood ratio

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)p_{k-1}(s_{k-1}|SN) \qquad (A.40)$$

So the likelihood ratio of the observation X_k can be written using Eqs. 3.8 and 5.12 as

$$f(X_k) = f(X_{k-1}) \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)p_{k-1}(s_{k-1}|SN) \qquad (A.41)$$

Since $f(X_{k-1})$ is independent of the summations over i, s_k and s_{k-1} one can bring $f(X_{k-1})$ within the summation signs and write Eq. A.36 as

$$f(X_k) = \sum_{i=1}^{b} \sum_{s_k=c_{i,0}}^{c_{i,n_i}} f(x_k|s_k) \sum_{s_{k-1}=c_{i,0}}^{c_{i,n_i}} g(s_k|s_{k-1}, SN)f(X_{k-1})p_{k-1}(s_{k-1}|SN) \qquad (A.42)$$

Comparing Eqs. A.37 and A.35 one sees that

$$Q_{i,j}(k) = f(X_k)p_k(s_k = c_{i,j}|SN) \qquad (A.43)$$

Recalling Eq. 5.16

$$b_{i,j}(k) = p_k(s_k = c_{i,j}|SN) \qquad (5.16)$$

one can write Eq. A.43 as

$$Q_{i,j}(k) = f(X_k)b_{i,j}(k) \qquad (A.44)$$

$Q_{i,j}(k)$ may be interpreted as the likelihood ratio of the observation X_k multiplied by the probability of the jth component sample of the ith component, under the condition SN, and the taking of k observations.

If the classification output, $b_{i,j}(k)$ is wanted, it can be determined from

$$b_{i,j}(k) = \frac{Q_{i,j}(k)}{f(X_k)} \tag{A.45}$$

If the updated component identification is desired, it can be derived from

$$p_k(C^i|SN) = \sum_{j=0}^{n_i} b_{i,j}(k) = \frac{\sum_{j=0}^{n_i} Q_{i,j}(k)}{f(X_k)} \tag{A.46}$$

The operations that the optimum receiver of Realization IV performs are summarized in Table 5.3 in Chapter V and block diagrams are presented in Figs. 5.5 through 5.8.

APPENDIX B

OPTIMUM ADAPTIVE RECEIVER REALIZATIONS, SYNCHRONOUS-POISSON TIME STRUCTURE

B. 1 Realization III

In Section 5. 2. 1 Realization I was presented for the Synchronous-Poisson Time Structure. Realization III is a "b" channel receiver. In this realization the likelihood ratio of the observation X_k may be written as

$$\ell(X_k) = \sum_{i=1}^{b} p_o(C^i|SN) \sum_{s \in S_i} \ell(X_k|s)\, p_o(s|C^i, SN) \tag{B. 1}$$

This conditional likelihood ratio is to be realized in a sequential fashion for each of the b possible components, so that

$$\ell(X_k|C^i) = \ell(X_{k-1}|C^i)\ell(x_k|X_{k-1}, C^i) \tag{B. 2}$$

In the Synchronous-Poisson Time Structure, the component conditional sequential likelihood ratio has been previously determined and is

$$\ell(x_k|X_{k-1}, C^i) = 1-\nu_i + \nu_i \ell(x_k|S_k = C_{i,1}) \tag{5. 60}$$

The classification output is obtained from

$$p_k(C^i|SN) = \frac{p_o(C^i|SN)\ell(X_k|C^i)}{\ell(X_k)} \tag{B. 3}$$

The design equations for this realization are summarized in Table B. 1. A block diagram is shown in Fig. B. 1 for added white Gaussian noise. A feature of this realization is the separate

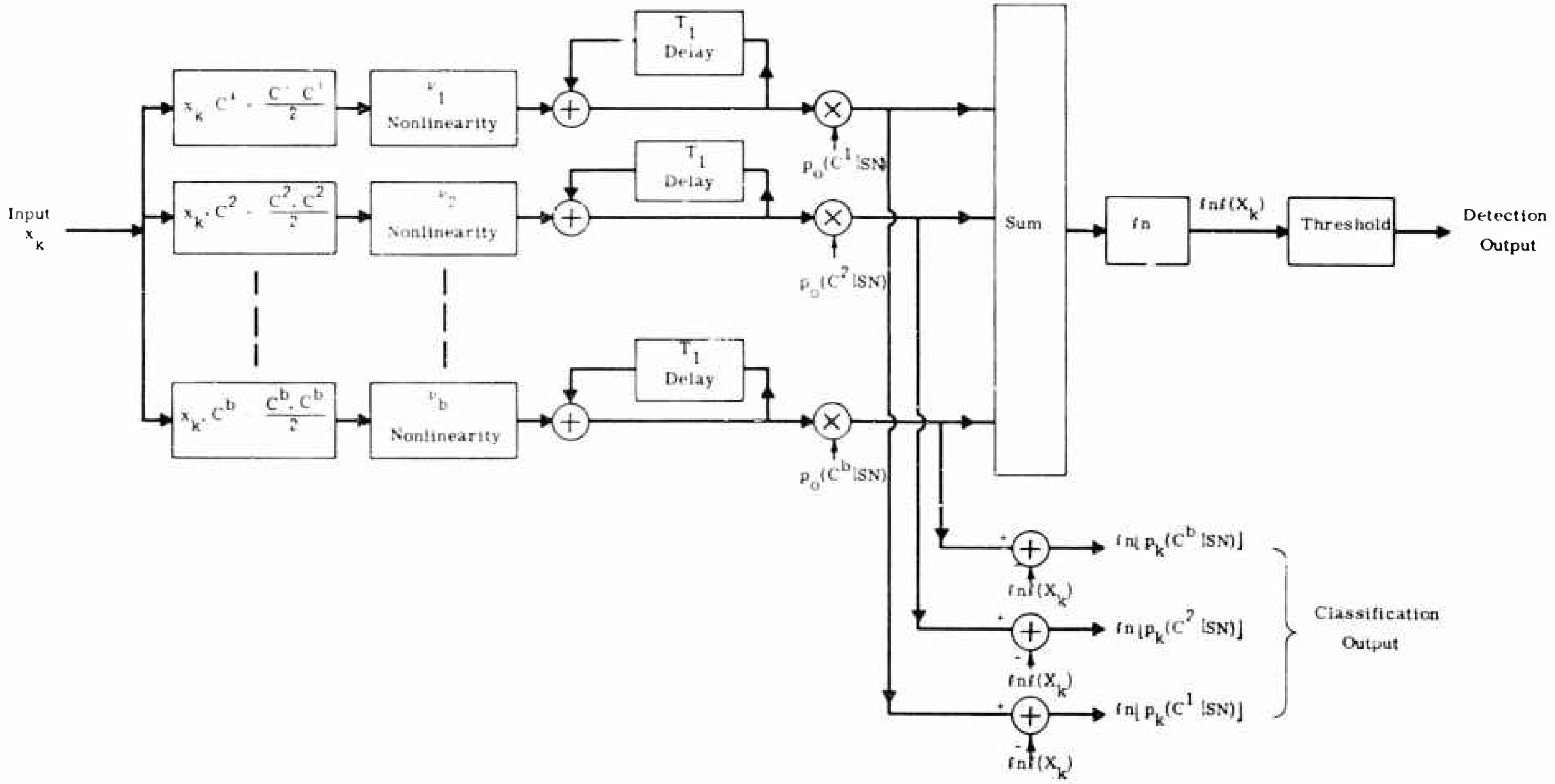

Fig. B. 1. Adaptive realization, Synchronous-Poisson Time Structure, Realization III channel allotted for each of the b components. The receiver input, x_k, is correlated with each possible component that could occur and the bias $C_{i,1} \cdot C_{i,1}/2$ is subtracted. These outputs are then passed through nonlinear functions which depend on the average duty factor of each component. The output of the nonlinear element is the logarithm of the component conditional sequential likelihood ratio, $f(x_k|X_{k-1}, C^i)$. These values are summed by means of the recirculating T_1 delay and exponentiated to form the component conditional likelihood ratio, $f(X_k|C^i)$. These component conditional likelihood ratios, weighted by the <u>a priori</u> probabilities of the possible components, are summed to form the detection output, $f(X_k)$. The classification output is obtained by taking the output of each channel and dividing it by the detection output, $f(X_k)$. This is done in this particular realization on a logarithmic basis.

TABLE B. 1

BASIC RECEIVER DESIGN EQUATIONS, SYNCHRONOUS-POISSON TIME STRUCTURE

REALIZATION III

Optimum Detection Output

$$f(X_k) = \sum_{i=1}^{b} f(X_k|C^i)p_o(C^i|SN) \qquad (A.\ 18)$$

Component Conditional Likelihood Ratio

$$f(X_k|C^i) = f(X_{k-1}|C^i)\, f(x_k|X_{k-1}, C^i) \qquad (B.\ 2)$$

Component Conditional Sequential Likelihood Ratio

$$f(x_k|X_{k-1}, C^i) = 1 - \nu_i + \nu_i f(x_k|S_k = C_{i,1}) \qquad (5.\ 60)$$

Classification - Component Identification

$$p_k(C^i|SN) = \frac{p_o(C^i|SN) f(X_k|C^i)}{f(X_k)} \qquad (B.\ 3)$$

B. 2 Realization IV

This realization is somewhat simpler than Realization III although the contrast is not so great as in the sporadic receiver. Let us begin the derivation of this realization by considering the likelihood ratio of the observation X_{k-1}. This is

$$f(X_{k-1}) = \sum_{i=1}^{b} \sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} f(X_{k-1}|S_1, S_2, \ldots, S_{k-1}) p_o(S_1, S_2, \ldots, S_{k-1}|SN) \tag{B. 4}$$

Summing with respect to i and S_{k-1} last one obtains

$$f(X_{k-1}) = \sum_{i=1}^{b} \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} \left[\sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_{k-2}=C_{i,0}}^{C_{i,1}} f(X_{k-1}|S_1, S_2, \ldots, S_{k-1}) \; p_o(S_1, S_2, \ldots, S_{k-1}|SN) \right] \tag{B. 5}$$

Denoting the quantity in brackets by $Q_{S_{k-1}}(k-1)$, Eq. B. 5 can be written as

$$f(X_{k-1}) = \sum_{i=1}^{b} \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} Q_{S_{k-1}}(k-1) \tag{B. 6}$$

Now, the likelihood ratio of the observation X_k is by definition

$$f(X_k) = \sum_{i=1}^{b} \sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_k=C_{i,0}}^{C_{i,1}} f(X_{k-1}|S_1, S_2, \ldots, S_{k-1}) f(x_k|S_k) p_o(S_1, S_2, \ldots, S_k|SN)$$

By definition of a ioint probability

$$p_o(S_1, S_2, \ldots, S_k|SN) = p_o(S_1, S_2, \ldots, S_k|SN) p_o(S_k|S_1, S_2, \ldots, S_{k-1}, SN) \tag{B. 8}$$

But in the Synchronous-Poisson Time Structure, the generator process is such that

$$p_o(S_k | S_1, S_2, \ldots, S_{k-1}, SN) = g(S_k | SN) \qquad (B.9)$$

Substituting Eq. B.9 into B.7 gives

$$f(X_k) = \sum_{i=1}^{b} \sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_k=C_{i,0}}^{C_{i,1}} f(X_{k-1} | S_1, S_2, \ldots, S_{k-1}) f(x_k | S_k) p_o(S_1, S_2, \ldots, S_{k-1}) g(s_k | SN) \qquad (B.10)$$

Interchanging the order of summation and factoring, Eq. B.10 can be written as

$$f(X_k) = \sum_{i=1}^{b} \sum_{S_k=C_{i,0}}^{C_{i,1}} f(x_k | S_k)\, g(S_k | SN) \left\{ \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} \left[\sum_{S_1=C_{i,0}}^{C_{i,1}} \sum_{S_2=C_{i,0}}^{C_{i,1}} \cdots \sum_{S_{k-2}=C_{i,0}}^{C_{i,1}} f(X_{k-1} | S_1, S_2, \ldots, S_{k-1})\, p_o(S_1, S_2, \ldots, S_{k-1} | SN) \right] \right\} \qquad (B.11)$$

But the multiple sum in brackets in Eq. B.11 is the same as the bracketed term in Eq. B.5 which has already been defined as $Q_{S_{k-1}}(k-1)$. Substituting $Q_{S_{k-1}}(k-1)$ into Eq. B.11 gives

$$f(X_k) = \sum_{i=1}^{b} \sum_{S_k=C_{i,0}}^{C_{i,1}} f(x_k | S_k) g(S_k | SN) \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} Q_{S_{k-1}}(k-1) \qquad (B.12)$$

Now Eq. B.6 can be written with the subscript k-1 advanced to k resulting in

$$f(X_k) = \sum_{i=1}^{b} \sum_{S_k=C_{i,0}}^{C_{i,1}} Q_{S_k}(k) \qquad (B.13)$$

Comparing Eqs. B.12 and B.13 one sees that

$$Q_{S_K}(k) = f(x_k | S_k) g(S_k | SN) \sum_{S_{k-1}=C_{i,0}}^{C_{i,1}} Q_{S_{k-1}}(k-1) \qquad (B.14)$$

Let us use the simpler notations $Q_{S_k = C_{i,0}}(k) = Q_{i,0}(k)$ and $Q_{S_k = C_{i,1}}(k) = Q_{i,1}(k)$. By definition of the Synchronous-Poisson generator process as given by Eqs. 4.7 and 4.8

$$g(S_k | SN) = 1 - \nu_i \quad \text{for} \quad S_k = C_{i,0} \tag{4.7}$$

and

$$g(S_k | SN) = \nu_i \quad \text{for} \quad S_k = C_{i,1} \tag{4.8}$$

Therefore Eq. B.14 can be written as

$$Q_{i,0}(k) = (1-\nu_i)\left[Q_{i,0}(k-1) + Q_{i,1}(k-1)\right] \tag{B.15}$$

and

$$Q_{i,1}(k) = \nu_i f(x_k | S_k = C_{i,1})\left[Q_{i,0}(k-1) + Q_{i,1}(k-1)\right] \tag{B.16}$$

Instead of updating $Q_{i,0}(k)$ and $Q_{i,1}(k)$ separately, one can update the sum, $Q_i(k) = Q_{i,0}(k) + Q_{i,1}(k)$

$$Q_{i,1}(k) = Q_i(k-1)\left[1-\nu_i + \nu_i f(x_k | S_k = C_{i,1})\right] \tag{B.17}$$

Using the notation introduced above for the $Q_{S_k}(k)$. Eq. B.13 can be written as

$$f(X_k) = \sum_{i=1}^{b} \left[Q_{i,0}(k) + Q_{i,1}(k)\right] \tag{B.18}$$

or

$$f(X_k) = \sum_{i=1}^{b} Q_i(k) \tag{B.19}$$

Equations B.17 and B.19 are the basic equations necessary to obtain the detection output. The interpretation of $Q_i(k)$ is similar to the interpretation given of $Q_{i,j}(k)$ in the sporadic case and is

$$p_k(C^i | SN) = \frac{Q_i(k)}{f(X_k)} \qquad (B.20)$$

Thus, a classification output is easily obtained. The design equations for this receiver realization are summarized in Table 5.5. Block diagrams of this realization are shown in Figs. 5.5 through 5.8.

APPENDIX C

OPTIMUM ADAPTIVE RECEIVER REALIZATIONS, PERIODIC TIME STRUCTURE

C. 1 Unknown Repetition Frequency, Realization II

A slight modification of Realization I, presented in Section 5. 3. 1 may be obtained by writing the joint probability, $b_{i,j}(k)$ in the form given in Eq. A. 1, $b_{i,j}(k) = b'_{i,j}(k)p_k(C^i|SN)$, and substituting into Eq. 5. 66 for the sequential average likelihood ratio. Factoring out $p_{k-1}(C^i|SN)$ gives

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN)\left[b'_{i,n_i}(k-1)f(x_k|s_k = c_{i,1}) + \sum_{j=2}^{n_i} b'_{i,j-1}(k-1)f(x_k|s_k = c_{i,j})\right] \tag{C. 1}$$

Defining a component conditional sequential likelihood ratio, $f(x_k|X_{k-1}, C^i)$ as

$$f(x_k|X_{k-1}, C^i) = b'_{i,n_i}(k-1)f(x_k|s_k = c_{i,1}) + \sum_{j=2}^{n_i} b'_{i,j-1}(k-1)f(x_k|s_k = c_{i,j}) \tag{C. 2}$$

one can put Eq. C. 1 into the form

$$f(x_k|X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i|SN)f(x_k|X_{k-1}, C^i) \tag{C. 3}$$

The updating equations for component positional information are obtained by making the substitution of the form $b_{i,j}(k) = b'_{i,j}(k)\, p_k(C^i|SN)$ in both sides of Eqs. 5. 68 and 5. 69. The result is

$$b'_{i,1}(k)p_k(C^i|SN) = \frac{b'_{i,n_i}(k-1)\; p_{k-1}(C^i|SN)f(x_k|s_k = c_{i,1})}{f(x_k|X_{k-1})} \tag{C. 4}$$

$$b'_{i,j}(k)\, p_k(C^i \mid SN) = \frac{b'_{i,j-1}(k-1)\, p_{k-1}(C^i \mid SN)\, f(x_k \mid s_k = c_{i,j})}{f(x_k \mid X_{k-1})} \tag{C. 5}$$

Instead of updating the products of the form $b'_{i,j}(k-1)\, p_{k-1}(C^i \mid SN)$, one can update $b'_{i,j-1}(k-1)$ and $p_{k-1}(C^i \mid SN)$ separately by observing that

$$b'_{i,1}(k) = \frac{b'_{i,n_i}(k-1)\, f(x_k \mid s_k = c_{i,1})}{f(x_k \mid X_{k-1})} \tag{C. 6}$$

$$b'_{i,j}(k) = \frac{b'_{i,j-1}(k-1)\, f(x_k \mid s_k = c_{i,j})}{f(x_k \mid X_{k-1})} \tag{C. 7}$$

for $j = 2, 3, \ldots, n_i$

and

$$p_k(C^i \mid SN) = \frac{p_{k-1}(C^i \mid SN)\, f(x_k \mid X_{k-1}, C^i)}{f(x_k \mid X_{k-1})} \tag{C. 8}$$

The design equations for this realization are summarized in Table C. 1

TABLE C. 1

BASIC RECEIVER DESIGN EQUATIONS, PERIODIC TIME STRUCTURE
UNKNOWN REPETITION FREQUENCY
REALIZATION II

Optimum Detection Output

$$f(X_k) = f(X_{k-1})\, f(x_k \mid X_{k-1}) \tag{3. 8}$$

Sequential Average Likelihood Ratio

$$f(x_k \mid X_{k-1}) = \sum_{i=1}^{b} p_{k-1}(C^i \mid SN)\, f(x_k \mid X_{k-1}, C^i) \tag{C. 3}$$

Component Conditional Sequential Likelihood Ratio

$$f(x_k|X_{k-1}, C^i) = b'_{i,n_i}(k-1)f(x_k|s_k = c_{i,1}) + \sum_{j=2}^{n_i} b'_{i,j-1}(k-1)f(x_k|s_k = c_{i,j}) \quad (C.2)$$

Classification - Component Position

$$b'_{i,1}(k) = \frac{b'_{i,n_i}(k-1)f(x_k|s_k = c_{i,1})}{f(x_k|X_{k-1})} \quad (C.6)$$

$$b'_{i,j}(k) = \frac{b'_{i,j-1}(k)f(x_k|s_k = c_{i,j})}{f(x_k|X_{k-1}, C^i)} \quad \text{for } j = 2, 3, \ldots, n_i \quad (C.7)$$

Classification - Component Identification

$$p_k(C^i|SN) = \frac{p_{k-1}(C^i|SN)f(x_k|X_{k-1}, C^i)}{f(x_k|X_{k-1})} \quad (C.8)$$

C.2 Unknown Repetition Frequency, Realization III

This is a "b" channel realization in which the likelihood ratio of the entire observation, X_k, is obtained sequentially under the condition the ith component is present. The b channels are then weighted by the a priori probability, before taking any observations, of each of the components. The detection output is

$$f(X_k) = \sum_{i=1}^{b} f(X_k|C^i)p_o(C^i|SN) \quad (A.18)$$

Each channel forms the likelihood ratio of the observation, X_k, conditional to the ith component and this is formed sequentially as

$$f(X_k|C^i) = f(X_{k-1}|C^i)f(x_k|X_{k-1}, C^i) \quad (A.19)$$

which becomes in the periodic case

$$f(x_k|X_{k-1}, C^i) = b'_{i,n_i}(k-1)f(x_k|s_k = c_{i,1}) + \sum_{j=2}^{n_i} b'_{i,j-1}(k-1)f(x_k|s_k = c_{i,j}) \quad (C.9)$$

and the updating equations on component identification and position are given by Eqs. C. 6 through C. 8. The receiver design equations for this realization are summarized in Table C. 2.

TABLE C. 2

BASIC RECEIVER DESIGN EQUATIONS, PERIODIC TIME STRUCTURE
UNKNOWN REPETITION FREQUENCY
REALIZATION III

Optimum Detection Output

$$f(X_k) = \sum_{i=1}^{b} f(X_k | C^i) p_o(C^i | SN) \tag{A. 17}$$

Component Conditional Likelihood Ratio

$$f(X_k | C^i) = f(X_{k-1} | C^i) f(x_k | X_{k-1}, C^i) \tag{A. 19}$$

Component Conditional Sequential Likelihood Ratio

$$f(x_k | X_{k-1}, C^i) = b'_{i, n_i}(k-1) f(x_k | s_k = c_{i, 1}) + \sum_{j=2}^{n_i} b'_{i, j-1}(k-1) f(x_k | s_k = c_{i, j}) \tag{C. 9}$$

Classification - Component Position

$$b'_{i, 1}(k) = \frac{b'_{i, n_i}(k-1) f(x_k | s_k = c_{i, 1})}{f(x_k | X_{k-1}, C^i)} \tag{C. 6}$$

$$b'_{i, j}(k) = \frac{b'_{i, j-1}(k-1) f(x_k | s_k = c_{i, j})}{f(x_k | X_{k-1}, C^i)} \tag{C. 7}$$

for $j = 2, 3, \ldots, n_i$

Classification - Component Identification

$$p_k(C^i | SN) = \frac{p_o(C^i | SN) f(X_k | C^i)}{f(X_k)} \tag{C. 8}$$

C. 3 Unknown Repetition Frequency, Realization IV

This realization is the least adaptive looking of the realizations but it is the simplest. It is a "b" channel receiver and its development follows Appendix A. 3 for the sporadic receiver up to Eq. A. 34 with the exception that the summations are over the states $c_{i,1}, c_{i,2}, \ldots, c_{i,n_i}$ rather than $c_{i,0}, c_{i,1}, c_{i,2}, \ldots, c_{i,n_i}$. Analagous to Eq. A. 31 one can write

$$Q_{s_k}(k) = f(x_k | s_k) \sum_{s_{k-1}=c_{i,1}}^{c_{i,n_i}} g(s_k | s_{k-1}, SN)\, Q_{s_{k-1}}(k-1) \qquad (C.\ 10)$$

Using the properties of the periodic generator process given by Eqs. 4. 9 and 4. 10 in Eq. C. 10 one can write

$$Q_{i,1}(k) = Q_{i,n_i}(k-1) f(x_k | s_k = c_{i,1}) \qquad (C.\ 11)$$

$$Q_{i,j}(k) = Q_{i,j-1}(k-1) f(x_k | s_k = c_{i,j}) \qquad \text{for } j = 2, 3, \ldots, n_i \qquad (C.\ 12)$$

where the likelihood ratio of the observation X_k is given by

$$f(X_k) = \sum_{i=1}^{b} \sum_{j=1}^{n_i} Q_{i,j}(k) \qquad (C.\ 13)$$

The classification output is obtained as before from Eqs. A. 44 and A. 45. The basic receiver design equations are summarized in Table 5. 7.

APPENDIX D

COMPUTER SIMULATION TECHNIQUE

The receiver realizations discussed in Chapter VII, from which the ROC data was obtained, were simulated on the IBM 7090 digital computer. The general computational method used throughout was to replace all continuous random variables with a discrete random variable. A 50-point discrete probability distribution was used to match the continuous probability distribution, each point being assigned 2 percent probability. That is, the two probability distribution functions were matched at values of .01, .03, . . . , .99. This method gives a rather good representation of the random variables within the middle 96 percent of the range, and a crude representation of the smallest 2 percent and the largest 2 percent of the range.

The optimum adaptive receiver realization discussed in Section 7.2 for one of eight orthogonal components, Synchronous-Poisson Time Structure, was simulated on the digital computer. The digital data was then converted to analog form on a digital-to-analog converter and plotted out on a Sanborn pen recorder.

APPENDIX E

DERIVATION OF PERFORMANCE OF ENERGY DETECTOR, PERIODIC AND SYNCHRONOUS-POISSON TIME STRUCTURE

In this section the detection performance of the energy detector is derived for the CKS (one of M orthogonal components), Periodic and Synchronous Time Structures. The detectability, d, can be expressed as

$$d = \eta \frac{2E}{N_o} \qquad \text{(E. 1)}$$

where η is the efficiency (See Reference 20). Lamphiear and Birdsall (Ref. 21) have shown that the efficiency for the energy detector is approximately

$$\eta = \frac{\left(2 + 2\frac{c^2}{n}\right)\left(\sqrt{1 + \frac{c^2}{n}} - 1\right)^2}{\left(2 + 3\frac{c^2}{n}\right)\left(\frac{c^2}{n}\right)} \qquad \text{(E. 2)}$$

where

$$n = 2WT$$

$$c^2 = \frac{2E}{N_o}$$

For $\frac{c^2}{n} \ll 1$,

$$\frac{\left(2 + 2\frac{c^2}{n}\right)}{\left(2 + 3\frac{c^2}{n}\right)} \approx 1 \qquad \text{(E. 3)}$$

and

$$\sqrt{1 + \frac{c^2}{n}} \approx 1 + \frac{1}{2}\,\frac{c^2}{n} \qquad \text{(E. 4)}$$

Using these approximations, Eq. E-2 becomes

$$\eta \approx \frac{1}{4}\,\frac{c^2}{n} = \frac{1}{4}\,\frac{\left(\frac{2E}{N_o}\right)}{2WT} \tag{E. 5}$$

For the Periodic Time Structure, $E = kE_c$ and $T = kT_1$, where E_c is the component energy and T_1 is the duration of a component. Therefore, Eq. E-5 can be written as

$$\eta = \frac{1}{4}\,\frac{\left(\frac{2E_c}{N_o}\right)}{2WT_1} \tag{E. 6}$$

For M orthogonal components $2WT_1$ is at least M. Using $M = 2WT_1$,

$$\eta = \frac{1}{4M}\left(\frac{2E_c}{N_o}\right) \tag{E. 7}$$

and the detectability as defined by Eq. E-1 becomes

$$d = \frac{1}{4M}\,k\left(\frac{2E_c}{N_o}\right)^2 \tag{E. 8}$$

which is Eq. 7. [illegible] in the text.

For the Synchronous-Poisson Time Structure, and the occurrence of the average number of components, $E = \nu kE_c$ and the efficiency becomes

$$\eta = \frac{1}{4M}\,\nu\left(\frac{2E_c}{N_o}\right) \tag{E. 9}$$

and d becomes

$$d = \eta\,\nu k\,\frac{2E_c}{N_o} = \frac{1}{4M}\,\nu^2 k\left(\frac{2E_c}{N_o}\right)^2 \tag{E. 10}$$

which is Eq. 7. 15 in the text.

APPENDIX F

DERIVATION OF PERFORMANCE, SUBOPTIMUM RECEIVER CKE, SYNCHRONOUS-POISSON TIME STRUCTURE

In this appendix the detection performance of a suboptimum receiver is derived which crosscorrelates the component, C, waveform with each unit observation, x_i, and integrates. In other words, the receiver forms

$$z = \sum_{i=1}^{k} x_i \quad C \tag{F. 1}$$

where x_i is an n_i-dimensional observation and C is an n_i-dimensional component. This derivation assumes that the detection performance is the same as that had the average number of components occurred.

When the receiver outputs under the hypotheses N and SN are normally distributed, the detectability, d, is given by (Ref. 20)

$$d = \frac{\left(\mu_{SN,k} - \mu_{N,k}\right)^2}{\sigma^2_{N,k}} \tag{F. 2}$$

where

$\mu_{SN,k}$	mean of the receiver output conditional to SN and k observations
$\mu_{N,k}$	mean of the receiver output conditional to N and k observations
$\sigma^2_{N,k}$	variance of the receiver output conditional to N and k observations

Let us now obtain expressions for $\mu_{SN,k}$, $\mu_{N,k}$ and $\sigma^2_{N,k}$. Since the sum of any number of independent normally distributed variables is itself normally distributed with mean the sum of the means and variance the sum of the variances, then

$$\mu_{SN,k} = (1 - \nu)k\mu_N + \nu k\mu_{CN} \tag{F. 3}$$

where

νk	average number of component occurrences
$(1 - \nu)k$	average number of no-component occurrences
μ_{CN}	mean of the observation, x_i, under component plus noise
μ_N	mean of the observation, x_i, under noise alone

The variance under noise alone after k observations is the sum of the variance under noise alone of each observation. Thus

$$\sigma^2_{N,k} = k\sigma^2_N \tag{F. 4}$$

Substituting Eqs. F-3 and F-4 into the definition for d, Eq. F-2, results in

$$d = \frac{\nu^2 k\left(\mu_{CN} - \mu_N\right)^2}{\sigma^2_N} \tag{F. 5}$$

When component plus noise is present the mean of the receiver output is $\mu_{CN} = 2WE_c$. When noise alone is present the mean of the correlator output is $\mu_N = 0$ and the variance is $\sigma^2_N = 2WE_cN$. Therefore, Eq. F-5 becomes

$$d = \nu^2 k \frac{2E_c}{N_o} \tag{F. 6}$$

which appears as Eq. 7.17.

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