A NOTE ON THE SOLUTION OF POLYNOMIAL CONGRUENCES

Richard Bellman

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. The present Memorandum makes a contribution to the theory of polynomial congruences.
SUMMARY

As is well known, the number of solutions of the congruence

\[(1) \quad f(x) = 0(p),\]

where \( f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \), can be expressed in the form

\[ N = \frac{1}{p} \sum_{t,x} e^{2\pi i t f(x)/p}, \]

where \( t \) and \( x \) run independently through the values \( 0, 1, 2, \ldots, p - 1 \). This result is an immediate consequence of the relation

\[ \sum_{t} e^{2\pi i ty/p} = 0, \quad y \neq 0(p), \]

\[ = p, \quad y = 0(p). \]

In this note we present an alternative expression for the number of solutions of (1).
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1. INTRODUCTION

It is well known that the number of solutions of the congruence

\[ f(x) = 0(p), \]

where \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \), can be expressed in the form

\[ N = \frac{1}{p} \sum_{t,x} e^{2\pi itf(x)/p}, \]

where \( t \) and \( x \) run independently through the values \( 0,1,2,\ldots,p-1 \). This result is an immediate consequence of the relation

\[ \sum_{t} e^{2\pi ity/p} = 0, \quad y \neq 0(p), \]

\[ = p, \quad y = 0(p). \]

In this note we present an alternative expression for the number of solutions of (1.1).

2. AN EQUIVALENT VECTOR–MATRIX CONGRUENCE

The equation \( f(x) = 0 \) is readily seen to be the characteristic equation of the matrix
\[\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1
\end{pmatrix};\]

(2.1)

Using arguments completely analogous to that for the complex field, we see that a necessary and sufficient condition for a nontrivial solution of the vector-matrix congruence

\[(2.2) \quad Ax \equiv \lambda x(p),\]

where \(x\) is now the \(n\)-dimensional column vector with components \(x_1, x_2, \ldots, x_n\), and \(\lambda\) is a scalar, is that

\[(2.3) \quad f(\lambda) = 0(p).\]

Each root of (2.3) generates a ray of solutions \(kx\), where \(k = 1, 2, \ldots, p - 1\).

3. MULTIDIMENSIONAL EXPONENTIAL SUM

Let \(t\) be an \(n\)-dimensional vector with components \(t_1, t_2, \ldots, t_n\), and let \((t, x)\) denote, as usual, the vector inner product. We can then write, as the number of nontrivial solutions of (2.2),
\[ (3.1) \quad \sum_{t} \sum_{\lambda} e^{\frac{2\pi i}{p}(t,Ax-\lambda x)} \]

where \((u,v)\) denotes the usual inner product and the prime denotes the fact that \(x = 0\) is omitted in the summation.

Since each solution of \(f(\lambda) = 0\) generates \(p - 1\) solutions of (2.2), we have

\[ (3.2) \quad N = \frac{1}{p^n(p-1)} \sum_{t} \sum_{\lambda} e^{\frac{2\pi i}{p}(t,Ax-\lambda x)} \]

We can eliminate the prime by writing

\[ (3.3) \quad N = \frac{1}{p^n(p-1)} \sum_{\lambda} \sum_{x} e^{\frac{2\pi i}{p}(t,Ax-\lambda x)} - \frac{p}{p-1}. \]

Summing over the scalar \(\lambda\) first, we have finally

\[ (3.4) \quad N = \frac{1}{p^{n-1}(p-1)} \sum_{(t,x)=0(p)} e^{2\pi i(t,Ax)/p} - \frac{p}{p-1}. \]

If \(A\) is symmetric, we write \(t = u + v, \quad x = u - v\), and obtain

\[ (3.5) \quad N = \frac{1}{p^{n-1}(p-1)} \sum_{(u,u) = (v,v)(p)} e^{\frac{2\pi i}{p}[(u,Au)-(v,Av)]} \]

\[ - \frac{1}{p-1}. \]
This, in turn, may be written

\[
N = \frac{1}{p^{(n-1)(p-1)}} \sum_k \sum_{u,u^* k(p)}^{2\pi i (u,Au)} \sum_{(u,u)} (u,u)^* \left| e^{\frac{2\pi i}{p} (u,Au)} \right|^2 - \frac{p}{p-1},
\]

an interesting formula.

4. EXAMPLE

Consider the congruence

\[
\lambda^3 + a = 0(p).
\]

The corresponding matrix is

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a & 0 & 0
\end{bmatrix}.
\]

Hence, the number of solutions of (4.1) is given by

\[
N = \frac{1}{p^{2(p-1)}} \sum_{S} \frac{2\pi i}{p} (t_1 x_2 + t_2 x_3 - at_3 x_1) - \frac{p}{p-1},
\]

where the set of values \( S \) is determined by

\[
t_1 x_1 + t_2 x_2 + t_3 x_3 = 0.
\]
REFERENCE