GTC
TR-1-21

Technical Report 1-21

Stress Analysis of Conical Shells With Linearly Varying Wall Thickness

September, 1964

COUNTED IN

TECHNICAL LIBRARY
BLDG 313
ABERDEEN PROVING GROUND, MD.
STEAP TL

GTC
TR-1-21

GENERAL TECHNOLOGY CORPORATION

THE GENERAL TECHNOLOGY CORPORATION is engaged in the application of science and scientific methods to industrial problems.

Because of its unique structure, the Corporation is in a position to supply the services of outstanding scientists having university positions. These men work in teams as consultants to solve problems undertaken by the Corporation. The following are typical examples of the fields of study in which the staff has been actively engaged:

- Elasticity
- Plasticity
- Viscoelasticity
- Seismic Waves
- Thermoelasticity
- Porous Media
- Linear Programming
- Plates and Shells
- Magnetohydrodynamics
- Aerodynamics
- Electromagnetics
- Systems Analysis
- Instrumentation
- Ordnance
- Noise and Random Vibrations

Inquiries for the arrangements of exploratory discussions of problems may be directed to either of the following offices:

Research and Development: 402 Northwestern Avenue
West Lafayette, Indiana
Phone: RI 3-3307

Business Office: 474 Summit Street
Elgin, Illinois
Phone: 695-1600

GTC TR-1-21

GENERAL TECHNOLOGY CORPORATION

CONICAL SHELLS WITH LINEARLY VARYING WALL THICKNESS

40p.

BY

HUO-HSI PAN

September 1964

Approved

R. C. Geldmacher

R. C. Geldmacher
Supervising Consultant

Submitted to Picatinny Arsenal under Contract No. DA-11-022
ORD-2917 AMCMS Code 5210.12 13200

AD 610032

TABLE OF CONTENTS

20060223343

GTC TR-1-21

I. DIFFERENTIAL EQUATIONS, BOUNDARY CONDITIONS AND JUNCTION CONDITIONS

The shell under consideration is in the form of a truncated right cone. Its thickness varies linearly along the length of the generator of the cone, thinner at one end and thicker at the other end (Fig. 1). The load applied to the shell is a distributed load $Z = p(s)$ normal to the middle surface of the shell and acting over the whole surface. On the boundary, that is, along the edges at both ends, axially symmetric forces and moments may be prescribed, but the forces cannot be entirely arbitrary as the equilibrium along the direction of the axis of the cone should be observed.

The shell is considered to be thin, that is, its thickness is small in comparison with other dimensions and with its radii of curvature ($r_x = \infty$, r_y).

The Stresses Let a local coordinate system be set up in the shell with the origin placed at the unstrained middle surface. The x-axis is placed on the generator of the middle surface and is pointing away from the apex, the y-axis is set tangent to the principle curvature, and the z-axis is set normal to the middle surface and is pointing inward.

Consider the stress components at a point in the shell. From the assumed symmetry, it is clear that $\tau_{xy} = \tau_{yx} = \tau_{yz} = \tau_{zy} = 0$. As the shell is considered to be thin, σ_z may be neglected.

Hence the remaining non-zero components needed to be considered are the normal stresses σ_x, σ_y and the shear stress $\tau_{xz} = \tau_{zx}$. For simplification, the normal stresses σ_x and σ_y are considered to be the sums of two parts, namely, the membrane stress.

(1a) $$\sigma_x = \sigma_{xm} + \sigma_{xb}$$

(1bO $$\sigma_y = \sigma_{ym} + \sigma_{yb}$$

The resultant forces and moments per unit length of the normal sections (Fig. 2) are obtained by integrations of these stresses over the thickness h.

(2a) $$N_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \, dz = \sigma_{xm} h$$

(2b) $$N_\theta = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y \, dz = \sigma_{ym} h$$

(2c) $$Q_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xz} \, dz$$

(2d) $$M_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x z \, dz = \frac{1}{6} h^2 (\sigma_{xb})_{max.}$$

(2e) $$M_\theta = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y \, zdz = \frac{1}{6} h^2 \, (\sigma_{yb})_{max.}$$

Equations of Equilibrium An infinitely small element $d\theta ds$ is defined by two adjacent meridian planes $d\theta$ apart and the distance ds along the generator of the cone. Consider the equilibrium of this element.

In the x - direction the equilibrium of forces requires

$$\frac{d}{ds} (N_\varphi s \cos\varphi \, d\theta) ds - N_\theta ds \, d\theta \cos\varphi = 0$$

or

(3) $$\frac{d}{ds} (sN_\varphi) - N_\theta = 0$$

In the z - direction the equilibrium of forces requires

$$\frac{d}{ds} (Q_\varphi s \cos\varphi \, d\theta) ds + N_\theta ds \, d\theta \sin\varphi + Zs \cos\varphi \, d\theta \, ds = 0$$

or

$$\frac{d}{ds}(sQ_\varphi) + N_\theta \tan\varphi + sZ = 0 \tag{4}$$

The condition that the summation of all moments about the y - axis be zero requires

$$\frac{d}{ds}(M_\varphi s \cos\varphi \, d\theta)ds - M_\theta ds(d\theta \cos\varphi) - Q_\varphi s \cos\varphi \, d\varphi \, ds = 0$$

or

$$\frac{d}{ds}(sM_\varphi) - M_\theta - sQ_\varphi = 0 \tag{5}$$

In deriving this equation, it has been assumed that the effect of the membrane force on the bending moment is negligible.

Combination of (3) and (5) gives

$$\frac{d}{ds}(sN_\varphi \sin\varphi + sQ_\varphi \cos\varphi) = - sZ \cos\varphi$$

which after integration becomes

$$\begin{aligned} sN_\varphi \sin\varphi + sQ_\varphi \cos\varphi = -\int_{s_1}^{s} Z \cos\varphi \, s ds + [sN_\varphi \sin\varphi \\ +sQ_\varphi \cos\varphi]_{s=s_1} = - F(s) \end{aligned} \tag{6}$$

This equation can be derived directly from the condition of equilibrium of the portion of the shell above the cross section s = s.

<u>Deformations and Stress-Strain Relationships</u> Let u_m and w_m be the displacement from its unstrained position of a point on the middle surface in the x and z directions respectively. The strains at the middle surface are found to be

$$(7a) \qquad \epsilon_{xm} = \frac{du_m}{ds} = u_m'$$

$$(7b) \qquad \epsilon_{ym} = \frac{u_m}{s} - \frac{w_m \tan\varphi}{s}$$

The second term in the expression for ϵ_{ym} is due to the deflection of the middle surface generator. The prime in (7a) denotes differentiation with respect to s.

The strains at a point at a distance z from the middle surface may be approximated as follows:

$$(8a) \qquad \epsilon_x = \epsilon_{xm} - z\frac{d^2 w_m}{ds^2} = u_m' - zw_m''$$

$$(8b) \qquad \epsilon_y = \epsilon_{ym} - \frac{z}{s}\frac{dw_m}{ds} = \frac{u_m - w_m \tan\varphi}{s} - \frac{z}{s} w_m'$$

The second terms in both (8a) and (8b) are due to the rotation of the cross section caused by the deflection of the middle surface generator.

From Hooke's law,

(9a) $$\sigma_x = \sigma_{xm} + \sigma_{xb} = \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y)$$

$$= \frac{E}{1 - \nu^2}[\epsilon_{xm} + \epsilon_{xb} + \nu(\epsilon_{ym} + \epsilon_{yb})]$$

(9b) $$\sigma_y = \sigma_{ym} + \sigma_{yb} = \frac{E}{1 - \nu^2}(\epsilon_y + \nu\epsilon_x)$$

$$= \frac{E}{1 - \nu^2}[\epsilon_{ym} + \epsilon_{yb} + \nu(\epsilon_{xm} + \epsilon_{xb})]$$

Four equations relating the shell forces and shell moments to the shell deformation are:

(10a) $$N_\varphi = \frac{Eh}{1 - \nu^2}(u_m' + \nu \frac{u_m - w_m \tan\varphi}{s})$$

(10b) $$N_\theta = \frac{Eh}{1 - \nu^2} (\frac{u_m - w_m \tan\varphi}{s} + \nu u_m')$$

(10c) $$M_\varphi = - \frac{Eh^3}{12(1 - \nu^2)} (w_m'' + \nu \frac{w_m'}{s})$$

(10d) $$M_\theta = - \frac{Eh^3}{12(1 - \nu^2)} (\frac{w_m'}{s} + \nu\, w_m')$$

Differential Equations Differentiating (5) and substituting $\frac{d}{ds}(sQ_\varphi)$ from (4), gives

(11) $$\frac{d^2}{ds^2}(sM_\varphi) - \frac{dM_\theta}{ds} + N_\theta \tan\varphi + sZ = 0$$

Using (10b), (10c), and (10d), (11) becomes

(12) $$- \frac{Eh^3}{12(1 - \nu^2)} \left[(2 + \nu)w_m''' + sw_m''''\right] + \frac{Eh^3}{12(1 - \nu^2)} \left[\frac{w_m''}{s} - \frac{w_m'}{s^2} + \nu w_m'''\right] - \frac{Eh^2h'}{4(1 - \nu^2)} \left[2(1 + \nu)w_m'' + 2sw_m''' - \frac{w_m'}{s} + \nu w_m'' \right] - \frac{E}{4(1 - \nu^2)} [2h(h')^2 + h^2h''][sw_m'' + \nu w_m']$$

$$+ \frac{Eh}{1 - \nu^2} \tan\varphi \left(\frac{u_m - w_m \tan\varphi}{s} + \nu\, u_m' \right) + sZ = 0$$

Multiplying (12) by s^3 gives

$$- \frac{Eh^3}{12(1 - \nu^2)} \left[s^4 w_m'''' + 2s^3 w''' + 2sw'' + sw' \right]$$

$$+ \frac{Eh^2 h' s}{4(1 - \nu^2)} \left[(2 + 3\nu) s^2 w_m'' + 2s^3 w_m''' - sw_m' \right]$$

(12a)

$$+ \frac{Es^2}{4(1 - \nu^2)} (2hh'^2 + h^2 h'') \left[s^2 w_m'' + \nu w_m' \right]$$

$$- \frac{Ehs^2}{1 - \nu^2} \tan\varphi \left(u_m - w_m \tan\varphi + \nu s u_m' \right) + s^4 Z = 0$$

From (3) after the substitution of N_φ and N_θ from (10a) and (10b) respectively one gets

$$\frac{Eh}{1 - \nu^2} \left(s\, u_m'' + u_m' + \nu\, u_m' - \nu w_m' \tan\varphi \right)$$

$$- \frac{Eh}{1 - \nu^2}\left(\frac{u_m - w_m \tan\varphi}{s} + \nu\, u_m{}'\right)$$

(13)

$$+ \frac{Eh'}{1 - \nu^2}[s\, u_m{}' + \nu(u_m - w_m \tan\varphi)] = 0$$

After multiplication by s, (13) becomes

$$s^2 u_m{}'' \quad s u_m{}' - \nu s w_m{}' \tan\varphi - u_m + \tan\varphi\, w_m$$

(13a)

$$+ \frac{h' s}{h} [s u_m{}' + \nu u_m - \nu w_m \tan\varphi] = 0$$

Equations (12a) and (13a) form a pair of coupled differential equations in displacements for the given conical shell.

If the thickness of the shell is constant then h is not a function of s and (12a) and (13a) reduce to

$$\frac{Eh^3}{12(1 - \nu^2)} (s^4 w_m{}'''' + 2s^3 w''' - 2sw'' + s w_m{}')$$

(12b)

$$- \frac{Ehs^2}{1 - \nu^2} \tan\varphi\, (u_m - w_m \tan\varphi + \nu s u_m{}') + s^4 Z = 0$$

and

(13b) $$s^2 u_m'' + s u_m' - u_m - \nu \tan\varphi \, s w_m' + \tan\varphi \, w_m = 0$$

respectively.

Two special cases, for $\varphi = 0$ (corresponds to a flat plate) and for $\varphi = \frac{\pi}{2}$ (corresponds to a straight cylinder) may be considered.

For $\varphi = 0$, $\tan\varphi = 0$, (12b) reduces to

(14) $$\frac{Eh^3}{12(1 - \nu^2)} \left(s^4 w_m^{iv} + 2s^3 w_m''' - 2s w_m'' + s w_m' \right) = s^4 Z$$

Using the notation D to denote $\frac{Eh^3}{12(1 - \nu^2)}$, (14) becomes

(14a) $$w_m'''' + 2\frac{1}{s} w_m''' - 2\frac{1}{s^2} w_m'' + \frac{1}{s^3} w_m' = \frac{Z}{D}$$

which agrees with the circular plate equation.

For $\varphi = 0$, (13b) reduces to

(15) $$s^2 u_m'' + s u_m' - u_m = 0$$

which is the equation for a circular plate subject to axially symmetric radial in-plane force.

It is noted that the decoupling of (14a) and (15) is in line with the assumption that the membrane forces have negligible effect on the bending moment. However this will not be true if the membrane forces are large compared to the normal distributed force Z.

Let r be the perpendicular distance from the point at s = s on the generator of the cone to the axis of the cone. Write $\tan\varphi = \frac{[s^2 - r^2]^{\frac{1}{2}}}{r}$. Let $s \to \infty$, then $\tan\varphi \to \frac{s}{r}$, $\varphi \to \frac{\pi}{2}$. Under such circumstance, the cone becomes a cylinder.

Substitute $\frac{s}{r}$ for $\tan\varphi$ in (12b),

$$\frac{Eh^3}{12(1 - \nu^2)}(s^4 w_m'''' + 2s^3 w_m''' - 2s w_m'' + s w_m')$$

(16)

$$- \frac{Eh\ s^3}{1 - \nu^2}(u_m - w_m \frac{s}{r} + \nu u_m') - s^4 Z = 0$$

Substitute $\frac{s}{r}$ for $\tan\varphi$ in (13b),

$$s^2 u_m'' + s u_m' - u_m - \nu \frac{s^2}{r} w_m' + \frac{s}{r} w_m = 0 \tag{17}$$

Let $s \to \infty$, (18) reduces to

$$u_m'' = \frac{\nu}{r} w_m'$$

or

(18) $$u_m' = \frac{\nu}{r} w_m + c$$

where c is the integration constant.

Setting c = 0, by (3), which is equivalent to setting $N_\varphi = 0$.[1]

(18a) $$u_m' = \frac{\nu}{r} w_m$$

Upon substitution of u_m' from (18a) into (16) and letting $s \to \infty$, (16) becomes

(19) $$w_m'''' - \frac{12(1-\nu^2)}{r^2h^2} w_m = \frac{Z}{D}$$

which agrees with the equation for a circular cylindrical shell loaded symmetrically with respect to its axis.

The pair of differential equations (12a) and (13a), as noted before, are in terms of displacements. If one wishes, one could proceed in a different manner.

Let S denote $\frac{sN_\varphi}{h^2}$. Hence

(20a) $$N_\varphi = \frac{h^2 S}{s}$$

1. See Timoshenko and Woinowsky-Krieger, "Theory of Plates and Shells" P. 467, 2nd Edition, McGraw Hill, 1959

From (3)

(20b) $$N_\theta = 2hh'S + h^2S'$$

and from (6)

(20c) $$Q_\varphi = -\frac{F(s)}{s\cos\varphi} - \frac{h^2}{s} S \tan\varphi$$

Upon substitution of M_φ from (10c), M_θ from (10d), and Q_φ from (20c) and using the notation $\theta' = \frac{d\,w_m}{ds}$, (5) becomes

(21)
$$-\frac{Eh^3}{12(1-\nu^2)}\left(\theta' + \nu\frac{\theta}{s}\right) + s\,\frac{-Eh^3}{12(1-\nu^2)}$$
$$\left(\theta'' + \nu\frac{\theta'}{s} - \nu\frac{\theta}{s^2}\right) + s\,\frac{-3Eh^2h'}{12(1-\nu^2)}\left(\theta' + \nu\frac{\theta}{s}\right)$$
$$-\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\theta}{s} + \nu\theta'\right) = s\left(-\frac{F(s)}{s\cos\varphi} - \frac{h^2}{s} S\tan\varphi\right)$$

which after simplification becomes

(21a)
$$s\theta'' + \left(1 + 3s\frac{h'}{H}\right)\theta' + \left(3s\frac{h'}{h}\nu - 1\right)\frac{\theta}{s}$$
$$= \frac{12\tan\varphi}{Eh}(1-\nu^2)S + \frac{12F(s)}{Eh^3\cos\varphi}(1-\nu^2)$$

To derive a second equation, solve for w_m in (7b).

(22) $$w_m = (u_m - \epsilon_{my} s) \cot\varphi$$

Differentiate (22) with respect to s.

(23) $$w_m' = (\epsilon_{xm} - \epsilon_{ym} - s\,\epsilon_{ym}') \cot\varphi$$

From Hooke's law,

(24a) $$\epsilon_{xm} = \frac{1}{E}(\sigma_{xm} - \nu\sigma_{ym}) = \frac{1}{Eh}(N_\varphi - \nu N_\theta)$$

(24b) $$\epsilon_{ym} = \frac{1}{E}(\sigma_{ym} - \nu\sigma_{xm}) = \frac{1}{Eh}(N_\theta - N_\varphi)$$

Using the values of N_φ and N_θ from (20a) and (20b), (24a) and (24b) become

(25a) $$\epsilon_{xm} = \frac{1}{E}\left[\frac{hS}{s} - \nu(2h'S + hS')\right]$$

(25b) $$\epsilon_{ym} = \frac{1}{E}\left[2h'S + hS' + \nu\frac{hS}{s}\right]$$

respectively.

Differentiate (25b),

(26) $$\epsilon'_{ym} = \frac{1}{E}\left[2h'S' - 2h''S + hS'' + h'S' - \nu\left(\frac{h}{s}S' + \frac{h'S}{s} - \frac{hS}{s^2}\right)\right]$$

Upon the substitution of the values of ϵ_{xm}, ϵ_{ym} and ϵ'_{ym} from (25a), (25b), and (26) respectively and after some simplification (23) becomes

(27) $$sS'' + (1 + 3\frac{h'}{h}s)S' + \left[s(2+\nu)\frac{h'}{h} + 2s^2\frac{h''}{h} - 1\right]\frac{S}{s} = -\frac{E\theta}{h\cdot\cot\varphi}$$

Equations (21a) and (27) were first derived by Honegger.[2] They are the alternate forms of a pair of coupled equations for the conical shell. If Meissner's operator and the following notations are used:

(29)
$$L(U) = h\cot\varphi\left[SU'' + (1 + 3s\frac{h'}{h})U' - \frac{U}{s}\right]$$

$$f_1 = 3\nu h'\cot\varphi$$

$$f_2 = \left[(2+\nu)h' + 2sh''\right]\cot\varphi$$

$$\lambda_1 = \frac{12(1-\nu^2)}{E}$$

$$\lambda_2 = -E$$

$$\bar{F}(s) = \frac{12F(s)}{Eh^2\sin\varphi}(1-\nu^2)$$

2. Honneger, E., "Festigkeitsberechnung von Kegelschalen mit linear veranderlicher Wandstärke," Doctoral Thesis, Zurich, 1919

equations (21a) and (27) may be rewritten as

(30) $$L(\theta) + f_1\theta = \lambda_1 S + F(s)$$

(31) $$L(S) + f_2 S = \lambda_2\theta$$

Boundary Conditions

On the boundaries, forces N_φ, Q_φ and moment M_φ may be prescribed. The moment M_φ may be prescribed completely arbitrarily on both edges. The forces N_φ, Q_φ may be prescribed completely arbitrarily on one end, but if at the other end the forces are also prescribed, only one of them can be arbitrary, the other one must satisfy (6), namely,

$$sN_\varphi \sin\varphi + sQ_\varphi \cos\varphi = - F(s)$$

Expressed in the dependent variables in the two alternate pairs of coupled differential equations (12a, 13a) and (21a, 27), the shell forces and the shell moment are as follows:

(32a) $$N_\varphi = \frac{Eh}{1 - \nu^2}\left(u_m{}' + \nu \frac{u_m - w_m \tan\varphi}{s}\right) = \frac{h^2 S}{s}$$

(32b) $$Q_\varphi = -\frac{Eh^3}{12(1 - \nu^2)}\left[w_m{}''' + \left(\frac{3h'}{h} s + 1\right)\frac{w_m{}''}{s} + \left(3s\frac{h'}{h} - 1\right)w_m{}'\right]$$

$$(32c) \qquad M_\varphi = -\frac{Eh^3}{12(1-\nu^2)}\left(w_m'' - \nu\frac{w_m'}{s}\right) = -\frac{Eh}{12(1-\nu^2)}\left(\theta' + \nu\frac{\theta}{s}\right)$$

If the forces and moment are not prescribed, displacements u_m, w_m and slope $\frac{d\,w_m}{ds}$ then should be prescribed. Instead of prescribing u_m and w_m an alternate way is to prescribe the quantities $(u_m \sin\varphi + w_m \cos\varphi)$ and $(u_m \cos\varphi - w_m \sin\varphi)$ which are the displacements in the axial direction and in the radial direction respectively The latter quantity may be expressed through radial strain ϵ_{ym} by dividing it by $s \cos\varphi$. Expressed in the dependent variable of the two alternate pairs of coupled differential equations, these displacements and slope are as follows:

$$(33a) \qquad d = u_m \sin\varphi + w_m \cos\varphi$$

$$(33b) \qquad \epsilon_{ym} = \frac{u_m}{s} - \frac{w_m \tan\varphi}{s} = \left(2hh' - \nu\frac{h^2}{s}\right)S + h^2 S'$$

$$(33c) \qquad w' = \theta$$

If the pair of coupled differential equations in θ and S (21a, 27) are used, the appropriate boundary conditions from the first group are to prescribe (32a) $N_\varphi = \frac{h^2 S}{s}$ and (32c) $M_\varphi = -\frac{Eh^3}{12(1-\nu^2)}\left(\theta' + \nu\frac{\theta}{s}\right)$. The appropriate boundary conditions from the second group are to prescribe (33b) $\epsilon_{ym} = \left(2hh' - \nu\frac{h^2}{s}\right)S + h^2 S'$ and (33c) $w' = 0$.

In some instances it may be found more convenient to prescribe the radial force ($P = N_\varphi \sin\varphi + Q_\varphi \cos\varphi$) and the axial force ($V = Q_\varphi \sin\varphi - N_\varphi \cos\varphi$) instead of N_φ and Q_φ on the boundary. With P and V given, N_φ and Q_φ can be solved as follows:

(34a) $$N_\varphi = \frac{h^2 S}{s} = P \sin\varphi - V \cos\varphi$$

(34b) $$Q_\varphi = -\frac{Eh^3}{12(1-\nu^2)}\left[w_m''' + \left(\frac{3h'}{h}s + 1\right)\frac{w_m''}{s} + \left(3s\frac{h'}{h} - 1\right)w_m'\right] = P \cos\varphi - V \sin\varphi$$

Junction Conditions

If two sections of different shells are joined together without misfit and are put under loads, by the condition of compatibility, the following conditions should hold at the joined ends:

(35) $$\left.\begin{array}{l} (u_m)_1 = (u_m)_2 \\ (w_m)_1 = (w_m)_2 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} d_1 = d_2 \\ (\epsilon_{ym})_1 = (\epsilon_{ym})_2 \end{array}\right.$$

$$(w_m')_1 = (w_m')_2$$

where subscripts "1" and "2" denote section "1" and "2".

For the equilibrium of a thin ring section containing the junction (Fig. 3), and with the second order effects ignored, the following condition must hold:

$$(M_\varphi)_1 = (M_\varphi)_2$$

$$P_1 = P_2 \tag{36}$$

$$V_1 = V_2$$

where P_1 or P_2 can be expressed through the given distributed loading and the end loads (possibly including the not yet determined end reactions). Considering P_1 and P_2 known, the last condition $V_1 = V_2$ can be written as

$$(P \tan\varphi - \frac{N_\varphi}{\cos\varphi})_1 = (P \tan\varphi - \frac{N_\varphi}{\cos\varphi})_2 \tag{37}$$

If the second pair of the coupled differential equations (21a, 27) are used, the appropriate junction conditions are

$$(\epsilon_{ym})_1 = (\epsilon_{ym})_2$$

$$w_1' = w_2'$$

$$(M_\varphi)_1 = (M_\varphi)_2 \tag{38}$$

$$(P \tan\varphi - \frac{N_\varphi}{\cos\varphi})_1 = (P \tan\varphi - \frac{N_\varphi}{\cos\varphi})_2$$

Expressed in the dependent variables in equations (21a, 27), they are

TECHNICAL LIBRARY BLDG. 313 ABERDEEN PROVING GROUND, MD. STEAP-TL

$$\left[(2hh' - \nu \frac{h^2}{s})S + h^2S'\right]_1 = \left[(2hh' - \nu \frac{h^2}{s})S + h^2S'\right]_2$$

$$\theta_1 = \theta_2$$

(38a)

$$\left[\frac{Eh}{12(1 - \nu^2)}(\theta' + \nu \frac{\theta}{s})\right]_1 = \left[\frac{Eh}{12(1 - \nu^2)}(\theta' + \nu \frac{\theta}{s})\right]_2$$

$$(P \tan\varphi - \frac{h^2S}{s\cos\varphi})_1 = (P \tan\varphi - \frac{h^2S}{s\cos\varphi})_2$$

The last two conditions in (38a) are more explicit and simpler to apply than those given by Tsui.[3]

---

3. Tsui, E. Y. W. "Analysis of Tapered Conical Shells" Proceedings of 4th U. S. National Congress of Applied Mechanics, p. 813

II. SOLUTIONS TO THE DIFFERENTIAL EQUATIONS

Referring to (30) and (31), the Honneger's coupled equations for the conical shell with linearly varying thickness subject to normal loading only are

(30) $$L(\theta) + f_1 \theta = \lambda_1 S + \bar{F}(s)$$

(31) $$L(S) + f_2 S = \lambda_2 \theta$$

with the adopted notations defined before in (29):

$$L(U) = h \cot\varphi \left[sU'' + (1 + 3s \frac{h'}{h})U' - \frac{U}{s} \right]$$

$$f_1 = 3\nu h' \cot\varphi$$

$$f_2 = \left[(2 + \nu)h' \right] \cot\varphi$$

(29)

$$\lambda_2 = - E$$

$$\bar{F}(s) = \frac{12\ F(s)}{Eh^2 \sin\varphi} (1 - \nu^2)$$

The linearity of the wall thickness is expressed by the equation

(32) $$h = a_o + b_o s$$

Recall that in (6)

$$F(s) = \int_{s_1}^{s} Z \cos\varphi \, s ds - \left[sN_{\varphi} \sin\varphi + Q_{\varphi} \cos\varphi \right]_{s = s_1} \tag{6}$$

For uniform normal pressure, Z = p.

$$\begin{aligned} F(s) &= \int_{s_1}^{s} p \cos\alpha \, s ds - \left[sN_{\varphi} \sin\varphi + sQ_{\varphi} \cos\varphi \right]_{s = s_1} \\ &= p \cos\alpha \frac{s^2}{2} - g(s_1) \end{aligned} \tag{33}$$

where

$$g(s_1) = \left[p \cos\alpha \frac{s^2}{2} + sN_{\varphi} \sin\varphi + sQ_{\varphi} \cos\varphi \right]_{s = s_1} \tag{34}$$

It follows that

$$F(s) = \frac{12(1 - \nu^2)}{Eh^2 \sin\varphi} \left[p \cos\alpha \frac{s^2}{2} - g(s_1) \right] \tag{35}$$

and (30) and (31) may be written more explicitly as

$$L(\theta) + f_1\theta = \lambda_1 S + \frac{12(1 - \nu^2)}{Eh^2 \sin\varphi} \left[p \cos\alpha \frac{s^2}{2} - g(s_1) \right] \tag{36}$$

$$L(S) + f_2 S = \lambda_2 S \tag{37}$$

The general solution to (36) and (37) consists of a particular solution and the solution to the reduced homogeneous equations by omitting the non-homogeneous term in (36).

Particular Solution

It can be verified by direct substitutions that the particular solution to (30) and (31) due to the term $\frac{12(1 - \nu^2)}{Eh^2 \sin^2\varphi}$ $\left[p \cos\alpha \frac{s^2}{2} - g(s_1)\right]$ is

(38)
$$\theta_p = \theta_{1p} + \theta_{2p}$$
$$S_p = S_{1p} + S_{2p}$$

where

(39)
$$\theta_{1p} = \frac{\alpha_1 s + \alpha_2 s^2}{h^2}$$
$$S_{1p} = \frac{\beta_1 s + \beta_2 s^2}{h^2}$$

with

$$\beta_2 = \frac{- 6p(1 - \nu) \cot\varphi}{12(1 - \nu) + b_o^2 (3\nu - 1) \cot^2\varphi}$$

(40)
$$\alpha_2 = - \frac{(1 - \nu)b_o \cot\varphi}{E} \beta_2$$
$$\alpha_1 = \frac{a_o}{b_o(1 - \nu)} \alpha_2 - \frac{4(1 + \nu)}{Eb_o \cot\varphi} \beta_1$$

(40) contd.

$$\beta_1 = \frac{3a_o b_o \cot^2\varphi}{4(1 + \nu) + b_o^{\,2} \cot^2\varphi\,(1 - \nu)}\,\beta_2$$

$$+ \frac{Ea \cot\varphi}{(1 - \nu)\left[4(1 + \nu) + b_o^{\,2}(1 - \nu)\cot^2\varphi\right]}\,\alpha_2$$

and

$$\theta_{2p} = \frac{\alpha_{-1}}{h^2 s} + \frac{\alpha_o}{h^2}$$

(41)

$$S_{2p} = \frac{\beta_{-1}}{h^2 s} + \frac{\beta_o}{h^2}$$

with

$$\beta_o = \frac{4g(s_1)(1 - \nu^2)}{4 \sin\varphi(1 - \nu^2) + b_o^{\,2} \cot\varphi \cos\varphi\,(1 - \nu)^2}$$

$$\cdot\ \frac{12(1 - \nu^2)}{Eh^2 \sin\varphi}$$

$$\alpha_o = \frac{b_o \cot\varphi(1 - \nu)}{E}\,\beta_o$$

(42)

$$\beta_{-1} = \frac{2a_o b_o \cot^2\varphi(2\nu - 1)}{(3\nu - 1)(1 + \nu)b_o^{\,2} \cot^2\varphi + 12(1 - \nu^2)}\,\beta_o$$

$$\alpha_{-1} = \frac{a_o \cot\varphi\,(1 - \nu^2)(12 + b_o^{\,2} \cot^2\varphi)}{E\left[(3\nu - 1)(1 + \nu)b_o^{\,2} \cot^2\varphi + 12(1 - \nu^2)\right]}\,\beta_o$$

Solution to the Reduced Homogeneous Equations

From (30), (31) or (36), (37) the reduced homogeneous equations are

(43) $$L(\theta) + f_1\theta = \lambda_1 S$$

(44) $$L(S) + f_2 S = \lambda_2 \theta$$

Eliminating S from (43) and (44) gives

(45) $$LL(\theta) + L(f_1\theta) + f_2\, L(\theta) + (f_1 f_2 - \lambda_1\lambda_2)\theta = o$$

A similar equation is obtained by eliminating θ,

(46) $$LL(S) + L(f_2 S) + f_1 L(S) + (f_1 f_2 - \lambda_1\lambda_2)S = o$$

Assume the following is true,

(47) $$\left[L + (c_1 + f_1)\right]\left[L + (c_2 + f_1)\right]\theta = 0$$

where c_1 and c_2 are some constants, then Equation (47) may be rewritten as

(47a) $$LL(\theta) + L(c_2 + f_1)\theta + (c_1 + f_1)L(\theta)$$

$$+ (c_1 c_2 + c_1 f_1 + c_2 f_1 + f_1 f_1) = o$$

Subtracting (45) from (47a) gives

$$(48) \qquad (c_1 + c_2)L(\theta) + (f_1 - f_2)L(\theta) + (c_1 + c_2)f_1 + (f_1 - f_2)f_1 + c_1c_2 + \lambda_1\lambda_2 = 0$$

which can be satisfied if

$$(49) \qquad \begin{aligned} c_1 + c_2 &= -(f_1 - f_2) \\ c_1\, c_2 &= -\lambda_1\lambda_2 \end{aligned}$$

Equation (49) is equivalent to stating that c_1 and c_2 are the roots of

$$(50) \qquad c^2 + (f_1 - f_2)c - \lambda_1\lambda_2 = 0$$

and solving (50),

$$(51) \qquad c_{1,2} = \frac{-(f_1 - f_2)}{2} \pm \left[\left(\frac{f_1 - f_2}{2}\right)^2 + \lambda_1\lambda_2 \right]^{\frac{1}{2}}$$

Hence (45) can be written in the form of (47) and since the operators in (47) are commutative, (45) can be split into two second order equations as follows:

$$L(\theta) + (c_1 + f_1)\theta = 0 \tag{52a}$$

$$L(\theta) + (c_2 + f_1)\theta = 0 \tag{52b}$$

with c_1 and c_2 given by (51).

For the case of linearly varying wall thickness according to (32), from (29) it is found

$$f_1 = 3\nu h' \cot\varphi = 3\nu b_o \cot\varphi \tag{53}$$

$$f_2 = \left[(2 + \nu)h' + 2sh''\right] \cot\varphi = (2 + \nu)b_o \cot\varphi \tag{54}$$

and from (51)

$$c_{1,2} = (1 - \nu)b_o \cot\varphi \pm \left[(1 - \nu)^2 b_o \cot^2\varphi - 12(1 - \nu^2)\right]^{\frac{1}{2}} \tag{55}$$

With these values of f_1, f_2 and c_1, c_2, (52a),(52b) become

$$h\cot\alpha \left[s\theta'' + (1 + 3s\frac{b_o}{(a_o + b_o s)}\theta' - \frac{\theta}{s}\right]$$

(56a,b)

$$+ \left[(1 - 2\nu)b_o \cot\varphi \pm \left[(1 - \nu^2)b_o{}^2\cot^2\varphi - 12(1 - \nu^2)\theta\right]^{\frac{1}{2}}\right] = 0$$

Making a change of variable

$$s = -\frac{a_o}{b_o}t, \qquad h = a_o(1 - t) \tag{57}$$

equations (56a,b) become

$$\ddot{\theta} + \left(\frac{3}{t - 1} + \frac{1}{t} \right)\dot{\theta} + \left(\frac{1}{t} + \sigma_{1,2} \right) \frac{\theta}{t(t - 1)} = 0 \tag{58a,b}$$

where

$$\sigma_{1,2} = 2\nu \pm \left[(1 - \nu)^2 - \frac{12(1 - \nu^2)}{b_o^{\,2}} \tan^2\alpha \right]^{\frac{1}{2}} \tag{59}$$

Comparing (58a,b) to the standard form of generalized hypergeometric equation[4]:

$$\begin{aligned} & Y'' + \left(\frac{1 - \alpha - \alpha'}{X} + \frac{1 - \gamma - \gamma'}{X - 1} \right) Y' \\ & + \left(\frac{-\alpha\alpha'}{X} + \frac{\gamma\gamma'}{X} + \beta\beta' \right) \frac{Y}{X(X - 1)} = 0 \end{aligned} \tag{60}$$

it is found

$$\begin{aligned} \alpha &= 1 & \gamma &= 0 \\ \alpha' &= -1 & \gamma' &= -2 \\ \beta_{1,2} &= \frac{3}{2} + \left[\frac{9}{4} - \sigma_{1,2} \right]^{\frac{1}{2}} \\ \beta'_{1,2} &= \frac{3}{2} - \left[\frac{9}{4} - \sigma_{1,2} \right]^{\frac{1}{2}} \end{aligned} \tag{61}$$

4. W. Magnus and F. Oberhattinger, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea, 1954, P. 12

(58) Let β and β' stand either for β_1, β_1' or β_2, β_2'

expressed in Reimann's symbol is

$$\theta = P\left\{\begin{matrix} 0 & 1 & \infty & \\ +1 & 0 & \beta & t \\ -1 & 2 & \beta' & \end{matrix}\right\} = tP\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & 1+\beta & t \\ -2 & -2 & 1+\beta' & \end{matrix}\right\}$$

(62)

$$= tP\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & t \\ 1-c & c-a-b & b & \end{matrix}\right\} = ty$$

where y satisfies the hypergeometric equation:

(63) $$t(1 - t)y + \left[c - (a + b + 1)t\right] y - aby = 0$$

From (62) it is recognized that

$$c = 3$$

(64) $$a = 1 + \beta = \frac{5}{2} + \left[\frac{9}{4} - \sigma_{1,2}\right]^{\frac{1}{2}}$$

$$b = 1 + \beta' = \frac{5}{2} - \left[\frac{9}{4} - \sigma_{1,2}\right]^{\frac{1}{2}}$$

Solution at t = 0

One of the independent solutions to (63) is

$$y_1 = F(a,b,c;t) = 1 + \frac{ab}{c1!}t + \frac{a(a+1)b(b+1)}{c(c+1)2!}t^2 + \cdots$$

$$+ \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!\,c(c+1)\cdots(c+n-1)}\,t^n + \cdots$$

(65) $$= 1 + \sum_{n=1}^{\infty} \frac{[a]_n[b]_n}{n!\,[c]_n}\,t^n$$

where

(66) $$[a]_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$$

It is noted that $1 - c = -2$, but neither a or b is equal to 2. The other independent solution is [5]

(66) $$y_2 = y_1 \log t + F_1\,(a,b,c;t)$$

where

(67) $$F_1(a,b,c;t) = (-1)^c\, t^{1-c} \sum_{n=0}^{c-2} (-1)^n \cdot \left\{ \frac{(c-1)!\,(c-n-2)!\,t^n}{n!(a-1)(a-2)\cdots(a-c+n+1)(b-1)(b-2)\cdots(b-c+n+1)} \right\}$$

5. T. M. MacRobert, Functions of a Complex Variable, MacMillan, 1954, P. 230.

(67) (Contd)

$$+\sum_{n=0}^{\infty} \frac{[a]_n[b]_n}{n!\,[c]_n} \left(\sum_{r=0}^{n-1} \frac{1}{a+r} + \sum_{r=0}^{n-1} \frac{1}{b+r} - \sum_{1}^{n} \frac{1}{r} - \sum_{r=0}^{n-1} \frac{1}{c+r} \right) t^n$$

Solution at $t = 1$

Making a substitution $t = 1 - \xi$, (63) becomes

$$\xi(1-\xi)\frac{d^2y}{d\xi^2} + \left[(a+b+1-c) - (a+b+1)\xi\right]\frac{dy}{d\xi} - aby = 0$$

The two independent solutions are respectively

(68) $\quad y_1 = F(a,b,c'; 1-t)$

(69) $\quad y_2 = y_1 \log t + F_1\,(a,b,c'; 1-t)$

where

(70) $\quad c' = a + b + 1 - c = 3$

Solution at $t = \infty$

Put $t = \frac{1}{\xi}$ and $y = \xi^a \overline{W}$. It is found from (63) that $\overline{W}$ satisfies

(71) $$\xi(1-\xi)\frac{d^2\overline{W}}{d\xi^2} + \left[(1+a-b) - (2a+2-c)\xi\right]\frac{d\overline{W}}{d\xi}$$

$$- a(a+1-c)\overline{W} = 0$$

Hence

(72) $$y_1 = t^{-a} F(a,\ 1+a-c,\ 1+a-b;\ \tfrac{1}{t})$$

by symmetry of a and b in (63),

(73) $$y_2 = t^{-b} F(b,\ 1+b-c,\ 1+b-a;\ \tfrac{1}{t})$$

Other Solution

Put $t = \frac{\xi - 1}{\xi}$ or $\xi = \frac{1}{1-t}$ and $y = \xi^a\overline{W}$. From (63) it is found that $\overline{W}$ satisfies

(74) $$\xi(1-\xi)\frac{d^2\overline{W}}{d\xi^2} + \left[(a+1-b) - (a+c+1-b)\xi\right]\frac{d\overline{W}}{d\xi}$$

$$- a(c-b)\overline{W} = 0$$

Hence

(75) $$y_1 = (1-t)^{-a} F(a,\ c-b,\ a-b+1,\ \frac{1}{1-t})$$

(76) $$y_2 = (1-t)^{-b} F(c,\ c-a,\ b-a+1,\ \frac{1}{1-t})$$

The range of convergence for (65), (66) is $-1 < t < +1$; for (68), (69) is $0 < t < 2$; for (72), (73) is $-\infty < t < -1$ and $1 < t < +\infty$; and for (75), (76) is $-\infty < t < 0$ and $2 < t < +\infty$.[6] The overlapping solutions form analytic continuation to one another.

Let the solutions to (56a) and (56b) be denoted respectively by

(77) $$\theta_I = A\theta_1 + B\theta_2$$

(78) $$\theta_{II} = C\theta_3 + D\theta_4$$

where A, B, C, D are arbitrary constants; θ_1, θ_2 are the two independent solutions to (56a) and θ_3, θ_4 are the two independent solutions to (56b). The corresponding S may be obtained through (43) and (52a) and (43) and (52b)

(79) $$S_I = -\frac{c_1\theta_I}{\lambda_1}$$

(80) $$S_{II} = -\frac{c_2\theta_{II}}{\lambda_2}$$

From (32) and (57) it is seen that if the wall thickness tapers off as the section moves away from the apex of the cone, t will be positive and increasing. When the wall thickness

6. A. R. Forsyth, A Treatise on Differential Equations, MacMillan, 1956, P. 218

approaches zero, t approaches + 1. On the other hand, if the thickness grows as the section moves away from the apex, t will be negative and decreasing. Appropriate solutions should be used which are convergent for the range of t in the problem on hand.

Though presented in somewhat different forms, part of the results in this section could be obtained indirectly by specializing Honneger's results. Reference is made to Honneger's original thesis.

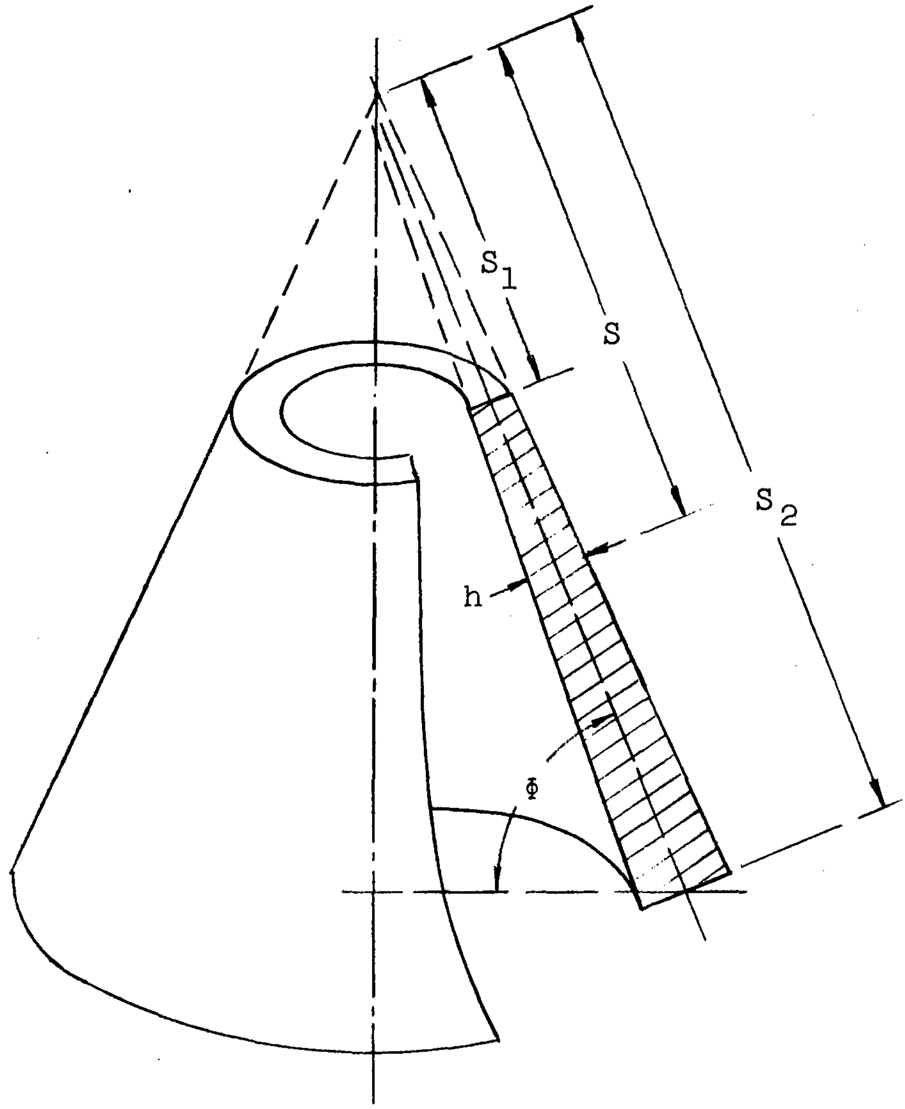


Fig. 1

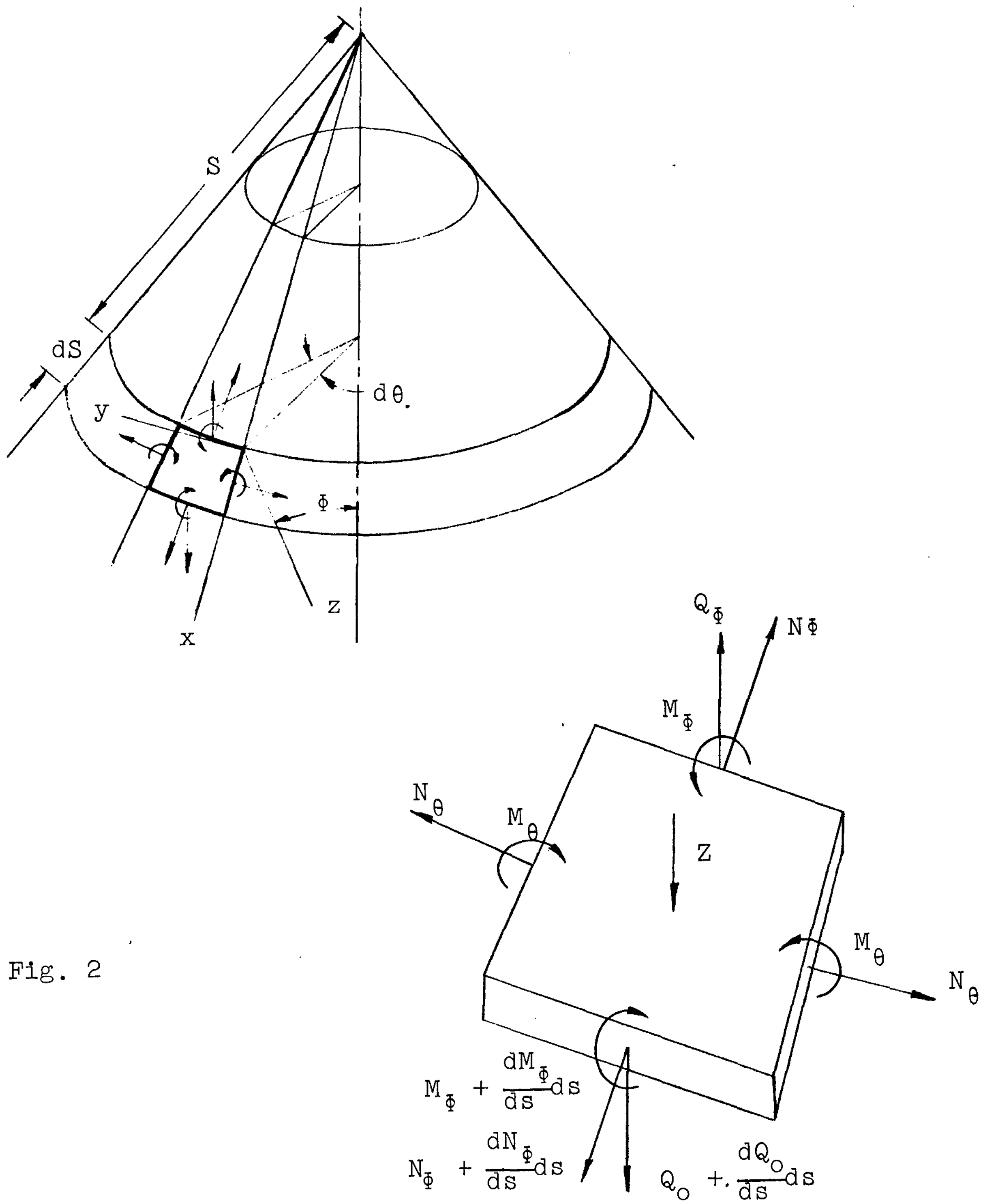
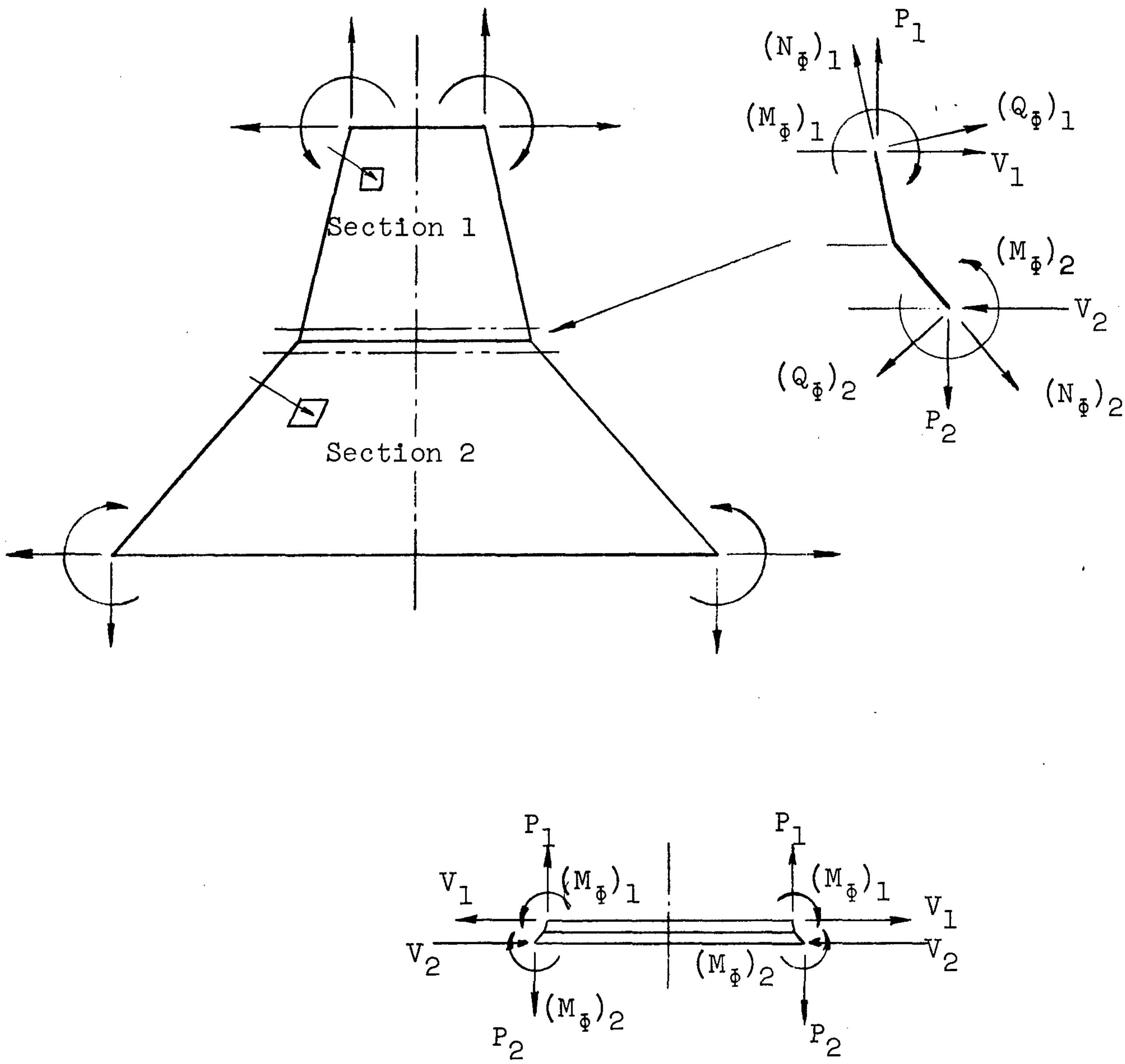

S
dS
dθ
y
Φ
z
x
Q_Φ
$N\Phi$
M_Φ
N_θ
M_θ
Z
M_θ
N_θ
$M_\Phi + \frac{dM_\Phi}{ds}ds$
$N_\Phi + \frac{dN_\Phi}{ds}ds$
$Q_o + \frac{dQ_o}{ds}ds$

Fig. 2



Section 1
Section 2
P_1
$(N_\Phi)_1$
$(M_\Phi)_1$
$(Q_\Phi)_1$
V_1
$(M_\Phi)_2$
V_2
$(Q_\Phi)_2$
$(N_\Phi)_2$
P_2
P_1
V_1
$(M_\Phi)_1$
V_2
$(M_\Phi)_2$
P_2

Fig. 3

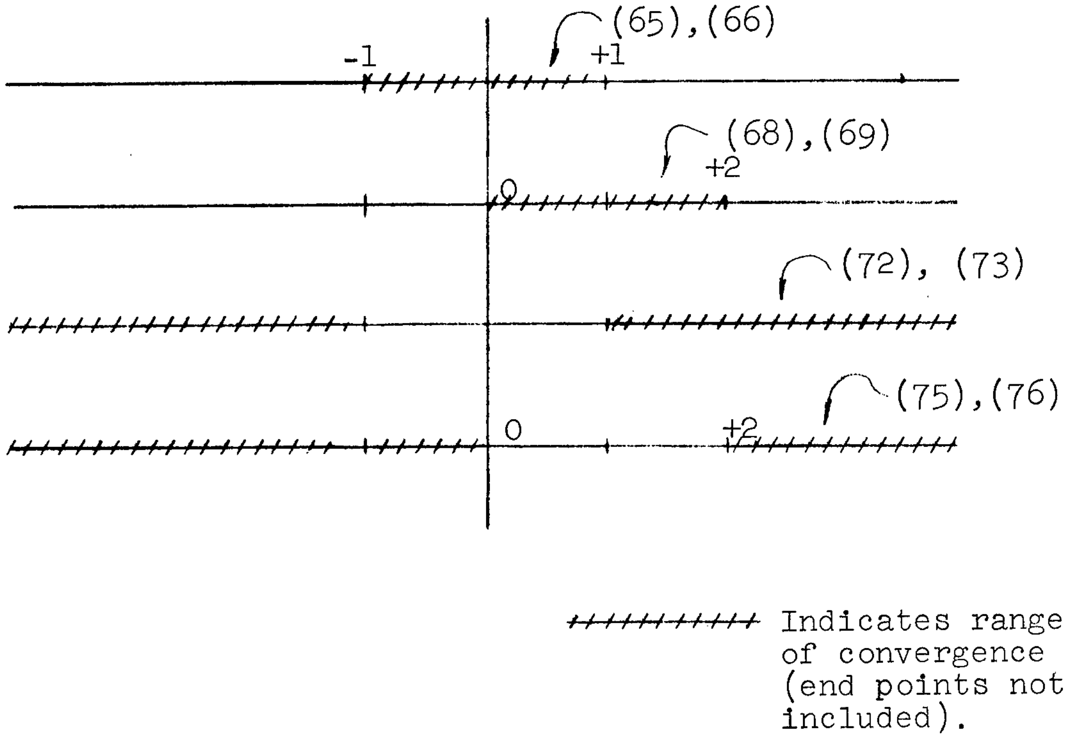


Fig. 4 Range of Convergence

for Various Solutions.

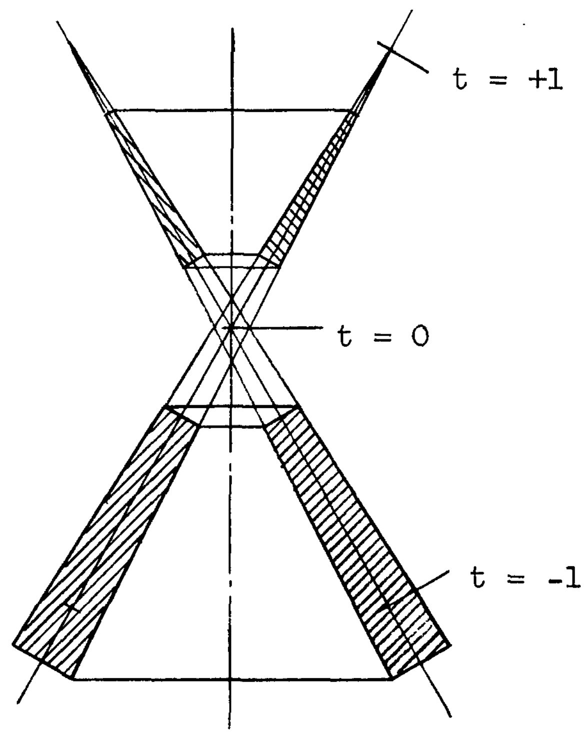


Fig. 5 Variation of Wall Thickness
and Values of t.

REFERENCES

[1] Timoshenko and Woinowsky-Krieger, Theory of Plates and Shells, P. 467, 2nd Edition, McGraw Hill, 1959

[2] Honneger, E., Festigkeitsberechnung von Kegelshalen mit linear veränderlicher Wandstärke, Doctoral Thesis, Zürich, 1919

[3] Tsui, E. Y. W. Analysis of Tapered Conical Shells, Proceedings of 4th U. S. National Congress of Applied Mechanics, P. 813

[4] Magnus, W., and Oberhattinger, F., Formulas and Theorems for the Functions of Mathematical Physics, Chelsea, 1954, P. 12

[5] MacRobert, T. M., Functions of a Complex Variable, MacMillan, 1954, P. 230

[6] Forsyth, A. R., A Treatise on Differential Equations, MacMillan, 1956, P. 218

DISTRIBUTION

8 - Commanding Officer
Picatinny Arsenal
Dover, New Jersey
ATTN: Purchasing Office
SMUPA-PB1

1 - Commanding Officer
Chicago Procurement District, U. S. Army
623 South Wabash Avenue
Chicago 5, Illinois
ATTN: AMXCH-DV

1 - Headquarters
U. S. Army Materiel Command
Washington 25, D. C.
ATTN: AMCCG

10 - Armed Services Technical Information Agency
Document Service Center
Knott Building
Dayton 2, Ohio

1 - Bureau of Ordnance
Department of the Navy
Washington 25, D. C.

1 - Commanding Officer
Watertown Arsenal
Watertown 72, Mass.
ATTN: Mr. J. E. Bluhm

1 - Commanding General
Aberdeen Proving Ground, Maryland
ATTN: Ballistic Res Laboratories