AD 607932

CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION CFSTI
DOCUMENT MANAGEMENT BRANCH 410.11

LIMITATIONS IN REPRODUCTION QUALITY

ACCESSION # AD 607 937

- [x] 1. WE REGRET THAT LEGIBILITY OF THIS DOCUMENT IS IN PART UNSATISFACTORY. REPRODUCTION HAS BEEN MADE FROM BEST AVAILABLE COPY.
- [ ] 2. A PORTION OF THE ORIGINAL DOCUMENT CONTAINS FINE DETAIL WHICH MAY MAKE READING OF PHOTOCOPY DIFFICULT.
- [ ] 3. THE ORIGINAL DOCUMENT CONTAINS COLOR, BUT DISTRIBUTION COPIES ARE AVAILABLE IN BLACK-AND-WHITE REPRODUCTION ONLY.
- [ ] 4. THE INITIAL DISTRIBUTION COPIES CONTAIN COLOR WHICH WILL BE SHOWN IN BLACK-AND-WHITE WHEN IT IS NECESSARY TO REPRINT.
- [ ] 5. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED, DOCUMENT WILL BE AVAILABLE IN MICROFICHE ONLY.
- [ ] 6. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED DOCUMENT WILL NOT BE AVAILABLE.
- [ ] 7. DOCUMENT IS AVAILABLE IN MICROFICHE ONLY.
- [ ] 8. DOCUMENT AVAILABLE ON LOAN FROM CFSTI (TT DOCUMENTS ONLY).
- [ ] 9.

PROCESSOR: jm

TSL-107-10 64

STL/TM-61-0000-19008

THE ANALYSIS OF PHASE DETECTOR
PERFORMANCE FOR A SIGNAL PLUS
NOISE INPUT

by

Arthur R. Kraiman

11 January 1961

STL/TM-61-0000-19008

SPACE TECHNOLOGY LABORATORIES, INC.
P. O. Box 95001
Los Angeles 45, California

ACKNOWLEDGEMENT

The figures and supporting calculations were made by Gene A. Gesch.

ABSTRACT

Expressions were derived for the mean and mean square value of the output of a phase detector to which is applied a sinusoidal signal corrupted by narrow-band Gaussian noise, together with an oscillator reference of the same frequency as the signal but displaced in phase. The analysis was based on the model of a balanced phase detector composed of peak-detecting diodes producing a difference output from the two halves. Based on the above output parameters, a measure of phase detector performance was defined in terms of a "noise-to-noise ratio," the square root of the output variance without signal divided by the same quantity with the signal present. The case where signal and reference are in quadrature was treated in detail, yielding curves of this ratio under varying input and reference conditions. In addition, the voltage output "signal-to-noise ratio" was calculated as a function of the phase angle between signal and reference, and the resulting graphs show the effects of reference level and input signal-to-noise power ratio.

CONTENTS

Page No.

Acknowledgement ii

Abstract. iii

1. Introduction. 1

2. Model of Phase Detector Operation 2

3. Statistical Properties of the Output. 6

   3.1 The Output Mean Value. 8

   3.2 The Output Mean Square Value 9

4. Calculation of Output Noise-to-Noise Ratio. 11

5. Calculation of Output Signal-to-Noise Ratio 16

6. Conclusion. 19

7. References. 20

8. Appendices. . . . .

   A. Evaluation of the Output Mean Value 1A

   B. Evaluation of the Output Mean Square Value. 1B

   C. Approximate Evaluations for the Output Noise-to-Noise Ratio for the Quadrature Case 1C

   D. Limiting Formulas for the Output Signal-to-Noise Ratio for the General Case 1D

1. INTRODUCTION

With the increasing use of phase-comparison techniques in the design of guidance and communication equipment, it is important to know how devices providing such operation behave under varying conditions of the signal and corrupting noise environment. Ideally, a phase "comparator" or "detector" combines a received signal and a locally generated reference waveform in some manner to produce an output voltage which is proportional to the phase difference between these inputs. However, in practice this desired linear operation does not continue indefinitely as the phase difference increases in magnitude, instead resulting in the familiar leveling off and subsequent decrease in detector output voltage.[1,2] In cases where the received and reference waveforms are relatively unperturbed by noise and differ little in their time behavior, the non-linear portion of the phase-detector characteristic can be ignored and a linear analysis of its performance carried out.[3] But with the advent of systems working in a high noise environment, large fluctuations are superimposed on the received signal which cause corresponding phase deviations from the normally steady reference waveform, resulting in excursions beyond linear detector operation.[4]

It was for the purpose of considering the performance of phase detectors under such high level noise conditions that the present study was undertaken. In analysis previously carried out, the simplifying assumption was made that the reference amplitude was much larger than the received signal magnitude, resulting in the phase detector output voltage being independent of this reference amplitude.[5] While this is a valid approximation in practice for relatively unperturbed signals, it no longer holds when the added noise fluctuations are sufficient to make the over-all received amplitude comparable to the reference magnitude. The present investigation proceeded under general signal, noise, and reference parameter conditions.

2. MODEL OF PHASE DETECTOR OPERATION

The noise performance of the phase detector is studied here on the basis of a model of its operation which has been used in previous applications to automatic frequency and phase-control systems.[6] It is assumed to be a balanced phase detector consisting of two peak-detecting diodes and associated filter circuits, as shown below in Figure 1, together with the usual

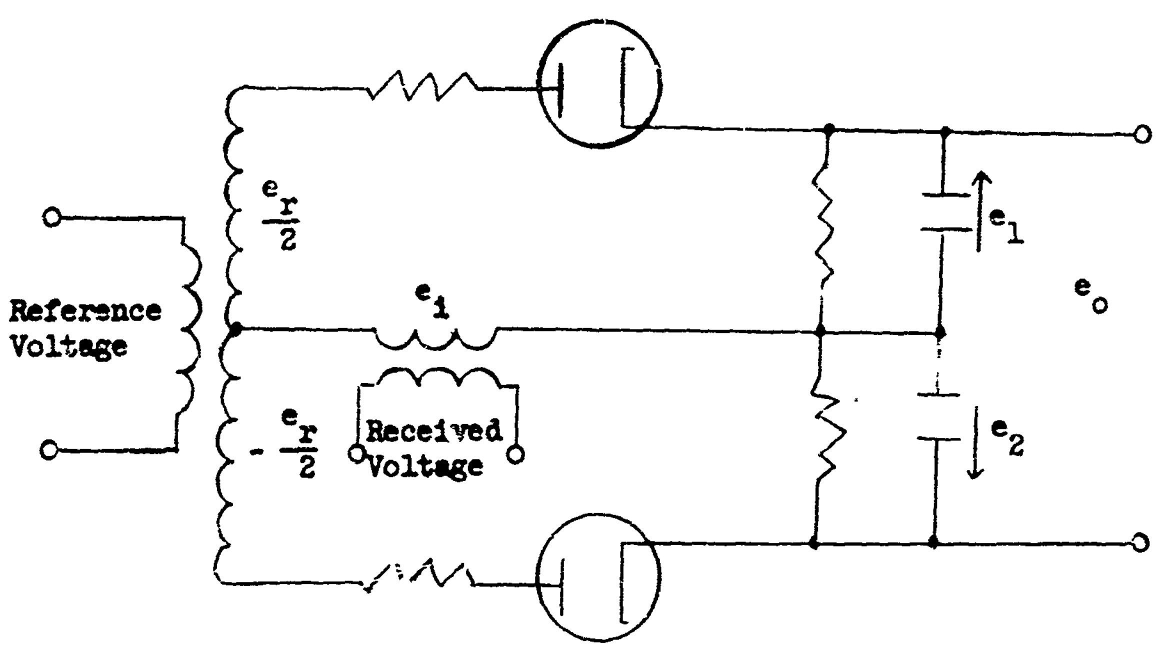

FIGURE 1 - SCHEMATIC DIAGRAM OF PHASE DETECTOR

transformer windings to introduce the received and reference voltages.[1,2] The expression for the output voltage e_o of the phase detector can readily be obtained from a vector diagram of the circuit voltages, once the latter are written in terms of amplitude and phase.[6]

For the present noise analysis, the received voltage e_i appearing within the diode circuits is taken to consist of a cosinusoidal signal of angular frequency ω, upon which is superimposed narrow-band Gaussian noise centered at ω. By the well-known Rice-Bennett representation, e_i can be expressed in the desired polar form by:[7,8]

$$e_i = \rho \cos \varphi_1 \quad , \qquad (1)$$

where ρ is the amplitude of signal plus noise:

$$\rho = \left[\left\{A + x(t)\right\}^2 + y^2(t)\right]^{1/2} \quad , \qquad (2)$$

A = the amplitude of the received signal,

$x(t)$, $y(t)$ = the slowly time-varying in-phase and quadrature amplitudes, respectively, of the noise centered at ω, which are independent Gaussian random variables with mean value zero and standard deviation σ, and

σ^2 = the total noise power (for unit resistance);

$$\varphi_1 = \omega t + \varphi_i \quad , \qquad (3)$$

φ_i being the phase of signal plus noise:

$$\varphi_i = \tan^{-1} \left\{\frac{y(t)}{A + x(t)}\right\} \quad . \qquad (4)$$

The reference voltage e_r within the diode circuits is a sine wave, assumed to have the same angular frequency as the signal portion of e_i and an arbitrary phase:

$$e_r = B \sin \varphi_2 \quad , \qquad (5)$$

where B = the amplitude of the reference voltage,

$$\varphi_2 = \omega t - \varphi_0 \quad , \tag{6}$$

φ_0 being the arbitrary phase angle. Thus, when $\varphi_0 = 0$, e_r is in quadrature with the signal portion of e_1, a condition typical of many phase detector applications.

The filter circuits associated with the diodes (See Figure 1) are made with sufficient bandwidth to pass the narrow-band noise components, and at the same time reproduce the crests of the half-wave rectified voltage. Thus, the peak-detected contributions of both halves of the balanced circuit are seen from the vector diagram of the circuit voltages in Figure 2, where their positions take into account the quadrature relation.[6] To one of the diode

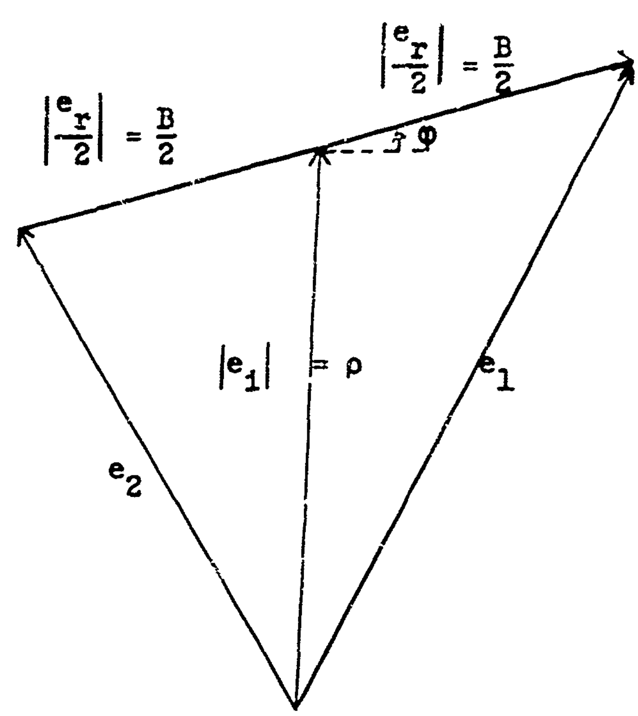

FIGURE 2 - VECTOR DIAGRAM OF CIRCUIT VOLTAGES

circuits is applied the vector sum of e_1 and half of the reference voltage $e_r/2$ during its conduction interval, while the other receives the vector difference. Upon peak detection, each half produces a rectified filtered output voltage equal to the magnitude of the total applied voltage vector. Noting the magnitudes of the individual vectors on Figure 2 (obtained from Equations (1) and (5)), and defining the phase difference φ between e_1 and e_r in addition to that of quadrature:

$$\varphi = \varphi_1 - \varphi_2 \quad , \qquad (7)$$

then the output voltages e_1 and e_2 of the two halves are obtained from the triangle law:

$$e_1 = \left[\rho^2 + \frac{B^2}{4} + B\rho \sin \varphi\right]^{1/2} \quad , \qquad (8)$$

$$e_2 = \left[\rho^2 + \frac{B^2}{4} - B\rho \sin \varphi\right]^{1/2} \quad , \qquad (9)$$

where the positive square root is always understood, and from Equations (3), (6), and (7):

$$\varphi = \varphi_1 + \varphi_o \quad . \qquad (10)$$

Since the individual peak voltages appear across the output terminals in opposite directions, the phase detector output voltage e_o consists of their difference:

$$e_o = e_1 - e_2 \quad . \qquad (11)$$

Equations (8) through (11) give the analytical representation of phase detector behavior to be used in subsequent calculations.

3. STATISTICAL PROPERTIES OF THE OUTPUT

The expressions derived for the phase detector show that the output voltage depends on the statistical nature of the noise superimposed upon the received signal. As a result, the performance of the phase detector can only be described in terms of probability distributions and their associated moments. Two of the most useful factors related to these are the "mean" and "mean square" values of the output, which are treated in this analysis in that order.

Noting from Equations (8) through (11) that the output voltage e_o is expressed in terms of the amplitude ρ and phase φ_1 of the signal plus noise, it is necessary to know the joint probability distribution of these random variables in order to formulate the desired output properties. Such a density function is obtained from the corresponding description of the in-phase and quadrature amplitude variables of the corrupting narrow-band noise, x and y, respectively. It was assumed that they were independent Gaussian random variables with mean value zero and standard deviation σ, where σ^2 is the sum of the equal individual noise powers $\sigma^2/2$ of the two components, called the "total noise power." Thus, the joint distribution $W_2(x,y)$ takes the simplified form:[8]

$$W_2(x,y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x^2+y^2)}{2\sigma^2}\right] . \qquad (12)$$

By means of Equations (2) and (4) treated as transformation relations, the desired joint distribution $W_2(\rho,\varphi_1)$ of amplitude and phase of the signal plus noise is obtained[8]:

$$W_2(\rho,\varphi_1) = \frac{\rho}{2\pi\sigma^2} \exp\left[-\frac{(\rho^2+A^2-2A\rho\cos\varphi_1)}{2\sigma^2}\right] . \qquad (13)$$

The application of Equation (13) and the expression for e_o in terms of ρ and φ_1 to the well-known statistical averaging formulas yields the mean value $\langle e_o \rangle$ and mean square value $\langle e_o^2 \rangle$ of the phase detector output:

$$\langle e_o \rangle = \int_0^\infty \int_0^{2\pi} e_o(\rho,\varphi_1)\ W_2(\rho,\varphi_1) d\varphi_1 d\rho \quad , \qquad (14)$$

$$\langle e_o^2 \rangle = \int_0^\infty \int_0^{2\pi} \left[e_o(\rho,\varphi_1)\right]^2 W_2(\rho,\varphi_1) d\varphi_1 d\rho \quad , \qquad (15)$$

where the integrations are carried out over the appropriate ranges of the amplitude and phase random variables.

While the desired calculations can be carried out directly with Equations (14) and (15), it is more convenient to convert these expressions back into the original in-phase and quadrature noise variables. As a result, the following equivalent pair of formulas is obtained:

$$\langle e_o \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e_o(x,y) \quad W_2(x,y) dx dy \quad , \qquad (16)$$

$$\langle e_o^2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[e_o(x,y)\right]^2 W_2(x,y) dx dy \quad , \qquad (17)$$

where x and y have infinite ranges, and the phase detector output voltage is expressed in terms of them with the aid of Equations (2), (4), and (8) through (11):

$$e_o(x,y) = \left[\left(x + A + \frac{B}{2}\sin\varphi_o\right)^2 + \left(y + \frac{B}{2}\cos\varphi_o\right)^2\right]^{1/2}$$

$$-\left[\left(x + A - \frac{B}{2}\sin\varphi_o\right)^2 + \left(y - \frac{B}{2}\cos\varphi_o\right)^2\right]^{1/2} . \qquad (18)$$

Together with Equation (12), Equations (16) through (18) form the basis for the calculation of $\langle e_o\rangle$ and $\langle e_o^2\rangle$ in this study.

3.1 THE OUTPUT MEAN VALUE

To aid in the application of Equation (16), it was found advantageous to express the phase detector output voltage $e_o(x,y)$ in integral form. The desired expression results from a table of Laplace transforms:[9]

$$e_o(x,y) = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{dt}{t^{3/2}}\sinh\left\{Bt\left[(x+A)\sin\varphi_o + y\cos\varphi_o\right]\right\} \cdot$$

$$\cdot \exp\left\{-t\left[(x+A)^2 + y^2 + \frac{B^2}{4}\right]\right\} , \qquad (19)$$

which is valid over the infinite ranges of integration for x and y. Based on the above representation, the detailed evaluation of $\langle e_o\rangle$ is carried out in Appendix A, which yields the following expression for the output mean value:

$$\langle e_o\rangle = \sigma\sqrt{\frac{\pi}{2}}\left\{\exp\left\{-\frac{(\alpha+\beta)}{2}\right\}\left[(1+\alpha+\beta)I_o\left(\frac{\alpha+\beta}{2}\right) + (\alpha+\beta)I_1\left(\frac{\alpha+\beta}{2}\right)\right]\right.$$

$$\left. - \exp\left\{-\frac{(\alpha-\beta)}{2}\right\}\left[(1+\alpha-\beta)I_o\left(\frac{\alpha-\beta}{2}\right) + (\alpha-\beta)I_1\left(\frac{\alpha-\beta}{2}\right)\right]\right\}, \qquad (20)$$

where $I_o(x)$ and $I_1(x)$ are modified Bessel functions of the first kind, and α,β are the normalized parameters:

$$\alpha = \frac{1}{2\sigma^2}\left(A^2 + \frac{B^2}{4}\right); \qquad \beta = \frac{AB}{2\sigma^2}\sin\varphi_o . \qquad (21)$$

Examination of Equation (20) with the aid of Equation (21) shows that $\langle e_o \rangle$ has the expected behavior for extreme values of the above parameters, based on the model used. Thus, when either the signal or reference amplitude vanishes, or these two sinusoidal components are in quadrature ($\varphi_o = 0$), the output mean value is zero. On the other hand, in the absence of noise ($\sigma = 0$), the phase detector yields a mean value:

$$\langle e_o \rangle_{\sigma=0} = \left| A^2 + AB \sin \varphi_o + \frac{B^2}{4} \right|^{1/2} - \left| A^2 - AB \sin \varphi_o + \frac{B^2}{4} \right|^{1/2} , \qquad (22)$$

which for the case when signal and reference are in phase ($\varphi_o = \pi/2$), reduces to just the reference amplitude B.

3.2 THE OUTPUT MEAN SQUARE VALUE

As in the case of the mean value, the square of the phase detector output voltage was expressed in integral form to facilitate the evaluation of $\langle e_o{}^2 \rangle$ from Equation (17). This is obtained from the following equivalent expression for $e_o{}^2(x,y)$, derived from Equations (8) through (11):

$$e_o^2(x,y) = 2B\rho \sin (\varphi_i + \varphi_o) \left\{ \frac{e_1 - e_2}{e_1 + e_2} \right\} . \qquad (23)$$

From the table of Laplace transforms, the resulting integral is:[9]

$$e_o^2(x,y) = 2B \left[(x + A) \sin \varphi_o + y \cos \varphi_o \right] \cdot$$

$$\cdot \int_0^{\infty} \frac{dt}{t} I_1 \left\{ Bt \left[(x+A) \sin \varphi_o + y \cos \varphi_o \right] \right\} \cdot$$

$$\cdot \exp \left\{ - t \left[(x+A)^2 + y^2 + \frac{B^2}{4} \right] \right\} , \qquad (24)$$

which is again valid over the infinite ranges of integration for x and y. As derived in Appendix B, the resulting expression for

$\langle e_o^2 \rangle$ is a complicated power series in β^2 [see Equation (21)], whose coefficients contain integrals involving confluent hypergeometric functions, for which no exact closed form evaluation can be found. Consideration of this general formula will not be undertaken here, since of prime interest are calculations of phase detector performance which are treated in the following sections.

4. CALCULATION OF OUTPUT NOISE-TO-NOISE RATIO

While the mean and mean square values previously determined give a partial statistical description of the phase detector output, these factors in themselves do not provide a criterion for the performance of such a device in the presence of noise. In general, a detection system has two states of interest, one when a signal is not present and one when a signal is present. From this it is natural to seek some parameter with which to compare the effect of the ever-present noise on both states. Since in each case there are output fluctuations about some mean value (which may be zero), a suitable choice would be the "variance of the output," denoted by σ_o^2. In terms of the moments previously discussed, this quantity is defined by:

$$\sigma_o^2 = \langle e_o^2 \rangle - \left[\langle e_o \rangle\right]^2 \quad . \tag{25}$$

For the case where the signal is present, the expressions derived in Appendices A and B are substituted directly into Equation (25), and the resulting variance is denoted by σ_s^2. When the signal is not present, an evaluation of these formulas at A = 0 must first be made, with the variance thus obtained being denoted by σ_n^2, since only noise is received here. From these two definitions, the criterion of performance for the phase detector is taken to be the noise-to-signal plus noise output variance ratio, to the one-half power. This quantity, σ_n/σ_s, is called more briefly the "noise-to-noise ratio."[10,11] Since the output mean value was seen to be zero for A = 0, this ratio can be written in terms of the moments from Equation (25) in the following form:

$$\frac{\sigma_n}{\sigma_s} = \left[\frac{\langle e_o^2 \rangle_{A=0}}{\langle e_o^2 \rangle - \left[\langle e_o \rangle\right]^2}\right]^{1/2} \quad , \tag{26}$$

where the evaluation at zero signal amplitude is indicated above. Equation (26) is now applied to an operating case of interest for the phase detector, namely when the signal and reference are in quadrature, i.e. $\varphi_o = 0$. This is additionally desirable because the output signal-to-noise ratio is not meaningful where no measure of signal exists.

In this case of null operation in the absence of noise, the output mean value $\langle e_o \rangle = 0$, even with signal present. From Equation (21), $\beta = 0$, so that from Equation (B41) of Appendix B the expression for $\langle e_o{}^2 \rangle$ reduces to the single integral:

$$\langle e_o{}^2 \rangle = \frac{B^2}{2} \int_0^1 dx \, \epsilon^{-\alpha x} \, {}_1F_1 \left(\frac{1}{2}; \; 2; \; -\frac{\gamma x^2}{1-x} \right) dx \quad , \tag{27}$$

where ${}_1F_1$ stands for the confluent hypergeometric function, α is defined in Equation (21), and γ is the dimensionless ratio:

$$\gamma = \frac{B^2}{8\sigma^2} \quad . \tag{28}$$

Setting A = 0 in Equation (27) and substituting it and the original form into Equation (26) with $\langle e_o \rangle = 0$ yields an exact expression for the output noise-to-noise ratio.

To relate the calculations made with Equation (26) for this quadrature case to laboratory measurements on actual devices, the above parameters are expressed instead in terms of the total received power P_i, the reference power P_r, the input signal-to-noise power ratio "a," and the reference-to-total received power ratio "b". These are given by the relations (for unit resistance):

$$P_i = \frac{A^2}{2} + \sigma^2 \quad ; \quad P_r = \frac{B^2}{2} \quad , \tag{29}$$

$$a = \frac{A^2}{2\sigma^2} \quad ; \quad b = \frac{P_r}{P_i} \quad . \tag{30}$$

As a result, the parameters α and γ appearing in Equation (27) take the form:

$$\alpha = a + \frac{b}{4}(1 + a) \quad , \tag{31}$$

$$\gamma = \frac{b(1 + a)}{4} \quad . \tag{32}$$

The evaluation of the output noise-to-noise ratio using the exact integral formula of Equation (27) led to involved time-consuming numerical procedures, so instead various approximation methods were considered, the details of which appear in Appendix C. A simplified functional form which fitted the confluent hypergeometric function over a range of the parameter γ led to an approximate result in terms of the modified exponential integral.[12] But since the latter's numerical evaluation became quite involved, further simplifications were sought. As a result, it was possible to obtain reasonable closed-form expressions for two asymptotic cases. These are:

<u>Small Reference Amplitude Limit ($b \to 0$):</u>

$$\frac{\sigma_n}{\sigma_s} \longrightarrow \left(\frac{a}{1 - \epsilon^{-a}}\right)^{1/2} \quad , \tag{33}$$

<u>Large Reference Amplitude:</u>

$$\frac{\sigma_n}{\sigma_s} \cong \left[\frac{1 - \frac{2}{b}}{1 - \frac{1 + 2a}{2\gamma} + \frac{15 - 9(3+a) + 2(6+6a+a^2)}{2\gamma^2}}\right]^{1/2} \quad , \tag{34}$$

where γ is expressed in terms of "a" and "b" by Equation (32).

The preceding formulas are presented graphically in Figure 3 in the form of curves of σ_n/σ_s for the quadrature case ($\varphi_o = 0$) as a function of "a" for the lower limit and large levels of "b". Intermediate reference values would result in plots falling in between the above extremes, and these would be appropriate for subsequent comparison with experimental data. But the curves calculated here do reveal the principal effect of the signal on the phase detector in decreasing the output noise over that in its absence, as seen in Figure 3 by the σ_n/σ_s values staying above unity. This is opposite to the behavior of amplitude detectors,[13] so judging from the output noise-to-noise ratio it can be concluded that the phase detector does not operate in the manner normally ascribed to detection processes.

The spread between the curves of Figure 3 illustrates the relative merits of the two extreme modes of operation. For large reference levels, the σ_n/σ_s ratio becomes insensitive to input signal-to-noise ratio changes, but never rises much above unity. On the other hand, the use of a low-reference level results in significant output noise decreases with the signal present, which could serve as the basis for some type of detection scheme, but here the σ_n/σ_s ratio is affected more by varying input conditions.

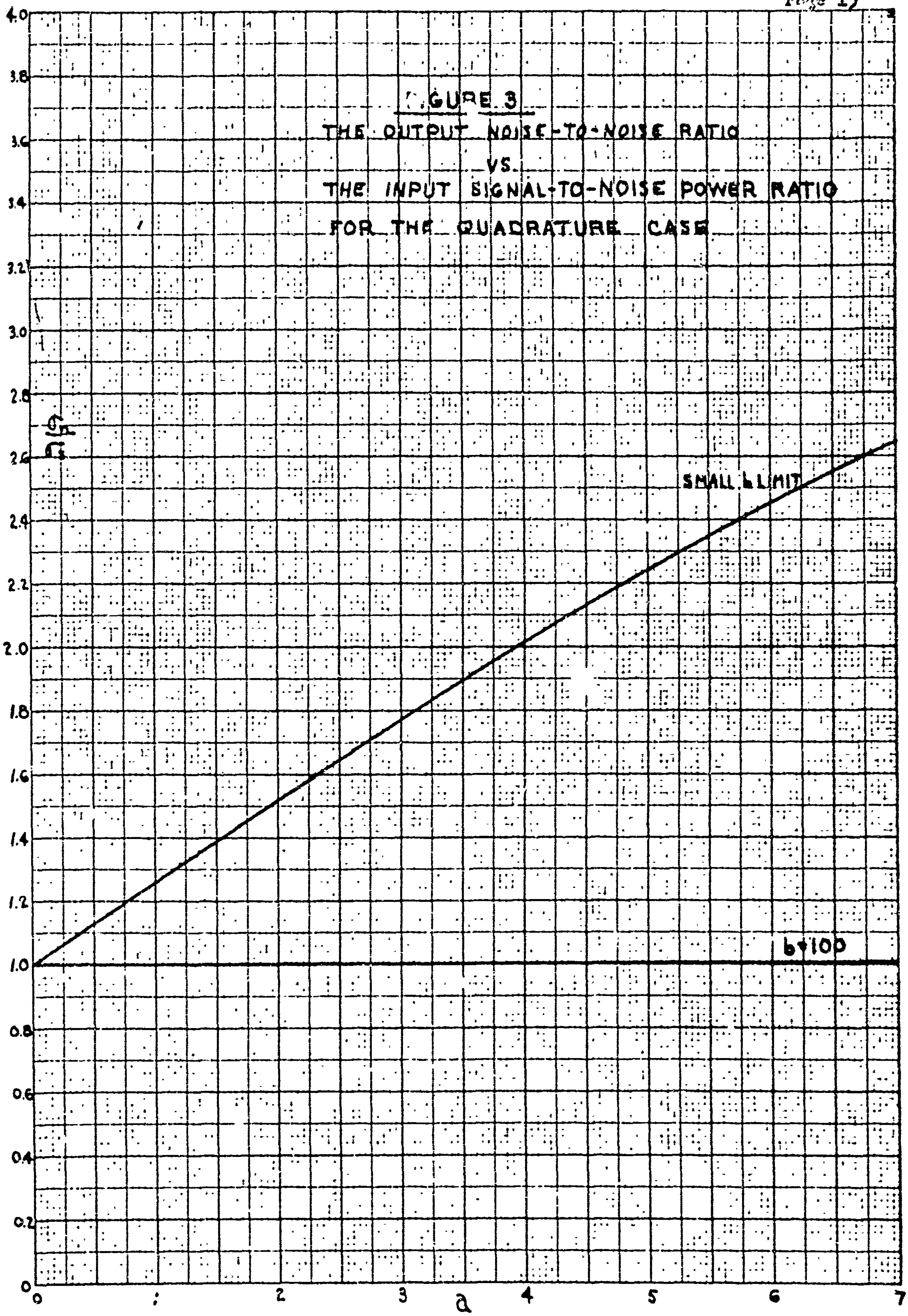
FIGURE 3
THE OUTPUT NOISE-TO-NOISE RATIO
VS.
THE INPUT SIGNAL-TO-NOISE POWER RATIO
FOR THE QUADRATURE CASE
SMALL b LIMIT
b→100
4.0
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
2
3
4
5
6
7
a

5. CALCULATION OF OUTPUT SIGNAL-TO-NOISE RATIO

For characterizing the response of a physical device to a combination of signal and noise waveforms, the most commonly used criterion of performance is the "output signal-to-noise power ratio," denoted by "a_o". For the type of input to the phase detector and with the output specified in terms of voltage, the square root of the above quantity is thus defined as:[14, 15]

$$\sqrt{a_o} = \frac{\begin{bmatrix}\text{average output for signal}\\ \text{plus noise input}\end{bmatrix} - \begin{bmatrix}\text{average output for}\\ \text{noise input only}\end{bmatrix}}{\begin{bmatrix}\text{standard deviation of the output about}\\ \text{the mean for signal plus noise input}\end{bmatrix}} . \tag{35}$$

Since from Equation (20) it is seen that for no input signal present, i.e. $A = 0$, $\langle e_o \rangle = 0$, then in terms of the moments previously discussed, this ratio is given by:

$$\sqrt{a_o} = \frac{\langle e_o \rangle}{\left\langle \langle e_o{}^2 \rangle - \left[\langle e_o \rangle\right]^2 \right\rangle^{1/2}} . \tag{36}$$

As indicated previously, the exact evaluation of $\langle e_o{}^2 \rangle$ does not have a form suitable for application to Equation (36), but more tractable approximate expressions were obtained for limiting cases of the reference parameter "b." Specifically, the derivations of Appendix D yielded for the voltage output signal-to-noise ratio in the general case of an arbitrary phase angle φ_o separating the signal-reference quadrature condition:

<u>Large Reference Amplitude Limit ($b \to \infty$)</u>:

$$\sqrt{a_o} \doteq \sqrt{2a} \sin \varphi_o , \tag{37}$$

Small Reference Amplitude Limit ($b \longrightarrow 0$):

$$\sqrt{a_o} \doteq \frac{\frac{\sqrt{\pi a}}{2}\,\epsilon^{-a/2}\left[I_o\left(\frac{a}{2}\right) + I_1\left(\frac{a}{2}\right)\right]\sin\varphi_o}{\left[\left\{1 - \frac{\pi}{4}\,a\epsilon^{-a}\left[I_o\left(\frac{a}{2}\right) + I_1\left(\frac{a}{2}\right)\right]^2\right\}\sin^2\varphi_o + \frac{1}{2a}\left(1 - \epsilon^{-a}\right)\cos 2\varphi_o\right]^{1/2}} , \quad (38)$$

where "a" is the input signal-to-noise power ratio defined in Equation (30), and I_o and I_1 are modified Bessel functions of the first kind.

Graphs of the calculations with Equations (37) and (38) are given in Figure 4 in the form of curves of $\sqrt{a_o}$ as a function of the sine of the phase angle φ_o for two values of "a" and the limiting reference levels. In spite of the radically different functional forms exhibited by the preceding two expressions, their resulting plots are seen to approach each other very closely over the initial angle range for each value of input signal-to-noise power ratio. Thus, such a relatively narrow spread shown in Figure 4 would appear to include the curves at intermediate reference levels, which should be the subject of computational verification in the future.

For the first 6^o off the signal-reference quadrature condition, the voltage output signal-to-noise ratio varies linearly with $\sin\varphi_o$, with the constant of proportionality being $\sqrt{2a}$ or slightly below over a wide range of reference levels. With larger phase offsets, the limiting curves will diverge more markedly due to the subsequent departure from linearity of Equation (38). But with the phase detector typically operating in quadrature, a small phase shift occurring between signal and reference will produce an increasing output signal-to-noise ratio which provides the means for correcting such waveform displacements.

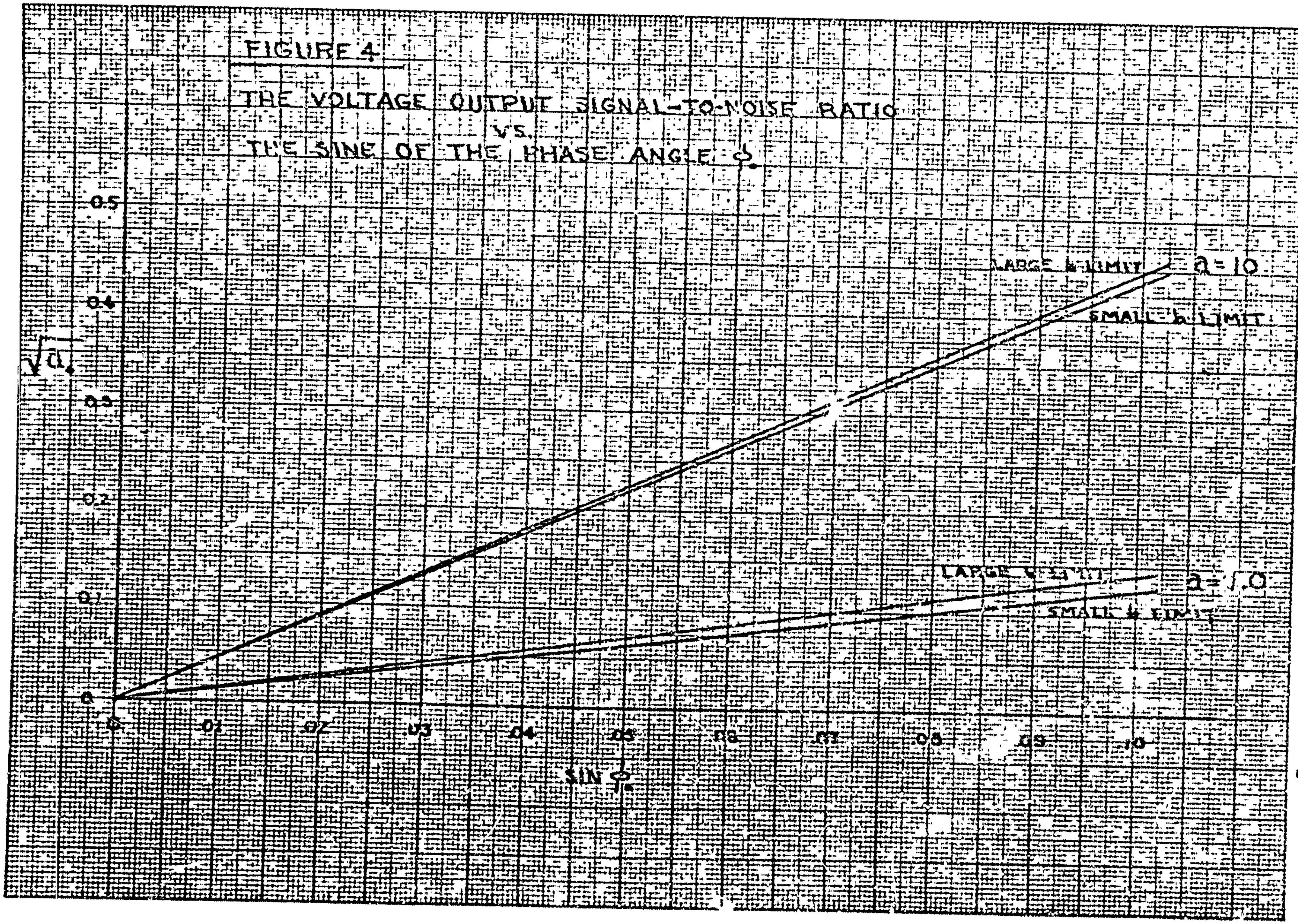

FIGURE 4

THE VOLTAGE OUTPUT SIGNAL-TO-NOISE RATIO VS. THE SINE OF THE PHASE ANGLE ϕ

6. CONCLUSION

The analysis of phase detector performance for a signal plus noise input was carried out for a balanced circuit with two peak-detecting diodes. The model of narrow-band Gaussian noise used was considered appropriate for most of the applications encountered. The criteria of "noise-to-noise ratio" and "signal-to-noise ratio" at the output were selected to evaluate the utility of this detection process. Although computational difficulties precluded obtaining a complete range of analytical results, the approximate formulas which were found served to illustrate the salient features of phase detector behavior.

For the signal and reference in quadrature, a relatively small decrease in output noise results when the signal is present for a wide range of input signal-to-noise power ratios at large reference levels, but a larger spread of output noise-to-noise ratios is exhibited at low reference levels under varying input conditions. When the signal and reference depart somewhat from the quadrature condition, the voltage output signal-to-noise ratio is roughly proportional to both the sine of this phase displacement and the corresponding input ratio (for values between unity and ten) over a wide range of reference levels.

It is the aim of this somewhat restricted analysis to encourage further efforts in investigating more general phase detector performance both from a theoretical and experimental viewpoint. One such study has considered the same type of circuit with noise present in both the signal and reference channels,[16] but the results are too complicated to yield practical information and in addition stress the square-law behavior of the diodes which in practice is a less realistic model. Since equipment applications are tending increasingly toward the use of more sophisticated phase detectors to improve performance from the noise and other standpoints, it becomes essential to extend the presently known analytical techniques in order to understand the behavior of such devices.

7. REFERENCES

1. L. I. Farren, "Phase Detectors, Some Theoretical and Practical Aspects," Wireless Eng., Vol.23, pp. 330-340, December, 1946.

2. R. H. Dishington, "Diode Phase-Discriminators," Proc. IRE, Vol. 37, No. 12, pp. 1401-1404, December, 1949.

3. T. S. George, "Analysis of Synchronizing Systems for Dot-Interlaced Color Television," Proc. IRE, Vol. 39, No. 2, pp. 124-131, February, 1951.

4. S. R. Margolis, "The Response of a Phase-Locked Loop to a Sinusoid Plus Noise," Jet Propulsion Laboratory, External Publication No. 348, August 1, 1956.

5. A. van der Ziel, "Noise," Prentice-Hall, Inc., 1954, pp.343-347.

6. W. J. Gruen, "Theory of AFC Synchronization," Proc. IRE, Vol.41, No. 2, pp. 1043-1048, August, 1953.

7. S. O. Rice, "Mathematical Analysis of Random Noise," Bell System Technical Journal, Vol. 23, pp. 282-332, July, 1944, and Vol. 24, pp. 46-156, January, 1945. Also: N. Wax, "Selected Papers on Noise and Stochastic Processes," Dover Publications, Inc., 1954, pp. 133-294.

8. W. R. Bennett, "Methods of Solving Noise Problems," Proc. IRE, Vol. 44, No. 5, pp. 609-638, May, 1956.

9. Bateman Manuscript Project, "Tables of Integral Transforms," Vol. 1, McGraw-Hill, Inc., 1954, pp. 125-301.

10. R. H. Dishington, "Threshold Probability," presented at the IRE-URSI Joint Fall Meeting, October 19-21, 1959, San Diego, California.

11. R. H. Dishington, "Analysis of Correlation Methods," The Ramo-Wooldridge Corporation, Report RW 1661.125, 30 April 1958.

12. Bateman Manuscript Project, "Higher Transcendental Functions," Vol. 2, McGraw-Hill, Inc., 1953, pp. 143-145.

13. R. A. Smith, "The Relative Advantages of Coherent and Incoherent Detectors: A Study of Their Output Noise Spectra under Various Conditions," IEE Monograph No. 6 - Radio Section, August 15, 1951.

14. J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," Radiation Laboratory Series No. 24, McGraw-Hill, Inc., 1950.

15. J. H. Van Vleck and D. Middleton, "A Theoretical Comparison of the Visual, Aural, and Meter Reception of Pulsed Signals in the Presence of Noise," J. Appl. Phys., Vol. 17, p. 940, 1946.

16. D. C. Miller, "Noise Analysis of Phase Detectors," Naval Avionics Facility, Report TP-103, December, 1959.

17. W. Gröbner and N. Hofreiter, "Integraltafel-Zweiter Teil-Bestimmte Integrale," Springer-Verlag (1950).

18. A. D. Wheelon and J. T. Robacker, "A Table of Integrals Involving Bessel Functions," The Ramo-Wooldridge Corporation, 1954.

19. H. B. Dwight, "Tables of Integrals and Other Mathematical Data," revised ed., The Macmillan Co., 1947.

20. N. W. McLachlan, "Bessel Functions for Engineers," second ed., Oxford at the University Press, 1955.

21. J. T. Robacker, "Reductions of the Hypergeometric Function," The Ramo-Wooldridge Corporation, 1956.

22. Bateman Manuscript Project, "Higher Transcendental Functions," Vol. 1, McGraw-Hill, Inc., 1953.

23. G. N. Watson, "A Treatise on the Theory of Bessel Functions," second ed., Cambridge at the University Press, 1958.

24. E. P. Adams, "Smithsonian Mathematical Formulae," second reprint, Publication 2672, Smithsonian Miscellaneous Collections, Vol. 74, No. 1, Smithsonian Institution, 1947.

25. R. S. Burington and R. C. May, Jr., "Handbook of Probability and Statistics with Tables," Handbook Publishers, Inc., 1953.

APPENDIX A

EVALUATION OF THE OUTPUT MEAN-VALUE

The integral representation for the phase detector output voltage $e_o(x,y)$, Equation (19), and Equation (12) for the joint distribution $W_2(x,y)$ are substituted into Equation (16) and the orders of integration reversed. By the addition formula for the hyperbolic sine:

$$\sinh(u + v) = \sinh u \cosh v + \cosh u \sinh v \quad , \qquad \text{(A1)}$$

the expression for $\langle e_o \rangle$ contains only products of integrals in x and y:

$$\langle e_o \rangle = \frac{1}{2\sigma^2 \pi^{3/2}} \int_0^\infty \frac{dt}{t^{3/2}} \exp\left(- \frac{B^2}{4} t\right) \cdot \qquad \text{(A2)}$$

$$\cdot \left\{ \int_{-\infty}^{\infty} \exp\left\{- \left[t(x+A)^2 + \frac{x^2}{2\sigma^2}\right]\right\} \sinh\left\{Bt(x+A) \sin \varphi_o\right\} dx \cdot \right.$$

$$\cdot \int_{-\infty}^{\infty} \exp\left\{- \left(t + \frac{1}{2\sigma^2}\right) y^2\right\} \cosh\left(Bty \cos \varphi_o\right) dy$$

$$+ \int_{-\infty}^{\infty} \exp\left\{- \left[t(x+A)^2 + \frac{x^2}{2\sigma^2}\right]\right\} \cosh\left\{Bt(x+A) \sin \varphi_o\right\} dx \cdot$$

$$\left. \cdot \int_{-\infty}^{\infty} \exp\left\{- \left(t + \frac{1}{2\sigma^2}\right) y^2\right\} \sinh\left(Bty \cos \varphi_o\right) dy \right\} \quad .$$

The second double integral in the bracket vanishes because of the integration with respect to y of an odd function over symmetrical limits. If, in the remaining part, the variable x is translated by an amount - A (the infinite limits remaining unchanged) and the square is completed in the resulting exponent:

$$\exp\left\{-\left[tx^2+\frac{(x-A)^2}{2\sigma^2}\right]\right\} = \exp\left\{-\frac{A^2}{2\sigma^2}\left[1-\frac{1}{2\sigma^2}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\right]\right\}\cdot \tag{A3}$$

$$\cdot\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)\left[x-\frac{A}{2\sigma^2}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\right]^2\right\},$$

then there results for $\langle e_o\rangle$:

$$\langle e_o\rangle = \frac{1}{2\sigma^2\pi^{3/2}}\int_o^\infty \frac{dt}{t^{3/2}}\exp\left(-\frac{B^2}{4}t\right)\exp\left\{-\frac{A^2}{2\sigma^2}\left[1-\frac{1}{2\sigma^2}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\right]\right\}X(t)Y(t) \;, \tag{A4}$$

where

$$X(t) = \int_{-\infty}^{\infty}\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)\left[x-\frac{A}{2\sigma^2}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\right]^2\right\}\sinh(Btx\sin\varphi_o)\,dx \quad , \tag{A5}$$

$$Y(t) = \int_{-\infty}^{\infty}\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)y^2\right\}\cosh(Bty\cos\varphi_o)\,dy \quad . \tag{A6}$$

The above integrals are evaluated by Reference 17, 353, p. 164, nos. (3) and (4), yielding:

$$X(t) = \pi^{1/2}\left(t+\frac{1}{2\sigma^2}\right)^{-1/2}\exp\left\{\frac{B^2t^2\sin^2\varphi_o}{4}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\right\}\cdot$$

$$\cdot\sinh\left\{\frac{ABt}{2\sigma^2}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\sin\varphi_o\right\} \quad , \tag{A7}$$

$$Y(t) = \pi^{1/2}\left(t+\frac{1}{2\sigma^2}\right)^{-1/2}\exp\left\{\frac{B^2t^2\cos^2\varphi_o}{4}\left(t+\frac{1}{2\sigma^2}\right)^{-1}\right\} \quad . \tag{A8}$$

Multiplying Equations (A7) and (A8) together, combining and simplifying the exponents, the following expression is obtained:

$$X(t)Y(t) = \kappa\left|t + \frac{1}{2\sigma^2}\right|^{-1} \exp\left\{\frac{B^2t^2}{4}\left|t + \frac{1}{2\sigma^2}\right|^{-1}\right\} \cdot \sinh\left\{\frac{ABt}{2\sigma^2}\left|t + \frac{1}{2\sigma^2}\right|^{-1} \sin\varphi_o\right\} . \tag{A9}$$

Noting from Equations (A4) and (A9) that in the exponents

$$1 - \frac{1}{2\sigma^2}\left|t + \frac{1}{2\sigma^2}\right|^{-1} = t\left|t + \frac{1}{2\sigma^2}\right|^{-1} ,$$

the substitution of $X(t)Y(t)$ into $\langle e_o \rangle$ and canceling terms yields the more compact form:

$$\langle e_o \rangle = \frac{1}{2\sigma^2\pi^{1/2}}\int_0^\infty \frac{dt}{t^{3/2}}\left|t + \frac{1}{2\sigma^2}\right|^{-1} \exp\left\{-\alpha t\left|t + \frac{1}{2\sigma^2}\right|^{-1}\right\} \sinh\left\{\beta t\left|t + \frac{1}{2\sigma^2}\right|^{-1}\right\}, \tag{A10}$$

where:

$$\alpha = \frac{1}{2\sigma^2}\left|A^2 + \frac{B^2}{4}\right| \ ; \ \beta = \frac{AB}{2\sigma^2}\sin\varphi_o \ . \tag{A11}$$

The evaluation of Equation (A10) proceeds from Equation (A9) by making the substitution:

$$\left|t + \frac{1}{2\sigma^2}\right|^{-1} = 2\sigma^2\left[1 - t\left|t + \frac{1}{2\sigma^2}\right|^{-1}\right] , \tag{A12}$$

which results in $\langle e_o \rangle$ as the difference of two integrals:

$$\left\langle e_o \right\rangle = \frac{1}{\pi^{1/2}}\left[\int_0^\infty \frac{dt}{t^{3/2}} \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \sinh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\}\right. \tag{A13}$$

$$\left. - \int_0^\infty \frac{dt}{t^{1/2}}\left|t+\frac{1}{2\sigma^2}\right|^{-1} \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \sinh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\}\right] .$$

In the first integral an integration by parts is performed, letting:

$$u = \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \sinh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} ; \quad dv = \frac{dt}{t^{3/2}} .$$

Taking the indicated differential and integral yields:

$$du = \frac{dt}{2\sigma^2} \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\}\left[\beta \cosh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} - \alpha \sinh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\}\right] \cdot$$

$$\cdot \left|t+\frac{1}{2\sigma^2}\right|^{-2} ;$$

$$v = -\frac{2}{t^{1/2}} .$$

Substituting into the parts formula, the evaluated part vanishes at $t = \infty$ due to the $t^{-1/2}$ factor and at $t = 0$ due to the hyperbolic sine factor being of order t. Putting the remaining integral part into Equation (A13) and arranging, the result is $\left\langle e_o \right\rangle$ expressed in three terms:

$$\left\langle e_o \right\rangle = \frac{1}{\pi^{1/2}}\left[\frac{\beta}{\sigma^2}\int_0^\infty \frac{dt}{t^{1/2}}\left|t+\frac{1}{2\sigma^2}\right|^{-2} \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \cosh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\}\right.$$

$$- \frac{\alpha}{\sigma^2}\int_0^\infty \frac{dt}{t^{1/2}}\left|t+\frac{1}{2\sigma^2}\right|^{-2} \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \sinh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \tag{A14}$$

$$\left. - \int_0^\infty \frac{dt}{t^{1/2}}\left|t+\frac{1}{2\sigma^2}\right|^{-1} \exp\left\{-\alpha t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\} \sinh\left\{\beta t\left|t+\frac{1}{2\sigma^2}\right|^{-1}\right\}\right] .$$

Next, the following change of variable is made:

$$u = t\left(t + \frac{1}{2\sigma^2}\right)^{-1} \quad ; \quad 2\sigma^2 du = dt\left(t + \frac{1}{2\sigma^2}\right)^{-2} \quad ;$$

$$t = \frac{u}{2\sigma^2(1-u)} \quad ; \quad \frac{1}{t^{1/2}} = \frac{\sqrt{2}\sigma(1-u)^{1/2}}{u^{1/2}} \quad ; \quad \left(t + \frac{1}{2\sigma^2}\right) = \frac{1}{2\sigma^2(1-u)} \quad ;$$

at $t = 0$, $u = 0$; at $t = \infty$, $u = 1$.

Substituting into Equation (A14) results in simpler integral forms:

$$\langle e_o \rangle = \sigma\sqrt{\frac{2}{\pi}}\left[2\beta\int_0^1 \frac{(1-u)^{1/2}}{u^{1/2}}\,\epsilon^{-\alpha u}\cosh\beta u\,du\right.$$

$$-\,2\alpha\int_0^1 \frac{(1-u)^{1/2}}{u^{1/2}}\,\epsilon^{-\alpha u}\sinh\beta u\,du$$

$$\left.-\int_0^1 \frac{1}{u^{1/2}(1-u)^{1/2}}\,\epsilon^{-\alpha u}\sinh\beta u\,du\right] . \qquad \text{(A15)}$$

Two basic integrals are to be evaluated in Equation (A15), one being defined by R(p):

$$R(p) = \int_0^1 \frac{1}{u^{1/2}(1-u)^{1/2}}\,\epsilon^{-pu}\,du \quad . \qquad \text{(A16)}$$

This is found directly in Reference 9, table 4.3, p. 138, no.(14), for $t = u$, $b = 1/2$, $\nu = 0$. Noting that $\Gamma(1/2) = \sqrt{\pi}$, there results the evaluation:

$$R(p) = \pi\,\epsilon^{-p/2}\,I_0\left(\frac{p}{2}\right) \quad , \qquad \text{(A17)}$$

where $I_o(p/2)$ is the modified Bessel function of the first kind and order zero. Next, the second basic integral to be evaluated is $S(p)$, defined by:

$$S(p) = \int_o^1 \frac{(1-u)^{1/2}}{u^{1/2}} \epsilon^{-pu} du \quad . \tag{A18}$$

First, differentiating under the integral sign with respect to p yields:

$$\frac{dS(p)}{dp} = -\int_o^1 u^{1/2}(1-u)^{1/2} \epsilon^{-pu} du \quad . \tag{A19}$$

Applying Reference 9, table 4.3, p. 138, No. (14), for t = u, b = 1/2, ν = 1, noting that $\Gamma(3/2) = \sqrt{\pi}/2$, there results the evaluation:

$$\frac{dS(p)}{dp} = -\frac{\pi}{2p} \epsilon^{-p/2} I_1\left(\frac{p}{2}\right) , \tag{A20}$$

where $I_1(p/2)$ is the modified Bessel function of first kind and order. Equation (A20) can be integrated directly by using Reference 18, p. 57, no. 3.105, for $\nu = -1$, $u = p/2$, noting that $I_{-1}(p/2) = I_1(p/2)$:

$$S(p) = \frac{\pi}{2} \epsilon^{-p/2} \left[I_o\left(\frac{p}{2}\right) + I_1\left(\frac{p}{2}\right)\right] + C \quad . \tag{A21}$$

The arbitrary constant is evaluated at p = 0, yielding:

$$C = S(0) - \frac{\pi}{2} \quad , \tag{A22}$$

where:

$$S(0) = \int_o^1 \frac{(1-u)^{1/2}}{u^{1/2}} du \quad . \tag{A23}$$

From Reference 19, p. 196, no. 855.1, for $x = u$, $m = 1/2$, $n = 3/2$, substituting in for the required gamma functions:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \ , \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \ , \quad \Gamma(2) = 1 \ ,$$

S(0) is found to have the value:

$$S(0) = \frac{\pi}{2} \ . \tag{A24}$$

Substituting Equations (A22) and (A24) into (A21) thus gives the complete evaluation of S(p):

$$S(p) = \frac{\pi}{2} \epsilon^{-p/2} \left[I_o\left(\frac{p}{2}\right) + I_1\left(\frac{p}{2}\right) \right] \ . \tag{A25}$$

By expressing the hyperbolic sine and cosine in terms of exponentials and combining them with the other exponent, Equation (A15) can be written in terms of the basic integrals just evaluated in the following rearranged form:

$$\langle e_o \rangle = \sigma \sqrt{\frac{2}{\pi}} \left[(\alpha + \beta) S(\alpha + \beta) - (\alpha - \beta) \ S(\alpha - \beta) \right.$$
$$\left. + \frac{1}{2} R(\alpha + \beta) - \frac{1}{2} R(\alpha - \beta) \right] \ . \tag{A26}$$

Substituting Equations (A17) and (A25) into Equation (A26) with the proper arguments and collecting terms, the evaluation of the mean value $\langle e_o \rangle$ of the phase detector output is complete:

$$\langle e_o \rangle = \sigma \sqrt{\frac{\pi}{2}} \left\{ \exp \left\{ - \frac{(\alpha+\beta)}{2} \right\} \left[(1+\alpha+\beta) I_o\left(\frac{\alpha+\beta}{2}\right) + (\alpha+\beta) I_1\left(\frac{\alpha+\beta}{2}\right) \right] \right.$$
$$\left. - \exp \left\{ - \frac{(\alpha-\beta)}{2} \right\} \left[(1+\alpha-\beta) I_o\left(\frac{\alpha-\beta}{2}\right) + (\alpha-\beta) I_1\left(\frac{\alpha-\beta}{2}\right) \right] \right\} . \tag{A27}$$

This result appears as Equation (20) of the main text.

APPENDIX B

EVALUATION OF THE OUTPUT MEAN SQUARE VALUE

The integral representation for the square of the phase detector output voltage $e_o^2(x,y)$, Equation (24), and Equation (12) for the joint distribution $W_2(x,y)$ are substituted into Equation (17) and the order of integration reversed. In this case, however, products of integrals in x and y are not immediately obtained because the modified Bessel function does not have the simple addition formula previously seen for the hyperbolic sine in Appendix A. Nevertheless, a separation of integrands can be effected if an integral representation for $I_1(z)$ is applied from Reference 20, p. 202, no. 179, for n = 1 and z = the given argument:

$$I_1 \left\langle Bt\left[(x + A) \sin \varphi_o + y \cos \varphi_o\right]\right\rangle$$
$$= -\frac{1}{\pi}\int_0^{\pi} d\theta \cos \theta \exp \left\langle - Bt \cos \theta\left[(x+A) \sin \varphi_o + y \cos \varphi_o\right]\right\rangle . \tag{B1}$$

Putting Equation (B1) into $\left\langle e_o^2\right\rangle$ and again reversing the order of integration, the following product integral form is obtained:

$$\left\langle e_o^2\right\rangle = -\frac{B}{\pi^2\sigma^2}\int_0^{\infty} \frac{dt}{t} \exp\left(-\frac{B^2}{4} t\right)\int_0^{\pi} d\theta \cos \theta \cdot \tag{B2}$$
$$\cdot\left[\sin \varphi_o \int_{-\infty}^{\infty} (x+A) \exp\left\{-\left[t(x+A)^2 + \frac{x^2}{2\sigma^2}\right] - Bt \cos \theta(x+A)\sin \varphi_o\right\} dx \cdot\right.$$
$$\cdot \int_{-\infty}^{\infty} \exp\left\{-\left(t + \frac{1}{2\sigma^2}\right) y^2 - Bt \cos \theta(y) \cos \varphi_o\right\} dy$$
$$+ \cos \varphi_o \int_{-\infty}^{\infty} \exp\left\{-\left[t(x+A)^2 + \frac{x^2}{2\sigma^2}\right] - Bt \cos \theta(x+A) \sin \varphi_o\right\} dx \cdot$$
$$\left. \cdot \int_{-\infty}^{\infty} y \exp\left\{-\left(t + \frac{1}{2\sigma^2}\right) y^2 - Bt \cos \theta(y) \cos \varphi_o\right\} dy\right] .$$

If in the integrations with respect to x, this variable is translated by an amount - A (the infinite limits remaining unchanged) and terms in powers of x are collected in the integrand, then there results for $\left\langle e_o^{\,2}\right\rangle$:

$$\left\langle e_o^{\,2}\right\rangle = \frac{B}{\pi^2\sigma^2}\int_0^{\infty}\frac{dt}{t}\exp\left(-\frac{B^2}{4}t\right)\int_0^{\pi}d\theta\ \cos\theta\ \cdot \tag{B3}$$

$$\cdot\ \exp\left(-\frac{A^2}{2\sigma^2}\right)\left\{X_2(\theta,t)Y_1(\theta,t)\sin\varphi_o + X_1(\theta,t)Y_2(\theta,t)\cos\varphi_o\right\},$$

where:

$$X_1(\theta,t) = \int_{-\infty}^{\infty}\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)x^2 + \left(\frac{A}{\sigma^2} - Bt\cos\theta\sin\varphi_o\right)x\right\}dx, \tag{B4}$$

$$X_2(\theta,t) = \int_{-\infty}^{\infty}x\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)x^2 + \left(\frac{A}{\sigma^2} - Bt\cos\theta\sin\varphi_o\right)x\right\}dx, \tag{B5}$$

$$Y_1(\theta,t) = \int_{-\infty}^{\infty}\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)y^2 - (Bt\cos\theta\cos\varphi_o)y\right\}dy, \tag{B6}$$

$$Y_2(\theta,t) = \int_{-\infty}^{\infty}y\exp\left\{-\left(t+\frac{1}{2\sigma^2}\right)y^2 - (Bt\cos\theta\cos\varphi_o)y\right\}dy. \tag{B7}$$

The above integrals are evaluated by Reference 17, 314, p. 65, nos.(6a) and (6c), suppressing the exp (-c) factor on both sides:

$$\int_{-\infty}^{\infty}\exp\left[-ax^2 - 2bx\right]dx = \frac{\pi^{1/2}}{a^{1/2}}\exp\left(\frac{b^2}{a}\right), \tag{B8}$$

$$\int_{-\infty}^{\infty}x\exp\left[-ax^2 - 2bx\right]dx = -\frac{b\pi^{1/2}}{a^{3/2}}\exp\left(\frac{b^2}{a}\right), \tag{B9}$$

both integrals being valid for $a > 0$, which is satisifed in this derivation for

$$a = t + \frac{1}{2\sigma^2} \quad . \tag{B10}$$

Making the appropriate substitutions for the constant b from Equations (B4) through (B7), there result the evaluations:

$$X_1(\theta,t) = \pi^{1/2}\left(t + \frac{1}{2\sigma^2}\right)^{-1/2} \exp\left\{\frac{1}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\left(\frac{A}{\sigma^2} - Bt\cos\theta\sin\varphi_o\right)^2\right\} , \tag{B11}$$

$$X_2(\theta,t) = \pi^{1/2}\left(t + \frac{1}{2\sigma^2}\right)^{-3/2} \frac{1}{2}\left(Bt\cos\theta\sin\varphi_o - \frac{A}{\sigma^2}\right) \cdot$$

$$\cdot \exp\left\{\frac{1}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\left(\frac{A}{\sigma^2} - Bt\cos\theta\sin\varphi_o\right)^2\right\} , \tag{B12}$$

$$Y_1(\theta,t) = \pi^{1/2}\left(t + \frac{1}{2\sigma^2}\right)^{-1/2} \exp\left\{\frac{1}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}(-Bt\cos\theta\cos\varphi_o)^2\right\} , \tag{B13}$$

$$Y_2(\theta,t) = \pi^{1/2}\left(t + \frac{1}{2\sigma^2}\right)^{-3/2} \frac{(Bt\cos\theta\cos\varphi_o)}{2} \cdot$$

$$\cdot \exp\left\{\frac{1}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}(-Bt\cos\theta\cos\varphi_o)^2\right\} . \tag{B14}$$

Multiplying Equations (B12) and (B13), and (B11) and (B14) together in pairs and simplifying the exponents, the following expressions are obtained:

$$X_2(\theta,t)Y_1(\theta,t) = \frac{\pi}{2}\left(t + \frac{1}{2\sigma^2}\right)^{-2}\left(Bt\cos\theta\sin\varphi_o - \frac{A}{\sigma^2}\right) \cdot$$

$$\cdot \exp\left\{\frac{1}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\left(\frac{A^2}{\sigma^4} - \frac{2ABt}{\sigma^2}\cos\theta\sin\varphi_o + B^2t^2\cos^2\theta\right)\right\} , \tag{B15}$$

$$X_1(\theta,t)Y_2(\theta,t) = \frac{\pi}{2}\left(t + \frac{1}{2\sigma^2}\right)^{-2}(Bt\cos\theta\cos\varphi_o)\cdot$$

$$\cdot\exp\left\{\frac{1}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\left(\frac{A^2}{\sigma^4} - \frac{2ABt}{\sigma^2}\cos\theta\sin\varphi_o + B^2t^2\cos^2\theta\right)\right\}. \qquad \text{(B16)}$$

Next, in the exponents the following combination is seen:

$$\exp\left(-\frac{A^2}{2\sigma^2}\right)\exp\left\{\frac{A^2}{4\sigma^4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right\} = \exp\left\{-\frac{A^2}{2\sigma^2}\,t\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right\}. \qquad \text{(B17)}$$

Thus, upon substituting Equations (B15) and (B16) into Equation (B3), and making use of Equation (B17), there results for $\langle e_o{}^2\rangle$ upon rearrangement and noting that $\sin^2\varphi_o + \cos^2\varphi_o = 1$:

$$\langle e_o{}^2\rangle = \frac{B}{2\pi\sigma^2}\int_0^\infty \frac{dt}{t}\left(t + \frac{1}{2\sigma^2}\right)^{-2}\exp\left\{-B^2 t - \frac{A^2 t}{2\sigma^2}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right\}\int_0^\pi\left(Bt\cos\theta - \frac{2\beta}{B}\right)\cdot$$

$$\cdot\cos\theta\exp\left\{\frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\cos^2\theta - \beta t\left(t + \frac{1}{2\sigma^2}\right)^{-1}\cos\theta\right\}d\theta, \qquad \text{(B18)}$$

where again the parameter β is:

$$\beta = \frac{AB}{2\sigma^2}\sin\varphi_o. \qquad \text{(B19)}$$

The integration with respect to θ is carried out in terms of the following basic integral, denoted by $S(p,q)$:

$$S(p,q) = \int_0^\pi \exp\left(p\cos^2\theta - q\cos\theta\right)d\theta. \qquad \text{(B20)}$$

It is seen from Equation (B18) that the desired integrals will be obtained from Equation (B20) by differentiation with respect to p and q. The evaluation of Equation (B20), which does not appear obtainable in closed form, proceeds by expanding each exponent in a power series:

$$\exp(p\cos^2\theta) = \sum_{n=0}^{\infty} \frac{p^n}{n!} \cos^{2n}\theta \quad , \tag{B21}$$

$$\exp(-q\cos\theta) = \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{m!} \cos^m\theta \quad . \tag{B22}$$

Putting into Equation (B20) and interchanging the order of operations, there results:

$$S(p,q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m p^n q^m}{m!\, n!} \int_0^{\pi} \cos^{2n+m}\theta \, d\theta \quad . \tag{B23}$$

Now, breaking up the integral into two parts, one with a range $0 - \pi/2$ and the other $\pi/2 - \pi$, translating θ in the latter by $-\pi/2$, it can be evaluated by Reference 19, p. 196, no.854.1, for $m \rightarrow 2n+m$, $x = \theta$, yielding upon collecting terms:

$$\int_0^{\pi} \cos^{2n+m}\theta \, d\theta = \int_0^{\pi/2} \cos^{2n+m}\theta \, d\theta + \int_{\pi/2}^{\pi} \cos^{2n+m}\theta \, d\theta$$

$$= \int_0^{\pi/2} \cos^{2n+m}\theta \, d\theta + \int_0^{\pi/2} \cos^{2n+m}\left(\theta + \frac{\pi}{2}\right) d\theta$$

$$= \int_0^{\pi/2} \cos^{2n+m}\theta \, d\theta + (-1)^{2n+m} \int_0^{\pi/2} \sin^{2n+m}\theta \, d\theta \quad ,$$

$$\int_0^{\pi} \cos^{2n+m}\theta \, d\theta = \left[1 + (-1)^m\right] \frac{\pi^{1/2}}{2} \frac{\Gamma\left(n + \frac{m+1}{2}\right)}{\Gamma\left(n + 1 + \frac{m}{2}\right)} \quad . \tag{B24}$$

But it is seen that

$$1 + (-1)^m = \begin{cases} 0; \text{ for } m = 1,3,5, \ldots\ldots \text{ odd} , \\ 2; \text{ for } m = 0,2,4, \ldots\ldots \text{ even} . \end{cases} \tag{B25}$$

Thus setting $m = 2k$, for $k = 0, 1, 2, \ldots$, there results instead of Equation (B24):

$$\int_0^{\pi} \cos^{2(n+k)}\theta \, d\theta = \pi^{1/2} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+k+1)} \quad . \tag{B26}$$

Upon substituting Equations (B24) through (B26) into Equation (B23), it is seen that only the even terms in the summation over m contribute. Thus, setting m = 2k here as well, the following form for S(p,q) is obtained:

$$S(p,q) = \pi^{1/2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{p^n q^{2k}}{(2k)!n!} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+k+1)} \quad . \tag{B27}$$

Next, the summation over n is performed first, yielding:

$$S(p,q) = \pi^{1/2} \sum_{k=0}^{\infty} \frac{q^{2k}}{(2k)!} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+k+1)} p^n \, . \tag{B28}$$

The latter series is recognized in terms of the expansion for the confluent hypergeometric function from Reference 21, II, p. 6, no. (32), for $a = k + 1/2$, $c = k + 1$, $z = p$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(n+k+\frac{1}{2})}{\Gamma(n+k+1)} p^n = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \, {}_1F_1\left(k+\frac{1}{2} ; \; k+1 ; \; p\right) \quad . \tag{B29}$$

Putting Equation (B29) into Equation (B28) gives the expansion formula:

$$S(p,q) = \pi^{1/2} \sum_{k=0}^{\infty} \frac{q^{2k}}{(2k)!} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \, {}_1F_1\left(k+\frac{1}{2}; k+1; \; p\right) \, . \tag{B30}$$

The desired integral evaluations in Equation (B18) are obtained from Equation (B30) by suitable differentiation. These are:

$$\int_0^{\pi} \exp\left(p \cos^2\theta - q \cos\theta\right) \cos\theta \, d\theta = -\frac{\partial}{\partial q} S(p,q) \quad , \tag{B31}$$

$$\int_0^{\pi} \exp\left(p \cos^2\theta - q \cos\theta\right) \cos^2\theta \, d\theta = \frac{\partial}{\partial p} S(p,q) \quad . \tag{B32}$$

The required derivatives of Equation (B30) are obtained with the aid of Reference 22, 6.4, p. 254, no. (8), for $a = k + 1/2$, $c = k + 1$, $x = p$, Φ stands for ${}_1F_1$, and making use of gamma function relations:

$$\frac{\partial}{\partial q} S(p,q) = \pi^{1/2} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(2k-1)!} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \, {}_1F_1(k+\tfrac{1}{2};\, k+1;\, p)$$

$$= \pi^{1/2} \sum_{k=0}^{\infty} \frac{q^{2k+1}}{(2k+1)!} \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+2)} \, {}_1F_1(k+\tfrac{3}{2};\, k+2;\, p) \quad , \tag{B33}$$

$$\frac{\partial}{\partial p} S(p,q) = \pi^{1/2} \sum_{k=0}^{\infty} \frac{q^{2k}}{(2k)!} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \frac{(k+\frac{1}{2})}{(k+1)} \, {}_1F_1(k+\tfrac{3}{2};\, k+2;\, p)$$

$$= \pi^{1/2} \sum_{k=0}^{\infty} \frac{q^{2k}}{(2k)!} \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+2)} \, {}_1F_1(k+\tfrac{3}{2};\, k+2;\, p) \quad . \tag{B34}$$

Finally, making the substitutions

$$p = \frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1} \quad ; \quad q = \beta t\left(t + \frac{1}{2\sigma^2}\right)^{-1} \quad ,$$

in Equations (B31) through (B34) and substituting appropriately into Equation (B18) and arranging, there results the following expansion formula for $\left\langle e_o{}^2 \right\rangle$:

$$\left\langle e_o{}^2 \right\rangle = \frac{B}{2\pi^{1/2}\sigma^2} \int_0^{\infty} dt \left(t + \frac{1}{2\sigma^2}\right)^{-2} \exp\left\{ -\frac{B^2}{4}t - \frac{A^2}{2\sigma^2} t \left(t + \frac{1}{2\sigma^2}\right)^{-1} \right\} \cdot$$

$$\cdot \sum_{k=0}^{\infty} \frac{(\beta t)^{2k}}{(2k+1)!} \left(t + \frac{1}{2\sigma^2}\right)^{-2k} \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+2)} \, {}_1F_1\left(k+\tfrac{3}{2};\, k+2;\, \frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right) \cdot$$

$$\cdot \left[(2k+1)\, B + \frac{2\beta^2}{B}\left(t + \frac{1}{2\sigma^2}\right)^{-1} \right] \quad . \tag{B35}$$

Applying the gamma function relation in Reference 22, 1.2, p. 5, no. (15) for $z = k + 1$, it is found:

$$(2k+1)! = \Gamma\left[2(k+1)\right] = \frac{2^{2(k+1)-1}}{\pi^{1/2}} \Gamma(k+1)\Gamma(k+1+\tfrac{1}{2})$$

$$= \frac{2^{2k+1}}{\pi^{1/2}} k! \; \Gamma(k + \tfrac{3}{2}) \quad . \tag{B36}$$

Putting into Equation (B35), cancelling, and collecting terms, noting that $\Gamma(k+2) = (k+1)!$, $\langle e_o{}^2 \rangle$ takes the form:

$$\langle e_o{}^2 \rangle = \frac{B}{4\sigma^2} \int_o^\infty dt \left| t + \frac{1}{2\sigma^2} \right|^{-2} \exp \left\{ - \frac{B^2}{4} t - \frac{A^2}{2\sigma^2} t \left| t + \frac{1}{2\sigma^2} \right|^{-1} \right\} \cdot$$

$$\cdot \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left| \frac{\beta t}{2} \right|^{2k} \left| t + \frac{1}{2\sigma^2} \right|^{-2k} {}_1F_1\left(k + \frac{3}{2};\; k + 2;\; \frac{B^2 t^2}{4} \left| t + \frac{1}{2\sigma^2} \right|^{-1} \right) \cdot$$

$$\cdot \left[(2k+1) B + \frac{2\beta^2}{B} \left| t + \frac{1}{2\sigma^2} \right|^{-1} \right] \quad . \tag{B37}$$

While the above form has been considered for approximate evaluations, it is more convenient for exact numerical integration to make the change of variable:

$$x = t \left| t + \frac{1}{2\sigma^2} \right|^{-1} \; ; \qquad dx = \frac{dt}{2\sigma^2} \left| t + \frac{1}{2\sigma^2} \right|^{-2} \; ;$$

$$t = \frac{1}{2\sigma^2} \left(\frac{x}{1-x} \right) \; ; \qquad \left| t + \frac{1}{2\sigma^2} \right|^{-1} = 2\sigma^2 (1 - x) \quad . \tag{B38}$$

At the same time the Kummer transformation is applied to the confluent hypergeometric function from Reference 22, 6.3, p. 253, no. (7), for $a = k + 3/2$, $c = k + 2$, $x = (B^2 t^2/4) \left| t + 1/_{2\sigma^2} \right|^{-1}$, and Φ standing for ${}_1F_1$:

$$
{}_1F_1\left(k + \frac{3}{2};\ k + 2;\ \frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right)
$$

$$
= \exp\left\{\frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right\} {}_1F_1\left(\frac{1}{2};\ k + 2;\ -\frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right) \quad . \qquad \text{(B39)}
$$

Combining the exponents containing B^2 to yield:

$$
\exp\left(-\frac{B^2}{4}t\right) \exp\left\{\frac{B^2t^2}{4}\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right\} = \exp\left\{-\frac{B^2}{8\sigma^2}t\left(t + \frac{1}{2\sigma^2}\right)^{-1}\right\} \quad , \qquad \text{(B40)}
$$

and then applying Equation (B38) to Equations (B39) and (B40) when substituted into Equation (B37), the new expression for $\left\langle e_o^2 \right\rangle$ contains integrals over the range 0 to 1:

$$
\left\langle e_o^2 \right\rangle = \frac{B}{2}\int_0^1 dx\, \epsilon^{-\alpha x} \sum_{k=0}^{\infty} \left[(2k + 1)\, B + \frac{4\beta^2\sigma^2}{B}(1 - x)\right] \cdot
$$

$$
\cdot \frac{1}{k!(k + 1)!}\left(\frac{\beta x}{2}\right)^{2k} {}_1F_1\left(\frac{1}{2};\ k + 2;\ -\frac{B^2}{8\sigma^2}\ \frac{x^2}{(1-x)}\right) \quad , \qquad \text{(B41)}
$$

where the parameter α is defined as in Appendix A:

$$
\alpha = \frac{1}{2\sigma^2}\left(A^2 + \frac{B^2}{4}\right) \quad . \qquad \text{(B42)}
$$

Equation (B41) is used as the starting point for the evaluation of $\left\langle e_o^2 \right\rangle$ in the quadrature case discussed in the main text and treated in Appendix C.

APPENDIX C

APPROXIMATE EVALUATIONS FOR THE OUTPUT NOISE-TO-NOISE RATIO FOR THE QUADRATURE CASE

The calculation of σ_n/σ_s from Equation (27) by numerical integration for a range of reference, signal, and noise parameters becomes quite time consuming, so it is desirable to seek accurate approximate evaluations where possible. The derivations presented here cover the cases of small and large reference amplitudes.

<u>Small Reference Amplitude</u>

Examination of Equation (27) for $\left\langle e_o^2 \right\rangle$ would indicate at first look that for $B << 2^{3/2}\sigma$, it is merely enough to take the first few terms in the series expansion for the confluent hypergeometric function, found in Reference 22, <u>6.1</u>, p. 248, No. (1) for a = 1/2, c = 2, x replaced by $-(B^2/8\sigma^2)(x^2/1-x)$:

$$
{}_1F_1\left(\frac{1}{2};\ 2;\ -\frac{B^2}{8\sigma^2}\ \frac{x^2}{(1-x)}\right) = 1 - \frac{1}{4}\left[\frac{B^2}{8\sigma^2}\ \frac{x^2}{(1-x)}\right] + \frac{1}{16}\left[\frac{B^2}{8\sigma^2}\ \frac{x^2}{(1-x)}\right]^2 - \frac{5}{384}\left[\frac{B^2}{8\sigma^2}\ \frac{x^2}{(1-x)}\right]^3 + \dots \qquad (C1)
$$

But an integration between the limits 0 and 1 must subsequently be carried out, and the above series obviously diverges as $x \to 1$.

However, closer examination of Equation (C1) reveals that the square of the second term gives the third term. With the above series alternating in sign, and denoting by y the quantity:

$$
y = \frac{\gamma}{4}\frac{x^2}{(1-x)} \quad , \qquad (C2)
$$

where γ is the normalized ratio:

$$\gamma = \frac{B^2}{8\sigma^2} \quad , \tag{C3}$$

then it is seen that the first three terms of Equation (C1) are identical with corresponding ones of the alternating series found in Reference 19, p. 3, no. 9.04, for y = x:

$$1 - y + y^2 - y^3 + \ldots\ldots = \frac{1}{1 + y} \quad . \tag{C4}$$

Further, the fourth term of Equation (C4), which in terms of x from Equations (C3) and (C4) becomes

$$- \frac{1}{64} \left[\frac{B^2}{8\sigma^2} \; \frac{x^2}{(1-x)} \right]^3 \quad ,$$

differs from the corresponding term of Equation (C1) by about 13%, which for small γ would be negligible over a considerable range of y. Ignoring for the moment that the series in Equation (D4) does not converge for y $\geq$ 1, it would appear that a possible approximation to ${}_1F_1$ for small γ is:

$${}_1F_1 \left(\frac{1}{2}; \; 2; \; - \frac{\gamma x^2}{1-x} \right) \cong \frac{1}{1 + \frac{\gamma}{4} \frac{x^2}{(1-x)}} \quad . \tag{C5}$$

It still remains to see how the approximation behaves in the range 1 $\leq$ y $\leq$ ∞ corresponding to values of x approaching unity. At x = 1, the denominator of the right side of Equation (C5) becomes infinite, so the over-all expression is zero. From Reference 22, 6.13.1, p. 278, no.(3), it is seen that as the argument of ${}_1F_1$ approaches $-\infty$ as $x \rightarrow 1$, the asymptotic behavior of the confluent hypergeometric function for the parameters 1/2, 2 is:

$$ {}_1F_1\left(\frac{1}{2};\ 2;\ -\frac{\gamma x^2}{1-x}\right) \approx \frac{2}{\sqrt{\pi}}\ \frac{(1-x)^{1/2}}{\gamma^{1/2}\ x} \longrightarrow 0 \text{ as } x \longrightarrow 1 \ . $$

Thus, the original function and the proposed approximation agree in this important respect.

To conclusively check the accuracy of Equation (C5), it is plotted against the actual function for two values of γ, 0.1 and 1, on Figure C1, over the range $0 \leqq x \leqq 1$. The case $\gamma = 1$ is considered the upper limit for the approximation, for which Equation (C5) reduces to the special form:

$$ {}_1F_1\left(\frac{1}{2};\ 2;-\frac{x^2}{1-x}\right) \cong \frac{4(1-x)}{(2-x)^2} \ . \qquad (C6) $$

The exact values of ${}_1F_1$ were obtained with the aid of tables in Reference 23, pp. 698-713, since it can alternatively be expressed in terms of exponentials and modified Bessel functions by:

$$ {}_1F_1\left(\frac{1}{2};\ 2;\ -t\right) = \epsilon^{-t/2}\ \left\langle I_o\left(\frac{t}{2}\right) + I_1\left(\frac{t}{2}\right)\right\rangle \ , \qquad (C7) $$

which can be derived from relations given in Reference 22, Chapter 6. The very close fit for the $\gamma = 0.1$ case plus the good agreement for the limiting case gives confidence that the approximation will be quite accurate over the range $.1 < \gamma < 1$, and even better for smaller γ.

Equation (C5) is applied to the evaluation of $\langle e_o^{\,2}\rangle$ by Equation (27) by rewriting the right-hand side in normal ratio form:

$$ \frac{1}{1 + \frac{\gamma}{4}\frac{x^2}{(1-x)}} = \frac{4(1-x)}{\gamma x^2 - 4x + 4} \ . \qquad (C8) $$

The roots for the denominator quadratic in x are:

$$ \mu_{\pm} = \frac{2}{\gamma}\left[1 \pm (1-\gamma)^{1/2}\right] \ , \qquad (C9) $$

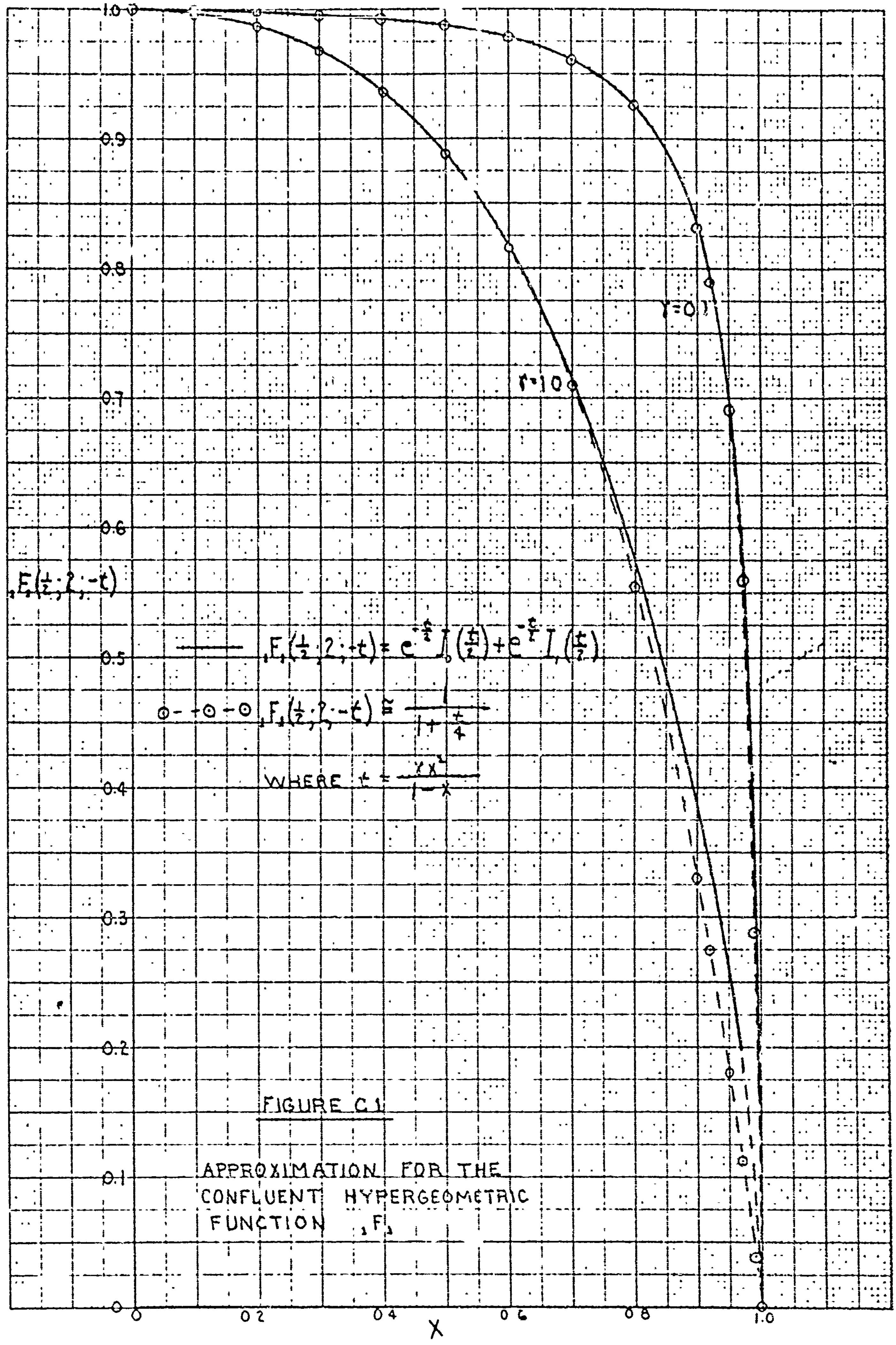

FIGURE C1

APPROXIMATION FOR THE CONFLUENT HYPERGEOMETRIC FUNCTION $_1F_1$

which are real and distinct for $\gamma < 1$, with equality considered as a special case by Equation (C6). The above restriction for real roots thus coincides with the previously considered upper limit for the approximation. Now, the right side of Equation (C8) can be written with the factored denominator, using Equation (C9):

$$\frac{1}{1 + \frac{\gamma}{4}\frac{x^2}{(1-x)}} = \frac{4(1-x)}{\gamma(x-\mu_+)(x-\mu_-)} \quad . \tag{C10}$$

This expression can be expanded further by partial fractions, which after some computation and combining with Equation (C5) yields the desired form of approximation:

$${}_1F_1\left(\frac{1}{2};\ 2;\ -\frac{\gamma x^2}{(1-x)}\right) \approx \frac{\gamma}{4(1-\gamma)^{1/2}}\left[\frac{\mu_-^{\,2}}{(x-\mu_-)} - \frac{\mu_+^{\,2}}{(x-\mu_+)}\right] \quad . \tag{C11}$$

Upon substitution of Equation (C11) into Equation (27) with Equation (C3) for the parameter γ, there results for $\left\langle e_o^{\,2}\right\rangle$ upon arrangement:

$$\left\langle e_o^{\,2}\right\rangle \approx \frac{B^2\gamma}{8(1-\gamma)^{1/2}}\left\{\mu_-^{\,2}\int_0^1 \frac{\epsilon^{-\alpha x}dx}{(x-\mu_-)} - \mu_+^{\,2}\int_0^1 \frac{\epsilon^{-\alpha x}dx}{(x-\mu_+)}\right\} \quad . \tag{C12}$$

Since for $0 < \gamma < 1$ the inequality $\mu_+ > 1$ holds, the denominators of the integrands in Equation (C12) do not vanish over the range of integration, and thus the integrals exist. They can be evaluated in terms of the modified exponential integral $\overline{Ei}(x)$, defined in Reference 12, 9.7, p. 143, no.(3), as a Cauchy principal value:

$$\overline{Ei}(x) = -\int_{-x}^{\infty} \frac{\epsilon^{-t}}{t}\,dt \quad , \tag{C13}$$

i.e., $\lim \left\{\int_{-x}^{-\epsilon} + \int_{\epsilon}^{\infty}\right\}$ as $\epsilon \rightarrow 0$.

By making the translational changes of variable

$$t = \alpha(x - \mu_{\pm})$$

in the appropriate integrals of Equation (C12), breaking them up into two components with infinite upper limits and applying Equation (C13), there results upon arrangement the evaluation:

$$\int_0^1 \frac{\epsilon^{-\alpha x} dx}{(x-\mu_{\pm})} = \exp(-\alpha\mu_{\pm}) \left[\int_{-\alpha\mu_{\pm}}^{\infty} \frac{\epsilon^{-t}}{t} dt - \int_{-\alpha(\mu_{\pm}-1)}^{\infty} \frac{\epsilon^{-t}}{t} dt \right]$$

$$= -\exp(-\alpha\mu_{\pm}) \left[\overline{Ei}(\alpha\mu_{\pm}) - \overline{Ei} \left\langle \alpha(\mu_{\pm}-1) \right\rangle \right] . \qquad (C14)$$

Substitution of Equation (C14) into Equation (C12) yields the expression for $\langle e_o^2 \rangle$ for $0 < \gamma < 1$.

For the limiting case $\gamma = 1$, the approximation to ${}_1F_1$ is given by Equation (C6), which takes the expanded form:

$${}_1F_1\left(\frac{1}{2}\, ;\, 2\, ;-\frac{x^2}{1-x}\right) \cong 4\left[\frac{1}{(x-2)^2} - \frac{1}{(x-2)}\right] . \qquad (C15)$$

Putting into Equation (27) gives for $\langle e_o^2 \rangle$ in this special case:

$$\langle e_o^2 \rangle_1 \cong 2B^2 \left\{ \int_0^1 \frac{\epsilon^{-\alpha_1 x} dx}{(x-2)^2} - \int_0^1 \frac{\epsilon^{-\alpha_1 x} dx}{(x-2)} \right\} , \qquad (C16)$$

where $\alpha_1 = \alpha\big]_{\gamma=1}$ and the integrals again exist. Integrating by parts in the first term, with

$$u = \epsilon^{-\alpha_1 x} , \quad dv = \frac{dx}{(x-2)^2} ,$$

yields upon combining terms the form:

$$\left\langle e_o^2 \right\rangle_1 = 2B^2 \left\{ \epsilon^{-\alpha_1} - \frac{1}{2} - (1+\alpha_1) \int_0^1 \frac{\epsilon^{-\alpha_1 x} dx}{(x-2)} \right\} . \qquad \text{(C17)}$$

The latter integral is evaluated again in terms of modified exponential integrals as before by the change of variable $t = \alpha_1(x-2)$ to give:

$$\int_0^1 \frac{\epsilon^{-\alpha_1 x} dx}{(x-2)} = \epsilon^{-2\alpha_1} \left[\int_{-2\alpha_1}^{\infty} \frac{\epsilon^{-t}}{t} dt - \int_{-\alpha_1}^{\infty} \frac{\epsilon^{-t}}{t} dt \right]$$

$$= -\epsilon^{-2\alpha_1} \left[\overline{Ei}(2\alpha_1) - \overline{Ei}(\alpha_1) \right] . \qquad \text{(C18)}$$

Substituting Equation (C18) into Equation (C17) gives for $\left\langle e_o^2 \right\rangle_1$:

$$\left\langle e_o^2 \right\rangle_1 \cong 2B^2 \left\{ (1+\alpha_1)\, \epsilon^{-2\alpha_1} \left[\overline{Ei}(2\alpha_1) - \overline{Ei}(\alpha_1) \right] + \epsilon^{-\alpha_1} - \frac{1}{2} \right\} , \qquad \text{(C19)}$$

where from the definition of α by Equation (21) and γ by Equation (C3):

$$\alpha_1 = 1 + \frac{A^2}{2\sigma^2} . \qquad \text{(C20)}$$

To complete the calculation of the noise-to-noise ratio, it is necessary to evaluate the expression for the mean square value for zero signal amplitude, A = 0. Under this condition, the following parameter reductions occur, denoted by zero subscript:

$$\alpha_o = \gamma \;; \quad (\alpha_1)_o = 1 . \qquad \text{(C21)}$$

Putting appropriately into Equations (C12) and (C14), the formula for $\left\langle e_o^2 \right\rangle$ at A = 0 is obtained. Finally, the substitutions into Equations (26) are made with $\left\langle e_o \right\rangle = 0$ to yield the approximate expression for σ_n/σ_s .

Small Reference Amplitude Limit

Here, the appropriate formula for $\left\langle e_o^2 \right\rangle$ can be found from the limiting form found in Appendix D, Equation (D30), for the general case, using Equation (30) and noting from Equation (21) that as $B \rightarrow 0$:

$$\alpha \longrightarrow \frac{A^2}{2\sigma^2} = a \quad . \tag{C22}$$

Making the appropriate substitution in the above-mentioned expression and setting $\varphi_o = 0$, there results the expression appearing in Equation (33) of the main text for σ_r/σ_s, since as $A \rightarrow 0$, the indeterminate evaluation yields the factor $B^2/2$:

$$\left\langle e_o^2 \right\rangle \longrightarrow \frac{B^2}{2a}\left(1 - \epsilon^{-a}\right) \quad . \tag{C23}$$

Large Reference Amplitude

The use of Equation (27) as a starting point for attempting an approximate evaluation of $\left\langle e_o^2 \right\rangle$ for large B led to difficulty in assessing the validity of the assumptions made. Further, a reduction of the corresponding limiting formula in Appendix D for $\varphi_o = 0$ gave an oversimplified result of no value. Thus, to obtain useful expressions, an asymptotic expansion of the original phase detector output voltage was taken as the basis for the following derivation.

Noting from Equations (8) and (9) that the voltages e_1 and e_2 from the circuit halves differ only by a sign in one term, they can be rewritten in combined form as $e_{1,2}$ after factoring out B/2:

$$e_{1,2} = \frac{B}{2}\left[1 + \frac{4\rho}{B}\left(\frac{\rho}{B} \pm \sin\varphi\right)\right]^{1/2} \quad . \tag{C24}$$

It is now assumed that the reference amplitude B is sufficiently larger than the input signal plus noise amplitude ρ for all values of φ to make the infinite series expansion of $(1 + z)^{1/2}$ valid, where:

$$z = \frac{4\rho}{B}\left(\frac{\rho}{B} \pm \sin\varphi\right) \quad . \tag{C25}$$

From Reference 19, p.2, no. 5.3, for x = z, the condition is

$$z^2 \leqq 1 \quad ,$$

and the expansion is:

$$(1 + z)^{1/2} = 1 + \frac{1}{2} z - \frac{1\cdot 1}{2\cdot 4} z^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6} z^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} z^4 + \cdots \tag{C26}$$

Reference 24, p.117, no. 6.33, for z = x gives the general term in this expansion for $k \geqq 2$:

$$(-1)^{k-1} \quad \frac{1\cdot 1\cdot 3\cdot 5\cdot\cdots\cdots(2k-3)}{k!\ 2^k} z^k \ .$$

But since

$$1\cdot 3\cdot 5\cdots\cdots (2k-3) = \frac{(2k-3)!}{2^{k-2}(k-2)!} z^k \quad ,$$

Equation (C26) can be rewritten in summation form as:

$$(1 + z)^{1/2} = 1 + \frac{1}{2} z - \sum_{k=2}^{\infty} \frac{(-1)^k (2k-3)!}{2^{2k-2} k!(k-2)!} z^k \quad . \tag{C27}$$

Raising Equation (C25) to the k-th power and applying the binominal expansion of Reference 19, p. 1, nos. 1 and 4, for $a = \rho/B$, $x = \sin\varphi$ and $n = k$:

$$z^k = \frac{2^{2k}\rho^k}{B^k}\left(\frac{\rho}{B} \pm \sin\varphi\right)^k$$

$$= \frac{2^{2k}}{B^k} \sum_{m=0}^{k} \frac{k!(\pm 1)^m}{m!(k-m)!}\rho^k \left(\frac{\rho}{B}\right)^{k-m} \sin^m\varphi \qquad \text{or}$$

$$z^k = \frac{2^{2k}}{B^{2k}} k! \sum_{m=0}^{k} \frac{(\pm 1)^m}{m!(k-m)!} B^m \rho^{2k-m} \sin^m\varphi \ . \tag{C28}$$

Substituting Equations (C25) and (C28) into Equation (C27) and arranging, there results for $e_{1,2}$ according to Equation (C24):

$$e_{1,2} = \frac{B}{2}\Bigg[1 + \frac{2\rho}{B}\left(\frac{\rho}{B} \pm \sin\varphi\right)$$

$$- \sum_{k=2}^{\infty} \frac{(-1)^k(2k-3)!}{(k-2)!} \frac{4}{B^{2k}} \sum_{m=0}^{k} \frac{(\pm 1)^m}{m!(k-m)!} B^m \rho^{2k-m} \sin^m\varphi\Bigg] \ . \tag{C29}$$

Thus, upon putting Equation (C29) into Equation (11) and noting the sign correspondence, the following expression for the phase detector output voltage is obtained after suitable canceling and combining of initial terms:

$$e_o = 2\rho\sin\varphi$$

$$- \frac{B}{2} \sum_{k=2}^{\infty} \frac{(-1)^k(2k-3)!}{(k-2)!} \frac{4}{B^{2k}} \sum_{m=0}^{\infty} \frac{\left[1-(-1)^m\right]}{m!(k-m)!} B^m \rho^{2k-m} \sin^m\varphi \ . \tag{C30}$$

But it is readily seen that

$$1 - (-1)^m = \begin{cases} 0 \ ; & \text{for m even,} \\ 2 \ ; & \text{for m odd.} \end{cases}$$

Setting $m = 2n + 1$, so that for

$$m = 1,3,5,\cdots,k \quad ; \quad n = 0,1,2,\cdots, \left[\frac{k-1}{2}\right] ,$$

where the latter notation means that if k is odd, it is the upper limit of the sum, but if k is even, then the sum terminates at the next lower integer index, $(k-2)/2$. Thus, in place of Equation (C30), there results with the index n:

$$e_o = 2\rho \sin\varphi - \sum_{k=2}^{\infty} \frac{4}{B^{2k-1}} \frac{(-1)^k(2k-3)!}{(k-2)!} \sum_{n=0}^{\left[\frac{k-1}{2}\right]} \frac{B^{2n+1}}{(2n+1)!(k-2n-1)!} \rho^{2(k-n)-1} \sin^{2n+1}\varphi \quad . \qquad (C31)$$

Writing out the first few terms of the expansion yields:

$$\begin{aligned} e_o = {} & 2\rho\sin\varphi - \frac{4}{B^3}\frac{1!}{0!}\left|\frac{B}{1!1!}\ \rho^3\sin\varphi\right| \\ & + \frac{4}{B^5}\frac{3!}{1!}\left|\frac{B}{1!2!}\rho^5\sin\varphi + \frac{B^3}{3!0!}\rho^3\sin^3\varphi\right| \\ & - \frac{4}{B^7}\frac{5!}{2!}\left|\frac{B}{1!3!}\rho^7\sin\varphi + \frac{B^3}{3!1!}\rho^5\sin^3\varphi\right| \qquad (C32) \\ & + \frac{4}{B^9}\frac{7!}{3!}\left|\frac{B}{1!4!}\rho^9\sin\varphi + \frac{B^3}{3!2!}\rho^7\sin^3\varphi + \frac{B^5}{5!0!}\rho^5\sin^5\varphi\right| \\ & - \frac{4}{B^{11}}\frac{9!}{4!}\left|\frac{B}{1!5!}\rho^{11}\sin\varphi + \frac{B^3}{3!3!}\rho^9\sin^3\varphi + \frac{B^5}{5!1!}\rho^7\sin^5\varphi\right| \\ & + \frac{4}{B^{13}}\frac{11!}{5!}\left|\frac{B}{1!6!}\rho^{13}\sin\varphi + \frac{B^3}{3!4!}\rho^{11}\sin^3\varphi + \frac{B^5}{5!2!}\rho^9\sin^5\varphi + \frac{B^7}{7!0!}\rho^7\sin^7\varphi\right| \\ & - \cdots\cdots \end{aligned}$$

Multiplying out the numerical factors and collecting terms in inverse powers of B up to B^{-6}, the following expression for e_o is obtained after arrangement:

$$
\begin{aligned}
e_o &= 2\rho \sin\varphi - \frac{4}{B^2}\left(\rho^3 \sin\varphi - \rho^3 \sin^3\varphi\right) \\
&\quad + \frac{4}{B^4}\left(7\rho^5 \sin^5\varphi - 10\rho^5 \sin^3\varphi + 3\rho^5 \sin\varphi\right) \qquad \text{(C33)} \\
&\quad + \frac{8}{B^6}\left(33\rho^7 \sin^7\varphi - 63\rho^7 \sin^5\varphi + 35\rho^7 \sin^3\varphi - 5\rho^7 \sin\varphi\right) + \cdots \quad .
\end{aligned}
$$

In order to compute the output noise-to-noise ratio for the quadrature case, only the ensemble average of the square of the output, $\left\langle e_o^2 \right\rangle$, is required. So, upon squaring Equation (C33) and collecting terms up to $1/B^4$ only for the asymptotic evaluation considered here, the approximate expression for e_o^2 becomes:

$$
\begin{aligned}
e_o^2 &\cong 4\left[\rho^2 \sin^2\varphi - \frac{4}{B^2}\left(\rho^4 \sin^2\varphi - \rho^4 \sin^4\varphi\right)\right. \\
&\quad \left. + \frac{16}{B^4}\left(2\rho^6 \sin^6\varphi - 3\rho^6 \sin^4\varphi + \rho^6 \sin^2\varphi\right)\right] \quad ,
\end{aligned}
$$

which further trigonometric simplification reduces to a form more convenient for averaging. Noting also from Equation (10) that for the quadrature case, $\varphi = \varphi_i$, it is found:

$$
\begin{aligned}
e_o^2 &\cong 4\left[\rho^2 \sin^2\varphi_i - \frac{4}{B^2}\left(\rho^2 \sin^2\varphi_i\right)\left(\rho^2 \cos^2\varphi_i\right)\right. \\
&\quad \left. + \frac{16}{B^4}\rho^6 \sin^2\varphi_i\left(2 \sin^4\varphi_i - 3 \sin^2\varphi_i + 1\right)\right] \quad . \qquad \text{(C34)}
\end{aligned}
$$

The ensemble averaging of Equation (C34) is simplified considerably by the fact that some of the factors can be expressed directly in terms of the in-phase and quadrature noise components and the signal amplitude by referring to Equations (2) and (4).

From them the following relations are found:

$$\rho^2 \sin^2\varphi_1 = y^2 \; ; \; (\rho^2 \sin^2\varphi_1)(\rho^2 \cos^2\varphi_1) = y^2(A+x)^2 \; ;$$

$$\rho^6 \sin^6\varphi_1 = y^6 \quad . \tag{C35}$$

Making the appropriate substitutions in Equation (C34) and taking the average on both sides, there results for $\langle e_o{}^2 \rangle$:

$$\langle e_o{}^2 \rangle \cong 4 \left[\langle y^2 \rangle - \frac{4}{B^2} \langle y^2(A+x)^2 \rangle \right. \tag{C36}$$
$$\left. + \frac{16}{B^4} \left\{ 2 \langle y^6 \rangle - 3 \langle \rho^6 \sin^4\varphi_1 \rangle + \langle \rho^6 \sin^2\varphi_1 \rangle \right\} \right] \quad .$$

The terms expressed in rectangular components are treated first. Since x and y are independent Gaussian random variables with mean value zero and standard deviation σ, then with the aid of Reference 25, p. 83, no. 10.5, the following averages are found:

$$\langle x \rangle = 0 \; ; \; \langle x^3 \rangle = 0; \; \langle x^5 \rangle = 0 \; ; \; \langle y \rangle = 0 \; ;$$
$$\langle x^2 \rangle = \sigma^2; \; \langle y^2 \rangle = \sigma^2; \; \langle y^6 \rangle = 15\sigma^6 \quad . \tag{C37}$$

Further, since A is a constant, it is clear that:

$$\langle A^k \rangle = A^k \quad . \tag{C38}$$

Computing the average indicated in Equation (C36) with the aid of Equations (C37) and (C38) yields:

$$\langle y^2(A+x)^2 \rangle = \langle y^2 A^2 \rangle + 2 \langle y^2 A x \rangle + \langle y^2 x^2 \rangle$$
$$= A^2 \langle y^2 \rangle + 2A \langle y^2 \rangle \langle x \rangle + \langle y^2 \rangle \langle x^2 \rangle \qquad \text{or}$$

$$\langle y^2(A+x)^2 \rangle = A^2\sigma^2 + \sigma^4 = \sigma^2(A^2 + \sigma^2) \quad . \tag{C39}$$

The remaining two averages to be found in Equation (C36) must be determined in a probability sense from $W_2(\rho,\varphi_i)$, the joint density function of amplitude and phase of the signal plus noise given by Equation (13). The resulting double integrals are therefore:

$$\left\langle \rho^6 \sin^4\varphi_i \right\rangle = \int_0^\infty d\rho \cdot \rho^6 \int_0^{2\pi} \sin^4\varphi_i \, W_2(\rho,\varphi_i)\, d\varphi_i \quad , \tag{C40}$$

$$\left\langle \rho^6 \sin^2\varphi_i \right\rangle = \int_0^\infty d\rho \cdot \rho^6 \int_0^{2\pi} \sin^2\varphi_i \, W_2(\rho,\varphi_i)\, d\varphi_i \quad . \tag{C41}$$

Putting in Equation (13) and arranging yields:

$$\left\langle \rho^6 \sin^4\varphi_i \right\rangle = \frac{\exp\left(-A^2/2\sigma^2\right)}{2\pi\sigma^2} \int_0^\infty d\rho \cdot \rho^7 \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \cdot$$

$$\cdot\, 2\int_0^\pi \sin^4\varphi_i \exp\left(\frac{A\rho\cos\varphi_i}{\sigma^2}\right) d\varphi_i \quad ,$$

$$\left\langle \rho^6 \sin^2\varphi_i \right\rangle = \frac{\exp\left(-A^2/2\sigma^2\right)}{2\pi\sigma^2} \int_0^\infty d\rho \cdot \rho^7 \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \cdot$$

$$\cdot\, 2\int_0^\pi \sin^2\varphi_i \exp\left(\frac{A\rho\cos\varphi_i}{\sigma^2}\right) d\varphi_i \quad .$$

Applying Reference 20, p. 202, no. 175, for $\nu = 1,2$, $z = A\rho/\sigma^2$, there results upon cancelling terms:

$$\left\langle \rho^6 \sin^4\varphi_i \right\rangle = \frac{3\sigma^2 \exp\left(-A^2/2\sigma^2\right)}{A^2} \int_0^\infty \rho^5 \exp\left(-\frac{\rho^2}{2\sigma^2}\right) I_2\left(\frac{A\rho}{\sigma^2}\right) d\rho \quad , \tag{C42}$$

$$\left\langle \rho^6 \sin^2\varphi_i \right\rangle = \frac{\exp\left(-A^2/2\sigma^2\right)}{A} \int_0^\infty \rho^6 \exp\left(-\frac{\rho^2}{2\sigma^2}\right) I_1\left(\frac{A\rho}{\sigma^2}\right) d\rho \quad , \tag{C43}$$

where I_1 and I_2 are the modified Bessel functions of first kind of first and second order, respectively.

Next, a change of variable $t = \rho^2$ preserving the limits results in:

$$\left\langle \rho^6 \sin^4 \varphi_1 \right\rangle = \frac{3\sigma^2 \exp(-A^2/2\sigma^2)}{2A^2} \int_0^\infty t^2 I_2 \left(\frac{A}{\sigma^2} t^{1/2} \right) \exp \left(- \frac{t}{2\sigma^2} \right) dt \quad , \qquad (C44)$$

$$\left\langle \rho^6 \sin^2 \varphi_1 \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2A} \int_0^\infty t^{5/2} I_1 \left(\frac{A}{\sigma^2} t^{1/2} \right) \exp \left(- \frac{t}{2\sigma^2} \right) dt \quad . \qquad (C45)$$

Integrals of the above type can be evaluated generally in terms of confluent hypergeometric functions by Reference 9, table 4.16, p. 197, no. 20, and then reducing the results for special parameter cases. But a more direct determination is seen by suitable differentiation of no. 18 of the same table for $\nu = 1, 2$. This gives the following formulas:

$$\begin{aligned} \int_0^\infty t^2 I_2(2\alpha^{1/2} t^{1/2}) \epsilon^{-pt} dt &= - \frac{d}{dp} \left\{ \int_0^\infty t \, I_2(2\alpha^{1/2} t^{1/2}) \epsilon^{-pt} dt \right\} \\ &= - \alpha \frac{d}{dp} \left\{ \frac{1}{p^3} \exp \left(\frac{\alpha}{p} \right) \right\} \\ &= \frac{\alpha}{p^5} (3p + \alpha) \exp \left(\frac{\alpha}{p} \right) \quad , \qquad (C46) \end{aligned}$$

$$\begin{aligned} \int_0^\infty t^{5/2} I_1(2\alpha^{1/2} t^{1/2}) \epsilon^{-pt} dt &= \frac{d^2}{dp^2} \left\{ \int_0^\infty t^{1/2} \, I_1(2\alpha^{1/2} t^{1/2}) \epsilon^{-pt} dt \right\} \\ &= \alpha^{1/2} \frac{d^2}{dp^2} \left\{ \frac{1}{p^2} \exp \left(\frac{\alpha}{p} \right) \right\} \\ &= \frac{\alpha^{1/2}}{p^6} (6p^2 + 6\alpha p + \alpha^2) \exp \left(\frac{\alpha}{p} \right) \quad . \qquad (C47) \end{aligned}$$

Setting the parameter values

$$p = \frac{1}{2\sigma^2} \quad ; \qquad \alpha = \frac{A^2}{4\sigma^4}$$

in Equations (C46) and (C47), simplifying, and then substituting appropriately into Equations (C44) and (C45), there results upon cancelling terms the ensemble averages:

$$\left\langle \rho^6 \sin^4 \varphi_1 \right\rangle = 6\sigma^6 \left(3 + \frac{A^2}{2\sigma^2}\right) , \tag{C48}$$

$$\left\langle \rho^6 \sin^2 \varphi_1 \right\rangle = 4\sigma^6 \left[6 + 6\left(\frac{A^2}{2\sigma^2}\right) + \left(\frac{A^2}{2\sigma^2}\right)^2\right] . \tag{C49}$$

Finally, applying Equations (C37), (C39), (C48), and (C49) to Equation (C36) and arranging, it is found for the mean square value:

$$\left\langle e_o^{\,2} \right\rangle \cong 4\sigma^2 \left\{1 - \frac{4\sigma^2}{B^2}\left[1 + 2\left(\frac{A^2}{2\sigma^2}\right)\right] + \frac{32\sigma^4}{B^4}\left[15 - 9\left(3 + \frac{A^2}{2\sigma^2}\right) + 2\left(6 + 6\left(\frac{A^2}{2\sigma^2}\right) + \left(\frac{A^2}{2\sigma^2}\right)^2\right)\right]\right\} . \tag{C50}$$

Making use of Equations (28) and (30) where the dimensionless ratio γ and the input signal-to-noise power ratio "a" are defined respectively, the formula for $\left\langle e_o^{\,2} \right\rangle$ for large reference amplitudes takes the simplified form:

$$\left\langle e_o^{\,2} \right\rangle \cong 4\sigma^2 \left[1 - \frac{1+2a}{2\gamma} + \frac{15 - 9(3+a) + 2(6+6a+a^2)}{2\gamma^2}\right] . \tag{C51}$$

From this, the formula for $\left\langle e_o^{\,2} \right\rangle$ at A = 0 is obtained by setting a = 0, which when substituted together with the original expression into Equation (26) along with $\left\langle e_o \right\rangle = 0$ yields the approximate expression for σ_η / σ_s appearing as Equation (34) of the main text.

APPENDIX D

LIMITING FORMULAS FOR THE OUTPUT SIGNAL-TO-NOISE RATIO FOR THE GENERAL CASE

The approximate evaluations of Equation (B37) considered for $\langle e_o^2 \rangle$ resulted from a number of computational steps which tended to obscure the nature of the approximations made and thus cast doubt on the range of their validity. Therefore, it appeared desirable to find some alternative derivation to obtain useful formulas. A very direct method presented itself in which the starting point, instead of involving the integrated results, is the original phase detector output voltage expression. Taking Equations (8) and (9) as the voltages from the circuit halves, squaring, subtracting, factoring, and using Equations (10) and (11), there results the following form for e_o:

$$e_o = \frac{2B\rho}{(e_1 + e_2)} \sin (\varphi_1 + \varphi_o) \quad . \tag{D1}$$

From this relation, two limiting cases in terms of the reference amplitude are derived for the output signal-to-noise ratio $\sqrt{a_o}$.

Large Reference Amplitude Limit

If the reference amplitude B is much larger than the amplitude ρ of signal plus noise, then with the aid of Figure 2 it is seen that:

$$e_1 + e_2 \doteq B \quad . \tag{D2}$$

Thus, for the limiting case, Equation (D1) reduces to:

$$e_o \doteq 2\rho \sin (\varphi_1 + \varphi_o) \quad , \tag{D3}$$

independent of B. Expanding and grouping terms yields:

$$e_o \doteq 2\left[(\rho \sin \varphi_1)\cos \varphi_o + (\rho \cos \varphi_1) \sin \varphi_o\right] \quad . \tag{D4}$$

But the quantities in parentheses can be expressed directly in terms of the in-phase and quadrature noise components and the signal amplitude by referring to Equations (2) and (4), which results in the following simplified form for e_o:

$$e_o \doteq 2\left[y \cos \varphi_o + (A+x) \sin \varphi_o\right] \quad . \tag{D5}$$

Upon taking the ensemble average of Equation (D5) and its square, noting that factors depending only on φ_o are independent of this operation, it is found:

$$\langle e_o \rangle \doteq 2\left[\langle y \rangle \cos \varphi_o + \left\langle \langle A \rangle + \langle x \rangle \right\rangle \sin \varphi_o\right] \quad , \tag{D6}$$

$$\begin{aligned} \langle e_o{}^2 \rangle \doteq 4\Big[&\langle y^2 \rangle \cos^2\varphi_o + 2\left\langle \langle Ay \rangle + \langle xy \rangle \right\rangle \sin \varphi_o \cos \varphi_o \\ &+ \left\langle \langle A^2 \rangle + 2\langle Ax \rangle + \langle x^2 \rangle \right\rangle \sin^2 \varphi_o\Big] \quad . \end{aligned} \tag{D7}$$

From the facts that x and y are independent Gaussian random variables with mean value zero and standard deviation σ, and that A is a constant, the following relations hold:

$$\begin{aligned} &\langle x \rangle = 0 \; ; \; \langle y \rangle = 0 \; ; \; \langle xy \rangle = 0 \; ; \\ &\langle Ax \rangle = A\langle x \rangle = 0 \; ; \; \langle Ay \rangle = A\langle y \rangle = 0 \; ; \\ &\langle A \rangle = A \; ; \; \langle A^2 \rangle = A^2 \; ; \; \langle x^2 \rangle = \sigma^2 \; ; \; \langle y^2 \rangle = \sigma^2 \quad . \end{aligned} \tag{D8}$$

Substituting into Equations (D6) and (D7) yields:

$$\langle e_o \rangle \doteq 2A \sin \varphi_o \quad , \tag{D9}$$

$$\langle e_o{}^2 \rangle \doteq 4(\sigma^2 + A^2 \sin^2\varphi_o) \quad . \tag{D10}$$

Applying these results to Equation (36), simplifying and making use of Equation (30) where the input signal-to-noise power ratio "a" is defined, $\sqrt{a_o}$ is found to have the limiting formula in the large reference amplitude case:

$$\sqrt{a_o} \doteq \sqrt{2a} \sin \varphi_o \quad . \tag{D11}$$

This is found as Equation (37) of the main text.

<u>Small Reference Amplitude Limit</u>

If the reference amplitude B is much smaller than the amplitude ρ of signal plus noise, then with the aid of Figure 2 it is seen that:

$$e_1 + e_2 \doteq 2\rho \quad . \tag{D12}$$

Thus for this limiting case, Equation (D1) reduces to:

$$e_o \doteq B \sin (\varphi_1 + \varphi_o) \quad , \tag{D13}$$

independent of ρ. Upon expanding Equation (D13):

$$e_o \doteq B(\sin \varphi_1 \cos \varphi_o + \cos \varphi_1 \sin \varphi_o) \quad , \tag{D14}$$

it is seen that the factors involved in the statistical averaging cannot be reduced into simple rectangular forms as before. So, carrying out the averaging on Equation (D14) and its square, it is found:

$$\langle e_o \rangle \doteq B\left[\langle \sin \varphi_1 \rangle \cos \varphi_o + \langle \cos \varphi_1 \rangle \sin \varphi_o\right] \quad , \tag{D15}$$

$$\langle e_o^2 \rangle \doteq B^2\left[\langle \sin^2 \varphi_1 \rangle \cos^2 \varphi_o + 2 \langle \sin \varphi_1 \cos \varphi_1 \rangle \sin \varphi_o \cos \varphi_o + \langle \cos^2 \varphi_1 \rangle \sin^2 \varphi_o\right] \quad . \tag{D16}$$

The desired averages are found from formulas analogous to Equations (14) and (15), where for a general function of φ_1, $f(\varphi_1)$:

$$\left\langle f(\varphi_1)\right\rangle = \int_0^{\infty} d\rho \int_0^{2\pi} f(\varphi_1)\, W_2(\rho,\varphi_1)\, d\varphi_1 \quad , \tag{D17}$$

where $W_2(\rho,\varphi_1)$, the joint probability distribution of amplitude and phase of the signal plus noise is given by Equation (13). Its application to Equation (D17) for the trigonometric functions indicated in Equations (D15) and (D16) follows in order of their appearance.

$$\left\langle \sin\varphi_1\right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2\pi\sigma^2}\int_0^{\infty} d\rho \cdot \rho \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \cdot \int_0^{2\pi} \sin\varphi_1 \exp\left(\frac{A\rho\cos\varphi_1}{\sigma^2}\right) d\varphi_1 \quad \text{or}$$

$$\left\langle \sin\varphi_1\right\rangle = 0 \ , \tag{D18}$$

since the integrand in φ_1 is an odd function over the 0 - 2π interval.

$$\left\langle \cos\varphi_1\right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2\pi\sigma^2}\int_0^{\infty} d\rho \cdot \rho \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \cdot \int_0^{2\pi} \cos\varphi_1 \exp\left(\frac{A\rho\cos\varphi_1}{\sigma^2}\right) d\varphi_1 \ .$$

Applying Reference 20, p. 202, no. 180, for $n = 1$, $z = A\rho/\sigma^2$:

$$\left\langle \cos\varphi_1\right\rangle = \frac{\exp(-A^2/2\sigma^2)}{\sigma^2}\int_0^{\infty} \rho\, I_1\left(\frac{A\rho}{\sigma^2}\right) \exp\left(-\frac{\rho^2}{2\sigma^2}\right) d\rho \quad , \tag{D19}$$

where I_1 is the modified Bessel function of first kind and order. Making the change of variable $x = \rho^2$ in Equation (D19) results in the form with the same limits:

$$\left\langle \cos \varphi_1 \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2\sigma^2} \int_0^\infty I_1 \left(\frac{A}{\sigma^2} x^{1/2} \right) \exp \left(- \frac{x}{2\sigma^2} \right) dx \quad . \tag{D20}$$

This integral is evaluated in Reference 18, p. 63, no. 3.431, for $p = 1/2\sigma^2$, $\lambda = 4\sigma^4/A^2$, $\nu = 1$, $\alpha = 3/2$, which upon substitution into Equation (D20) and simplification yields for $\langle \cos \varphi_1 \rangle$:

$$\left\langle \cos \varphi_1 \right\rangle = \frac{\sqrt{\pi} A}{2^{3/2}\sigma} \exp \left(- \frac{A^2}{2\sigma^2} \right) {}_1F_1 \left(\frac{3}{2} ; 2 ; \frac{A^2}{2\sigma^2} \right) \quad , \tag{D21}$$

where ${}_1F_1$ is again the confluent hypergeometric function. For the particular parameter values found above, this function can also be expressed in terms of exponentials and modified Bessel functions from relations given in Reference 22, Chapter 6, as was the case in Appendix C, Equation (C7). The result is:

$${}_1F_1 \left(\frac{3}{2}; 2; t \right) = \epsilon^{t/2} \left\{ I_0 \left(\frac{t}{2} \right) + I_1 \left(\frac{t}{2} \right) \right\} \quad . \tag{D22}$$

Substituting into Equation (D21) with $t = A^2/2\sigma^2$ thus gives:

$$\left\langle \cos \varphi_1 \right\rangle = \frac{\sqrt{\pi}\, A}{2^{3/2}\sigma} \exp \left(- \frac{A^2}{4\sigma^2} \right) \left\{ I_0 \left(\frac{A^2}{4\sigma^2} \right) + I_1 \left(\frac{A^2}{4\sigma^2} \right) \right\} \quad , \tag{D23}$$

which is a more convenient form for calculation.

$$\left\langle \sin^2 \varphi_1 \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2\pi\sigma^2} \int_0^\infty d\rho \cdot \rho \exp \left(- \frac{\rho^2}{2\sigma^2} \right) \cdot$$

$$\cdot\, 2 \int_0^\pi \sin^2 \varphi_1 \exp \left(\frac{A\rho \cos \varphi_1}{\sigma^2} \right) d\varphi_1 \quad .$$

Applying Reference 20, p. 202, no. 175, for $\nu = 1$, $z = A\rho/\sigma^2$, there results upon canceling terms:

$$\left\langle \sin^2 \varphi_1 \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{A} \int_0^\infty \exp \left(- \frac{\rho^2}{2\sigma^2} \right) I_1 \left(\frac{A\rho}{\sigma^2} \right) d\rho \quad . \tag{D24}$$

The change of variable $t = \rho^2$ preserving the limits yields:

$$\left\langle \sin^2\varphi_i \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2A} \int_0^\infty I_1\left(\frac{A}{\sigma^2}\, t^{1/2}\right) \exp\left(-\frac{t}{2\sigma^2}\right) \frac{dt}{t^{1/2}} \quad . \tag{D25}$$

This integral is evaluated in Reference 9, table 4.16, p. 197, no. 16 for $p = 1/2\sigma^2$, $\alpha = A^2/4\sigma^4$, which when put into Equation (D25) gives:

$$\left\langle \sin^2\varphi_i \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2A} \cdot \frac{2\sigma^2}{A}\left[\exp\left(\frac{A^2}{2\sigma^2}\right) - 1\right] \qquad \text{or}$$

$$\left\langle \sin^2\varphi_i \right\rangle = \frac{\sigma^2}{A^2}\left[1 - \exp\left(-\frac{A^2}{2\sigma^2}\right)\right] \quad , \tag{D26}$$

a form convenient for later parameter change.

$$\left\langle \sin\varphi_i \cos\varphi_i \right\rangle = \frac{\exp(-A^2/2\sigma^2)}{2\pi\sigma^2} \int_0^\infty d\rho \cdot \rho \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \cdot$$

$$\cdot \int_0^{2\pi} \sin\varphi_i \cos\varphi_i \exp\left(\frac{A\rho\cos\varphi_i}{\sigma^2}\right) d\varphi_i \qquad \text{or}$$

$$\left\langle \sin\varphi_i \cos\varphi_i \right\rangle = 0 \quad , \tag{D27}$$

since the integrand in φ_i is an odd function over the $0 - 2\pi$ interval.

Finally, the remaining ensemble average, $\left\langle \cos^2\varphi_i \right\rangle$, can be found directly from the result for $\left\langle \sin^2\varphi_i \right\rangle$ by trigonometric identity and the application of Equation (D26):

$$\left\langle \cos^2\varphi_i \right\rangle = \left\langle 1 - \sin^2\varphi_i \right\rangle = 1 - \left\langle \sin^2\varphi_i \right\rangle \quad \text{or}$$

$$\left\langle \cos^2\varphi_i \right\rangle = 1 - \frac{\sigma^2}{A^2}\left[1 - \exp\left(-\frac{A^2}{2\sigma^2}\right)\right] \quad . \tag{D28}$$

Substituting Equations (D18), (D23), and (D26) to (D28) appropriately into Equations (D15) and (D16), arranging and simplifying, there results the desired ensemble averages:

$$\left\langle e_o \right\rangle \doteq \frac{B\sqrt{\pi}}{2^{3/2}} \frac{A}{\sigma} \exp\left(-\frac{A^2}{4\sigma^2}\right) \left\{ I_o\left(\frac{A^2}{4\sigma^2}\right) + I_1\left(\frac{A^2}{4\sigma^2}\right) \right\} \sin \varphi_o \,, \tag{D29}$$

$$\left\langle e_o^{\,2} \right\rangle \doteq B^2 \left\{ \frac{\sigma^2}{A^2} \left[1 - \exp\left(-\frac{A^2}{2\sigma^2}\right)\right] \cos 2\,\varphi_o + \sin^2 \varphi_o \right\} \,. \tag{D30}$$

Applying these results to Equation (36) and making use of Equation (30) where the input signal-to-noise power ratio "a" is defined, the limiting formula for $\sqrt{a_o}$ in the small reference amplitude case is obtained:

$$\sqrt{a_o} \doteq \frac{\frac{\sqrt{\pi a}}{2}\, \epsilon^{-a/2} \left[I_o\left(\frac{a}{2}\right) + I_1\left(\frac{a}{2}\right)\right] \sin \varphi_o}{\left[\left\{1 - \frac{\pi}{4}\, a\, \epsilon^{-a} \left[I_o\left(\frac{a}{2}\right) + I_1\left(\frac{a}{2}\right)\right]^2 \right\} \sin^2\varphi_o + \frac{1}{2a}\left(1 - \epsilon^{-a}\right) \cos 2\varphi_o \right]^{1/2}} \,. \tag{D31}$$

This is found as Equation (38) of the main text.