ON A GENERALIZATION OF
A RESULT OF WINTNER

Richard Bellman

February 4, 1958

APPENDIX FOR OTS RELEASE

COPY 1 OF 1
HARD COPY $ .006
MICROFICHE $ .002

D D C
OCT 5 1964
DDC:IRA C

The RAND Corporation
1700 MAIN ST - SANTA MONICA - CALIFORNIA
SUMMARY

We obtain a generalization of the Hukuwara stability theorem analogous to a recent generalization for second order equations due to Wintner, Quarterly of Applied Mathematics, Vol. XV (1958), pp. 428–430.
ON A GENERALIZATION OF A RESULT OF WINTNER

Richard Bellman

In a recent note, [1], Wintner proved the following interesting result.

**Theorem 1.** Consider the two equations

\[
\begin{align*}
(a) \quad u'' + f(t)u &= 0, \\
(b) \quad v'' + g(t)v &= 0.
\end{align*}
\]

(1.1)

If there exist two linearly independent solutions of (1a), \( u_1 \) and \( u_2 \), such that

\[
\int_0^\infty (|u_1|^2 + |u_2|^2)|f - g| \, dt < \infty
\]

(1.2)

then every solution of (1b) can be written in the form

\[
v = c_1 u_1 + c_2 u_2 + O(|u_1| + |u_2|).
\]

(1.3)

This is an extension of known stability results, cf. [2], to which it reduces if we assume that all solutions of (1a) are bounded as \( t \to \infty \).

Let us now show that we can obtain a generalization of this result following the method used in our book, [2], to establish the Hukuhara stability theorem, of which this will be an extension.

**Theorem 2.** Consider the vector–matrix systems

\[
\begin{align*}
(a) \quad \frac{dx}{dt} &= A(t)x, \\
(b) \quad \frac{dy}{dt} &= B(t)y.
\end{align*}
\]

(1.4)

Let \( X(t) \) be the solution of
\[
\frac{dX}{dt} = A(t)X, \quad X(0) = I. \tag{1.5}
\]

If
\[
\int_0^\infty \|B(t) - A(t)\| \|X(t)\| \|X^{-1}(t)\| \, dt < \infty, \tag{1.6}
\]

then every solution of (1b) may be written
\[
y = Xc + O(\|X\|) \quad \text{as } t \to \infty. \tag{1.7}
\]

The norms of matrices and vectors are taken to be respectively \(\sum_{i,j} |x_{ij}|\) and \(\sum_{i} |x_i|\).

Proof. Write
\[
\frac{dy}{dt} = A(t)y + (B(t) - A(t))y. \tag{1.8}
\]

Then, if \(y(0) = b\), we have
\[
y = X(t)b + \int_0^t X(t)X^{-1}(s)(B(s) - A(s))y(s) \, ds. \tag{1.9}
\]

Hence
\[
\|y\| \leq \|X(t)\| \|b\| + \int_0^t \|X(t)\| \|X^{-1}(s)\| \|B(s) - A(s)\| \|y(s)\| \, ds. \tag{1.10}
\]

Thus, if we set
\[
u(t) = \|X^{-1}(t)\| \|B(t) - A(t)\| \|y(t)\|, \tag{1.11}
\]
\[
v(t) = \|B(t) - A(t)\| \|X(t)\| \|X^{-1}(t)\|,
\]
we obtain the scalar inequality

\[ u \leq c_1 v + v \int_0^t u ds. \] (1.12)

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

\[ \int_0^t u ds \leq c_1 \int_0^t v(s)e^s ds. \] (1.13)

By assumption, \( \int_0^\infty v ds < \infty \). Hence, the integral

\[ \int_0^\infty x^{-1}(s)(B(s) - A(s))y ds \] (1.14)

converges. This means that we can write (1.9) in the form

\[ y = X(t)b + X(t) \int_0^\infty x^{-1}(s)(P(s) - A(s))y ds \]

\[ - X(t) \int_t^\infty x^{-1}(s)(B(s) - A(s))y ds , \]

which yields the stated result.
REFERENCES
