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PROCESSOR:
DYNAMIC PROGRAMMING AND MEAN SQUARE DEVIATION

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P-1147

Revised September 13, 1957

Approved for OTS release

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SUMMARY

In this paper we wish to discuss some applications of the functional equation technique of dynamic programming to the treatment of some quadratic variational problems and the linear equations arising therefrom.

The first problem we shall consider is that of determining the minimum value of the quadratic deviation

$$\int_0^T (f - \sum_{k=1}^{N} x_k s_k')^2 dx,$$

where $f(x)$ is a given function of $\{s_k(x)\}$, a given sequence of real functions.

Next we consider the problem of minimizing the quadratic form

$$Q_{N,M}(x) = \sum_{k=0}^{N} (b_k - \sum_{j=0}^{M} x_j a_{k-j})^2$$

over all real $x_1$, where $\{a_k\}$ and $\{b_k\}$ are given real sequences. This problem arises in the Kolmogoroff-Wiener theory of linear predictors, and the limiting version of the problem, $N = \infty$, is discussed by Levinson in the appendix to Wiener's book.

Finally we discuss the problem of solving the linear system

$$Ax = b,$$

under the assumption that $A$ is positive definite.
Although it would seem that in all three cases the methods we present have some computational utility, we shall not discuss these matters here, since at the moment we are interested only in the purely analytic aspects of these questions.
DYNAMIC PROGRAMMING AND MEAN SQUARE DEVIATION

Richard Bellman

§1. INTRODUCTION

In this paper we wish to discuss some applications of the functional equation technique of dynamic programming [1] to the treatment of some quadratic variational problems and the linear equations arising therefrom.

The first problem we shall consider is that of determining the minimum value of the quadratic deviation

\[ \int_0^T (f - \sum_{k=1}^N x_k g_k)^2 dx, \]

where \( f(x) \) is a given function of \( \{g_k(x)\} \), a given sequence of real functions.

Next we consider the problem of minimizing the quadratic form

\[ Q_{N,M}(x) = \sum_{k=0}^N (b_k - \sum_{\ell=0}^M x_{k-\ell})^2 \]

over all real \( x_1 \), where \( \{a_k\} \) and \( \{b_k\} \) are given real sequences. This problem arises in the Kolmogoroff-Wiener theory of linear predictors, and the limiting version of the problem, \( N = \infty \), is discussed by Levinson in the appendix to Wiener’s book, [2].

Finally we discuss the problem of solving the linear system
(3) \[ Ax = b, \]

under the assumption that \( A \) is positive definite.

Although it would seem that in all three cases the methods we present have some computational utility, we shall not discuss these matters here, since at the moment we are interested only in the purely analytic aspects of these questions.

2. QUADRATIC DEVIATION

The problem of minimizing the quadratic form

\[ Q_N(u) = \int_0^T (f - \sum_{k=1}^N u_k \varphi_k)^2 dx \]

is, as we know, quite easily resolvable. Let \( \{\psi_k\} \) denote the orthonormal sequence formed from \( \{\varphi_k\} \) by means of the Gram-Schmidt orthogonalization procedure. The functions \( \varphi_k(x) \) are assumed to be linearly independent. Then

\[
\text{Min}_u Q_N(u) = \text{Min}_y \int_0^T (f - \sum_{k=1}^N y_k \psi_k)^2 dx \\
= \int_0^T f^2 dx - \sum_{k=1}^N (\int_0^T f \psi_k dx)^2 \\
= \int_0^T f^2 dx - \int_0^T \int_0^T f(x)f(y)K_N(x,y)dx dy,
\]

where

\[ K_N(x,y) = \sum_{k=1}^N \varphi_k(x)\psi_k(y). \]
We wish to obtain a recurrence relation for $K_N(x,y)$ without going through the orthogonalization procedure. To do this, we introduce the quadratic functional

$$P_N(f) = \min_{u_N} \int_0^T \left( f - \sum_{k=1}^N u_k \phi_k \right)^2 \, dx, \quad N = 1, 2, \ldots .$$

Then, using the principle of optimality,

$$P_N(f) = \min_{u_N} P_{N-1}(f - u_N \phi_N), \quad N = 2, 3, \ldots .$$

To utilize (5), we use the known form of $P_N(f)$ obtained in (2) above. We have

$$P_N(f) = \min_{u_N} \left[ \int_0^T \left( f - u_N \phi_N \right)^2 \, dx - \int_0^T \int_0^T (f(x) - u_N \phi_N(x))(f(y) - u_N \phi_N(y)) K_{N-1}(x,y) \, dx \, dy \right]$$

$$= \min_{u_N} \left[ \int_0^T \phi_N^2 \, dx - \int_0^T \int_0^T K_{N-1}(x,y) f(x)f(y) \, dx \, dy - 2u_N \left\{ \int_0^T f \phi_N \, dx - \int_0^T \int_0^T \phi_N(x)f(y)K_{N-1}(x,y) \, dx \, dy \right\} + \int_0^T \phi_N^2 \, dx - \int_0^T \int_0^T \phi_N(x) \phi_N(y) \, dx \, dy \right] .$$

It is easily seen that the coefficient of $u_N^2$ is positive.

The minimizing value is given by

$$u_N = \frac{\int_0^T f \phi_N \, dx - \int_0^T \int_0^T \phi_N(x)f(y)K_{N-1}(x,y) \, dx \, dy}{D_N} ,$$

where

$$D_N = \int_0^T \int_0^T \phi_N(x) \phi_N(y) \, dx \, dy .$$
where, to simplify the notation, we have set

\[ D_N = \int_0^T \varphi_N^2 \, dx - \int_0^T \int_0^T K_{N-1}(x, y) \varphi_N(x) \varphi_N(y) \, dx \, dy. \]

The minimum value is given by

\[ F_N(f) = \left[ \int_0^T f^2 \, dx - \int_0^T \int_0^T K_{N-1}(x, y) f(x) f(y) \, dx \, dy \right] - \left[ \int_0^T f \varphi_N \, dx - \int_0^T \int_0^T \varphi_N(x) f(y) K_{N-1}(x, y) \, dx \, dy \right]^2 / D_N. \]

To obtain the desired relation between \( K_N \) and \( K_{N-1} \), we compare (9) and (2), and equate coefficients of \( f(x)f(y) \).

Cancelling the term \( \int_0^T f^2 \, dx \), the result is

\[ K_N(x, y) = K_{N-1}(x, y) + \frac{\varphi_N(x) \varphi_N(y)}{D_N} - \frac{2 \int_0^T \varphi_N(z) K_{N-1}(z, y) \, dz \, dy}{D_N} \]

\[ + \frac{1}{D_N} \int_0^T \int_0^T \varphi_N(z) \varphi_N(w) K_{N-1}(z, x) K_{N-1}(w, y) \, dz \, dw, \]

a nonlinear integral equation.

\section*{53. LINEAR PREDICTORS}

Consider the quadratic form

\[ Q_{N, M}(x) = \sum_{k=0}^N (b_k - \sum_{\ell=0}^M x_{\ell+1}^k)^2, \quad N > M \geq 1, \]

and the sequence defined by

\[ f_{N, M} = \min_x Q_{N, M}(x). \]
We wish to determine a relation between $f_{N,M}$ and $f_{N-1,M-1}$ which will enable us to compute $f_{N,M}$ starting with $f_{N-M+1,1}$.

Consider the function of $N + 1$ real variables $y_0$, $y_1$, ..., $y_N$, and the integers $N$ and $M$ defined by

$$f_{N,M}(y_0, y_1, \ldots, y_N) = \min_x \left[ (y_0 - x_0 a_0)^2 + (y_1 - x_0 a_1 - x_1 a_0)^2 \right. \left. + \cdots + (y_N - x_0 a_N - x_1 a_{N-1} - \cdots - x_M a_{N-M})^2 \right],$$

where the sequences $\{y_j\}$ and $\{a_j\}$ are given and the minimization is over the $x_j$. In order for the problem to be non-trivial, we assume that $N > M > 1$.

Let us now obtain a recurrence relation. Assume that $x_0$ has been chosen. Then the quantities $x_1$, $x_2$, ..., $x_M$ are to be chosen to minimize the remaining sum

$$f_{N,M}(y_0, y_1, \ldots, y_N) = \min_x \left[ (y_1 - x_0 a_1 - x_1 a_0)^2 + (y_2 - x_0 a_2 - x_1 a_1 - x_2 a_0)^2 + \cdots \right. \left. + (y_N - x_0 a_N - x_1 a_{N-1} - \cdots - x_M a_{N-M})^2 \right].$$

This minimum is precisely, in accordance with the notation introduced above,

$$f_{N-1,M-1}(y_1 - x_0 a_1, y_2 - x_0 a_2, \ldots, y_N - x_0 a_N).$$

Hence we obtain the recurrence relation...
For $M = 1$, we have

$$f_{N,1}(y_0, y_1, \ldots, y_N) = \operatorname{Min}_{x_0} \left( \sum_{k=0}^{N} (y_k - x_0 a_k)^2 \right)$$

(7)

$$= \left( \sum_{k=0}^{N} y_k^2 \right) - (\sum_{k=0}^{N} a_k y_k)^2.$$

64. RECURRENCE RELATIONS

In order to use (3.5) to determine the functions $f_{N,M}(y)$ in a more constructive fashion, let us observe that these functions are quadratic forms in the $y_i$.

(1) $f_{N,M}(y_0, y_1, \ldots, y_N) = \sum_{i,j=0}^{N} c_{ij(N,M)} y_i y_j$.

Substituting in (3.5), we obtain the result

$$\sum_{i,j=0}^{N} c_{ij(N,M)} y_i y_j = \operatorname{Min}_{x_0} \left( (y_0 - x_0 a_0)^2 + \sum_{i=1}^{N-1} c_{ij(N-1,M-1)} (y_{i+1} - x_0 a_{i+1}) (y_{j+1} - x_0 a_j) \right).$$

(2)

Hence $x_0$ is determined by the relation

$$x_0 = \frac{a_0 y_0 + \sum_{i=1}^{N-1} c_{i,N-1} a_{i+1} (y_{i+1} a_j + y_{j+1} a_i) / 2}{\sum_{i,j=0}^{N-1} c_{ij(N-1,M-1)} a_i a_j}.$$
and we have the relation

\[ \sum_{i,j=0}^{N} c_{ij}(N,M)y_{i}y_{j} = y_{0}^2 + \sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)y_{i+1}y_{j+1} \]

\[ = \left\{ \sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1)a_{i}a_{j} \right\} \]

\[ - \left\{ a_{0}y_{0} + \sum_{i,j=0}^{N-1} c_{ij}(N-1,M-1) \right\} \]

\[ \left( y_{i+1}a_{j} + y_{j+1}a_{i} \right) / 2 \right\}^2. \]

Equating coefficients of \( y_{i}y_{j} \) on both sides of this equation, we will obtain a relation for \( c_{ij}(N,M) \) in terms of the set \( \{ c_{ij}(N-1,M-1) \} \). Once we have obtained the \( c_{ij}(N,M) \), we can then calculate the elements of the minimizing sequence \( \{ x_{i} \} \), using (3) repeatedly.

55. THE EQUATION \( Ax = b \)

Let us now sketch the application of the same techniques to the problem of solving the system of equations

\[ \sum_{j=1}^{N} a_{ij}x_{j} = y_{1}, \quad 1 = 1, 2, \ldots, N, \]

where \( A = (a_{ij}) \) is a positive definite matrix.

Introduce the function of \( N \) variables, \( f_{N}(y_{1},y_{2},\ldots,y_{N}) \), for \( N = 1, 2, \ldots, -\infty < y_{1} < \infty \), by means of the relation

\[ f_{N}(y_{1},y_{2},\ldots,y_{N}) = \min_{x_{1}} \left[ \sum_{i,j=1}^{N} a_{ij}x_{i}x_{j} - 2 \sum_{i=1}^{N} x_{i}y_{1} \right]. \]
To obtain an equation for \( f \), we write

\[
\sum_{i,j=1}^{N} a_{ij}x_i x_j - 2 \sum_{i=1}^{N} x_i y_i = a_{NN}x_N^2 + \sum_{i,j=1}^{N-1} a_{ij}x_i x_j - 2 \sum_{i=1}^{N-1} x_i (y_i - a_{iN}x_N) - 2x_N y_N.
\]

(3)

Once \( x_N \) has been chosen, it is clear that the remaining \( x_i, \ i = 1, 2, \ldots, N-1 \), will be chosen to minimize the expression

\[
\sum_{i=1}^{N-1} a_{ij}x_i x_j - 2 \sum_{i=1}^{N-1} x_i (y_i - a_{iN}x_N).
\]

(4)

In accordance with the above notation, this minimum is precisely

\[
f_{N-1}(y_1 - a_{1N}x_N, y_2 - a_{2N}x_N, \ldots, y_{N-1} - a_{(N-1)N}x_N).
\]

(5)

Hence, we obtain the recurrence relation

\[
f_N(y_1, y_2, \ldots, y_N) = \min_{x_N} \left[ a_{NN}x_N^2 + f_{N-1}(y_1 - a_{1N}x_N, y_2 - a_{2N}x_N, \ldots, y_{N-1} - a_{(N-1)N}x_N) - 2x_N y_N \right].
\]

(6)

In order to use this relation constructively, we use the fact that \( f_N \) is a quadratic form in the \( y_i \),

\[
f_N(y_1, y_2, \ldots, y_N) = \sum_{i,j=1}^{N} c_{ij}(N)y_i y_j,
\]

(7)

for \( N = 1, 2, \ldots \).
Returning to equation (6), we have the relation

$$\sum_{i,j=1}^{N} c_{ij}(N)y_{i}y_{j} = \min_{x_{N}} \left[ a_{NN}x_{N}^{2} - 2x_{N}y_{N} \right] + \sum_{i,j=1}^{N-1} c_{ij}(N-1)(y_{i}+a_{i1}x_{N})(y_{j}+a_{jN}x_{N})].$$

(8)

Collecting terms on the right, we have

$$\sum_{i,j=1}^{N} c_{ij}(N)y_{i}y_{j} = \min_{x_{N}} \left[ x_{N}^{2} \left\{ a_{NN} + \sum_{i,j=1}^{N-1} a_{i1}a_{jN}c_{ij}(N-1) \right\} + x_{N} \left\{ -2y_{N} + \sum_{i,j=1}^{N-1} [y_{i}a_{jN} + y_{j}a_{iN}]c_{ij}(N-1) \right\} + \sum_{i,j=1}^{N-1} c_{ij}(N-1)y_{i}y_{j} \right].$$

(9)

The minimization can now be performed readily and the recurrence relations connecting \( \{c_{ij}(N)\} \) with \( \{c_{ij}(N-1)\} \) read off.

**6. DISCUSSION**

The method discussed above might be particularly useful in connecting with the problem of solving an infinite system of equations of the form

$$\sum_{j=1}^{\infty} a_{ij}x_{j} = b_{i}, \quad i = 1, 2, \ldots,$$

where we solve successively the finite systems

$$\sum_{j=1}^{N} a_{ij}x_{j} = b_{i}, \quad i = 1, 2, \ldots, N,$$

to obtain approximations to the solutions of (1).
REFERENCES
