THE COMBINATION OF TIME SERIES AND CROSS-SECTION
DATA IN INTERINDUSTRY FLOW ANALYSIS

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THE COMBINATION OF TIME SERIES AND CROSS-SECTION DATA

IN INTERINDUSTRY FLOW ANALYSIS*

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O. Summary

The following problem arose in the course of a larger study which sought
to explain the variations of input-output ratios over time. (By the input-output
ratio of industry i to industry j is meant the ratio of that part of the
output of industry i used by industry j to the output of industry j; (see [1]).
For such a study, there are two types of data available. For all years, there
are available (ideally) outputs and final demands (the final demand for an
industry consists of all uses of its product other than in other industries or
itself) for all industries. The "balance equations" of input-output analysis
(see [2] below) form an (incomplete) system of simultaneous relations which
may be estimated by some version of the method of maximum likelihood (for computa-
tional reasons, the single-equation limited-information method is the only
one likely to be used). However, for some years, we have additional information
in the form of knowing the actual interindustry flows, which clearly should
substantially increase the accuracy of our estimates.

The simplest technique is, of course, to assume that the "true" input-output'
ratio for any year for which flow data are available is exactly equal to the
observed input-output ratio for that year. The assumption behind this is, however,
contradictory to the basic postulate that all the relations involved are valid
only up to a stochastic term.

It is therefore of interest to consider more explicitly the interindustry
flow model implied in the use of both time series and interindustry flow data for
estimating the parameters involved. (The maximum likelihood estimates for a single

* This paper arose out of a study conducted by the RAND Corporation. I am
indebted to Ronald W. Shephard, now of the Sandia Corporation, for raising
the problem.
equation are then derived, in the sense of being expressed as the solution of a system of simultaneous equations. Unfortunately, these equations are rather cumbersome, and it is hoped that some reasonably efficient method of solution can be found.

1. A Model With Constant Input-Output Coefficients

For expository reasons, we will start by assuming that the input-output ratios are constant over time. We consider the estimation of a single one of the balance equations of the input-output table. Let

\[ x_{ot} = \text{derived demand for commodity } O \text{ at time } t, \text{ i.e., total net output less final demand}, \]

\[ x_{jt} = \text{output of commodity } j \text{ at time } t \text{ for } j = 1, \ldots, N, \]

\[ x_{jt} = \text{input of commodity } O \text{ to industry } j \text{ at time } t. \]

We have the identity,

\[ x_{ot} = \sum_{j=1}^{N} x_{jt}, \tag{1} \]

where \( N \) is the number of commodities other than commodity \( O \). The assumption of input-output coefficients constant over time can be written,

\[ x_{jt} = \alpha_j x_{jt} + u_{jt}, \tag{2} \]

where \( u_{jt} \) is a disturbance, distributed normally with mean zero. From (1) and (2), we have the balance equation for industry \( O \),

\[ x_{ot} = \sum_{j=1}^{N} \alpha_j x_{jt} + u_t, \tag{3} \]

where

\[ u = \sum_{j=1}^{N} u_{jt}. \tag{4} \]

The disturbance \( u_t \) is therefore normally distributed with mean 0.
It is assumed that there are two types of observations, one in which only
the variables \( x_{jt} \) (\( j = 0, \ldots, N \)) are observed and one in which the variables
\( \overline{x}_{jt} \) are also observed. Let \( U \) be the set of years of the first type, \( V \) the set
of years of the second type; in application, \( V \) contains only the year 1947
plus possibly the year 1939. Let

\[
\overline{x}_{N+j,t} = x_{jt} \quad \text{for any } t \in V,
\]

\( x_t \) be the column vector with components \( x_{jt} \) (\( j = 0, \ldots, N \)),
\( \overline{x}_t \) the column vector with components \( \overline{x}_{jt} \) (\( j = 1, \ldots, 2M \)),
\( z_t \) the column vector of the predetermined variables,
\( \mathbf{r} \) the regression matrix of \( x_t \) on \( z_t \),
\( \overline{\mathbf{r}} \) the regression matrix of \( \overline{x}_t \) on \( z_t \).

The reduced forms for \( x_t \), \( \overline{x}_t \) can therefore be written,

\[
(5) \quad x_t = \mathbf{r} z_t + \mathbf{v}_t,
\]

\[
(6) \quad \overline{x}_t = \overline{\mathbf{r}} z_t + \overline{\mathbf{v}}_t,
\]

where the vectors \( \mathbf{v}_t \), \( \overline{\mathbf{v}}_t \) are each distributed normally with means zero and inde-
dependently of \( z_t \).

Let \( \mathbf{r}_j \) be the \( j^{th} \) row of \( \mathbf{r} \), \( \overline{\mathbf{r}}_j \) the \( j^{th} \) row of \( \overline{\mathbf{r}} \). By substituting (5)
and (6) into (1) and (2) and equating coefficients of \( z_t \) in the usual way (see (2.2.1)),

\[
(7) \quad \mathbf{r}_0 = \sum_{j=1}^{N} \overline{\mathbf{r}}_j,
\]

\[
(8) \quad \overline{\mathbf{r}}_j = \alpha_j \mathbf{r}_j \quad (j = 1, \ldots, N).
\]

From the definition of \( \overline{x}_{N+j,t} \),

\[
(9) \quad \mathbf{r}_j = \overline{\mathbf{r}}_{N+j}.
\]

Let \( \mathbf{A} \) be the covariance matrix of \( \mathbf{v}_t \), \( \overline{\mathbf{A}} \) that of \( \overline{\mathbf{v}}_t \). Then the distributions
of \( x_t \) and \( \bar{x}_t \) (for given \( z_t \)) are given by

\begin{align*}
(10) \quad f(x_t) &= k_1 \left| \Lambda \right|^\frac{1}{2} \exp \left[ -\left(\frac{1}{2}\right)(x_t - \overline{\pi} z_t)' \Lambda (x_t - \overline{\pi} z_t) \right], \\
(11) \quad g(\bar{x}_t) &= k_2 \left| \overline{\Lambda} \right|^\frac{1}{2} \exp \left[ -\left(\frac{1}{2}\right)(\bar{x}_t - \overline{\pi} z_t)' \overline{\Lambda} (\bar{x}_t - \overline{\pi} z_t) \right],
\end{align*}

where \( k_1 \) and \( k_2 \) are constants. For \( t \) in \( U \), \( x_t \) is observed; for \( t \) in \( V \), \( \bar{x}_t \) is observed. Hence, the likelihood function \( L \) is obtained (assuming serial independence of the disturbances) by multiplying together all expressions of form (10) with \( t \in U \) with all expressions of form (11) with \( t \in V \).

\begin{align*}
(12) \quad \log L &= C + (T/2) \log \left| \Lambda \right| - (\frac{1}{2}) \sum_{t \in U} (x_t - \overline{\pi} z_t)' \Lambda (x_t - \overline{\pi} z_t) \\
&\quad + (S/2) \log \left| \overline{\Lambda} \right| - (\frac{1}{2}) \sum_{s \in V} (\bar{x}_s - \overline{\pi} z_t)' \overline{\Lambda} (\bar{x}_s - \overline{\pi} z_t),
\end{align*}

where \( T \) and \( S \) are numbers of observations in \( U \) and \( V \), respectively.

The aim, then, is to maximize \( \log L \) with respect to \( \pi, \overline{\pi}, \alpha_1, \ldots, \alpha_M \), \( \Lambda \), and \( \overline{\Lambda} \), subject to the restraints (7-9). The resulting values of \( \alpha_1, \ldots, \alpha_M \) are the required maximum likelihood estimates.

2. The Model With Varying Input-Output Ratios

In general, each input-output ratio is presumed to vary linearly with a number of other variables. Let \( w_1, \ldots, w_k \) be the variables on which the input-output ratios \( \alpha_j \) depend. Any one coefficient may depend on only some of these variables; let \( K_j \) be the set of variables on which \( \alpha_j \) depends. For simplicity of notation, let \( w_0 \) be the constant 1, and assume that \( K_j \) contains
0 for all $j$.

(13) $\alpha_j = \sum_{k \in K_j} \alpha_{kj} w_k$.

(13) and (2) can be written,

(14) $\bar{x}_{jt} = \left( \sum_{k \in K_j} \alpha_{kj} w_{kt} \right) x_{jt} + u_{jt}$ ($j = 1, \ldots, N$).

Let

$$x_{Nk+j,t} = w_{kt} x_{jt} \quad (k = 0, \ldots, K; \ j = 1, \ldots, N).$$

For $k = 0$, this definition corresponds to the earlier one, since $w_0 = 1$ for all $t$. Also, let

(15) $\bar{x}_{Nk+j,t} = x_{N(k-1)+j,t}$ ($k = 1, \ldots, K+1; \ j = 1, \ldots, N$),

$x_t$ be the column vector whose components are $x_{jt}$ ($j = 0, \ldots, N(K+1)$),

$\bar{x}_t$ be the column vector whose components are $\bar{x}_{jt}$ ($j = 1, \ldots, N(K+2)$).

The vectors $x_t$ and $\bar{x}_t$ are the observables in the observations under $U$ and $V$, respectively; they are somewhat redefined from section 1. (14) can be written,

(16) $\bar{x}_{jt} = \sum_{k \in K_j} \alpha_{kj} x_{Nk+j,t}$ ($j = 1, \ldots, N$).

Define $\overline{\Pi}$ and $\overline{\Pi}$, as before, as the regression matrices of $x_t$ and $\bar{x}_t$, respectively, on the vector of predetermined variables $z_t$. Since most of the components of the two vectors are formed as a product of two variables, there is no reason for the regressions to be linear; however, as Anderson and Rubin have shown, acting as if they were linear normal leads to consistent estimates (see $\underline{3.7}$, p. 574). Hence, equations (5) and (6) will still be regarded as valid. Since (1) still holds, (7) is still valid. Substituting (5) and
(6) into (16) yields the following analogue of (5):

\[(17) \quad \overline{\pi}_j = \sum_{k \in k_j} k_j \overline{\pi}_{Nk+j} \quad (j = 1, \ldots, N).\]

Finally, (15) implies that,

\[(18) \quad \overline{\pi}_{Nk+j} = \pi_{N(k-1)+j} \quad (k = 1, \ldots, K+1; j = 1, \ldots, N).\]

The likelihood function has the same form as before. Hence the maximum likelihood estimates are obtained by maximizing (12) subject to the restraints (7), (17), and (18).

3. Restrictions on the Covariance Matrices of the Reduced Form Disturbances.

It can be argued that the above model implies certain relations between the covariance matrices of the reduced forms (5) and (6), i.e., \(\Lambda^{-1}\) and \(\Lambda^{-1}\).

Since (15) is an identity,

\[(19) \quad \overline{\pi}_{Nk+j, t} = \pi_{N(k-1)+j, t} \quad (k = 1, \ldots, K+1; j = 1, \ldots, N)\]

Let \(\sigma_{iJ}\) be the covariance of \(v_i\) and \(v_j\), \(\sigma_{11}\) the covariance of \(\overline{v}_1\) and \(\overline{v}_1\). Then, from (19)

\[\overline{\sigma}_{Nk+j, Nk'+j'} = \sigma_{N(k-1)+j, N(k'-1)+j'} \quad (k, k' = 1, \ldots, K+1; j, j' = 1, \ldots, N),\]

which can also be written

\[(20) \quad \sigma_{1j} = \sigma_{1-N, j-N} \quad (i = N+1, \ldots, N(K+2); j = N+1, \ldots, N(K+2)).\]

Similarly, since (1) is an identity,

\[(21) \quad \sigma_{i1} = \sum_{j=1}^{N} \overline{\sigma}_{j, i-N} \quad (i = 1, \ldots, N(K+1)).\]

It would be reasonable to impose the conditions (20) and (21) upon the maximization of the likelihood function. However, this would greatly complicate
the form of the estimation procedure. Further, if we wish to assume that the interindustry flows are observed only with error, the relation (1) becomes a stochastic relation (if \( \overline{z}_{jt} \) is taken to refer to the observed rather than the actual flow), and the argument leading to (21) breaks down. For these reasons, we will disregard the restrictions on the covariance matrices of the reduced form disturbances.

4. The Derivation of the Estimates

Rewrite (18) as

\[
(22) \quad \overline{\pi}_j = \overline{\pi}_{j-N} \quad (j = N+1, \ldots, \Pi(N+2)).
\]

We wish to maximize (12) subject to (7), (17), and (22). We use the method of Lagrange multipliers. There are altogether \( \Pi(N+2)+1 \) restrictions, each a vector restriction with as many components as there are predetermined variables. To each restriction we assign a Lagrange multiplier, itself a vector. Let \( \lambda \) correspond to (7), \( \mu^j \) to the \( j \)th restriction in (17), and \( \nu^j \) to the \( j \)th restriction in (22). The Lagrangian may then be written,

\[
(23) \quad \mathcal{L} = c + (\Pi/2) \log |\Lambda| + (3/2) \log |\Lambda| - (1/2) \sum_{t \in U} (x_t - \overline{\pi}_t z_t)^t \Lambda (x_t - \overline{\pi}_t z_t)
\]

\[
- (1/2) \sum_{s \in V} (\overline{x}_s - \overline{\pi}_s z_s)^t \Lambda (\overline{x}_s - \overline{\pi}_s z_s)
\]

\[
+ (\Pi_0 - \sum_{j=1}^{\Pi} \overline{\pi}_j) \lambda + \sum_{j=1}^{\Pi} (\sum_{k \in K_j} \alpha_{kj} \Pi_{Nk+j} - \overline{\pi}_j) \mu^j
\]

\[
+ \sum_{j=N+1}^{\Pi(N+2)} (\overline{\pi}_{j-N} - \overline{\pi}_j) \nu^j.
\]

Let \( \mathbf{M}_{xx} \) be the matrix of sums of cross-products of \( x \) and \( z \), summation extending over all observations in \( U \),

\[
\mathbf{M}_{zz} \quad \text{be the matrix of sums of squares and cross-products of the } z \text{'s, summed over observations in } U.
\]
\[ \overline{M}_{xz} \] be the matrix of sums of cross-products of \( \overline{x} \) and \( z \) over \( V \),

\[ \overline{M}_{zz} \] be the matrix of sums of squares and cross-products of the \( z \)'s, summed over \( V \).

Now differentiate \( A \) successively with respect to \( \overline{\pi}_o \), \( \overline{\pi}_{Nk+j} \) (\( k = 0, \ldots, K; j = 1, \ldots, N \)), \( \overline{\pi}_j \), and equate the results to 0.

\begin{align*}
(24) & \quad \partial \overline{\pi}_o = \bigwedge^0 (M_{xz} - \overline{\pi} M_{zz}) + \lambda' = 0. \\
(25) & \quad \partial \overline{\pi}_{Nk+j} = \bigwedge^{Nk+j} (M_{xz} - \overline{\pi} M_{zz}) + \sum_{j=1}^{N} M^j + \sqrt{N (k+1)+j'} = 0, \quad (k = 0, \ldots, K; j = 1, \ldots, N). \\
(26) & \quad \partial \overline{\pi}_j = \bigwedge^j (\overline{M}_{xz} - \overline{\pi} \overline{M}_{zz}) - \mu^j + \lambda' = 0 \quad (j = 1, \ldots, N). \\
(27) & \quad \partial \overline{\pi}_j = \bigwedge^j (\overline{M}_{xz} - \overline{\pi} \overline{M}_{zz}) - \nu^j = 0 \quad (j = N+1, \ldots, N(K+2)).
\end{align*}

Here, each equation represents a row vector of derivatives. In (25), it is understood that

\[ \alpha_{kj} = 0 \text{ if } k \notin K_j. \]

Define row vectors,

\begin{align*}
(29) & \quad p_o = -\lambda', \quad p_{Nk+j} = - (\alpha_{kj} M^j + \sqrt{N (k+1)+j'}) \quad (k = 0, \ldots, K; j = 1, \ldots, N), \\
(30) & \quad r_j = \lambda' + \nu^j \quad (j = 1, \ldots, N), \quad r_j = \nu^j \quad (j = N+1, \ldots, N(K+2)),
\end{align*}

and matrices,

\[ P = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N(K+1)} \end{bmatrix}, \quad R = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{N(K+2)} \end{bmatrix}. \]

Then (24) and (25) may be written,

\[ \bigwedge (M_{xz} - \overline{\pi} M_{zz}) = P, \]
(26) and (27) become

\[ \mathcal{L}(\mathbf{M}_{xz} - \mathbf{N}_{zz}) = R. \]

From (29) and (30),

\[ P_{Nk+j} = -\alpha_{kj} R_j - R_{N(k+1)+j} - \alpha_{kj} P_0. \]

Define

\[ Q_{eq} = 0, \quad Q_{Nk+j} = -\alpha_{kj}, \quad Q_{Nk+j,N(k+1)+j} = -1, \quad Q_{Nk+j,q} = 0 \quad \text{for all} \quad q \neq j, \]

\[ N(k+1)+j (q = 1, \ldots, N(K+2); k = 0, \ldots, K; j = 1, \ldots, N). \]

Let \( Q \) be the matrix with elements \( Q_{pq} \) (\( p = 0, \ldots, N(K+1); q = 1, \ldots, N(K+2) \)).

Let \( \bar{Q} \) be the column vector with \( \bar{Q}_0 = 1, \quad \bar{Q}_{Nk+j} = -\alpha_{kj} \) (\( k = 0, \ldots, K; j = 1, \ldots, N \)).

Then (34) can be written,

\[ P = Q R + \bar{Q} P_0. \]

If we combine (32) and (36),

\[ \mathcal{L}(\mathbf{M}_{xz} - \mathbf{N}_{zz}) = Q R + \bar{Q} P_0. \]

Note that \( Q \) and \( \bar{Q} \) depend only on the \( \alpha_{kj} \)'s. In effect, the unknown Lagrange parameters \( \lambda, \mu^J (j = 1, \ldots, N), \nu^J (j = N+1, \ldots, N(N+2)) \) have been transformed into the unknown matrix \( R \) and vector \( P_0 \). (33) and (37) are the results of the differentiation of the Lagrangian (23) with respect to the regression matrices \( \mathbf{N}, \mathbf{M} \). Now differentiate with respect to \( \alpha_{kj} \).

\[ \mathcal{L}_{\alpha_{kj}} X = 0 \quad (k = 0, \ldots, K; j = 1, \ldots, N). \]

With the aid of (29) and (30), (38) may be rewritten

\[ \mathcal{L}_{\alpha_{kj}} (R_j + P_0)^t = 0 \quad (k = 0, \ldots, K; j = 1, \ldots, N). \]
We can eliminate $P_0$ as follows: In (37), multiply on the left by $\overline{Q}^T \Sigma$, where $\Sigma = \Lambda^{-1}$. Let

$$
(40) \quad \overline{\sigma} = (\overline{Q}^T \Sigma \overline{Q})^{-1}.
$$

Note that $\overline{\sigma}$ is a scalar. Then,

$$
(41) \quad P_0 = \overline{\sigma} (\overline{Q}^T M_{xz} - \overline{Q}^T R M_{zz} \overline{Q}^T \Sigma R).
$$

Finally, we may differentiate $A$ with respect to the matrices $\Lambda, \overline{\Lambda}$, the inverses of the covariance matrices of reduced form disturbances in the equations in the observations of $U$ and $V$, respectively. Let $\overline{\Sigma} = \overline{\Lambda}^{-1}$. Then, as in Anderson and Rubin (see n. 2),

$$
(42) \quad \overline{\Sigma} = M_{xx} - M_{xz} \pi' - M_{xz} \pi' + M_{zz} \pi';
$$

$$
(43) \quad \overline{\Sigma} = \overline{M}_{xx} - \overline{M}_{xz} \overline{\pi'} - \overline{M}_{xz} \overline{\pi'} + \overline{M}_{zz} \overline{\pi'}.
$$

We have then to solve equations (33), (37), (39), (42), and (43) for the unknowns $\Lambda, \overline{\Lambda}, \overline{Q}, R, \overline{\sigma}, \pi, \overline{\pi}$, and $\overline{\pi}$, where $P_0$ is eliminated with the aid of (40) and (41). Actually, we are only interested in $\overline{Q}$ (note that the matrix $Q$ is completely defined by the vector $\overline{Q}$) which involves the structural coefficients to be estimated. These equations seem considerably more difficult to handle than the corresponding ones of Anderson and Rubin.
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