ON THE EXPANSIONS OF SOME INFINITE PRODUCTS

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SUMMARY

The purpose of this paper is to present an expansion for

$$\prod_{k,\ell=1}^{\infty} (1 - x^{k\ell})$$

analogous to the classical expansion of

$$\prod_{k=1}^{\infty} (1 - x^k).$$
ON THE EXPANSIONS OF SOME INFINITE PRODUCTS

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§1. INTRODUCTION

The technique used by Euler, Gauss, and Jacobi to obtain identities of the form

\[ \prod_{k=1}^{\infty} (1 - x^k) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n(n+1)/2}}{\prod_{k=1}^{n} (1 - x^k)} , \quad |x| < 1, \]

was the following. We begin with the function

\[ f(x, t) = \prod_{k=1}^{\infty} (1 - x^k t) , \quad |x| < 1, \]

and observe that it satisfies the functional equation

\[ (1 - xt) f(x, tx) = f(x, t). \]

Writing

\[ f(x, t) = \sum_{n=0}^{\infty} a_n(x) t^n , \]

the relation in (3) yields

\[ \sum_{n=0}^{\infty} a_n(x) t^n = (1 - xt) \sum_{n=0}^{\infty} a_n(x) x^n t^n , \]

whence, equating coefficients,

\[ a_n(x) = a_n(x) x^n - a_{n-1}(x) x^n . \]
This leads to the formula

\[ a_n(x) = \frac{(-1)^n x n(n+1)/2}{\prod_{k=1}^n (1 - x^k)} \]

If we attempt to follow the same method for the product

\[ f(x, y, t) = \prod_{k,l=0}^{\infty} (1 - x^k y^l t), \]

where the prime indicates that \( k \) and \( l \) are not simultaneously zero, we encounter a difficulty. Setting

\[ f(x, y, t) = \prod_{k,l=0}^{\infty} (1 - x^k y^l t), \]

we have

\[ f(x, y, t) = \prod_{k=0}^{\infty} (1 - x^k t) f(x, y, yt) \]

\[ - \prod_{l=0}^{\infty} (1 - y^l t) f(x, y, xt). \]

Neither of these functional equations yield a result corresponding to (1) above.

We wish consequently to pursue a different course, one which yields (1) in the one-dimensional case, and which is equally applicable to the multi-dimensional case. There are some interesting convergence questions connected with this method, which we shall bypass here.

\[ \S 2. \text{ THE ONE-DIMENSIONAL CASE} \]

Consider the function
(1) \[ f_N(z) = \prod_{k=1}^{n} (1 - x^k z)^{-1}, \quad |x| < 1, \]

which possesses the partial fraction decomposition

(2) \[ f_N(z) = \sum_{k=1}^{N} \frac{a_k(x)}{1 - x^k z}, \]

where \( a_k(x) \) is determined by the relation

(3) \[ a_k(x) = \lim_{z \to x^k} (1 - x^k z) f_N(z) \]

\[ = \prod_{k=1}^{K-1} (1 - x^{k-K}) \prod_{k=K+1}^{N} (1 - x^{k-K})^{-1}. \]

Thus we have

(4) \[ a_k(x) = \prod_{k=1}^{K-1} (1 - x^{-k})^{-1} \prod_{k=1}^{N-K} (1 - x^k)^{-1}. \]

Letting \( N \to \infty \), we see that

(5) \[ f_N(z) \to \prod_{k=1}^{\infty} (1 - x^k z)^{-1}, \]

\[ a_k(x) \to \prod_{k=1}^{K-1} (1 - x^{-k})^{-1} \prod_{k=1}^{\infty} (1 - x^k)^{-1}. \]

Thus, formally, we obtain

(6) \[ \frac{\prod_{k=1}^{\infty} (1 - x^k)}{\prod_{k=1}^{\infty} (1 - x^k z)} = \sum_{k=1}^{\infty} \frac{1}{1 - x^k z} = \sum_{k=1}^{\infty} \frac{1}{1 - x^k z}, \quad |x| < 1 \]

Setting \( z = 0 \), we obtain (1.1).
A number of similar identities may be obtained upon substituting other values of $z$. It is not difficult to justify the passage to the limits.

§3. THE TWO-DIMENSIONAL CASE

Let us now follow the same procedure starting with the function

$$f_N(z) = \prod_{k,l=1}^N \left(1 - x^{k,y z} \right)^{-1}, \quad |x|, |y| < 1,$$

and

$$x^r \neq y^s \quad \text{for} \quad r, s = 1, 2, \ldots .$$

We write

$$f_N(z) = \sum_{k,l=1}^N \frac{a_{k,l}(x, y)}{(1 - x^{k,y z})},$$

where

$$a_{k,l}(x, y) = \lim_{z \to y - L} f_N(z)(1 - x^{K,y L} z).$$

Thus

$$a_{k,l}(x, y) = \prod_{k,l=1}^{K-1,L-1} (1 - x^{k,y L-L})^{-1} \left( \prod_{k=K+1}^N \prod_{l=1}^L (1 - x^{k,y L-L})^{-1} \right).$$

.$$
Letting \( N \rightarrow \infty \), the coefficient approaches

\[
\begin{align*}
\alpha_{KL} &= \prod_{k,l=1}^{K-1} (1 - x^{-K_{y,-l}})^{-1} \prod_{k=1}^{\infty} \prod_{l=0}^{L-1} (1 - x^{K_{y,l}})^{-1} \\
&= \prod_{k=0}^{K-1} \prod_{l=0}^{\infty} (1 - x^{K_{y,l}})^{-1} \prod_{k,l=1}^{\infty} (1 - x^{K_{y,l}})^{-1}.
\end{align*}
\]

The formal equivalent of (2.6) is thus

\[
\prod_{k,l=1}^{\infty} (1 - x^{K_{y,l}}) = \sum_{k,l=1}^{\infty} \frac{b_{kl}}{(1 - x^{K_{y,l}})^{Z}}
\]

where

\[
\begin{align*}
b_{KL} &= \prod_{k,l=1}^{K-1} (1 - x^{-K_{y,-l}})^{-1} \prod_{k=1}^{\infty} \prod_{l=0}^{L-1} (1 - x^{K_{y,l}})^{-1} \\
&= \prod_{k=0}^{K-1} \prod_{l=0}^{\infty} (1 - x^{K_{y,l}})^{-1}.
\end{align*}
\]

Setting \( z = 0 \), we obtain a two-dimensional analogue of (1.1).

A rigorous proof of this identity requires a measure of the irrationality of \( \log x / \log y \). This we shall not discuss here.
§4. SOME RECENT WORK OF CARLITZ

In a recent paper, [1], Carlitz has studied the expansion problem for the products \( \prod_{m,n=0}^{\infty} (1 + x^m y^n t), \prod_{m,n=0}^{\infty} (1 - x^m y^n t). \)

Setting

\[
(1) \quad \prod_{m,n=0}^{\infty} (1 + x^m y^n t) = \sum_{m=0}^{\infty} t^m q_m(x, y)/(x)_m(y)_m,
\]

where \((x)_m = (1 - x)(1 - x^2) \cdots (1 - x^m)\), he derives some interesting recurrence relations for the coefficient functions \(q_m(x, y)\), together with other properties. Analogous results are obtained for the other product mentioned above.