CONSTRUCTIVE PROOF OF THE MIN-MAX THEOREM

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P-564

8 September 1954

Approved for OTS release

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CONSTRUCTIVE PROOF OF THE MIN-MAX THEOREM

by

George B. Dantzig

1. INTRODUCTION

The foundations of a mathematical theory of "games of strategy" were laid by John von Neumann between 1928 and 1944. The publication in 1944 of the book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern climaxed this pioneering effort. The first part of this paper is concerned with games with a finite number of pure strategies with particular emphasis on the "zero-sum two-person" type of game. There it is shown that in most instances a player is at a disadvantage if he always plays the same pure strategy and that it is better to "mix" his pure strategies by a chance device. The starting point of all discussions of this type of game is the celebrated "Main Theorem" or Min-Max Theorem which is concerned with existence and properties of mixed strategies for two players.

The first proof of this theorem, given by von Neumann, was rather indirect and used topology, functional calculus, and fixed point theorems of L. E. J. Brouwer. The first proof of an elementary nature was given by J. V. Luce and H. W. Kuhn. The von Neumann-Morgenstern theorem, the pure strategy showing a dead end which we are about to discuss will be illustrative of a less direct technique that often underlies the theme of

"The purest witness to the existence of such a theory."

"For all opportunities to the contrary, in this field are clever and intelligent approaches."

"This contribution of this paper is, this field sees little meth
elementarization further \([1]\). At this late date there still continues to be a need for a truly elementary proof; for example, the recent book of McKinsey on game theory \([3]\) omitted a self-contained proof because none was available.

Kuhn \([2]\) in his "Lectures on the Theory of Games" gives a bibliography of some of the better known proofs of the Min-Max Theorem, together with a discussion of their general characteristics which he broadly classifies into (1) those based on separation properties of convex sets and (2) those using some notion of a fixed point of a transformation. Kuhn in \([2]\) and McKinsey in \([3]\) provide proofs along the lines of von Neumann \([1]\) based on a separation theorem. Dresher in \([4]\) gives a self-contained proof along the lines of Ville. As was pointed out in Weyl \([5]\), the Min-Max Theorem is completely algebraic and should be given an algebraic proof. The purely algebraic proofs, when made self-contained and elementary, appear to be quite long, \([2]\), \([4]\), \([5]\), and, with the exception of Weyl's proof \([5]\), make use of non-algebraic concepts as the minimum of a continuous function on a closed bounded set is assumed on the set. All these proofs are either pure existence proofs or, from the viewpoint of practical computations, non-constructive.

The present proof has the following features: It is purely algebraic (in the spirit of Weyl) and elementary in the sense that it used nothing more advanced than the notion of an inverse of a matrix. It is short, self-contained, and non-inductive.
The very nature of the solution, if desired, could be used to advantage to establish well-known theorems regarding the structure of the class of optimal strategies. It is a special adaptation for games of the simplex method used for solving linear programming problems [6]. As such, it provides perhaps the most efficient means currently available for explicitly constructing optimal mixed strategies for both players.

2 THE MIN-MAX THEOREM

It has been found convenient in a part of the proof to compare certain vectors "lexicographically." The term is borrowed from an alphabetically ordering of words (as in a dictionary). Thus a vector A is greater than B (written \( A > B \)) if the first component of A is greater than the first component of B. If the first components are the same, then the second components are compared, etc. To be more precise we say \( A > B \) if \( (A - B) > 0 \), where by \( (A - B) > 0 \) is meant that \( (A - B) \) has non-zero components, the first of which is positive.

Let \( [a_{ij}] \) be the payoff matrix of a finite zero-sum two-person game where \( a_{ij} \) is the payoff to Player I (the maximizing player) when Player I plays pure strategy i and Player II (the minimizing player) plays pure strategy j. Player I (in order to gain from his strategy calls "mixed"), chooses a mixed strategy.
strategy \((x_1, x_2, \ldots, x_m)\) where \(x_i\) is the probability of playing strategy \(i\); accordingly, Player I's expected payoff becomes 
\[
\left(\sum_{i=1}^{m} a_{ij} x_i\right)
\]
if the minimizing player plays pure strategy \(j\). If Player I's mixed strategy is found out he can expect that Player II will choose \(j\) such that \(\sum_{i=1}^{m} a_{ij} x_i\) is minimum. Thus, Player I wishes to choose his \(x_i\) such that the smallest such sum (which we will denote by \(x_o\)) is a maximum. For similar reasons the Player II chooses a mixed strategy \(y_1, y_2, \ldots, y_n\) such that the largest sum \(\sum_{j=1}^{n} a_{ij} y_j\) (denoted by \(y_o\)) is minimum. The Min-Max Theorem states that there exists a choice for Player I of \(x_1 = \hat{x}_1\) and a choice for Player II of \(y_j = \hat{y}_j\) such that the corresponding \(x_o = \hat{x}_o\) is the maximum value for \(x_o\) and the corresponding \(y_o = \hat{y}_o\) is the minimum value for \(y_o\) and, moreover, \(\hat{x}_o = \hat{y}_o\). The common value of \(\hat{x}_o\) and \(\hat{y}_o\) is known as the "value" of the game.

To establish this result we shall consider, as is often done, a related linear inequality problem. Let \(x_i\) and \(y_j\) satisfy the system of relations

\[
\begin{align*}
(1) \quad x_i &\geq 0, \quad (i=1,\ldots,m); \\
(2) \quad \sum_{i=1}^{m} x_i &= 1 \\
(3) \quad x_o &\leq \sum_{i=1}^{m} a_{ij} x_i, \quad (j=1,\ldots,n); \\
(4) \quad y_j &\geq 0, \quad (j=1,\ldots,n); \\
(5) \quad \sum_{j=1}^{n} y_j &= 1 \\
(6) \quad \sum_{j=1}^{n} a_{ij} y_j &\leq y_o, \quad (i=1,\ldots,m).
\end{align*}
\]

If we multiply (3) through by any \(y_j\) satisfying (4), (5), and (6) and sum with respect to \(j\); similarly multiply through (6) by any \(x_i\) satisfying (1), (2), (3) and sum with respect to \(i\), one
obtains immediately

\[ x_0 - x_0 \sum_{j} y_j \leq \sum_{i} \sum_{j} x_i a_{ij} y_j \leq y_0 \sum_{i} x_i = y_0 \]

so that the lower bounds \( x_0 \) never exceed the upper bounds \( y_0 \).

We shall, however, construct a solution \( x_i = \hat{x}_i \) and \( y_j = \hat{y}_j \) with the property that

\[ \hat{x}_0 = \hat{y}_0 \]

In particular (7) holds for \( \hat{y}_0 \) and any \( x_0 \) and also for \( \hat{x}_0 \) and any \( y_0 \). It follows, therefore, that \( x_0 \leq \hat{x}_0 = \hat{x}_0 \leq y_0 \) and

\[ \hat{x}_0 = \text{Max} \ x_0 \quad \text{and} \quad \hat{y}_0 = \text{Min} \ y_0 \]

and the Min–Max Theorem would be demonstrated.

3. PROOF OF THE MIN–MAX THEOREM

We shall begin the proof by augmenting the matrix of the game \( a_{ij} \) and consider the matrix

\[
\begin{bmatrix}
0 & 1 & \ldots & 1 & 0 & \ldots & 0 \\
1 & a_{11} & \ldots & a_{1n} & 1 & \ldots & 1 \\
-1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
-1 & a_{m1} & \ldots & a_{mn} & 0 & \ldots & 1
\end{bmatrix}
\]

(10)

The columns of this matrix will be denoted \( P_0; P_1, \ldots, P_n; \) \( (P_{n+1} = U_1), \ldots, (P_{n+m} = U_m) \) where \( U_i \) are unit vectors with 1 in the \( (i+1)\)st component. It will be convenient to arrange
the rows of the matrix such that

\[(11) \quad a_{m1} = \text{Max} a_{11} \quad .\]

Let \( B \) (which we will call a basis) be a subset of \((m+1)\) columns of \((10)\) (including \(P_0\) as first column) which, considered as an \( m+1 \) square matrix, is non-singular and let the rows of the inverse of \( B \) be denoted by \( \beta_i \) where \( i = 0, 1, \ldots, m \). We shall further require that \( B \) to be a basis, have the property that each row (except \( i = 0 \)) of the inverse of \( B \) have its first non-zero component positive. Thus we are assuming in the lexicographic sense that

\[(12) \quad \beta_i > 0 \quad (i = 1, 2, \ldots, m) .\]

For example, we may choose \( B = B_0 \) as consisting of the first two columns of \((10)\) and the unit vectors \( U_1, \ldots, U_{m-1} \). This near identity matrix

\[
B_0 = [P_0, P_1, U_1, \ldots, U_{m-1}] = [P_0, P_1, P_{n+1}, \ldots, P_{n+m-1}]
\]

is obviously non-singular and possesses a simple inverse

\[
(13) \quad B_0^{-1} =
\begin{bmatrix}
  a_{m1} & 0 & 0 & 0 & -1 \\
  1 & 0 & 0 & 0 & 0 \\
  b_1 & 1 & 0 & \ldots & -1 \\
  b_2 & 0 & 1 & \ldots & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{m-1} & 0 & 0 & 1 & -1 
\end{bmatrix}
\]
where \( b_1 = a_{m1} - a_{11} \). Because of (11) it follows that \( b_1 \geq 0 \) and our special lexicographic assumption (12) holds.

Let the columns of a general basis be denoted by

\[
B = [P_0, P_1, \ldots, P_m]
\]

and note that the conditions \( \beta_k P_j = 0 \) for \( i \neq k \) and \( \beta_k P_j = 1 \) for \( i, k = 0, 1, \ldots, m \) \((j_0 = 0)\) must hold between \( B \) and its inverse. The \( 0 \)-row of \( B^{-1} \) is used to compute the scalar quantities \( \beta_0 P_j \) for \( j = 1, 2, \ldots, n, \ldots, n + m \). We shall now prove

**Theorem**

If for all \( j = 1, 2, \ldots, n + m \)

\[
\beta_0 P_j \leq 0
\]

then the components of the \( 0 \)-row and \( 0 \)-column of \( B^{-1} \) yield the required optimal strategies.

**Proof.** Denote the components of the \( 0 \)-row of \( B^{-1} \) by

\[
= [\hat{x}_0, -\hat{x}_1, \ldots, -\hat{x}_m];
\]

the components of the \( 0 \)-column of \( B^{-1} \) by

\[
\left\{\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_m\right\}.
\]

We shall now show that an optimum mixed strategy for Player I is obtained by setting \( x_1 = \hat{x}_i \) for \( i = 1, 2, \ldots, m \); and one for
Player II, by setting \( y_{j_1} = \hat{y}_{j_1} \) for \( j_1 \leq n \) and \( y_j = \hat{y}_j = 0 \) for all other \( j \leq n \). Moreover, the value of the game is \( \hat{x}_0 = \hat{y}_0 \).

Indeed, for Player I, it is easy to verify the condition 
\[ \beta_0 P_o = 1 \]

is the same as (2); moreover, \( \beta_0 P_j \leq 0 \) for \( 1 \leq j \leq n \) are the same as (3), while for \( n+1 \leq j \leq n+m \) they are the same as (1). For Player II, the lexicographic property of the rows of \( B^{-1} \), namely \( \beta_1 > 0 \) for \( 1 = 1, \ldots, m \) implies that the first component of \( \beta_1 \) (which by definition is \( \hat{y}_{j_1} \)) is non-negative; thus, (4) is satisfied. Multiplying \( B \) on the right by \( O \)-column of \( B^{-1} \) yields \((m+1)\) linear expressions in \((\hat{y}_0, \hat{y}_{j_1}, \ldots, \hat{y}_{j_m})\) which may be equated to unit vector \( U \).

The first of these \((m+1)\) linear equations yields (5) since the 1st components of \( P_{j_1} \) are unity for \( 1 \leq j_1 \leq n \) and zero otherwise. The remaining \( m \) equations yield the inequalities (6) if the terms involving \( j_1 > n \) are dropped (the latter are non-negative because \( \hat{y}_{j_1} \geq 0 \) and their coefficients are the components of the unit vectors \( P_{n+1} \)). Finally, the proof is completed by noting that (6) or \( \hat{x}_C = \hat{y}_0 \) holds since both are defined in (16) and (17) as the \((0,0)\) element of \( B^{-1} \).

Constructing an Optimal Basis

It is clear now that the central problem is one of constructing a basis \( B \) with the property that \( \beta_0 P_j \leq 0 \) for \( j = 1, 2, \ldots, n+m \) since this in turn yields an optimal mixed strategy for each player. We shall show that if some basis \( B \),
such as $B_0$, does not have the requisite property (15), then it is easy to construct from $B$ a new basis $B^*$ which differs from $B$ by only one column where 0-row of $B^*$ (which we denote by $\beta^*_o$) has the property that

$$\beta^*_o > \beta^*_o$$

i.e., the first non-zero component of $(\beta^*_o - \beta^*_o)$ is positive. If the new basis $B^*$ does not satisfy (15) then the algorithm just outlined for $B$ is iterated, with $B$ replaced by $B^*$, etc. This process generates a sequence of bases which terminates when a basis is obtained that has the required property. This must occur in a finite number of steps since the condition (18) is a strict inequality which insuresthat no basis can be repeated and the number of different bases cannot exceed the number of ways of choosing $m$ columns out of $n+m$ from (10). The 0-column of successive bases of the iterative process may be interpreted as a succession of improved mixed strategies for Player II for which his expected loss, $y_0$, if his opponent is playing optimally, is decreasing to a minimum. Indeed, the components of the first column of any basis (as in (17) and sequel) satisfy (4) and (5) independently of condition (15), while $y_0$, the first component of $\beta^*_o$, is non-increasing from basis to basis by virtue of (18).

*In practical computations with the simplex method, of which this is a variation, the number of iterations is usually very small. In a game case where, say, $m/2$ of pure strategies are used with positive probability in an optimal mixed strategy, something in the order of $m/2$ iterations might be expected before an optimal basis is obtained.*
To construct $B'$ from $B$ let $P$ denote the column of $(10)$ which replaces the $r$th column of $B$ where $P$ and $P_j$ are determined by the following rules: Choose $P$ such that

$$\beta_o P = \max_j \beta_o P_j > 0, \quad (j = 1, \ldots, n+m).$$

In case the choice of $s$ is not unique, then choose $s$ with the smallest index satisfying (23). Next, compute the column vector $V$ satisfying $BV = P$. It is clear that components of $V = \{v_c, v_1, \ldots, v_m\}$ are given by

$$v_1 = \beta_1 P$$

where, in particular $v_c = \beta_o P > 0$ from (19). We now choose to drop from $B$ that column $P_j$ such that the lexicographic minimum of the vectors $(1/v_1)\beta_1$ for $v_1 > 0$ is attained for $i = r$. Thus,

$$(1/v_1)\beta_r = \min (1/v_1)\beta_1 \quad (v_r > 0, v_1 > 0)$$

where $r, r \neq 0$, and where it is assumed for the moment that there is at least one $v_1 > 0$. The minimizing vector is easily obtained in practice by finding the vector whose first component is the least; if there is a tie, then one passes to the second components of the tying vectors and selects the least, etc. A relation which will be used later that follows from (21) is

$$v_1 - (v_1/v_r)\beta_r > 0 \quad (v_1 > 0).$$
It is clear from the structure of the augmented matrix \( [\mathcal{A}] \) that the first column \( \mathcal{A}_1 \) can only be formed as a positive linear combination of the other columns \( \mathcal{A}_j \). However, if we assume the contrary to the assumption of \( [\mathcal{A}] \), that \( v_1 \mathcal{A}_1 < 0 \) and write \( \mathcal{A}_1 = \mathcal{A}_1 \mathcal{A}_\mathcal{R} + \mathcal{A}_1 \mathcal{A}_b \mathcal{A}_j \), then, by eliminating to the left all terms other than \( \mathcal{A}_1 \mathcal{A}_\mathcal{R} \), we obtain a positive linear combination of columns \( \mathcal{A}_b \) and \( \mathcal{A}_j \) that yields \( v_1 \mathcal{A}_b > 0 \) which is a contradiction.

There remains only to show that \( \mathcal{A}_1 \) has the remaining properties (12) and (18). The proof, as well as the efficiency of the computational algorithm, is obtained by constructing \( [\mathcal{A}]^{-1} \) from \( \mathcal{A}^{-1} \) using the relations

\[
\begin{align*}
\beta^*_1 &= \beta_1 - \frac{v_1}{v_R} \beta_R, \\
\beta^*_R &= \frac{1}{v_R} \beta_R
\end{align*}
\]

where \( \beta^*_1 \) is the \( i \)th row of \( [\mathcal{A}]^{-1} \). To verify that (23) is indeed the inverse of \( \mathcal{A}_1 \), one notes from (23) that for \( i \neq r \) the values \( \beta^*_i \mathcal{A}_j \) are the same as \( \beta_i \mathcal{A}^*_j = 0 \) (or 1 if \( i = k \)); moreover, it follows readily from the definitions of \( v_1 \) given in (20) that \( \beta^*_R = 1 \) and \( \beta^*_i \mathcal{A}_R = 0 \) for \( i \neq r \).

The required properties of \( \beta^*_i \) are immediately evident: Thus, the first non-zero component of \( \beta^*_i \) is positive because \( \beta_R \) has this property and \( v_R > 0 \). Next, for all other \( (1 = 1, 2, \ldots, m) \) the property must hold if \( v_1 \leq 0 \) since \( \beta^*_i \) is
the sum of two vectors with this property. If \( v_1 > 0 \) then \( \beta^*_1 > 0 \) by (22) and (23). Finally we note that the relation \( \beta^*_1 > \beta^*_2 \) (and not \( \beta^*_1 > \beta^*_2 \)) holds because \( \beta^*_p \), a row of a non-singular matrix, possesses at least one non-zero component and \( \beta^*_p \) is formed by subtracting from \( \beta^*_p \) a vector \((v/v_1, \beta^*_p)\) where \( v > 0 \), \( v_1 > 0 \), hence, (14) holds and the proof is complete.

**EXAMPLE**

Solve the 3 x 6 game matrix \( M \)

\[
M = \begin{bmatrix}
4 & 3 & 3 & 2 & 2 & 6 \\
6 & 0 & 4 & 2 & 6 & 2 \\
0 & 7 & 3 & 6 & 2 & ?
\end{bmatrix}
\]

from J. D. Williams' The Compleat Strategyst, Chapter 3, Exercise 16. [5]. Element \((u_{21})\) of \( M \) has been starred. It will be noted that this is the maximal element in the first column.

For convenience, below, the second and third rows have been interchanged so that this element appears in the bottom position of this column in forming the augmented matrix, \([P_1, \ldots, P_9]\).

Given below:

\[
\begin{bmatrix}
P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \\
- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
- & 7 & 3 & 6 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 6 & 0 & 4 & 2 & 6 & 2 & 2 & 2 & 2
\end{bmatrix}
\]
Initial Iteration

The initial basis, \( B = B_0 \), consists of \( P_0, P_1, (P_7 = U_1), (P_8 = U_2) \). The inverse of \( B_0 \) (given below) is determined by formula (13). The entries \( v_i \) shown, for the moment, cannot be filled in until \( P_8 \) is first determined.

\[
B_0^{-1} = \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} = \begin{bmatrix}
6 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 \\
6 & 0 & 1 & -1
\end{bmatrix}; \quad v_0 = 6, v_1 = 1, v_2 = 5, v_3 = 13
\]

Next, \( P_8 = P_2 \) is determined by

\[
\beta_0 P_8 = \beta_0 P_2 = \max_{j \neq 0} \beta_0 P_j = 6 > 0
\]

so that the entries \( v_1 = \beta_1 P_8 \) (given above) can now be computed.

The column \( r \) to be dropped from the basis is determined by forming the lexicographic minimum of the vectors. See paragraph 2.

\[
1/v_r \beta_r = 1/v_1 \beta_2 = \min \text{ (lexico.) } (1/v_1)\beta_1 \quad (v_1 > 0, \ 1 \neq 0)
\]

Drop col \( r = 2 \); i.e., \( P_7 \).

1st Iteration

The next basis \( B' = B_1 \) is \( [P_0, P_1, P_2, P_8] \). To obtain its inverse set: \( \beta_1^1 = \beta_1 - (v_k/v_r)\beta_r \), \( (k \neq r) \) and \( \beta_r^1 = (1/v_r) \).
where \( r = 2 \) where the superscript (in place of \( \cdot \)) refers to the basis \( B = B_k \).

\[
B_1^{-1} = \begin{bmatrix}
\beta_0^1 \\
\beta_1^1 \\
\beta_2^1 \\
\beta_3^1 \\
\end{bmatrix} = \begin{bmatrix}
\frac{18}{5} & -\frac{6}{5} & 0 & + \frac{1}{5} \\
\frac{3}{5} & -\frac{1}{5} & 0 & + \frac{1}{5} \\
\frac{2}{5} & \frac{2}{5} & 0 & -\frac{1}{5} \\
\frac{4}{5} & -\frac{13}{5} & \frac{5}{5} & 8 \\
\end{bmatrix}
\]

\( v_c = \frac{12}{5} \)

\( v_2 = \frac{7}{5} \)

\( v_2 = -\frac{2}{5} \)

\( v_3 = \frac{36}{5} \)

where \( P_s = P_2 \) is determined by

\[
\beta_0^1 P_s = \beta_0^2 P_5 = \max_{j \neq 0} \beta_1^1 P_j = \frac{12}{5} > 0
\]

and \( P_{r_1} = P_{r_2} = P_r \) is determined by

\[
(1/v_r) \beta_1^1 = \frac{5}{15} \beta_3^1 = \min \{ \text{lexic.} \} \left( \frac{1}{v_1} \beta_1^1 \right) \\
(1/v_r) \beta_1^1 > 0, \quad i \neq 3
\]

**2nd (Final) Iteration**

\[
B_2 = \begin{bmatrix}
0 & 1 & -1 & 1 \\
-1 & 4 & 3 & 2 \\
-1 & 0 & 7 & 2 \\
-1 & 6 & 0 & 6 \\
\end{bmatrix}
\]

\[
B_2^{-1} = \begin{bmatrix}
\beta_0^2 \\
\beta_1^2 \\
\beta_2^2 \\
\beta_3^2 \\
\end{bmatrix} = \begin{bmatrix}
\frac{10}{15} & -\frac{5}{15} & -\frac{5}{15} & -\frac{5}{15} \\
\frac{10}{15} & \frac{10}{15} & -\frac{5}{15} & -\frac{5}{15} \\
\frac{16}{15} & \frac{2}{15} & \frac{2}{15} & \frac{16}{15} \\
\frac{4}{15} & -\frac{13}{15} & \frac{5}{15} & \frac{36}{15}
\end{bmatrix}
\]
where \( b \) cannot be determined since \( \beta_0^2 P_j < 0 \) for \( j \geq 1 \). Thus an optimal solution has been obtained (from top row) \( \hat{x}_1 = \frac{5}{15} \), \( \hat{x}_2 = \frac{5}{15} \), \( \hat{x}_3 = \frac{5}{12} \) and (from first column) \( \hat{y}_1 = \frac{16}{50} \), \( \hat{y}_2 = \frac{16}{50} \), \( \hat{y}_5 = \frac{4}{50} \) where all other \( \hat{y}_i = 0 \). The "value of the game" (from upper left corner) is \( \hat{x}_0 = \hat{y}_0 = \frac{50}{15} \). It will be noted that actually \( P_0^2 P_j = 0 \) for all \( j \geq 1 \), which means there exist other bases and corresponding solutions. Williams shows in his book, in all, eight such solutions.
REFERENCES


