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A MATHEMATICAL SOLUTION

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P-561

1 September 1954

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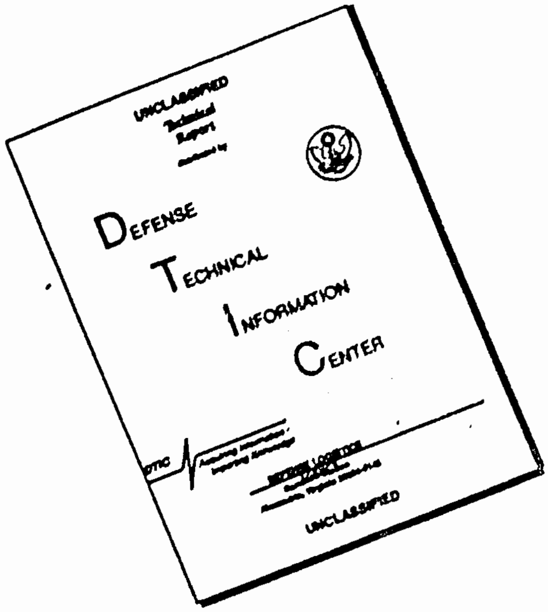

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THE PROBLEM OF ROUTING AIRCRAFT -- A MATHEMATICAL SOLUTION

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Mathematical techniques for the solution of many practical economic and operational problems have been developed very rapidly in recent years, stimulated in large part by the demands that World War II made upon military planners. A major portion of these new techniques, having basically similar mathematical characteristics, is called linear programming. Some types of transportation problems have been subjected to this kind of analysis over a period of several years, * and useful results have been obtained.

The purpose of the present paper is twofold. We want to present and explain a method for assigning a given fleet of several different types of aircraft to carry an anticipated traffic load over several routes at minimum cost. Secondly, we wish to illustrate the application of linear programming to transportation problems of this general type.

The problem which has been chosen, while simpler than the routing problem faced by an airline, is reasonably realistic. Much more complicated problems can be handled with basically the same techniques. However, they would not serve the present illustrative purposes.

The following problem has been chosen: A fixed fleet of four types of airplanes are to carry passengers over five routes. The objective is to develop a

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* See references at end of paper for information regarding theory and application of linear programming.

route pattern for the fleet which minimizes the cost of performing the transportation demanded, considering lost revenue as being equivalent to a cost.

Four types of airplanes have been used: Type A representing a post-war four-engine airplane, Type B representing a post-war two-engine airplane, Type C representing a pre-war two-engine airplane, and Type D representing a pre-war four-engine design. Variable costs per hour and per landing-take-off cycle have been roughly estimated to reflect the differences in out-of-pocket operating costs for the types of aircraft used.

The various types are assumed to operate at different speeds and to carry different loads on any given route. It is also assumed that certain types of aircraft cannot, because of range limitations, operate on particular routes, and further that some types of aircraft cannot carry their full payload on certain other routes.[1] To introduce a further element of heterogeneity into the problem it is assumed that the utilization obtained by each airplane type varies from route to route. It is assumed that on the two transcontinental routes 300 hours per month per airplane are obtainable. On the Dallas route, 285 and on the Boston route, 240 hours per month are assumed.

In this way we have set up a problem which, we hope, incorporates enough of the actual problems of routing to be interesting and one whose optimal solution is not obvious by inspection, while preserving enough simplicity to permit solution without large computing machines.

The assumed traffic demand and characteristics of the fleet and routes are shown in Tables 1 and 2.

1/ Specifically it is assumed that Type C cannot fly either Route 1 or Route 3; and Type B cannot fly Route 1. Also Type B is assumed to operate at 75 per cent of full payload capacity on Route 3 and Type D is assumed to operate at 80 per cent of full payload capacity on Route 1.

TABLE 1: ASSUMED AIRCRAFT FLEET AND TRAFFIC

Aircraft on Hand		Traffic Load		
Type	Number	Route	Number of Passengers 1/	Route Miles 2/
A	10	1. N.Y.-L.A. (1 stop)	25,000	2,475
B	19	2. N.Y.-L.A. (2 stop)	12,000	2,475
C	25	3. N.Y.-Dallas (0 stop)	18,000	1,381
D	15	4. N.Y.-Dallas (1 stop)	9,000	1,439
		5. N.Y.-Boston (0 stop)	60,000	185

1/ This is the anticipated number of full one-way trips per month to be carried on each route. If a passenger gets off en route and is replaced by another passenger, it is counted as one full trip.

2/ Official Airline Guide, July 1954, p. 276. The N.Y.-Los Angeles routes are via Chicago and via Chicago and Denver; the stop en route between New York and Dallas is at Memphis.

TABLE 2A: ASSUMED VARIABLE COSTS PER PASSENGER-MILE
(units = $.001)

Route / Aircraft Type	New York to Los Angeles 1-Stop	New York to Los Angeles 2-Stops	New York to Dallas 0-Stop	New York to Dallas 1-Stop	New York to Boston 0-Stop
A	4.5	5.7	4.5	4.7	6.4
B	—	6.4	8.3	6.3	8.8
C	—	9.2	—	9.3	11.3
D	7.4	6.1	5.9	6.2	8.1

TABLE 2B: ASSUMED VARIABLE COSTS AND PASSENGER CAPABILITIES
PER AIRCRAFT PER MONTH

Aircraft Type \ Route		New York to Los Angeles 1-Stop	New York to Los Angeles 2-Stops	New York to Dallas 0-Stop	New York to Dallas 1-Stop	New York to Boston 0-Stop
A	Passengers(00)	16	15	28	23	81
	Costs($000)	18	21	18	16	10
B	Passengers(00)	*	10	14	15	57
	Costs($000)		15	16	14	9
C	Passengers(00)	*	5	*	7	29
	Costs($000)		10		9	6
D	Passengers (00)	9	11	22	17	55
	Costs($000)	17	16	17	15	10

To recapitulate, the problem is to employ in an optimal fashion a fleet of 69 aircraft of four types to carry a total of 124,000 passengers per month over five routes.

The solution is accomplished in a simple mechanical fashion which in this case involves four approximations and the final solution. Each approximation embodies the following steps. A blank table is set up showing initially only the fleet of aircraft, the routes, and the passenger demand (in hundreds) on each route as given in Table 1. Then using Table 2, aircraft are assigned to each route until either all aircraft in the fleet are employed or all passengers wishing to travel are served. Then (as will be explained in a moment) the system is "costed" in implicit values or "shadow prices" and examined for situations in which a shift of aircraft between routes would reduce the costs. This process is repeated until no further opportunities for economizing remain. The resulting set of values can be proven to be optimal.

To illustrate in detail: Table 1 shows the demand on each route and the aircraft available to satisfy that demand. Table 2A shows which aircraft are most economical on each route. It will be noted that Type A is most economical in cost per passenger-mile on all routes. Hence, it was decided to allocate them first. Also since they are a long range airplane, they were assigned (Table 3) first to Route 1. The ten Type A's would, under our assumptions, carry less than the 25,000 passengers per month anticipated for that route. Hence, more aircraft are assigned, namely, enough Type D's (the only other airplane assumed capable of flying the route) to carry the remaining passengers. Table 3, thus, shows ten A's and ten D's assigned to the N.Y. - L.A. one-stop route.

Moving to Route 2, Type D is found (Table 2a) to be the most economical of the three types still not entirely committed. Therefore, the remaining five are assigned to that route and the remaining passengers are to be carried in

TABLE 3: FIRST APPROXIMATION

Route / Aircraft Type	New York to Los Angeles 1-Stop	New York to Los Angeles 2-Stops	New York to Dallas 0-Stop	New York to Dallas 1-Stop	New York to Boston 0-Stop
A	10				
B		6.5	12.5		
C				12.8	12.2
D	10	5			
Deficit (Passengers hundreds)			5		246

Type B's the more economical of the two types still available. Thus Table 3 shows 6.5 B's and 5 D's on the N.Y. - L.A. two-stop route. Aircraft assignments (such as 6.5 B's) need not come out in whole numbers since, for example, .5 B's is to be interpreted as assigning one aircraft to the route for one-half a month.

Turning to Route 3, there are no A's or D's left for assignment and the C is, by assumption, not capable of flying the route. Therefore, the remaining 12.5 Type B's are assigned to Route 3 and they accomodate all but 500 of the 18,000 passengers desiring to fly the route.

Only Type C's remain. 12.8 are assigned to Route 4, and this is adequate for the 9,000 passengers per month demanding travel on the route. The remaining 12.2 of that type are available for Route 5, but they are incapable of carrying all the N.Y. - Boston traffic, and 24,600 passengers per month are left unserved.

Note that the entire fleet of 69 aircraft are committed and that except on the N.Y. - Boston run the passenger demand is essentially filled. Further, although airplanes have been assigned in a very simple and somewhat arbitrary fashion the assignment is not obviously absurd: The longest range aircraft are assigned to the longest haul, the N.Y. - Los Angeles routes. The routes with the shortest critical legs have only Type C's. The N.Y. - L.A. two-stop and the N.Y. - Dallas no-stop routes with substantially equal critical legs have middle range Type C's and, on Route 2, D's.

Now the question is whether this is the best solution. If in any case costs of the whole system can be reduced by shifting aircraft between routes the solution is not optimal. To determine whether this is the case, the first tentative solution must be "priced out", and each of the activities to which no aircraft are assigned must be checked to determine whether their introduction into the solution would result in any economies.

The procedure for determining whether a particular set of activities is optimal may, upon first reading, sound rather complex, but it can be done by inspection very quickly for a small problem such as the one under discussion.

The process can best be explained by reference to Table 4-A. Each cell has three entries (zeros are represented by blanks). In the upper left corner of each cell is the number of aircraft assigned to the activity (from Table 3); in the center is the number of passengers (in hundreds) which one airplane of each type can carry on that route (from Table 2). In the bottom row is the monthly variable cost (in thousand of dollars) per aircraft assigned to the route (Table 2b). Thus the upper left cell indicates that ten Type A's will be alloted to Route 1 and that each will carry 1,600 passengers per month at a (variable) cost of $18,000. It will be convenient to denote the three entries in any row i, and in any column j, by the symbols.

X_{ij} (No. of Aircraft)

p_{ij} (100's passengers/Mo./A.C.)

c_{ij} (Cost, 1,000's dollars/Mo./A.C.

The variable cost of performing each activity, c_{ij}, is arbitrarily divided into two sets of implicit values, which may be construed to be the cost per airplane, u_i, column (7) and the cost per passenger, v_j, row (6). The u_i's may be considered as being the implicit opportunity cost of operating the i^{th} aircraft on any route and the v_j's the cost of transporting a passenger on the j^{th} route by any airplane. The determination of the assigned values of the v's and u's follows certain simple rules. In the first place the u_5's corresponding to surplus aircraft column (7) are always 0. All other u_i's are determined through the equations

$u_i + p_{ij} v_j = c_{ij}$ where only those (i, j) combinations are used that appear in that particular assignment of aircraft which is being costed. From Table 3 or 4, the permissible combinations are (1,1), (4,1), (2,2), (4,2), (2,3), (5,3), (3,4), (3,5), (5,5). Hence, $u_5 + p_{55} v_5 = c_{55}$ may be written

$$0 - v_5 = 1$$

$$v_5 = 1$$

In a similar fashion v_3 may be found to equal 7. Since there is only one cell in the fifth column to which aircraft are assigned and since the value of v_5 has been determined it is possible simply to ascertain the value of u_3,

$$u_3 = c_{35} - p_{35} v_5,$$

or

$$u_3 = 6 - 29 \cdot 1 = -23$$

Knowing u_3 it is possible to obtain the value of v_4 by solving for it with the data in cell (3,4). In a similar manner it is possible to determine the implicit costs ("shadow prices") in each cell.[1]

For the particular approximation shown in Table 4 the values of the u's and v's are, as shown in column 7 and row 6:

$u_1 = -174$	$v_1 = 12$
$u_2 = -82$	$v_2 = 9.7$
$u_3 = -23$	$v_3 = 7$
$u_4 = -91$	$v_4 = 4.6$
$u_5 = 0$	$v_5 = 1$
	$v_6 = 0$

Using these values it is possible to determine whether the introduction of any cell into the solution would increase the efficiency of the operation. In the event that for any cell the values $(u_i + p_{ij} v_j)$, computed as described, exceed the costs (c_{ij}) it is economical to introduce that cell. For the cell

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1/ It is not necessarily true that in all cases a row or column can be found in which only one remaining unknown appears. Situations may be encountered in which there are two unknowns with two equations which have to be solved.

TABLE 4A: INITIAL ASSIGNMENT
[Total Cost $1,197(000)]

Aircraft Type and Number	Passengers (1=100)	(1) N.Y.-L.A. 1-Stop	(2) N.Y.-L.A. 2-Stop	(3) N.Y.-Dallas 0-Stop	(4) N.Y.-Dallas 1-Stop	(5) N.Y.-Boston no-Stop	(6) Surplus Aircraft	(7) Implicit Values
		250	120	180	90	600		
(1) A 10		10 16 18	15 21	28 18	23 16	81 10	- -	$u_1 = -.74$
(2) B 19		*	6.5 10 15	12.5 14 16	15 14	57 9	- -	$u_2 = -82$
(3) C 25		*	5 10	*	12.8 7 9	12.2 29 6	- -	$u_3 = -23$
(4) D 15		10 9 17	5 11 16	22 17	17 15	55 10	- -	$u_4 = -91$
(5)	Deficit	1 13	1 13	5 1 7	1 7	246 1 1		$u_5 = 0$
(6) Implicit Values		$v_1 = 12$	$v_2 = 9.7$	$v_3 = 7$	$v_4 = 4.6$	$v_5 = 1$	$v_6 = 0$	

TABLE 4B: ADJUSTMENT OF INITIAL ASSIGNMENT

Aircraft Type and Number	Passengers (1=100)	(1) N.Y.-L.A. 1-Stop	(2) N.Y.-L.A. 2-Stop	(3) N.Y.-Dallas 0-Stop	(4) N.Y.-Dallas 1-Stop	(5) N.Y.-Boston no-Stop	(6) Surplus Aircraft
		250	120	180	90	600	
(1) A 10		10 16 18					- -
(2) B 19			6.5+1.10 10 15	12.5-1.10 14 16			- -
(3) C 25					12.8 7 9	12.2 29 6	
(4) D 15		10 9 17	5-0 11 16	0 22 17			
(5)				5-6.60 1 7		246 1 7	

(.,2) it can be seen by inspection that this is not the case, $(-174 + 15x9.7 < 21)$, and so on for the rest of the first row. Not until we come to the fourth row is it possible to introduce a new activity, in cell (4,3). There, the implicit values are -91 + 22·7 which considerably exceeds the cost of c_{43} = 17. In fact the difference of 45 (thousand dollars) represents the decrease in cost that will be acheived if one Type D can be devoted to the N.Y. - Dallas run and the corallary adjustments made in accordance with the requirements of the problem. It would also be profitable to introduce cell (4,4) since its implicit values total 27 as compared with a cost of 15. There is no other opportunity to introduce any activities at a profit.

Since cell (4,3) promises the greatest possible saving per unit we shall introduce it. It is not always desirable to introduce the cell which promises the greatest saving, but this is frequently a good guide. In any case where familiarity with the characteristics of the industry in question indicates that on common sense grounds some cell with an indicated saving less than the greatest is a more reasonable change in the existing set it may be well to try it. Such a procedure may result in a reduction in the number of iterations required to attain an optimum solution.

Refer to Table 4-B. How many units (aircraft) should be assigned to cell (4,3)? Let $X_{43} = \theta$ be the number of aircraft to be introduced. If θ Type D's are introduced into cell (4,3) these aircraft must come from other activities used in the initial assignment -- that is, by adjusting the values of X_{11}, X_{41}, X_{22}, X_{42}, X_{23}, X_{53}, X_{34}, X_{54}, X_{35}. From the first column with X_{11} = 10 it follows X_{41} = 10 to carry the 250 (hundred) passengers, and hence it follows that $X_{42} = 5 - \theta$. This in turn requires, in order to serve the 120 (hundred) passengers on route 2, that additional aircraft be assigned to route 2 from some other source, i.e., by increasing the aircraft assigned to cell (2,2). To provide the same passenger capacity in B's as has been withdrawn in D's it is necessary (under our assumptions) to add 1.1θ B's. These, in turn, must come from cell (2,3). The final adjustment is in cell (5,3) which follows from the known adjustments in (2,3) and (4,3) and the fact that the passengers carried by the B's and C's on route 3 plus those turned away must add to 180. The final step in this iteration is to determine the value of $X_{43} = \theta$. Since there will be a savings of \$45,000 for each unit increase in X_{43}, we shall make the assignment of aircraft to cell (4,3) as large as possible consistent with the obvious condition that the adjusted assignments in the other cells are not negative. For very large θ, the assignments in (2,3), (4,2), (5,3) would be negative and is clear that the first cell to go negative with increasing θ will be cell (5,3). Therefore, the maximum value that θ can have is that which provides sufficient additional capacity to carry 5 (hundred) passengers on route 3 when θ D's are added and 1.1θ B's are removed from the route. The value of θ is $5/(6.6) = .75$. For this assignment the value of X_{53} vanishes. However, the number of assigned values remaining are again 9. (Nine corresponds to four aircraft equations plus five passenger equations).

Enter into Table 5 in the upper left hand corner of each cell the value of X_{ij} obtained by setting $X_{43} = \theta = .75$ in Table 4. Leave entry X_{53} blank as well as other unassigned cells. The first iteration is now completed.

In a similar fashion it is possible to continue iterating as shown in Tables 5, 6, 7, and 8 until no cells offer an opportunity for further economy by introducing them into the solution. It can be shown rigorously that this is an optimum solution. In the present case an optimum was obtained after five iterations. The solution is presented in Table 8.

It is interesting to compare the original approximation (Table 4) with the final solution (Table 8). It is to be noted that the route with the longest stages (L.A. - N.Y. 1-stop) is served only by the 4-engine airplanes, Type A's and D's in both the first approximation and the solution. A significant change has been made, however, in the N.Y. - L.A., 2-stop route. In the first approximation this route is served by 6.5 B's and five D's. In the solution, no D's are used on this route, instead 7.8 C's and 8 B's are used. The D's have been transferred to the New York - Dallas 0-stop route, allowing a reduction in the B's on that route and their transfer to the New York - Dallas 1-stop route. The latter route, in turn, in the final solution is being served by Type B's exclusively rather than exclusively by C's as in the original set. All these transfers in effect allow an increase in the C's on Route 5 from 12.2 in the original to 17.2 in the optimal schedule. Consequently, the number of passengers not served on the New York - Boston route is reduced from 246 (hundred) to 100 (hundred). Similarly the passenger deficit on Route 3 has been eliminated.

Although little would be added to this exposition by describing the intermediate iterations in detail it may be interesting to review Tables 5 through 8 which describe them. Each iteration results in a reduction of costs as follows:

	Total "Variable Costs"
First Set	$1,197,000
Second Set	1,160,000
Third Set	1,088,000
Fourth Set	1,022,000
Fifth Set	1,000,000

The total savings then amount to $197,000.

It is possible for a person with some practice to perform each of the iterations in this problem in a matter of a few minutes. Some skill in such operations can be gained through practice and the use of a map. Some judicious

TABLE 5: SECOND ASSIGNMENT
[Total Cost $1,160,000], [$\epsilon$ = .7]

Aircraft Type and Number		(1) N.Y.-L.A. 1-Stop	(2) N.Y.-L.A. 2-Stop	(3) N.Y.-Dallas 0-Stop	(4) N.Y.-Dallas 1-Stop	(5) N.Y.-Boston no-Stop	(6) Surplus Aircraft	(7) Implicit Values
	Passengers (1=100)	250	120	180	90	600		
(1) A 10		10-0			0		-	u_1 = 18
(2) B 19			7.3+6.60	11.7-6.60			-	u_2 = 16
(3) C 25					12.8-3.30	12.2+3.30	-	u_3 = -23
(4) D 15		10+1.80	4.2-60	.8-4.20			-	u_4 = +17
(5)	Deficit					246-950		u_5 = 0
(6) Implicit Values		V_1 = 0	V_2 = -.1	V_3 = $\bar{o}$	V_4 = 4.6	V_5 = 1	V_6 = 0	

TABLE 6: THIRD ASSIGNMENT
[Total Cost $1,088,000], [$\theta = 1.95$]

Aircraft Type and Number	Passengers (1=100)	(1) N.Y.-L.A. 1-Stop	(2) N.Y.-L.A. 2-Stop	(3) N.Y.-Dallas 0-Stop	(4) N.Y.-Dallas 1-Stop	(5) N.Y.-Boston no-Stop	(6) Surplus Aircraft	(7) Implicit Values
		250	120	180	90	600		
(1) A 10	10	$9.3+.36\theta$			$.7-.36\theta$		-	$u_1 = -89.8$
(2) B 19	19		11.9	$7.1-\theta$	θ		-	$u_2 = -22$
(3) C 25	25				$10.5-.96\theta$	$14.5+.96\theta$	-	$u_3 = -23$
(4) D 15	15	$11.3-.64\theta$		$3.7+.64\theta$			-	$u_4 = -43$
(5)	Deficit					$180-29\theta$		$u_5 = 0$
(6) Implicit Values		$V_1 = 6.7$	$V_2 = 3.7$	$V_3 = 2.7$	$V_4 = 4.6$	$V_5 = 1$	$V_6 = 0$	

TABLE 7: FOURTH ASSIGNMENT
[Total Cost $1,022,000], [$\theta = 7.8$]

Aircraft Type and Number	Passengers (1=100)	(1) N.Y.-L.A. 1-Stop	(2) N.Y.-L.A. 2-Stop	(3) N.Y.-Dallas 0-Stop	(4) N.Y.-Dallas 1-Stop	(5) N.Y.-Boston no-Stop	(6) Surplus Aircraft	(7) Implicit Values
		250	120	180	90	600		
(1) A 10	10	10					-	$u_1 = 177$
(2) B 19	19		$11.9-.5\theta$	5.2	$1.9+.5\theta$		-	$u_2 = 54.55$
(3) C 25	25				$8.6-1.1\theta$	$16.4+.1\theta$	-	$u_3 = -23$
(4) D 15	15	10.1		4.9			-	$u_4 = -93$
(5)	Deficit					$123-2.9\theta$	-	$u_5 = 0$
(6) Implicit Values		$V_1 = 12.2$	$V_2 = 6.96$	$V_3 = 5.0$	$V_4 = 4.57$	$V_5 = 1$	$V_6 = 0$	

TABLE 8: FIFTH ASSIGNMENT (OPTIMAL)
[Total Cost $1,000,000]

Aircraft Type and Number	(1) N.Y.-L.A. 1-Stop	(2) N.Y.-L.A. 2-Stop	(3) N.Y.-Dallas 0-Stop	(4) N.Y.-Dallas 1-Stop	(5) N.Y.-Boston no-Stop	(6) Surplus Aircraft	(7) Implicit Values
Passengers (1=100)	250	120	180	90	600		
(1) A 10	10					-	$u_1 = 136$
(2) B 19		8	5	6		—	$u_2 = -51$
(3) C 25		7.8			17.2	-	$u_3 = -23$
(4) D 15	10		5			-	$u_4 = -94$
(5) Deficit					100	-	$u_5 = 0$
(6) Implicit Values	$V_1 = 8.55$	$V_2 = 6.6$	$V_3 = 5.5$	$V_4 = 4.33$	$V_5 = 1$	$V_6 = 0$	

guessing as to the most nearly optimal selection of the first set can reduce the number of iterations required, the exercise of common sense in selecting the cells to introduce in the process of iterating usually has the same result.

Considerably more complicated problems including more restrictions which provide greater realism can be handled with computing equipment, but this simple solution by inspection provides a useful method of handling small problems in transportation with a heterogeneous fleet of vehicles.

REFERENCES

1. Activity Analysis of Production and Allocation, T. C. Koopmans ed. John Wiley and Sons, 1951.

   (A) Koopmans, T. C., "A Model of Transportation", Chapt. XIV, p. 222.

   (B) Dantzig, George B., "Application of the Simplex Method to a Transportation Problem", Chapt. XXIII, p. 359.

2. Symposium on Linear Inequalities and Programming, Hq. USAF, Comptroller, 1 April 1952.

3. R. Dorfman, "Application of Linear Programming to the Theory of the Firm", University of California Press, 1951.

4. A. Charnes, W. W. Cooper, and A. Henderson, "An Introduction to Linear Programming", John Wiley and Sons, 1953.

5. A. Henderson and R. Schlaifer, "Mathematical Programming Better Information for Better Decision Making", Harvard Business Review, May-June, 1954.