NOTES ON MATRIX THEORY--IV
(An Inequality due to Bergström)

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P-469

21 December 1953

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SUMMARY

Two proofs are presented of an inequality due to Bergström.

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§1. Introduction.

In a recent note,* Bergstrom proved the following interesting inequality:

"Let A and B be positive definite matrices and let A_{ii}, B_{ii} denote the sub-matrices obtained by deleting the i^{th} row and column. Then

$$(1) \qquad \frac{|A+B|}{|A_{ii}+B_{ii}|} \geq \frac{|A|}{|A_{ii}|} + \frac{|B|}{|B_{ii}|} \, ,$$

where $| \ |$ represents the determinant."

Bergstrom's proof is essentially a verification. We present two proofs below, the second of which lays bare the origin of the result.

§2. First Proof:

The first proof is an immediate consequence of the result:

Lemma 1: If A is positive definite, then

$$(1) \qquad \phi(A) = \frac{|A|}{|A_{ii}|} = \operatorname*{Min}_{x} \sum_{i,j=1}^{N} a_{ij} x_i x_j \, ,$$

where x is constrained by

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* H. Bergstrom, "A Triangle Inequality for Matrices," Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949; Oslo, 1952; pp. 264-267.

(2) $\quad x_i = 1.$

From this it is clear that $\phi(A+B) \geq \phi(A) + \phi(B)$.

We shall not present the proof, which is easily obtained by the use of a Lagrange multiplier, since Lemma 1 is a special case of the more general result established in the next section.

§3. <u>Second Proof</u>:

We begin by establishing

<u>Lemma</u> 2: <u>If</u> A <u>is positive definite, then</u>

(1) $$(x, Ax)(y, A^{-1}y) \geq (x, y)^2,$$

<u>for all</u> x <u>and</u> y.

Here (x,y) denotes the inner product of x and y and (x, Ax) the quadratic form $\sum_{i,j} a_{ij}x_i x_j$. A^{-1} is the inverse of A.

<u>Proof of Lemma</u> 2: Reduce A to diagonal form by an orthogonal matrix T, <u>i.e.</u>, $T'AT = L$, $T' = T^{-1}$. Let $x = Tu$, $y = Tv$. Then (1) becomes

(2) $$\left(\sum_{i=1}^{N} \lambda_i u_i^2\right)\left(\sum_{i=1}^{N} v_i^2/\lambda_i\right) = \sum_{i=1}^{N} u_i v_i \,,$$

which is the Cauchy-Schwarz inequality.

Since the inequality becomes an equality for suitable choice of x, we have

(3) $$\operatorname*{Min}_{x} \frac{(x, Ax)}{(x, y)^2} = \frac{1}{(y, A^{-1}y)} = \psi(A).$$

From this it is immediate that

$$(4) \qquad \psi(A+B) \geq \psi(A) + \psi(B).$$

The case $y_i = 1$, $y_j = 0$, $j \neq i$ yields Bergstrom's result.

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12-21-53.