ON CERTAIN GAMES WITH TRANSCENDENTAL VALUES

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Let $\Gamma$ be a two person zero-sum game for which the compact pure strategy spaces, $S_1$ and $S_2$, and the payoff function $M$, defined over $S_1 \times S_2$, are definable in Tarski's system of "elementary algebra" (see [1]). Suppose, also, that $\Gamma$ has a value which is a transcendental number. We can then conclude that there is no optimal strategy for either player consisting of a step function of finitely many steps (i.e. a distribution in which the probabilities are all concentrated on a finite set of points). For, suppose the contrary for one of the players, say the maximizing one. Then, for some positive integer $m$, the value of $\Gamma$ is given by

$$ v = \max_{\langle \alpha_1, \ldots, \alpha_m \rangle \in \mathcal{N}_m} \max_{x_1, \ldots, x_m \in S_1} \min_{y \in S_2} \sum_{i=1}^{m} \alpha_i M(x_i, y), $$

where $\mathcal{N}_m$ is the set of all $m$-tuples $\langle \alpha_1, \ldots, \alpha_m \rangle$ such that $\alpha_i \geq 0$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \alpha_i = 1$. But, according to [1], $v$ would be algebraically definable, and it is a principal result of [1] that every algebraically definable number is algebraic.

In particular, our result applies to any game with transcendental value, in which $M$ is a continuous rational function with integral coefficients.
Example: Take $M(x,y) = \frac{(1+x)(1-y)(1-xy)}{(1+xy)^2}$, $S_1 = \{x | 0 \leq x \leq 1\}$, and $S_2 = \{y | 0 \leq y \leq 1\}$. Here, $v = \frac{1}{n}$, and a pair of distribution functions yielding this value is given by:

$$
\begin{align*}
F^*(x) &= \frac{x}{n} \arctan \sqrt{x} \\
G^*(y) &= \frac{x}{n} \arctan \sqrt{y}
\end{align*}
$$

Thus, in this game, there is no optimal strategy consisting of a step function of finitely many steps, for $n$ is a transcendental number.

Reference