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Phase Shift by Periodic Loading of Waveguide

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PHASE SHIFT BY PERIODIC LOADING OF WAVEGUIDE

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ABSTRACT

Periodic loading of a transmission line is considered in terms of a
discrete number of identical sections in cascade. For n sections there are
(n-1) discrete solutions, i.e., (n-1) spacings each less than half wavelength,
between identical susceptances, which produce input match. Formulas are
given for locating these (n-1) roots and for evaluating phase shifts. Some
numerical examples are worked out.

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PHASE SHIFT BY PERIODIC LOADING OF WAVEGUIDE

I. INTRODUCTION

The analysis arose out of the following problem: two linearly polarized signals at frequencies $f_1$, $f_2$ are to be launched in a common waveguide, are transmitted through a periodically loaded section, and emerge with the signals at $f_1$, say, right-hand circularly polarized while those at $f_2$ are left-hand circularly polarized; at each frequency the input match is to be of unity VSWR and the phase shifting section is to be short. Hardware-wise the solution has not been attempted. However, because the basic building block is somewhat different, and because the analysis stresses different aspects, it is believed that the theoretical results obtained are worthwhile.

The analysis is carried out in terms of the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - matrix approach and is based on a discrete number of basic units in cascade. The latter may appear to be a serious restriction — but it is not, since the solutions so obtained contain the others as special cases.

II. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - MATRIX OF THE BASIC UNIT

The basic unit is illustrated in Fig. 1. The susceptance, $jB$, is assumed to be lumped.

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* Suggested by L. J. Ricardi.
If \( \begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = (u) \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} \), then

\[
u = \begin{pmatrix} \cos \frac{\theta}{2} & jZ \sin \frac{\theta}{2} \\ jY \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ jB & 1 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & jZ \sin \frac{\theta}{2} \\ jY \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\] (1)

Performing multiplication of the matrices above, and using appropriate trigonometric identities, results in:

\[
A = \cos \theta - \frac{BZ}{2} \sin \theta,
\]

\[
B = j \left[ Z \sin \theta + 1/2 \ BZ^2 \cos \theta - 1/2 \ BZ^2 \right],
\]

\[
C = j \left[ Y \sin \theta + \frac{B}{2} \cos \theta + \frac{B}{2} \right],
\]

\[
D = A.
\]

Let the normalized characteristic impedance, \( Z \), equal unity. Then:

\[
A = D = \cos \theta - \frac{B}{2} \sin \theta,
\]

\[
B = j \left[ \sin \theta + \frac{B}{2} (\cos \theta - 1) \right],
\]

\[
C = j \left[ \sin \theta + \frac{B}{2} (\cos \theta + 1) \right].
\] (2)

III. \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) - MATRIX OF \( n \) SECTIONS IN CASCADE, \( \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \).

The guide, periodically loaded with normalized susceptance \( jB \), is shown in Fig. 2.
The over-all two-port equivalent for \( n \) sections in cascade is obtained by raising the matrix \((u)\) of Eq. (1) to the \( n \)-th power. This can be done in general terms. The results are:

\[
\begin{align*}
A_n &= \cosh an, \\
B_n &= B \frac{\sinh an}{\sinh a}, \\
C_n &= C \frac{\sinh an}{\sinh a}, \\
D_n &= A_n, \\
\end{align*}
\]  

(3)

where \( n \) = integral number of sections,

\( a = \cosh^{-1} A \), and

\( A, B, C \) are defined by Eq. (2).

IV. "ROOTS" OF INPUT IMPEDANCE

In terms of the generalized \( n \)-section matrix elements \( A_n, B_n, C_n, D_n \) with unity load termination the input impedance is:
\[ Z_{in(n)} = \frac{A_n + B_n}{D_n + C_n} = \frac{\cosh an + B_n \frac{\sinh an}{\sinh a}}{\cosh an + C_n \frac{\sinh an}{\sinh a}} \]  \hspace{1cm} (4)

Since it is desired to have \( Z_{in(n)} = 1 \), then \( B_n \) must equal \( C_n \). This leads to two solutions:

1. \( B = C \), giving the trivial solution \( B = 0 \), and
2. \( B_n = C_n = 0 \), requiring that

\[ \frac{\sinh an}{\sinh a} = 0. \]  \hspace{1cm} (5)

In Reference 3 it is shown that

\[ \frac{\sinh an}{\sinh a} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \]  \hspace{1cm} (6)

where \( \lambda_1, \lambda_2 \) are the two non-degenerate eigenvalues of the characteristic equation of the matrix Eq. (1), which in this application reduces to:

\[ \lambda_1 = \frac{A}{2} \pm \sqrt{\frac{A^2}{2} - 1}, \quad \text{with} \]  \hspace{1cm} (7)

\[ A = \cos \theta - \frac{B}{2} \sin \theta \]

In Eq. (6) it is possible to cancel \( \lambda_1 - \lambda_2 \), leaving a polynomial of the \((n-1)\) order, which is then set equal to zero. For \( n \) sections in cascade then, Eq. (6) gives \((n-1)\) principal roots, so that there are \((n-1)\) values of \( A \), or of spacing \( \theta \), which will give an input impedance of unity. Naturally, \( n=1 \) is excluded, for this corresponds to the trivial solution \( B = 0 \).
Using Eq. (6), the polynomial for any \( n \) is:

\[
\lambda_1^{n-1} + \lambda_1^{n-2} \lambda_2 + \lambda_1^{n-3} \lambda_2^2 + \lambda_1^{n-4} \lambda_2^3 + \cdots + \lambda_1 \lambda_2^{n-2} + \lambda_2^{n-1}.
\]

This may be reduced by remembering that \( \lambda_1 \lambda_2 = 1 \), and finally factoring, where possible. Table I below lists the polynomials for \( n \) ranging from 2 to 10.

### Table I

Factors of \( \frac{\sinh an}{\sinh a} = 0 \) for \( n = 2 \) to 10.

<table>
<thead>
<tr>
<th>( n )</th>
<th>factors of ( \frac{\sinh an}{\sinh a} = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>((\lambda_1 + \lambda_2))</td>
</tr>
<tr>
<td>3</td>
<td>((\lambda_1^2 + 1 + \lambda_2^2))</td>
</tr>
<tr>
<td>4</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^2 + \lambda_2^2))</td>
</tr>
<tr>
<td>5</td>
<td>((\lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4))</td>
</tr>
<tr>
<td>6</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^2 + 1 + \lambda_2^2) (\lambda_1^2 - 1 + \lambda_2^2))</td>
</tr>
<tr>
<td>7</td>
<td>((\lambda_1^6 + \lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4 + \lambda_2^6))</td>
</tr>
<tr>
<td>8</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^2 + \lambda_2^2) (\lambda_1^4 + \lambda_2^4))</td>
</tr>
<tr>
<td>9</td>
<td>((\lambda_1^2 + 1 + \lambda_2^2) (\lambda_1^6 - 1 + \lambda_2^6))</td>
</tr>
<tr>
<td>10</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4) (\lambda_1^4 - \lambda_1^2 + 1 - \lambda_2^2 + \lambda_2^4))</td>
</tr>
</tbody>
</table>

Study of the table reveals some interesting and obvious facts:

1. For \( n \) sections there are \((n-1)\) solutions.
2. For \( n = 4 \), the \( n = 2 \) solution is contained, as well as two others.
not directly related to either \( n = 2 \) or 3.

3. If \( n \) can be factored into products of lesser integers, then there will be roots correspondingly specified by the lesser integers, i.e., if \( n = 6 = 2 \times 3 \), one root coincides with \( n = 2 \), two roots are given by \( n = 3 \), and two others.

Figure 3 illustrates the distribution of the roots of Table I for a specific case of \( B = +1 \) (capacitive susceptance). It is quite clear that the single root of \( n = 2 \) appears in \( n = 4, 6, 8, 10 \); the two roots of \( n = 3 \) are repeated in \( n = 6, 9 \); \( n = 10 \) contains roots corresponding to \( n = 2, 5 \). Furthermore, for any \( n > 2 \), the \((n-1)\) roots seem to appear in "mirror-image" pairs about the \( n = 2 \) root. That this is true will be proved in Section V.

Although the preceding gives usable results, a more meaningful method for evaluation of roots was suggested by Dr. R. N. Assaly.

\[
\sinh an = \frac{\lambda_1^n - \lambda_2^n}{2} = \frac{e - e^{-1}}{2} = 0. \quad \text{This equation is satisfied by}
\]

\[
a = j\pi \frac{m}{n} \quad (m = \pm 1, 2, 3, \ldots).
\]

Similarly, from \( \sinh a = 0 \), \( a = j\pi \ell \quad (\ell = 1, 2, 3, \ldots). \) But these values must be excluded, otherwise the ratio is infinite.

Hence all values of \( a \) which give integer values of \( j\pi \) are to be excluded, or, the roots are given by

\[
a = j\pi \frac{m}{n} \quad (m = \pm 1, 2, 3, \ldots, n - 1).
\]

Furthermore, \( \cosh (\pm x') = \cosh x \), if \( x \) real, or \( \cosh (\pm jx') = \cos (\pm x') = \cos x' \), if \( x' \) real, therefore only the + sign need be used. Finally, then,
the principal roots are given by:

\[ a = j\pi \frac{m}{n} \ (m = 1, 2, 3, \ldots, n - 1) = ja'_{nm}. \]  \hspace{1cm} (8)

If \( m \) were allowed to exceed \( n \), say \( m' = n + m \), then \( a_{nm'} = j\pi \left(1 + \frac{m}{n}\right)\)

\[ = j\pi + a_{nm}, \] which are the principal roots augmented by \( \pi \).

In Eq. (8), the roots are specified in terms of \( a = \cosh^{-1} \frac{A}{A} = \cosh^{-1} \)

\([\cos \theta - \frac{B}{2} \sin \theta] \). To specify the roots in terms of \( \theta \) the following may be done:

\[ \cosh a = \cosh a_{nm} = \cosh ja'_{nm} = \cos a'_{nm}. \] Therefore,

\[ \cos \theta - \frac{B}{2} \sin \theta = \sqrt{1 + \frac{B^2}{4}} \cos (\theta + \tan^{-1} \frac{B}{2}) = \cos a'_{nm}, \] or

\[ \theta_{nm} = \cos^{-1} \left[ \frac{\cos a'_{nm}}{\sqrt{1 + \frac{B^2}{4}}} \right] - \tan^{-1} \frac{B}{2}. \] \hspace{1cm} (9)

\[ \theta_{nm} = \cos^{-1} \left[ \frac{\cos a'_{nm}}{\sqrt{1 + \frac{B^2}{4}}} \right] - \tan^{-1} \frac{B}{2}. \] \hspace{1cm} (10)

In Eq. (10) principal values only are included.

V. SOME RELATIONS AMONG ROOTS

It will now be shown that the root-pair \( \theta_{nm} \) and \( \theta_{n-n-m} \) are mirror-images in the root \( \theta_{21}' \) corresponding to \( n = 2 \).

\[ a'_{n,m} = \frac{\pi m}{n} = \frac{\pi}{2} \left(\frac{2m}{n}\right) = \frac{\pi}{2} \left[1 - \frac{n - 2m}{n}\right], \ (2m \leq n). \]

\[ a'_{n,n-m} = \frac{\pi}{2} \left[1 - \frac{n - 2(n-m)}{n}\right] = \frac{\pi}{2} \left[1 + \frac{n - 2m}{n}\right]. \]

\[ \cos a'_{n,m} = 0 + \sin \frac{\pi}{2} \left(\frac{n - 2m}{n}\right), \]

\[ \cos a'_{n,n-m} = 0 - \sin \frac{\pi}{2} \left(\frac{n - 2m}{n}\right). \] \hspace{1cm} (11)
From Eqs. (10) and (11) it may be shown that:

\[ \theta_{n,m} + \tan^{-1} \left( \frac{B}{2} \right) = \frac{\pi}{2} - \cos^{-1} \sqrt{1 - \frac{\sin \frac{2\pi}{2} \left( \frac{n}{n} - \frac{2m}{n} \right)}{1 + \frac{B^2}{4}}}, \]

\[ \theta_{n,n-m} + \tan^{-1} \left( \frac{B}{2} \right) = \frac{\pi}{2} + \cos^{-1} \sqrt{1 - \frac{\sin \frac{2\pi}{2} \left( \frac{n}{n} - \frac{2m}{n} \right)}{1 + \frac{B^2}{4}}}, \]

and

\[ \theta_{n,m} + \cos^{-1} \sqrt{\frac{\cos \frac{2\pi}{2} \left( \frac{n}{n} - \frac{2m}{n} \right) + \frac{B^2}{4}}}{1 + \frac{B^2}{4}} = \theta_{n,n-m} - \cos^{-1} \sqrt{\frac{\cos \frac{2\pi}{2} \left( \frac{n}{n} - \frac{2m}{n} \right) + \frac{B^2}{4}}}{1 + \frac{B^2}{4}} = \frac{\pi}{2} - \tan^{-1} \left( \frac{B}{2} \right) = \theta_{21}. \]

In Eq. (10) it is tacitly assumed that the susceptance, B, is positive.

How are the roots distributed if B were negative?

Equation (8) is not altered by reversal in the sign of B, whereas Eq. (10) is. Using Eq. (12),

For B positive, write

\[ \theta_{nm} + \tan^{-1} \left( \frac{B}{2} \right) = \frac{\pi}{2} - \cos^{-1} \sqrt{1 - \frac{\sin \frac{2\pi}{2} \left( \frac{n}{n} - \frac{2m}{n} \right)}{1 + \frac{B^2}{4}}}, \]
For \( B \) negative,

\[
\theta_{nm} = \tan^{-1} \left( \frac{B}{2} \right) = \pi - \cos^{-1} \sqrt{1 - \frac{\sin^2 \frac{2\pi (n - 2m)}{2}}{n} \cdot \frac{1 + B^2}{4}}. \tag{14}
\]

From Eq. (14) it is obvious that \( \theta_{nm}^+ \) and \( \theta_{nm}^- \) form a root-pair which are mirrored in

\[
\tan^{-1} \left( \frac{B}{2} \right) = \pi + \cos^{-1} \sqrt{1 - \frac{\sin^2 \frac{2\pi (n - 2m)}{2}}{n} \cdot \frac{1 + B^2}{4}}. \tag{15}
\]

Now, if \( m = n - m \) in \( \theta_{nm}' \), then:

\[
\theta_{nm}, n-m = \tan^{-1} \left( \frac{B}{2} \right) = \pi + \cos^{-1} \sqrt{1 - \frac{\sin^2 \frac{2\pi (n - 2m)}{2}}{n} \cdot \frac{1 + B^2}{4}}. \tag{16}
\]

From Eqs. (14) and (15), it is easily shown that the root-pair \( \theta_{nm}^+ \) and \( \theta_{nm}^- \) are supplementary,

\[
\theta_{nm}^+ + \theta_{n, n-m}^- = \pi. \tag{16}
\]

Equation (16) is of importance in considerations of use of a guide supporting two modes in space quadrature where a single discontinuity may set up susceptances of different sign in each mode. In such a case, if the magnitude of the susceptance be equal for each mode, since the number of sections, \( n \), is identical for both modes, an ideal situation exists only when
\[ \theta = \theta_m^{+} = \theta_m^{-} = \frac{\pi}{2} \text{ for } \lambda^+ = \lambda^- \text{.} \]

From Fig. 3, it is clear that this can occur only when \( n \) is even and \( B \) approaches zero.

**VI. INSERTION PHASE OF \( n \) CASCADED SECTIONS, AS FUNCTION OF \( \theta \).**

The complex insertion voltage ratio \( R \) between matched generator and load is given by

\[
R = \frac{1}{2} \left[ \frac{1}{A_n^2} + \frac{1}{B_n^2} + \frac{1}{C_n^2} + \frac{1}{D_n^2} \right] = \cos \alpha + j \left[ \sin \theta + \frac{B}{2 \cos \theta} \right] \frac{\sinh \alpha}{\sinh \alpha},
\]

and the insertion phase shift, \( \beta \), is given by,

\[
\tan \beta = \frac{\im R}{\re R} = \left[ \sin \theta + \frac{B}{2 \cos \theta} \right] \frac{\tanh \alpha}{\sinh \alpha}.
\]

(17)

In view of Eq. (9) it is easily shown that

\[
\sin \theta + \frac{B}{2 \cos \theta} = \sqrt{1 + \frac{B^2}{4} - A^2} = 0. \text{ Hence,}
\]

\[
\tan \beta = \sqrt{1 + \frac{B^2}{4} - A^2} \frac{\tanh \alpha}{\sinh \alpha}.
\]

(18)

**Case A:** \( |A| < 1 \).

When \( |A| < 1 \), \( \cosh \alpha = \cosh j a' = \cos \alpha' = A = \cos \theta - \frac{B}{2} \sin \theta \), so that

\[
a' = \cos^{-1} A; \quad \frac{\tanh \alpha}{\sinh \alpha} = \frac{\tan a'n}{\sin a'} \quad \text{and}
\]

10
\[ \tan P = \sqrt{1 + \frac{B^2}{4} - \frac{A^2 \tan a'n}{\sin a'}}. \]  \hspace{1cm} (19)

**Case B** \( A = +1. \)

\[ a' = \cos^{-1} 1 = 2\pi p \quad (p = 0, 1, 2-\cdots). \]

\[ \frac{\tan a'n}{\sin a'} \to n. \]

\[ \therefore \tan P = \frac{B}{2} \cdot n. \]  \hspace{1cm} (20)

**Case C** \( A = -1. \)

\[ a' = \cos^{-1} (-1) = \pi q \quad (q = 1, 3, 5-\cdots). \]

\[ \frac{\tan a'n}{\sin a'} \to -n. \]

\[ \therefore \tan P = -\frac{B}{2} \cdot n. \]  \hspace{1cm} (21)

**Case D** \( A = 0. \)

\[ a' = \cos^{-1} 0 = \frac{\pi}{2} r \quad (r = 1, 3, 5-\cdots). \]

\[ \tan a'n = \pm \tan \left(\frac{\pi}{2} n\right) = 0 \text{ if } n \text{ even}, \quad \pm \infty \text{ if } n \text{ odd}. \]

\[ \therefore P = \begin{cases} \pi q \quad (q = 0, 1, 2, \cdots) & \text{for } n \text{ even}, \\ \frac{\pi}{2} r \quad (r = 1, 3, 5-\cdots) & \text{for } n \text{ odd}. \end{cases} \]  \hspace{1cm} (22)
Case E \( A > 1 \).

\[
\cosh a = A, \text{ which is equivalent to } e^{a} = A + \sqrt{A^2 - 1}, \text{ or }
\]

\[
a_1, 2 = \ln (A + \sqrt{A^2 - 1}) + j 2\pi p. \text{ But } a_2 = -a_1, \text{ therefore select } a_1.
\]

\[
\tan P = \sqrt{1 + \frac{B^2}{4} - A^2} \frac{\tanh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]}{\sinh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]},
\]

(23)

Since \( 1 + \frac{B^2}{4} > A^2 \), the radical is > 0, therefore \( 0 < P < \frac{\pi}{2} \), or, augmented by \( \pi p \).

Case F \( A < -1 \)

Let \( \underline{A} = -A \). Then \( e^{a} = (A + \sqrt{A^2 - 1}) e^{j\pi} \).

Selecting the positive radical, as before,

\[
\tan P = -\sqrt{1 + \frac{B^2}{4} - A^2} \frac{\tanh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]}{\sinh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]},
\]

(24)

where \( \frac{\pi}{2} < P > \pi \); or, augmented by \( \pi p \).

VII. PHASE SHIFT AT ROOT VALUES

For any \( n > 1 \), the root-values are specified by Eq. (8); \( a_{nm} = ja_{nm} \)

\[
= j\pi \frac{m}{n} \quad (m = 1, 2, \ldots, n - 1). \text{ Therefore, since } |A| = |\cos a'| \leq 1, \text{ Eq. (19)}
\]

is applicable:

\[
\tan P = \sqrt{1 + \frac{B^2}{4} - A^2} \frac{\tan a'_{nm} \cdot n}{\sin a'_{nm}}.
\]

(19)

But \( \tan a'_{nm} \cdot n = \tan \pi m = 0 \), whereas \( \sin a'_{nm} \neq 0 \). Therefore \( P = \pi p \) \((p = 0, 1, 2, \ldots)\). It is impossible to say which value of \( p \) is to be
chosen for a particular $a'_{nm}$.

However, in the Appendix it is shown that when any symmetrical, loss-less two-port is equated to a transmission line of characteristic impedance $Z_0 = \frac{1}{Y_0}$ and electrical length $\phi$, then

$$\cos \phi = A_n = \cosh an = \cosh ja'n = \cos a'_{nm} \cdot n. \quad (25)$$

From the above

$$\phi_{nm} = a'_{nm} \cdot n. \quad (26)$$

Consequently, in ascending order, $\phi_{nm} = \pi, 2\pi, 3\pi, \ldots, (n-1)\pi$, or, returning to Eq. (19),

$$P_{nm} = \pi, 2\pi, 3\pi, \ldots, (n-1)\pi.$$

The physical interpretation of this is the following:

Given a configuration as shown in Fig. 2, if the frequency is varied above cut-off of the guide, the first "resonance" for the entire structure occurs when the insertion phase, $P$, is $\pi$; the second "resonance" is at $P = \pi \cdot 2$, etc., up to $\pi (n - 1)$.

It should perhaps be pointed out that the roots $a'_{nm}$ are determined by the number of discontinuities, $n$, only. $\phi_{nm}$, on the other hand, is a function of both $a'_{nm}$ and the susceptance, $B$.

If it is assumed that the susceptance, $B$, is invariant with frequency, and that the loading is only for one of two orthogonal modes of an otherwise symmetric guide (i.e., $\lambda g_1 = \lambda g_2$), then the incremental phase shift, phase shift in
loaded mode minus phase shift in unloaded mode, for \( n \) sections in cascade, is given by:

\[
\Delta \theta = P_{nm} - n \cdot \theta_{nm}.
\]  

(27)

Figure 4 shows a plot of \( \Delta \theta \) for \( B = +1 \), and \( n \) ranging from 2 to 10. From the figure it is clear that it is impossible to get a \( \Delta \theta \) greater than one wavelength, except for \( n = 10 \) and \( m = 9 \), or operation at root \( a_{10} \). The figure also shows that if \( n = 2 \) is used as a unit, then cascading five such units, gives \( 0.74\lambda \)g incremental phase shift, for the root \( \theta_{10, 5} = \theta_{21} \) (See Fig. 3).

In Fig. 4 it should not be concluded that the incremental phase varies linearly when moving from one root to the next. In the next section the input voltage reflection coefficient as a function of \( \theta \) is evaluated. Since \( \Gamma \neq 0 \) at \( \theta = \theta_{nm} \), between roots the phase shift will be non-linear.

VIII. INPUT VOLTAGE REFLECTION COEFFICIENT, \( \Gamma_{in} \), FOR n CASCADED SECTIONS, AS FUNCTION OF \( \theta \).

\[
\Gamma_{in} = \frac{Z_{in} - 1}{Z_{in} + 1} = \frac{B_n - C_n}{A_n + B_n + C_n + D_n} = \Gamma e^{j\gamma}
\]

\[
\Gamma = \sqrt{\frac{\cosh a \cdot \cosh b}{\sinh a \cdot \sinh b}} \cdot \exp \left[ \frac{\pi}{2} \tan^{-1} \left( \frac{\sin \theta + \frac{B}{2} \cos \theta}{\sinh^2 \frac{\gamma}{2}} \right) \right].
\]

(28)

The input VSWR, \( q = \frac{1 + \Gamma}{1 - \Gamma} \).

(29)
Case A: \( |A| < 1 \).

\( a' = \cos^{-1} A \), as in Section VI-A, and

\[
\Gamma = \frac{B \tan a'n}{2 \sin a'} \sqrt{\left[ 1 + \left( 1 + \frac{B^2}{4} - \frac{A^2}{\sin^2 a'} \right) \frac{\tan^2 a'n}{\sin^2 a'} \right]^{1/2}}.
\]

Case B: \( A = 0 \).

\( a' = \cos^{-1} 0 = \pm \frac{\pi}{2} \).

\[
\tan a'n = \begin{cases} 0 & \text{if } n \text{ even,} \\ \pm \infty & \text{if } n \text{ odd.} \end{cases}
\]

\( \sin a' = \pm 1 \).

\( \Gamma = 0 \), when \( n \) is even, and

\[
\Gamma = \frac{B/2}{1/2}, \text{ when } n \text{ is odd.}
\]

The above only corroborates the fact that when \( n \) is even the root corresponding to \( A = 0 \) is identical to the single root for \( n = 2 \).

Case C: \( A = \pm 1 \).

\( a' = 2\pi p (p = 0, 1, 2---) \).

\[
\frac{\tan a'n}{\sin a'} \rightarrow n, \text{ and}
\]
\[ \Gamma = \frac{B n}{2^{\frac{n}{2}}} \cdot \frac{1}{\sqrt{1 + \frac{B^2}{4n^2}}} \]  
(32)

**Case D:** \( A = -1 \)

\[ a' = \pi q \quad (q = 1, 3, 5, \ldots) \]

\[ \frac{\tan a'n}{\sin a'} \to -n, \text{ and} \]

\[ \Gamma = \frac{-n \frac{B}{2}}{\sqrt{1 + \frac{B^2}{4n^2}}} \cdot \]  
(33)

**Case E:** \( |A| > 1 \).

\( A > 1, \quad a = a_1 = \ln (A + \sqrt{A^2 - 1}) + j 2\pi p \quad (p = 0, 1, 2, \ldots) \)

\( A < -1, \quad a = a_2 = \ln (A - \sqrt{A^2 - 1}) + j \pi q \quad (q = 1, 3, 5, \ldots) \)

\[ \frac{\tanh a_1n}{\sinh a_1} = \frac{\tanh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]}{\sinh \left[ \ln (A + \sqrt{A^2 - 1}) \right]} \]

\[ \frac{\tanh a_2n}{\sinh a_2} = -\frac{\tanh \left[ n \ln (A - \sqrt{A^2 - 1}) \right]}{\sinh \left[ \ln (A - \sqrt{A^2 - 1}) \right]} \]
\[ \Gamma (A > 1) = \frac{\tanh \left[ \frac{n \ln (A + \sqrt{A^2 - 1})}{\sinh \left[ \ln (A + \sqrt{A^2 - 1}) \right]} \right] B}{2} \]

\[ \left[ 1 + \left( 1 + \frac{B^2}{4} - A^2 \right) \tanh^2 \left[ \frac{n \ln (A + \sqrt{A^2 - 1})}{\sinh^2 \left[ \ln (A + \sqrt{A^2 - 1}) \right]} \right] \right]^{1/2} \]

and

\[ \Gamma (A < -1) = - \Gamma (A > 1) \] (35)

Figures 5, 6 and 7 illustrate the behavior of input VSWR VS. \( \theta \) for \( n = 10, 11, \) and 17, each for two constant values of \( B, \) i.e., .9 and 1.0. In each case, with \( n \) constant, increasing \( B \) compresses the roots and increases the VSWR. The minimum VSWR peak occurs between roots \( \theta_n \) and \( \theta_{n - 1} \); on either side the VSWR peaks are symmetrically distributed, i.e.,

\[ \theta_n = \frac{n}{2} + 1 \]

the peak between first and second root is equal to that between next to last and last. With \( B \) constant and \( n \) increasing, there is a tendency to lower the minimum VSWR peak slightly, while increasing the side peaks.

Figure 8 shows a plot of the incremental phase shift, \( \Delta \phi \), as a function of \( \theta \), for the conditions of Figs. 5, 6 and 7. In each case \( \frac{\Delta \phi}{\Delta \theta} \) is maximum at root values \( \theta_{nm} \) (where the VSWR is unity) and is minimum (zero, practically) over a wide range in between roots (where the VSWR is greatest). This behavior would pose a serious problem in implementing a matched, low axial ratio, orthogonal-circularly polarized dual frequency polarizer. As an
example, consider that at a frequency $f_1$ the polarizer (dual-mode guide) is to give circular polarization (say $\Delta \phi = 270^\circ$) of low VSWR and circularity, and at a frequency $f_2$ the polarization should correspond to a $\Delta \phi$ of $450^\circ$. At the lower frequency $f_1$, $\lambda g_1$, $B$, $n$ and $\theta_{nm}$ must be selected to give a $\Delta \phi$ of $270^\circ$. For small $\theta_{nm}$ values interaction between sections can be expected. At the higher frequency $f_2$, $n$ is already pre-selected; $B$ is no longer constant; $\lambda g_2$ must be exactly right to give a root $\theta_{nm}$ $-\lambda g_1$ otherwise the attendant mismatch sets up an $H/V$ ratio which is not unity and $\Delta \phi$ is no longer $450^\circ$.

IX. CONCLUSION

A basic unit, as in Fig. 1, when cascaded $n$ times gives a wide choice of element spacings, which are less than $180^\circ$, to produce unity input VSWR and a wide range of insertion phase.
Figure 3

$\theta_{nm}$ vs. $n$. 

$n =$ NO. SECTIONS

$\theta =$ SEPARATION BETWEEN SECTIONS

$B =$ +1

$\theta/\Delta \theta =$ SPACING TO GIVE INPUT MATCH

POINT NUMBERS CORRESPOND TO ROOT NUMBER

NUMBER OF SECTIONS, $n$

19
SECTIONS SPACED TO GIVE INPUT MATCH

B = 1

\[ \theta/2 \quad \theta/2 \]

+ jB

SECTION

POINT NUMBERS
CORRESPOND TO ROOT NUMBER

\[ \Delta \phi, \text{INCREMENTAL PHASE SHIFT, FRACTION OF } \lambda_0 \]

NUMBER OF SECTIONS, \( n \)

Figure 4  Incremental phase shift for \( n \) sections.
Figure 6  Input VSWR vs. Separation Angle, $\theta$, for 11 sections.
Figure 7
Input VSWR vs. Separation Angle, $\theta$, for 17 sections.
"CHARACTERISTIC IMPEDANCE" AND "LINE LENGTH" EQUIVALENT OF ANY LOSS-LESS, SYMMETRICAL TWO-PORT WHEN OPERATED BETWEEN UNIT GENERATOR AND LOAD IMPEDANCES

Let the two-port be characterized by an \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) - matrix:

\[
\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}.
\]

A and D are real, B and C are imaginary; \( AD - BC = 1 \) by reciprocity.

Also, in general:

Complex insertion voltage ratio, \( R = \frac{1}{2} (A + B + C + D) = \text{Re} \, R + j \text{Im} \, R \).

Insertion Phase, \( P = \tan^{-1} \frac{\text{Im} \, R}{\text{Re} \, R} \).

Insertion Loss, \( L = |R|^2 \).

Input Impedance, \( Z_{in} = \frac{A + B}{C + D} \).

When the network is symmetrical, \( A = D \).

If the two-port is to be equivalent to a transmission line of characteristic
impedance $Z_o$ and electrical length $\phi$, the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - matrix for a section of line must equal the matrix for the two-port.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{line}} = \begin{pmatrix} \cos \phi & jZ_o \sin \phi \\ jY_o \sin \phi & \cos \phi \end{pmatrix}$$

To establish equivalence the following must hold:

$$\cos \phi = A, \text{ and}$$

$$Y_o = \frac{C}{j \sin \phi} \tag{A-2}$$

By definition, $A$ is real. Let $-1 < A < 1$. Then Eq. (A-1) gives

$$0 < \phi < \pi, \text{ and } \sin \phi = \sqrt{1 - A^2}.$$ From Eq. (A-2), $Y_o$ is real since $C$ is imaginary.

Therefore, the two-port is equivalent to a line of real characteristic impedance and real length, $l = \frac{\phi \lambda_g}{2\pi}$, when $-1 < A < 1$.

The insertion phase, $P = \tan^{-1} \frac{\text{Im} R}{\text{Re} R}$, with $R = A + j\frac{B + C}{2j}$.

$$\tan P = \frac{B + C}{2j A} = \frac{\sin \phi (Z_o + Y_o)}{2 \cos \phi} = \frac{1}{2} \left( Y_o + \frac{1}{Y_o} \right) \tan \phi \tag{A-3}$$

It is only when $Y_o = 1$ that $P = \phi$, in general. However, if $\phi = \pi p$ ($p = 1, 2, -\ldots$), then also $P = \pi p = \phi$.

When $A$ falls outside of the limits $\pm 1$, $\cos \phi$ does also, hence $\phi$ must become imaginary, $j\phi'$, for $\cos j\phi' = \cosh \phi' > 1$. This corresponds to a line below cut-off, for the guide wavelength is imaginary, $\lambda_g = j\lambda_L$.
Electrical length, \( b_L = \frac{2\pi L}{\lambda g} = -j \frac{2\pi L}{\lambda L_i} = -j \psi \).

\[
\begin{align*}
\cos \theta & \rightarrow \cos \theta_L = \cosh \psi \\
\sin \theta & \rightarrow \sin \theta_L = -j \sinh \psi.
\end{align*}
\]

Then,

\[
\begin{pmatrix} \cos \theta & j Z_0 \sin \theta \\ j Y_0 \sin \theta & \cos \theta \end{pmatrix} \rightarrow \begin{pmatrix} \cosh \psi & Z \sinh \psi \\ Y \sinh \psi & \cosh \psi \end{pmatrix} = \begin{pmatrix} \cosh \psi & j Z_o' \sinh \psi \\ -j Y_o' \sinh \psi & \cosh \psi \end{pmatrix},
\]

where \( Y_o', \psi \) real.

Now,

\[
\cosh \psi = A, \text{ and}
\]

\[
Y_o' = \frac{jC}{\sinh \psi}.
\]

Since

\[
R = \cosh \psi + j \frac{(Z_o' - Y_o')}{2} \sinh \psi,
\]

\[
\tan P = \frac{1}{2} (Z_o' - \frac{1}{Z_o'}) \tanh \psi.
\]  

(A-4)
ACKNOWLEDGMENT

Numerical computations for Figures 5 through 8 were carried out by Mr. W. C. Danforth.
REFERENCES


3. Internal Publication, not generally available.
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**Phase Shift by Periodic Loading of Waveguide**

Periodic loading of a transmission line is considered in terms of a discrete number of identical sections in cascade. For \( n \) sections there are \((n-1)\) discrete solutions, i.e., \((n-1)\) spacings each less than half wavelength, between identical susceptances, which produce input match. Formulas are given for locating these \((n-1)\) roots and for evaluating phase shifts. Some numerical examples are worked out.
Electronics
Periodic Loading
Phase Shift
Waveguides
Mathematical Analysis
Equations