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Phase Shift by Periodic Loading of Waveguide

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PHASE SHIFT BY PERIODIC LOADING OF WAVEGUIDE

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ABSTRACT

Periodic loading of a transmission line is considered in terms of a discrete number of identical sections in cascade. For n sections there are (n-1) discrete solutions, i.e., (n-1) spacings each less than half wavelength, between identical susceptances, which produce input match. Formulas are given for locating these (n-1) roots and for evaluating phase shifts. Some numerical examples are worked out.

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PHASE SHIFT BY PERIODIC LOADING OF WAVEGUIDE

I. INTRODUCTION

The analysis arose out of the following problem: two linearly polarized signals at frequencies $f_1$, $f_2$ are to be launched in a common waveguide, are transmitted through a periodically loaded section, and emerge with the signals at $f_1$, say, right-hand circularly polarized while those at $f_2$ are left-hand circularly polarized; at each frequency the input match is to be of unity VSWR and the phase shifting section is to be short. Hardware-wise the solution has not been attempted. However, because the basic building block is somewhat different, ¹ and because the analysis stresses different aspects, ² it is believed that the theoretical results obtained are worthwhile.

The analysis is carried out in terms of the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ -matrix approach and is based on a discrete number of basic units in cascade. The latter may appear to be a serious restriction — but it is not, since the solutions so obtained contain the others as special cases.

II. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - MATRIX OF THE BASIC UNIT

The basic unit is illustrated in Fig. 1. The susceptance, $jB$, is assumed to be lumped.

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* Suggested by L. J. Ricardi.
If \( \begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = (u) \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} \), then
\[
\begin{pmatrix}
\cos \frac{\theta}{2} & jZ \sin \frac{\theta}{2} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \frac{\theta}{2} & jZ \sin \frac{\theta}{2} \\
0 & 1
\end{pmatrix}
= \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
(1)

Performing multiplication of the matrices above, and using appropriate
trigonometric identities, results in:

\[
A = \cos \theta - \frac{BZ}{2} \sin \theta,
\]
\[
B = j \left[ Z \sin \theta + \frac{1}{2} BZ^2 \cos \theta - \frac{1}{2} BZ^2 \right],
\]
\[
C = j \left[ Y \sin \theta + \frac{B}{2} \cos \theta + \frac{B}{2} \right],
\]
\[
D = A.
\]

Let the normalized characteristic impedance, \( Z \), equal unity. Then:

\[
A = D = \cos \theta - \frac{B}{2} \sin \theta,
\]
\[
B = j \left[ \sin \theta + \frac{B}{2} \cos \theta \right] - \frac{B}{2} (\cos \theta - 1),
\]
\[
C = j \left[ \sin \theta + \frac{B}{2} \cos \theta \right] + \frac{B}{2} (\cos \theta + 1).
\]
(2)

III. \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \)- MATRIX OF \( n \) SECTIONS IN CASCADE, \( \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \).

The guide, periodically loaded with normalized susceptance \( jB \), is shown
in Fig. 2.
The over-all two-port equivalent for \( n \) sections in cascade is obtained by raising the matrix \( (u) \) of Eq. (1) to the \( n \)-th power. This can be done in general terms. The results are:

\[
\begin{align*}
A_n &= \cosh an, \\
B_n &= B \frac{\sinh an}{\sinh a}, \\
C_n &= C \frac{\sinh an}{\sinh a}, \\
D_n &= A_n,
\end{align*}
\]  

(3)

where \( n \) = integral number of sections,

\( a = \cosh^{-1} A \), and

\( A, B, C \) are defined by Eq. (2).

IV. "ROOTS" OF INPUT IMPEDANCE

In terms of the generalized \( n \)-section matrix elements \( A_n, B_n, C_n, D_n \) with unity load termination the input impedance is:
\[ Z_{\text{in}(n)} = \frac{A_n + B_n}{D_n + C_n} = \frac{\cosh an + B_n}{\sinh a} \left( \frac{\sinh an}{\cosh an + C_n} \right) \]

Since it is desired to have \( Z_{\text{in}(n)} = 1 \), then \( B_n \) must equal \( C_n \). This leads to two solutions:

1. \( B_n = C_n \), giving the trivial solution \( B = 0 \), and
2. \( B_n = C_n = 0 \), requiring that

\[ \frac{\sinh an}{\sinh a} = 0. \]

In Reference 3 it is shown that

\[ \frac{\sinh an}{\sinh a} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \]

where \( \lambda_1, \lambda_2 \) are the two non-degenerate eigenvalues of the characteristic equation of the matrix Eq. (1), which in this application reduces to:

\[ \lambda_1 = \frac{A \pm \sqrt{A^2 - 1}}{2}, \text{ with} \]

\[ A = \cos \theta - \frac{B}{2} \sin \theta \]

In Eq. (6) it is possible to cancel \( \lambda_1 - \lambda_2 \), leaving a polynomial of the \((n-1)\) order, which is then set equal to zero. For \( n \) sections in cascade then, Eq. (6) gives \((n-1)\) principal roots, so that there are \((n-1)\) values of \( A \), or of spacing \( \theta \), which will give an input impedance of unity. Naturally, \( n=1 \) is excluded, for this corresponds to the trivial solution \( B = 0 \).
Using Eq. (6), the polynomial for any \(n\) is:

\[
\lambda_1^{n-1} + \lambda_1^{n-2} \lambda_2 + \lambda_1^{n-3} \lambda_2^2 + \lambda_1^{n-4} \lambda_2^3 + \ldots + \lambda_1 \lambda_2^{n-2} + \lambda_2^{n-1}.
\]

This may be reduced by remembering that \(\lambda_1 \lambda_2 = 1\), and finally factoring, where possible. Table I below lists the polynomials for \(n\) ranging from 2 to 10.

**TABLE I**

Factors of \(\frac{\sinh an}{\sinh a} = 0\) for \(n = 2\) to 10.

<table>
<thead>
<tr>
<th>(n)</th>
<th>factors of (\frac{\sinh an}{\sinh a} = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>((\lambda_1 + \lambda_2))</td>
</tr>
<tr>
<td>3</td>
<td>((\lambda_1^2 + 1 + \lambda_2^2))</td>
</tr>
<tr>
<td>4</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^2 + \lambda_2^2))</td>
</tr>
<tr>
<td>5</td>
<td>((\lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4))</td>
</tr>
<tr>
<td>6</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^2 + 1 + \lambda_2^2) (\lambda_1^2 - 1 + \lambda_2^2))</td>
</tr>
<tr>
<td>7</td>
<td>((\lambda_1^6 + \lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4 + \lambda_2^6))</td>
</tr>
<tr>
<td>8</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^2 + \lambda_2^2) (\lambda_1^4 + \lambda_2^4))</td>
</tr>
<tr>
<td>9</td>
<td>((\lambda_1^2 + 1 + \lambda_2^2) (\lambda_1^6 - 1 + \lambda_2^6))</td>
</tr>
<tr>
<td>10</td>
<td>((\lambda_1 + \lambda_2) (\lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4) (\lambda_1^4 - \lambda_1^2 + 1 - \lambda_2^2 + \lambda_2^4))</td>
</tr>
</tbody>
</table>

Study of the table reveals some interesting and obvious facts:

1. For \(n\) sections there are \((n-1)\) solutions.

2. For \(n = 4\), the \(n = 2\) solution is contained, as well as two others
not directly related to either $n = 2$ or 3.

3. If $n$ can be factored into products of lesser integers, then there will be roots correspondingly specified by the lesser integers, i.e., if $n = 6 = 2 \times 3$, one root coincides with $n = 2$, two roots are given by $n = 3$, and two others.

Figure 3 illustrates the distribution of the roots of Table I for a specific case of $B = +1$ (capacitive susceptance). It is quite clear that the single root of $n = 2$ appears in $n = 4, 6, 8, 10$; the two roots of $n = 3$ are repeated in $n = 6, 9$; $n = 10$ contains roots corresponding to $n = 2, 5$. Furthermore, for any $n > 2$, the $(n-1)$ roots seem to appear in "mirror - image" pairs about the $n = 2$ root. That this is true will be proved in Section V.

Although the preceding gives usable results, a more meaningful method for evaluation of roots was suggested by Dr. R. N. Assaly.

$$\sinh a_n = \frac{\lambda_1^n - \lambda_2^n}{2} = \frac{e - e^{-a_n}}{2} = 0.$$  This equation is satisfied by

$$a = j \pi \frac{m}{n} \left( m = \pm 1, 2, 3, \ldots \right).$$

Similarly, from $\sinh a = 0$, $a = j \pi \ell \left( \ell = 1, 2, 3, \ldots \right)$. But these values must be excluded, otherwise the ratio is infinite.

Hence all values of $a$ which give integer values of $j \pi$ are to be excluded, or, the roots are given by

$$a = j \pi \frac{m}{n} \left( m = \pm 1, 2, 3, \ldots, n - 1 \right).$$

Furthermore, $\cosh (\pm x) = \cosh x$, if $x$ real, or $\cosh (\pm jx') = \cos (\pm x') = \cos x'$, if $x'$ real, therefore only the $+$ sign need be used. Finally, then,
the principal roots are given by:

\[ a = j \pi \frac{m}{n} \quad (m = 1, 2, 3, \ldots, n - 1) = ja'_{nm}. \quad (8) \]

If \( m \) were allowed to exceed \( n \), say \( m' = n + m \), then \( a'_{nm} = j\pi \left(1 + \frac{m}{n}\right) = j\pi + a'_{nm} \), which are the principal roots augmented by \( \pi \).

In Eq. (8), the roots are specified in terms of \( a = \cosh^{-1} \frac{A}{a} = \cosh^{-1} (\cos \theta - \frac{B}{2} \sin \theta) \). To specify the roots in terms of \( \theta \) the following may be done:

\[
\cosh a = \cosh a'_{nm} = \cosh ja'_{nm} = \cos a'_{nm}. \quad \text{Therefore,} \]

\[
\cos \theta - \frac{B}{2} \sin \theta = \sqrt{1 + \frac{B^2}{4}} \cos \left(\theta + \tan^{-1} \frac{B}{2}\right) = \cos a'_{nm}, \quad \text{or} \]

\[
\theta_{nm} = \cos^{-1} \left[ \frac{\cos a'_{nm}}{\sqrt{1 + \frac{B^2}{4}}} \right] - \tan^{-1} \frac{B}{2}. \quad (9) \]

In Eq. (10) principal values only are included.

V. SOME RELATIONS AMONG ROOTS

It will now be shown that the root-pair \( \theta_{nm} \) and \( \theta_{n, n-m} \) are mirror-images in the root \( \theta_{21'} \) corresponding to \( n = 2 \).

\[ a'_{n, m} = \frac{\pi m}{n} = \frac{\pi}{2} \left(\frac{2m}{n}\right) = \frac{\pi}{2} \left[1 - \frac{n-2m}{n}\right], \quad (2m \leq n). \]

\[ a'_{n, n-m} = \frac{\pi}{2} \left[1 - \frac{n - 2(n-m)}{n}\right] = \frac{\pi}{2} \left[1 + \frac{n - 2m}{n}\right]. \]

\[
\begin{align*}
\cos a'_{n, m} &= 0 + \sin \frac{\pi}{2} \left(\frac{n - 2m}{n}\right), \\
\cos a'_{n, n-m} &= 0 - \sin \frac{\pi}{2} \left(\frac{n - 2m}{n}\right).
\end{align*} \quad (11)
\]
From Eqs. (10) and (11) it may be shown that:

\[ \theta_{n, m} + \tan^{-1} \frac{B}{2} = \frac{\pi}{2} - \cos^{-1} \sqrt{1 - \frac{\sin^2 \frac{\pi}{2} \left( \frac{n - 2m}{n} \right)}{1 + \frac{B^2}{4}}} \]

\[ \theta_{n, n-m} + \tan^{-1} \frac{B}{2} = \frac{\pi}{2} + \cos^{-1} \sqrt{1 - \frac{\sin^2 \frac{\pi}{2} \left( \frac{n - 2m}{n} \right)}{1 + \frac{B^2}{4}}} \]

and

\[ \theta_{n, m} + \cos^{-1} \sqrt{\frac{\cos^2 \frac{\pi}{2} \left( \frac{n - 2m}{n} \right) + \frac{B^2}{4}}{1 + \frac{B^2}{4}}} = \theta_{n, n-m} - \cos^{-1} \sqrt{\frac{\cos^2 \frac{\pi}{2} \left( \frac{n - 2m}{n} \right) + \frac{B^2}{4}}{1 + \frac{B^2}{4}}} \]

\[ = \frac{\pi}{2} - \tan^{-1} \frac{B}{2} = \theta_{21} \]  

In Eq. (10) it is tacitly assumed that the susceptance, B, is positive.

How are the roots distributed if B were negative?

Equation (8) is not altered by reversal in the sign of B, whereas Eq. (10) is. Using Eq. (12),

For B positive, write

\[ \theta_{nm} + \tan^{-1} \frac{B}{2} = \frac{\pi}{2} - \cos^{-1} \sqrt{1 - \frac{\sin^2 \frac{\pi}{2} \left( \frac{n - 2m}{n} \right)}{1 + \frac{B^2}{4}}} \]
For $B$ negative,

$$\theta_{nm} = \tan^{-1} \frac{B}{2} = \frac{\pi}{2} - \cos^{-1} \sqrt{1 - \frac{\sin \frac{2\pi(n-2m)}{n}}{1 + \frac{B^2}{4}}}.$$

From Eq. (14) it is obvious that $\theta^+_{nm}$ and $\theta^-_{nm}$ form a root-pair which are mirrored in $\cos^{-1} \sqrt{1 - \frac{\sin \frac{2\pi(n-2m)}{n}}{1 + \frac{B^2}{4}}}.$

Now, if $m = n - m$ in $\theta'_{nm}$, then:

$$\theta'_{n, n-m} = \tan^{-1} \frac{B}{2} = \frac{\pi}{2} + \cos^{-1} \sqrt{1 - \frac{\sin \frac{2\pi(n-2m)}{n}}{1 + \frac{B^2}{4}}}.$$

From Eqs. (14) and (15), it is easily shown that the root-pair $\theta^+_{nm}$ and $\theta^-_{n, n-m}$ are supplementary,

$$\theta^+_{nm} + \theta^-_{n, n-m} = \pi.$$

Equation (16) is of importance in considerations of use of a guide supporting two modes in space quadrature where a single discontinuity may set up susceptances of different sign in each mode. In such a case, if the magnitude of the susceptance be equal for each mode, since the number of sections, $n$, is identical for both modes, an ideal situation exists only when
\[ \theta = \theta^+ = \pi \text{ for } \lambda^+ = \lambda^- \text{.} \] From Fig. 3, it is clear that this can occur only when \( n \) is even and \( B \) approaches zero.

VI. INSERTION PHASE OF \( n \) CASCADED SECTIONS, AS FUNCTION OF \( \theta \).

The complex insertion voltage ratio between matched generator and load is given by

\[
\begin{align*}
R &= \frac{1}{2} \left[ A_n + B_n + C_n + D_n \right] = \cos \alpha n + j \left[ \sin \theta + \frac{B}{2} \cos \theta \right] \frac{\sinh \alpha n}{\sinh \alpha},
\end{align*}
\]

and the insertion phase shift, \( P \), is given by,

\[
\tan P = \frac{\text{Im} R}{\text{Re} R} = \left[ \sin \theta + \frac{B}{2} \cos \theta \right] \frac{\tanh \alpha n}{\sinh \alpha}. \tag{17}
\]

In view of Eq. (9) it is easily shown that

\[
\sin \theta + \frac{B}{2} \cos \theta = \sqrt{1 + \frac{B^2}{4} - A^2} > 0. \text{ Hence,}
\]

\[
\tan P = \sqrt{1 + \frac{B^2}{4} - A^2} \frac{\tanh \alpha n}{\sinh \alpha}. \tag{18}
\]

**Case A:** \( |A| < 1 \).

When \( |A| < 1 \), \( \cosh \alpha = \cosh j \alpha' = \cos \alpha' = A = \cos \theta - \frac{B}{2} \sin \theta \), so that

\[
a' = \cos^{-1} A; \quad \frac{\tanh \alpha n}{\sinh \alpha} = \frac{\tan a'n}{\sin a'} \text{ and}
\]
\[ \tan P = \sqrt{1 + \frac{B^2}{4} - \frac{A^2}{\sin a'}} \tan a'n. \] \hspace{1cm} (19)

**Case B** \( A = +1. \)

\[ a' = \cos^{-1} 1 = 2\pi p \ (p = 0, 1, 2--). \]

\[ \frac{\tan a'n}{\sin a'} \rightarrow n. \]

\[ \therefore \tan P = \frac{B}{2} \cdot n. \] \hspace{1cm} (20)

**Case C** \( A = -1. \)

\[ a' = \cos^{-1} (-1) = \pi q \ (q = 1, 3, 5--). \]

\[ \frac{\tan a'n}{\sin a'} \rightarrow -n. \]

\[ \therefore \tan P = -\frac{B}{2} \cdot n. \] \hspace{1cm} (21)

**Case D** \( A = 0. \)

\[ a' = \cos^{-1} 0 = \frac{\pi}{2} r \ (r = 1, 3, 5--). \]

\[ \tan a'n = \pm \tan \left( \frac{\pi}{2} n \right) = 0 \text{ if } n \text{ even}, \]

\[ \pm \infty \text{ if } n \text{ odd}. \]

\[ \therefore P = \begin{cases} \pi q \ (q = 0, 1, 2, ----) \text{ for } n \text{ even}, \\
\frac{\pi}{2} r \ (r = 1, 3, 5----) \text{ for } n \text{ odd}. \end{cases} \] \hspace{1cm} (22)
Case E \( A > 1 \).

cosh a = A, which is equivalent to \( e^a = A \pm \sqrt{A^2 - 1} \), or

\[
a_1, 2 = \ln (A \pm \sqrt{A^2 - 1}) + j 2\pi p. \quad \text{But } a_2 = -a_1, \text{ therefore select } a_1.
\]

\[
\tan P = \sqrt{1 + \frac{B^2}{4} - \frac{A^2}{A}} \tan \left[ \frac{n \ln (A + \sqrt{A^2 - 1})}{\sinh \left( \ln (A + \sqrt{A^2 - 1}) \right)} \right]
\]  

(23)

Since \( 1 + \frac{B^2}{4} \geq A^2 \), the radical is \( \geq 0 \), therefore \( 0 < P < \frac{\pi}{2} \), or, augmented by \( \pi p \).

Case F \( A < -1 \)

Let \( A = -A \). Then \( e^a = (A \pm \sqrt{A^2 - 1}) e^{j\pi} \).

Selecting the positive radical, as before,

\[
\tan P = -\sqrt{1 + \frac{B^2}{4} - \frac{A^2}{A}} \tan \left[ \frac{n \ln (A + \sqrt{A^2 - 1})}{\sinh \left( \ln (A + \sqrt{A^2 - 1}) \right)} \right]
\]  

(24)

where \( \frac{\pi}{2} < P > \pi \); or, augmented by \( \pi p \).

VII. PHASE SHIFT AT ROOT VALUES

For any \( n > 1 \), the root-values are specified by Eq. (8); \( a_{nm} = ja'_{nm} = j\pi \frac{m}{n} \) \( (m = 1, 2, \ldots, n - 1) \). Therefore, since \( |A| = |\cos a'| \leq 1 \), Eq. (19) is applicable:

\[
\tan P = \sqrt{1 + \frac{B^2}{4} - \frac{A^2}{A}} \tan a'_{nm} \cdot \frac{n}{\sin a'_{nm}}
\]  

(19)

But \( a'_{nm} \cdot n = \tan \pi m = 0 \), whereas \( \sin a'_{nm} \neq 0 \). Therefore \( P = \pi p \) \( (p = 0, 1, 2, \ldots) \). It is impossible to say which value of \( p \) is to be
chosen for a particular \( a'_{nm} \).

However, in the Appendix it is shown that when any symmetrical, loss-less two-port is equated to a transmission line of characteristic impedance \( Z_o = \frac{1}{Y_o} \) and electrical length \( \phi \), then

\[
\cos \phi = A_n = \cosh an = \cosh ja'n = \cos a'_{nm} \cdot n.
\]  

(25)

From the above

\[ \phi_{nm} = a'_{nm} \cdot n. \]  

(26)

Consequently, in ascending order, \( \phi_{nm} = \pi, 2\pi, 3\pi, \ldots, (n - 1)\pi \), or, returning to Eq. (19),

\[ P_{nm} = \pi, 2\pi, 3\pi, \ldots, (n - 1)\pi. \]

The physical interpretation of this is the following:

Given a configuration as shown in Fig. 2, if the frequency is varied above cut-off of the guide, the first "resonance" for the entire structure occurs when the insertion phase, \( P \), is \( \pi \); the second "resonance" is at \( P = \pi \cdot 2 \), etc., up to \( \pi(n - 1) \).

It should perhaps be pointed out that the roots \( a'_{nm} \) are determined by the number of discontinuities, \( n \), only. \( \theta_{nm} \), on the other hand, is a function of both \( a'_{nm} \) and the susceptance, \( B \).

If it is assumed that the susceptance, \( B \), is invariant with frequency, and that the loading is only for one of two orthogonal modes of an otherwise symmetric guide (i.e., \( \lambda g_1 = \lambda g_2 \)), then the incremental phase shift, phase shift in
loaded mode minus phase shift in unloaded mode, for \( n \) sections in cascade, is given by:

\[
\Delta \phi = P_{nm} - n \cdot \theta_{nm}.
\]  \hspace{1cm} (27)

Figure 4 shows a plot of \( \Delta \phi \) for \( B = +1 \), and \( n \) ranging from 2 to 10. From the figure it is clear that it is impossible to get a \( \Delta \phi \) greater than one wavelength, except for \( n = 10 \) and \( m = 9 \), or operation at root \( a_1^{10} \), \( 9 \). The figure also shows that if \( n = 2 \) is used as a unit, then cascading five such units, gives \( 0.74 \lambda_g \) incremental phase shift, for the root \( \theta_{10}, 5 = \theta_{21} \) (See Fig. 3).

In Fig. 4 it should not be concluded that the incremental phase varies linearly when moving from one root to the next. In the next section the input voltage reflection coefficient as a function of \( \theta \) is evaluated. Since \( \Gamma \neq 0 \) at \( \Theta_{nm} \), between roots the phase shift will be non-linear.

VIII. INPUT VOLTAGE REFLECTION COEFFICIENT, \( \Gamma_{in} \), FOR \( n \) CASCADED SECTIONS, AS FUNCTION OF \( \theta \).

\[
\Gamma_{in} = \frac{Z_{in} - 1}{Z_{in} + 1} = \frac{B_n - C_n}{A_n + B_n + C_n + D_n} = \Gamma e^{iY}
\]

\[
= \frac{B \tanh an}{2 \sinh a} \exp\left[\frac{\pi}{2} + \tan^{-1}\left(\sin \theta + \frac{B}{2 \cos \theta}\right) \frac{\tanh an}{\sinh a}\right] \left[1 + \left(\sin \theta + \frac{B}{2 \cos \theta}\right)^2 \frac{\tanh \frac{an}{2}}{\sinh \frac{a}{2}}\right]^{1/2}.
\]  \hspace{1cm} (28)

The input VSWR, \( q = \frac{1 + \Gamma}{1 - \Gamma} \).  \hspace{1cm} (29)
\textbf{Case A:} \( |A| < 1 \).
\[ a' = \cos^{-1} A, \text{ as in Section VI-A, and} \]
\[ \Gamma = \frac{\frac{B}{2} \tan a'n}{\sin a'} \cdot \left[ 1 + \left(1 + \frac{B^2}{4} - A^2 \right) \frac{\tan^2 a'n}{\sin^2 a'} \right]^{1/2}. \]  \hfill (30)

\textbf{Case B:} \( A = 0 \).
\[ a' = \cos^{-1} 0 = \pm \frac{\pi}{2}. \]
\[ \tan a'n = \begin{cases} 0 & \text{if } n \text{ even,} \\ \pm \infty & \text{if } n \text{ odd.} \end{cases} \]
\[ \sin a' = \pm 1. \]
\[ \therefore \Gamma = 0, \text{ when } n \text{ is even, and} \]
\[ \Gamma = \frac{B/2}{1/2}, \quad \text{when } n \text{ is odd.} \]  \hfill (31)

The above only corroborates the fact that when \( n \) is even the root corresponding to \( A = 0 \) is identical to the single root for \( n = 2 \).

\textbf{Case C:} \( A = \pm 1 \).
\[ a' = 2\pi p \ (p = 0, 1, 2, \ldots). \]
\[ \frac{\tan a'n}{\sin a'} \rightarrow n, \text{ and} \]

15
\[
\Gamma = \frac{\frac{B}{2} n}{\left[1 + \frac{B^2}{4} n^2\right]^{1/2}}.
\]

**Case D:** \( A = -1 \)

\[ a' = \pi q \ (q = 1, \ 3, \ 5--). \]

\[
\frac{\tan a'n}{\sin a'} \rightarrow -n, \ \text{and}
\]

\[
\Gamma = -n \frac{\frac{B}{2}}{\left[1 + \frac{B^2}{4} n^2\right]^{1/2}}. \tag{33}
\]

**Case E:** \( |A| > 1 \).

\( A > 1, \ a = a_1 = \ln (A + \sqrt{A^2 - 1}) + j 2\pi p \ (p = 0, \ 1, \ 2--), \)

\( A < -1, \ a = a_2 = \ln (A + \sqrt{A^2 - 1}) + j \pi q \ (q = 1, \ 3, \ 5--). \)

\[
\frac{\tanh a_2'n}{\sinh a_2'} = \frac{\tanh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]}{\sinh \left[ \ln (A + \sqrt{A^2 - 1}) \right]}. \]

\[
\frac{\tanh a_1'n}{\sinh a_1'} = -\frac{\tanh \left[ n \ln (A + \sqrt{A^2 - 1}) \right]}{\sinh \left[ \ln (A + \sqrt{A^2 - 1}) \right]}. \]
\[ \Gamma (A > 1) = \frac{\tanh \left[ \frac{n \ln (A + \sqrt{A^2 - 1})}{\sinh \left[ \ln (A + \sqrt{A^2 - 1}) \right]} \right]}{\sinh \left[ \ln (A + \sqrt{A^2 - 1}) \right]} \cdot \frac{B}{2} \]

\[ 1 + \left( 1 + \frac{B^2}{4} - A^2 \right) \tanh^2 \left[ \frac{n \ln (A + \sqrt{A^2 - 1})}{\sinh^2 \left[ \ln (A + \sqrt{A^2 - 1}) \right]} \right]^{-1/2} \]

and

\[ \Gamma (A < -1) = -\Gamma (A > 1) \quad (35) \]

Figures 5, 6 and 7 illustrate the behavior of input VSWR VS. \( \theta \) for \( n = 10, 11, \text{ and } 17 \), each for two constant values of \( B \), i.e., \( 0.9 \) and \( 1.0 \). In each case, with \( n \) constant, increasing \( B \) compresses the roots and increases the VSWR. The minimum VSWR peak occurs between roots \( \theta_n \) and \( \theta_n \); on either side the VSWR peaks are symmetrically distributed, i.e.,

\[ \theta_n \frac{n}{2} + 1 \]

the peak between first and second root is equal to that between next to last and last. With \( B \) constant and \( n \) increasing, there is a tendency to lower the minimum VSWR peak slightly, while increasing the side peaks.

Figure 8 shows a plot of the incremental phase shift, \( \Delta \phi \), as a function of \( \theta \), for the conditions of Figs. 5, 6 and 7. In each case \( \frac{\Delta \phi}{\Delta \theta} \) is maximum at root values \( \theta_{nm} \) (where the VSWR is unity) and is minimum (zero, practically) over a wide range in between roots (where the VSWR is greatest). This behavior would pose a serious problem in implementing a matched, low axial ratio, orthogonal-circularly polarized dual frequency polarizer. As an
example, consider that at a frequency $f_1$ the polarizer (dual-mode guide) is to give circular polarization (say $\Delta \phi = 270^\circ$) of low VSWR and circularity, and at a frequency $f_2$ the polarization should correspond to a $\Delta \phi$ of $450^\circ$. At the lower frequency $f_1$, $\lambda g_1$, $B$, $n$, and $\theta_{nm}$ must be selected to give a $\Delta \phi$ of $270^\circ$. For small $\theta_{nm}$ values interaction between sections can be expected. At the higher frequency $f_2$, $n$ is already pre-selected; $B$ is no longer constant; $\lambda g_2$ must be exactly right to give a root $\theta_{nm}$ — otherwise the attendant mismatch sets up an $\frac{H}{V}$ ratio which is not unity and $\Delta \phi$ is no longer $450^\circ$.

IX. CONCLUSION

A basic unit, as in Fig. 1, when cascaded $n$ times gives a wide choice of element spacings, which are less than $180^\circ$, to produce unity input VSWR and a wide range of insertion phase.
Figure 3: $\theta_{nm}$ vs. $n$. 

- $n$: number of sections
- $\theta$: separation between sections
- $B = +1$
Figure 4 Incremental phase shift for n sections.

Figure 4 Incremental phase shift for n sections.
Figure 6  Input VSWR vs. Separation Angle, $\theta$, for 11 sections.
Figure 7  Input VSWR vs. Separation Angle, $\theta$, for 17 sections.
APPENDIX

"CHARACTERISTIC IMPEDANCE" AND "LINE LENGTH" EQUIVALENT OF ANY LOSS-LESS, SYMMETRICAL TWO-PORT WHEN OPERATED BETWEEN UNIT GENERATOR AND LOAD IMPEDANCES

Let the two-port be characterized by an \((A\ B)\quad(C\ D)\) - matrix:

\[
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} E_2 \\
I_2
\end{pmatrix}.
\]

A and D are real, B and C are imaginary; \(AD - BC = 1\) by reciprocity.

Also, in general:

Complex insertion voltage ratio, \(R = \frac{1}{2} (A + B + C + D) = \text{Re} \ R + j \text{Im} \ R\).

Insertion Phase, \(P = \tan^{-1} \frac{\text{Im} \ R}{\text{Re} \ R}\).

Insertion Loss, \(L = |R|^2\).

Input Impedance, \(Z_{\text{in}} = \frac{A + B}{C + D}\).

When the network is symmetrical, \(A = D\).

If the two-port is to be equivalent to a transmission line of characteristic
impedance $Z_0$ and electrical length $\phi$, the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ matrix for a section of line must equal the matrix for the two-port.

$$\text{But } \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{line}} = \begin{pmatrix} \cos\phi & jZ_0 \sin\phi \\ jY_0 \sin\phi & \cos\phi \end{pmatrix}$$

To establish equivalence the following must hold:

$$\cos\phi = A, \text{ and } Y_0 = \frac{C}{j \sin\phi} \quad (A-1)$$

By definition, $A$ is real. Let $-1 < A < 1$. Then Eq. (A-1) gives $0 < \phi < \pi$, and $\sin\phi = \sqrt{1 - A^2}$. From Eq. (A-2), $Y_0$ is real since $C$ is imaginary.

Therefore, the two-port is equivalent to a line of real characteristic impedance and real length, $\ell = \frac{\phi \lambda g}{2\pi}$, when $-1 < A < 1$.

The insertion phase, $P = \tan \frac{-1 \text{ Im } R}{\text{ Re } R}$, with $R = A + j \frac{B + C}{2j}$.

$$\therefore \tan P = \frac{B + C}{2j A} = \frac{\sin\phi (Z_0 + Y_0)}{2 \cos\phi} = \frac{1}{2} \left( \frac{Y_0 + \frac{1}{Y_0}}{Y_0} \right) \tan\phi \quad (A-3)$$

It is only when $Y_0 = 1$ that $P = \phi$, in general. However, if $\phi = p\pi$ ($p = 1, 2, \ldots$), then also $P = p\pi = \phi$.

When $A$ falls outside of the limits $\pm 1$, $\cos\phi$ does also, hence $\phi$ must become imaginary, $j\phi'$, for $\cos j\phi' = \cosh \phi' > 1$. This corresponds to a line below cut-off, for the guide wavelength is imaginary, $\lambda g = j\lambda_L$. 

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Electrical length, \( \theta_L = \frac{2\pi L}{\lambda g} = -j \frac{2\pi L}{\lambda_L} = -j \psi. \)

\[
\begin{align*}
\cos \theta & \rightarrow \cos \theta_L = \cosh \psi \\
\sin \theta & \rightarrow \sin \theta_L = -j \sinh \psi.
\end{align*}
\]

Then,
\[
\begin{pmatrix}
\cos \theta & j Z_o \sin \theta \\
-j Y_o \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cosh \psi & Z \sinh \psi \\
Y \sinh \psi & \cosh \psi
\end{pmatrix}
= \begin{pmatrix}
\cosh \psi & j Z_o' \sinh \psi \\
-j Y_o' \sinh \psi & \cosh \psi
\end{pmatrix},
\]

where \( Y_o', \psi \) real.

Now,
\[
\cosh \psi = A, \text{ and } Y_o' = \frac{j C}{\sinh \psi}.
\]

Since
\[
R = \cosh \psi + j \frac{(Z_o' - Y_o')}{2} \sinh \psi,
\]
\[
\tan P = \frac{1}{2} (Z_o' - \frac{1}{Z_o'}) \tanh \psi. \tag{A-4}
\]
ACKNOWLEDGMENT

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3. Internal Publication, not generally available.
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