FINAL REPORT

OPERATIONS RESEARCH STUDY OF COMMAND AND CONTROL
TRADE-OFFS IN AN AMPHIBIOUS ENVIRONMENT

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This study was performed in response to an ONR Contract No. N00-3983(00) awarded to Space Technology Laboratories, Inc. in November 1962. The proposal for this study was motivated by a paper titled "A Model for the Survival of Tactical Units on the Battlefield" that was presented at the Military Operations Research Symposium (MORS) at Ft. Monroe, Virginia, in April 1962, by Paul D. Chaiken. This paper demonstrated the use of analytical techniques to quantitatively describe the inter-relationships between mobility, dispersion, surveillance, and fire power as it affects the survival of tactical units on the battlefield. The Office of Naval Research indicated interest in the methods utilized in the paper and requested a repetition of the MORS presentation in Washington. It was after this briefing that the proposal was generated. The resulting study attempts to analyze significant problems from both Navy/Marine Corps amphibious operations and Marine Corps tactical operations. In order to help STL formulate such problems, Mr. Irving Dow of the Naval Warfare Research Center of Stanford Research Institute (Pasadena) and the staff members of the Office of Naval Research (Washington) were consulted.

The problems were formulated in game-theoretic terms in an attempt to obtain basic mathematical structure rather than specific numerical analyses. By mathematical structure, it is meant the inherent strategies, mixed or pure (in a game-theoretic sense), which appear in the course of the analysis that allude to the decision making events associated with the military scenario from which the problem was defined. Hopefully, these strategies can then be functionally related to real world inputs in order to obtain better insight into the nature of the specific requirements to be made of the limited warfare system used as a basis for the model. The definitizing of tactical system requirements affecting surveillance, communications, command and control, logistics, fire power, force size, dispersion, defense, and ancillary support equipment, utilizing mathematical models, was the major objective of the approach described above.
The approach taken by this study will be to mathematically abstract specific problems (see Figure 1) from the area of Navy/Marine Corps amphibious operations and Marine Corps tactical operations. This approach presents the analyst with a dilemma. If the problem abstracted reflects too many real world variables, a successful solution to the problem can become impossible. On the other hand, if the problem is too much of a simplification of the real world situation, the solution will yield results that are trivial. Significant analyses, i.e., non-trivial solvable problems, can only be accomplished by a full understanding of the problem area. This understanding allows the analyst to determine the important parameters affecting the structure of the solution and even more important allows him to pick measures of effectiveness or criteria such that these parameters can be related.

There are many parameters or factors which have a great deal of importance in the real world situation. However, the true test of their importance when abstracting this real world situation into a mathematical model depends upon how these parameters, factors or variables affect the structure of the solution. An example of this was illustrated in the MORS\(^3\) paper where the parameters of mobility, dispersion and surveillance completely defined the structure of the mathematics developed whereas fire power, a very important real world parameter, could not be introduced except as a scaler quantity completely independent of the mathematical structure. On the other hand, the same military situation utilizing the Lanchester Model completely inverts the above parameter relationships to mathematical structure and yields an entirely different approach to the same problem. Which model formulation should be used can only be answered by asking what type of parameter trade-offs (outputs) is the analyst interested in. The formulation that involves parameters that define the mathematical structure usually (but not always) yields more hidden or less obvious results concerning these parameters than those formulations where the same parameters are independent of the mathematical structure. Which model formulation is the best is simply answered by stating that the model that works the best is the best! By 'best' it is meant the formulation which yields outputs or answers that more closely correspond to the real world outputs in a similar real world situation.
Figure 1. The Mathematical Abstraction of Physical Process.
This study demonstrates the use of analytical techniques to quantitatively describe the interrelationships between mobility, dispersion, surveillance, and fire power as they affect the survival of tactical units on the battlefield. The purpose of such a study was to emphasize the possible use of analytical models to explore areas of Marine Corps/Navy advanced warfare military systems and operations in which outputs obtained from such analyses could lead, by implication, to recommendations for requirements for surveillance, fire power, force size, logistics, and command and control subsystems.

The basic problem analyzed in this study can be summarized as follows:

An amphibious landing force $x$, at sea, is to assault a limited area defended by a force $y$. The landing force $x$ is to be split up into an air mobile force $x_a$ and a surface mobile force $x_s$. The defending force $y$, in turn, commits its forces by allocating part of $y$, $\beta_1 y$, in defense of the beach and the remaining force, $\beta_2 y$, to defend against the air mobile attack.

Some pertinent questions based upon this amphibious operations scenario were:

- What strategic allocation of forces $\{\beta_1, \beta_2, x_a, x_s\}$ should each side use against the other during the ensuing engagement?

- What is the mathematical structure of the strategic decisions made by both sides (i.e., $\{\beta_1, \beta_2, x_a, x_s\}$) as a function of initial conditions, such as force levels $(x,y)$ available to each side at the beginning of the battle, mobility of the assault and defense units on the battlefield, fire power available to each fighting unit, and the time sequencing or staging of force units into battle?

- How does the analysis relate back to the real world in terms of logistics, equipment, force levels, operational plans, etc.?

The above amphibious operation problem was presented mathematically in a game theoretic context which attempts to relate the interplay of the many parameters enumerated above as they affect the outcome of the battle. The payoff of the game was defined in terms of time dependent solutions of equations reflecting the above scenario. The significant outputs of the mathematical model formulated in this study were:
The number of survivors at the end of the battle.

The duration of the battle for any threshold of defeat of one side.

The optimum allocation strategies available to each side. That is, what fraction of the total forces goes across the beach and/or is deployed air mobile as opposed to the defender's allocation when splitting his force in defense against the amphibious operation?

The natural strategic discontinuity levels. That is, the areas of the strategy domain beyond which both sides must play to obtain strategic optimality.

Sub-optimal strategies (restricted within discontinuity levels). That is, areas of the strategy domain in which strategic optimality cannot be achieved and sub-optimal strategies become important.

The last two outputs reflect the constraining nature of the real-world situation in that mathematically optimal strategies may not always be possible to achieve because of the physical constraints of the tactical systems employed, e.g., landing craft and helicopter capacity, speed of operation and duty cycle of the logistic support systems, etc. However, it should be noted that the model does not delineate these constraints directly but rather defines the mathematical structure of the decisions made in terms of natural discontinuity levels in the decision domain. These discontinuities in turn define the areas in which optimal strategies shift violently if the real world were for any reason so constrained to operate only in these areas. The resulting strategies derived from the shift from optimality are sub-optimal.

A sensitivity analysis of the mathematical model formulated using the above scenario as a basis was performed by taking the important input parameters and perturbing them away from what was considered a standard model configuration. Because of the multi-dimensional complexity of the model, it was felt that the best way to explore the structure of the problem was to set up a standard case, which would represent a typical amphibious operational exercise. Then, based upon the nature of the results obtained from solving this standard case, other input parameters were perturbed sequentially in order to test for their sensitivity to the overall measure of merit defined by the model.
The results of the standard case indicated that the landing force's optimal strategy is to send all of its forces air mobile, even though such a strategy would represent a severe constraint upon the fire power available for an air mobile operation. In other words, the value of quick deployment of landing forces via the air mobile mode more than compensates (in terms of effectiveness) for the inherent fire power weakness associated with the air mobile mode.

The disadvantage of the surface assault mode for the deployment of attacking forces lies in the fact that the assault units that contribute most to the heavy fire power characteristics of the assault could not be deployed until the beachhead was first secured by the rifle and close-support artillery units, which have relatively the same fire power characteristics as the air mobile mode. In other words, to deploy heavy fire power via the surface mobile mode requires the assault units to deploy sequentially in time, thus losing the effectiveness derived from the quick deployment as demonstrated by the air mobile mode. Of course, if all fire power of both sides in the standard case were substantially reduced (by a factor of five and kept in the same proportion), the optimal strategy reverts to the surface mobile mode, which indicates that time sequencing of assault units relative to quick deployment is no longer detrimental to the overall optimality of the operation. More significantly, however, the analysis of the standard case also reveals a natural strategic discontinuity in the decision allocation domain, which severely limits the commander's ability to make an optimal decision.

The results of this analysis indicate that the amphibious task force commander is constrained to send greater than 90% of his forces via the air mobile mode if his decision is to remain optimal. If for any reason such percentage of total forces cannot be sent via this mode, then optimality reverts to a decision requiring 100% of the commander's forces going across the beach in the surface mobile mode.
The natural discontinuity level of 90% is called the decision threshold level for air mobile deployment and represents in the real world the amphibious operation commander's 'flip-flog' decision level in which, based upon long years of experience in command, he determines the mode of attack during the planning stage of the operation. The current vertical envelopment doctrine deploys about 20% to 30% of the landing force via the air mobile mode, which is considerably less than the threshold level resulting from the standard case.

Although our standard case model is not purported to be an accurate representation of the real world, it does yield combat results in the range of values characteristic of the World War II Pacific Island Campaigns. Also, this study does not attempt to develop absolute evaluations of amphibious operations via exercising mathematical models. What this study does attempt to establish functional relationships between various important parameters that are characteristic of the amphibious operation, such as mobility, fire power, time dependent deployment of assault units, and order of battle, within the framework of optimizing an overall measure of merit for the operation. Concerning the amphibious commander's decision threshold of 90%, we can only say that it is high, not only in its range of possible values but also with respect to reality. The primary objective of the sensitivity analysis performed on the standard case was to determine the parameter or set of parameters that will influence this threshold of 90% in such a way as to reduce its value so it conforms to present doctrine.

The results of the sensitivity analysis indicate that the air mobile decision threshold can be reduced. The perturbation of the mobility factor assigned to assault units on the battlefield caused the decision threshold to approach values consistent with reality. That is, it was determined that under conditions of amphibious landing force superiority (force ratios of four-to-one) the decision threshold would drop to zero if the mobility of the assault units on the battlefield was extremely large. The mathematical model indicated that distance between separate and distinct battle engagements on the battlefield had to be traversed by the individual landing force assault units at infinite speed. In other words, if a finite period of time was necessary for the amphibious assault units (completing their local battle engagement) to link up with other amphibious assault units at a
different geographical location on the battlefield, then the air mobile
decision threshold remained high. Only infinite speed (a physical im-
possibility) allowed for the air mobile deployment of forces in percentage
of total forces consistent with current doctrine.

Next, perturbations of various fire powers associated with assault
and defensive units were tried. Increasing all the assault units' fire
power relative to the defense did not seem to have any effect upon the air
mobile decision threshold level. However, increasing the defender's fire
power relative to the amphibious force reduced the amphibious commander's
decision threshold to 60% (equivalent to a four-fold increase in defender
fire power). Unfortunately, the level of superiority of the amphibious
operation (two-to-one order of battle ratio superiority) over the defense
was insufficient in the face of heavier fire power to gain the objective
of the operation, that is, a successful amphibious landing. Increasing
the order of battle ratio superiority of the amphibious operation to four
to-one over the defense achieved victory for the landing force, but it als-
had the surprising effect of raising the air mobile threshold back to 90%.
Finally, only the amphibious operation's air mobile force's fire power was
perturbed upward relative to the fire power of all other battlefield units.
For two-to-one superiority, it was possible to reduce the air mobile
decision threshold to 70% - still not compatible with real-world doctrine.
Again, increasing the order of battle ratio in favor of the landing force
to four-to-one causes the air mobile decision threshold to return to the
90% level.

Based upon these results, one might ask whether the commander's low
air mobile decision threshold level of 20% to 30% is consistent with the
optimal achieving of the amphibious operation objective, that objective
being the winning of the battle? What the mathematical model indicates is
that the air mobile mode of quick deployment of troops is not as effective
as the time sequencing of the various surface assault units, which in-
herently deploy greater fire power, unless: either almost all of the
amphibious force can go via the air mobile mode, or the assault units can
move about the battlefield with infinite speed, or the margin of victory
(overall superiority) of the amphibious operation is low or nonexistent.
Still another possible interpretation of the results is that the mathematical model used in the completed study was too abstract a representation of the amphibious operation and therefore did not adequately reflect all the important parameters affecting the outcome of the battle. This presumes that a more realistic presentation of the amphibious operation, taking into consideration parametric effects omitted from the completed study because they were considered of second-order of effect, would yield results more compatible with reality. Such a study certainly bears consideration in view of the importance of the information potentially available from the mathematical models and techniques demonstrated by the completed study.
II. THE AMPHIBIOUS OPERATION

A. INTRODUCTION

A typical fleet/air/amphibious operation is shown in Figure 2 illustrating the complexity of a MEF landing operation envisioned for the early 1970 time period. The geographic area of involvement is over a hundred thousand square miles, both on land and sea. The initial assault includes a beachhead landing which contains the bulk of the total MEF and a vertical-envelopment inland landing designed either to secure the beachhead or to achieve some objective that will aid in successfully getting the total force ashore. Once the beachhead is secure and control of the air accomplished, command is transferred ashore and inland objectives are pursued. The area of involvement in establishing the beachhead depends upon the nature of the threat and the geo-political environment. The beachhead and its geographic neighborhood (shown by the dashed area) can be as low as 600 square miles in area for a non-nuclear threat, and as high as 2500 square miles for a nuclear threat (or a high probability of such a threat)\(^4\).

Air warfare such as interdiction, air-to-air/ground, surveillance, etc., extend beyond the neighborhood of the beachhead for at least a radius of 300 n. mi. Air-to-air detection, acquisition, and kill extends another 150 n. mi. Surface-to-air missile systems will be deployed on the beachhead with effective ranges up to 50 n. mi. The MEF vertical envelopment activity ranges are up to 25 n. mi. beyond the beachhead area. This distance can be extended to 50 n. mi. under exceptional circumstances. The Naval ship-to-shore fire support includes guns and missiles at ranges from 10, 20, 40 and even perhaps to 75 n. mi. inland from the beachhead. Figure 3 summarizes the geometry of the battlefield as visualized in reference 4.

B. PROBLEM DEFINITION, FORMULATION, AND SYNOPIS

Based upon a simple scenario derivable from Figure 2, the following problem has been abstracted:
Figure 2. A Typical Fleet/Air/Amphibious Operation
Figure 3. Geometric analysis of fleet operations:

- Fleet Surface-to-Surface
- Beachhead Surface-to-Air
- Upper Limit Vertical Air
- Fleet Bombardment
- Vertical Envelopment
- Fleet Bombardment
- Fleet Bombardment
- Interdiction, Air-to-Air/Ground, Detection, Acquisition, Kill Air-to-Air-

Scale 1:1
Radius 75 nm

Radius 50 nm

Radius 40 nm

dius 25 nm
dius 20 nm
dius 10 nm

llance-Radius 300 nm

30 nm

= 20 nm

e Battlefield
An amphibious landing force, \( x \), at sea, is to assault a limited area defended by a force \( y \). The landing force, \( x \), is to be split up into air mobile, \( x_a \), and a surface mobile, \( x_s \). The defending force \( y \) in turn commit its defense by allocating \( \beta_i \), \( i = 1 \) to \( 2 \) of its force to each element of \( x \) where \[ \sum_{i=1}^{2} \beta_i = 1. \]

The questions to be asked are:

a) What allocation for forces should each side use against the other during the ensuing engagement?

b) What is the mathematical structure of the tactical decisions made by both sides (as defined by a) as a function of initial conditions (e.g., force levels \( x, y \) at time \( t = 0 \)) and constraints (e.g., spatial and temporal limitations when allocating forces)?

c) How does the analysis relate back to the real world in terms of logistics, equipment, force levels, operational plans, etc.?

To abstract the analytic nature of combat during an amphibious operation from a scenario based upon the above descriptive analysis, one is tempted at first to start simply with a "Lanchester Equation" model approach. This model is by far the oldest analytic approach to land warfare and has great flexibility in its generalized form.\(^5\)

The form of the Lanchester Model which seems applicable is

\[
\frac{dx}{dt} = -by \\
\frac{dy}{dt} = -ax \\
x(0) = x_0 \\
y(0) = y_0
\]
where \( x_0, y_0 \) represent the force levels of both sides at time \( t = 0 \) and \( a, b \) reflects each side's normalized rate of attrition of the opposing side. Lanchester's Square Law can be deduced from (1) by taking the ratio of the two differential equations and integrating.

\[
\frac{dx}{dy} = \frac{by}{ax} \tag{2}
\]

\[
\int_a^x \frac{ax}{x_0} \, dx = \int_a^y \frac{by}{y_0} \, dy \tag{3}
\]

\[
a(x_o^2 - x^2) = b(y_o^2 - y^2) \tag{4}
\]

At this point it might be useful to test the above square law relationship with some of the available data on past battles. Weiss has indicated that reasonable agreement with Lanchester's Square Law exists for the Pacific Island Campaign of World War II. In this paper eleven island campaigns were analyzed in terms of United States and enemy strengths before the operation and total losses after the islands were successfully taken. In each campaign the United States initiated the action with superior forces and if Lanchester's Square Law were applicable certain characteristics of the outcome of the battle can be deduced from equation (4). If one rewrites equation (4) as follows:

\[
\frac{(y_o - y)(y_o + y)}{(x_o - x)(x_o + x)} = \frac{a}{b} \tag{5}
\]

for the factors on the left hand side can be definable in real world terms. For example,

\[
\left(\frac{y_o - y}{x_o - x}\right)
\]

is the casualty ratio at time \( t \).
\[ \frac{y_0 + y}{x_0 + x} \]

is the average force ratio from the start of battle to time

\[ \frac{a}{b} \]

is a constant representing the average fire power ratio between sides.

If one converts the battle statistics available in Table I of reference 7 to the above ratios and inserts these ratios into the following logarithmic version of equation (5):

\[
\log \left( \frac{y_0 - y}{x_0 - x} \right) + \log \left( \frac{y_0 + y}{x_0 + x} \right) = \text{constant}
\]

(6)

the plot shown in figure 4 results. The dispersion of all the battle points does reasonably confirm the minus-one slope of the line defined in equation (6). Assuming the validity of the square law in this plot, an estimate of the superiority of the attacking forces' fire power over the defending forces' fire power can be made by noting the ordinate-intercept of the band about the dispersed battle points. This intercept varies from 3 to 18 indicating the extreme variance in fire power superiority over the enemy for different amphibious operations of the last war. This begs the question as to why the effective fire power varies so much relative to the defender. Does the answer lie in the differences in tactical systems employed or operational doctrine used, or the enemy capabilities encountered etc? Certainly this wide dispersion of relative fire power found in real world situations suggests to the analyst the importance of being able to functionally relate the specific amphibious operational characteristics of the campaign such as systems employed, military doctrine used, and enemy capability encountered to an overall measure of effectiveness of the campaign. This type of analysis is one of the objectives of this study.

In another plot of these same statistics, United States loss ratios have been plotted (see figure 5) against Order of Battle Ratios \( \left( \frac{B/R}{y_0 + y} \right) \) indicating the effect of superior forces in keeping the loss ratios low. In fact for a constant defending force, the attackers absolute loss goes down when
Figure 4. CASUALTY VS. AVERAGE FORCE DURING BATTLES (ON BOTH SIDES)
RATIOS FOR WORLD WAR II PACIFIC ISLAND CAMPAIGNS
order of battle ratio \( \left[ \frac{B}{R} \right] \) goes up if the square law is applicable to the situation. Also shown in figure 5 as solid lines are some theoretical battles taken from a study computer runs. These battles were defined analytically in terms of Lanch Square Law and the results show an amazing similarity to the actual island camp. The reason for the difference between the two runs is due to the fact that each curve is based upon a different scenario defining the amphibious operation battle. Also the individual campaigns plotted from reference 9 represent different scenarios, however, their scatter on this particular chart does conform to a characteristic horn-shaped channel outlined by the envelope lines enclosing the battles shown in figure 5.

Let us now proceed with the development of the Lanchester Model. Equation 4 indicates that the normalized attrition rate varies inversely as the square of the force level. This suggests the following transformation which will allow one to consider a force level and its attrition rate as an effective force level only.

\[
\begin{align*}
  x & \rightarrow \frac{x}{\sqrt{a}} \\
  y & \rightarrow \frac{y}{\sqrt{b}} \\
  t & \rightarrow \frac{t}{\sqrt{ab}}
\end{align*}
\]

(7)

This reduces equation (1) to

\[
\begin{align*}
  \frac{dx}{dt} &= -y \\
  \frac{dy}{dt} &= -x \\
  x(0) &= x_0 \\
  y(0) &= y_0
\end{align*}
\]

(8)
We are now in a position to relate this model formulation directly to the scenario of the amphibious operation described above. However, before we describe the mathematical model actually programmed, it might be useful to give a brief mathematical synopsis of this model utilizing the above notation in order to indicate the overall direction of the analysis.

Using the dot notation for time derivatives, the amphibious operation can be described mathematically by the following equations:

\[
\begin{align*}
\dot{x}_a &= -\beta y \\
\dot{x}_b &= -(1-\beta)y \\
\dot{\beta}y &= -x_a \\
\dot{y} &= -x_b \\
x &= x_a + x_b \\
x(0) &\text{ given at } t = 0 \\
y(0)
\end{align*}
\]

(9)

where \( \beta; 0 < \beta < 1 \) is the allocation of that part of \( y \)'s force to \( x_a \) and \( (1-\beta) \) to \( x_b \), and \( \{x_b, x_a\} \) represents an allocation of all of the amphibious landing force \( x \) to either the surface or air mobile mode of attack.

If any of the units on either side are reduced to zero during the battle, the opposing side's remaining allocation against the vanishing unit will be assumed to be transferable for combat against the other side's non-vanishing unit. The winner will be the side that survives the battle. To relate force levels and normalized attrition rates, the following scaling laws apply:

\[
\begin{align*}
x_a &\xrightarrow{\sqrt{a}} x_a \\
x_b &\xrightarrow{\sqrt{a}} x_b
\end{align*}
\]
The model actually programmed can be extended to include x's reserve for
which can be thrown into the action at some later time. The computerized
also takes into consideration the differences in fire power available to x when
projected through a vertical envelopment assault or via a surface beachhead lan
That is, x's Regimental Landing Teams, RLT's, are projected ashore as quickly -
possible with the heavy ordnance coming after. In the above model it is assume-
x has many units, each with its own characteristic fire power. The greater nor
attrition rate a, as associated with the surface force can only be projecte-
and at a given rate. The vertical envelopment with the lower attrition rate, a,
assumed to be projected ashore instantaneously. In this way x is faced with an
allocation choice of projecting into battle low fire power forces x at a high
(instantaneously) or a high fire power force x at a much lower rate.

It is felt that these limiting assumptions are justified in that we are
cerned only with the essence of the problem at this stage of the study; that is
the gross outcome of the battle as a function of input parameters \{ x(o); y(o) \}
decision parameters \{ x_1(o), x_2(o); \beta \} and constraints \{ a, b; time to ini
battle \}. Hopefully, this initial analysis will develop useful insight into the
structure of amphibious operations such that more significant models can be deve
later on.

' Summarizing, equation (9) represents a simple model of an amphibious ope
in a game theoretic context which attempts to relate the interplay of force lev
allocation decisions, and constraints as they affect the outcome of the battle.
Payoff of such a game will be defined by the solutions of the above equations as
function of time at the specific time when one side's force level is either reduced
to zero or any finite level. The remaining side's force level represents the va
the game, positive to blue if the winner and negative to blue if the loser. This
called a zero sum two person game.
Another possible zero-sum two-person game based upon the above model could be constructed by defining the payoff function or value of the game as the time $t^*$ it takes to complete the amphibious assault—either successfully or successfully. That is, the time it takes to reduce the opposing side either to zero or any predetermined level. This time would be minimized by the attacker (the amphibious landing force) while the defender would attempt to maximize the same time. Strategies for both sides defined by this type of payoff function could have a realistic interpretation in that one of the primary purposes of an amphibious operation is to secure a beachhead as quickly as possible such that the main body of attacking forces can be placed ashore safely. The defending forces will attempt to hinder the amphibious landing force as long as possible, such that superior reserves deployed elsewhere can be brought to bear upon the attacker while he is in the vulnerable position of attempting to establish a beachhead. While it is true that the attacking amphibious force would never knowingly go into an area under an adverse force posture, it is also true that if the defender knew the location of the beachhead in advance and marshalled all his forces against this beachhead, the attacker would be under an extreme disadvantage in concluding the landing successfully. Since the defending force is unaware of the landing point because it is kept secret by the attacker and the attacking force further hinders the ability of the defender to obtain this secret by employing deception tactics, such as false landing starts, etc., the defending force is forced to keep troops in reserves. This makes it possible for the amphibious landing force to gain local superiority at a beachhead provided, the time between committing the attacking forces and securing of the beachhead is less than the time necessary for the defender to effectively deploy his reserves against the known landing point determined by the actual landing. The outcome of such a battle using the above model can then define the effectiveness of the deception techniques employed by the attacker. The model could also determine the relationship and value of fire power, mobility, dispersion, reaction time, and surveillance to various proposed deception techniques.
Up to now we have been describing the mathematics and possible use of the above model in extreme generality; however, before we get involved with the actual use of the amphibious landing model, we should describe the model as it was actually programmed. The next section describes the actual model from the mathematical point of view, and Appendix A gives the detailed computer program used to implement the mathematics.
C. **THE AMPHIBIOUS OPERATION MODEL**

1. **Introduction**

In the construction of a model of an amphibious operation the entire operation is first formulated as a complicated abstract process in the physical world. Certain items, such as number of battle units, force levels, firepower, etc., may be considered as inputs to the physical process while other items such as casualties, duration of the battle, etc., may be viewed as outputs. In mathematical terms, the amphibious operation is an "operator" with the physical input variables as its domain and the physical output variables as its range. In view of the complexity of a general amphibious operation and the extremely large number of contingent possibilities that can arise during the execution of any individual operation, it would be exceedingly difficult if not impossible to completely represent the operator mathematically. Our task then is to construct a mathematical operator that is an approximation to the physical process. The nature of the approximation is determined by the uses to which the model will be put and the resultant simplifications that can be tolerated without materially affecting the significant results.

As the purpose of this study is to qualitatively determine the nature of optimal battle strategies under very general conditions, no attempt will be made to abstract the analytic nature of the amphibious operation via an all-inclusive model. The basic unit of force for the protagonists, Blue and Red, will be taken as a battle unit and the battle as a whole will be viewed as an aggregation of local conflicts among individual battle units. Thus, at any one instant of time, we need only consider a series of relatively simple local conflicts to determine the state of the battle as a whole. Since we will assume that the dynamics of any local conflict are governed by Lanchester's Equations, the only pertinent information is the composition of the local conflicts and the force levels and attrition constants for the individual battle units. This information may be summarized in a list of the following form:
<table>
<thead>
<tr>
<th>Unit</th>
<th>Force Level</th>
<th>Activity Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Local Conflict No. 1

| B1   |             |               |
| B2   |             |               |
| ...  |             |               |
| Bj   |             |               |
| R1+1 |             |               |
| R1+2 |             |               |
| ...  |             |               |
| Rk   |             |               |

Local Conflict No. 2

| Bj+1 |             |               |
| Bj+2 |             |               |
| ...  |             |               |
| Bm   |             |               |

etc
The composition of the local conflicts will be determined by giving an Opponent Priority List (OPL) for each battle unit. In order to reduce the complexity of the computer program that implements this model, the decision was made to limit the OPL to two levels, the first priority opponents and the second priority opponents. The following behavior is then postulated for the individual battle unit.

RULE 1: A battle unit will seek to engage his first priority opponents if they have not been already eliminated from the battle. If a unit is engaged with an opponent, that opponent is termed a direct opponent of the given unit.

RULE 2: If a unit's first priority opponents have been eliminated, he will, after a specified time delay, seek to engage his second priority opponents.

RULE 3: Any two units that share a common direct opponent are considered to be battle allies.

RULE 4: All the direct opponents of a given unit's battle allies are taken to be additional direct opponents of the unit itself.

These rules offer a great flexibility in the determination of the conflict, since a unit can be drawn into a given conflict in many ways - by attacking an opponent who is involved in that conflict, by being attacked by an opponent who is involved in that conflict, or by sharing a common direct opponent with an ally who is involved in that conflict.

In general, the application of Rules 1 and 2 and the repeated application of Rules 3 and 4 will completely determine the composition of all the local conflicts (a formal proof can be constructed that shows that Rules 3 and 4 need only be applied a limited number of times and that a unique splitting up of the units into local conflicts results. The proof, which will be omitted here, rests on the fact that the above rules define an equivalence relation on the set of all units and this relation completely partitions the set into equivalence classes.)

Within each local conflict, the progress of the battle may be measured by means of Lanchester's Equations. These differential equations can be solved analytically and evaluated to determine the force levels of any unit in a given local conflict.
conflict at any time in the future, as long as the basic composition of the conflict remains unchanged. Once the basic composition changes, the coefficients of the differential equations change and the original analytic solution is no longer valid. At this point we must redetermine the composition of the local conflict, recalculate the new coefficients for the differential equations, and begin a new series of analytic solutions to continue where the old solutions left off.

The composition of a local conflict can be changed one of two ways:

1. a unit is added to the battle
2. a unit is eliminated

The specification of a time of arrival for each unit determines the time at which the unit is added to the battle. A single time of arrival is given since it is specified that once a force is committed to the battle it will not be withdrawn and re-deployed at another time. Once a unit is added, it of course seeks out its opponents according to Rules 1 and 2.

The elimination of a unit occurs when its force level drops below a specified minimum force level. The time at which this takes place can be determined by inverting the analytic solutions of the differential equations and solving as a function of force levels. This inverse solution has several possible solutions depending on the coefficients and initial values of the differential equations. These solutions are given in detail in the next section.

After all the first priority opponents of a given unit are eliminated, it specifies a time delay before that unit may engage its second priority opponents (The rule also applies if the first priority opponents have been eliminated at another time.) This delay factor is intended to reflect the geographical location of the units and their relative mobilities.

In summary, the following items are the necessary input parameters for a unit:

1. initial force level
2. minimum force level
3. attrition constants
4. time of arrival
5. time delay factor
6. first priority opponents list
7. second priority opponents list
The application of Rules 1, 2, 3 and 4, in conjunction with the preceding items, determines the composition of the local conflicts while Lanchester's equations give the force level of each of the units as a function of time and indicate the time at which a unit is eliminated.

As analytic methods are used throughout and no time-step simulation is utilized, this furnishes us with an extremely rapid means of determining the expected outcome (or "payoff") of the battle determined by the given battle plans and initial force levels.

If various elements of the battle plans and/or initial force levels are considered as parameters we thus are in a position to generate trade-off tables showing the effect of a variation in one or more of the parameters, for example, the degradation of payoff due to increased time spacing of the various landing groups. Furthermore, game theoretic techniques may be used to simultaneously optimize the choice of parameters for both the attacker and defender.
2. **The Application of Lanchester's Equations to Local Conflicts**

As outlined in the previous section, Lanchester's Equations are used to measure the force levels of any unit at any given time. It should again be noted that the application of these equations depends upon the various coefficients of the differential equations remaining constants. To ensure this, we have specifically assumed that the composition of the local conflict remains unchanged during the time period to which we are limiting ourselves.

For the sake of simplicity in the following discussion we will begin by assuming that the local conflict occurs between two individual battle units, A and B, on opposing sides and then generalize to a larger number of units. The force level of each unit will be termed $n_A$ and $n_B$ respectively and the attrition rate by $a_A$ and $a_B$. The familiar Lanchester equations governing this simplified combat are:

\[
\begin{align*}
\dot{n}_A &= -a_B n_B \\
\dot{n}_B &= -a_A n_A
\end{align*}
\]

When the dot notation is used to denote differentiation with respect to time. These simultaneous equations may be solved to give $n_A$ and $n_B$ as functions of time:

\[
\begin{align*}
n_A(t) &= n_A(t_0) \cosh \left( \frac{\sqrt{a_A}}{a_B} (t-t_0) \right) - \frac{\sqrt{a_B}}{\sqrt{a_A}} n_B(t_0) \sinh \left( \frac{\sqrt{a_A}}{a_B} [t-t_0] \right) \\
n_B(t) &= n_B(t_0) \cosh \left( \frac{\sqrt{a_B}}{a_A} [t-t_0] \right) - \frac{\sqrt{a_A}}{\sqrt{a_B}} n_A(t_0) \sinh \left( \frac{\sqrt{a_B}}{a_A} [t-t_0] \right)
\end{align*}
\]
In general, however, a local conflict may have more than a single unit each side. If side A has \( k \) battle units involved and side B has \( \ell \) battle units involved we may identify the forces levels as \( n_{A1}^1, n_{A2}^1, \ldots, n_{Ak}^1, n_{B1}^2, n_{B2}^2, \ldots, n_B^\ell \) the attrition rates as \( a_{A1}, a_{A2}, \ldots, a_{Ak}, a_{B1}, a_{B2}, \ldots, a_B^\ell \). For simplicity sake, we may assume without any loss of generality, that the subscripts are assigned in a manner that

\[
a_{A1} \geq a_{A2} \geq \cdots \geq a_{Ak}
\]

and

\[
a_{B1} \geq a_{B2} \geq \cdots \geq a_B^\ell
\]

Under these circumstances it can be shown that optimum behavior on each side demands that fire be concentrated on units \( A_1 \) and \( B_1 \). (It is recognized that this type of behavior does not occur in practice. It is felt, however, that the optimum behavior postulated here offers a sufficiently close approximation to these purposes. If subsequent investigation should prove this not to be correct, then an alternate scheme of behavior could easily be postulated.)

The differential equations in this case, become:

\[
\dot{n}_{A1} = - a_{A1} n_{B1} - \sum_{i=2}^{k} a_{A1} n_{Bi}
\]

\[
\dot{n}_{Bj} = 0 \quad \text{(for } j = 2, \ell)\]

\[
\dot{n}_{B1} = - a_{A1} n_{A1} - \sum_{\alpha=2}^{k} a_{A1} n_{A\alpha}
\]

\[
\dot{n}_{B\beta} = 0 \quad \text{(for } \beta = 2, \ell)\]
Equations (8) and (10) indicate that:

\[
\begin{align*}
\eta_{A_j}(t) &= \eta_{A_j}(t_0) \quad &\text{(for } j = 2, 4) \\
\eta_{B_s}(t) &= \eta_{B_s}(t_0) \quad &\text{(for } s = 2, k)
\end{align*}
\]

and so equations (7) and (9) can be represented in the form:

\[
\begin{align*}
\dot{\eta}_A &= -a_{Bl} n_{Bl} + r \\
\dot{\eta}_N &= -a_{Al} n_{Al} + s
\end{align*}
\]

where \( r \) and \( s \) are both constants:

\[
\begin{align*}
r &= - \sum_{i=2}^{k} a_{Bl} n_{Bl} \\
s &= - \sum_{i=2}^{l} a_{Al} n_{Al}
\end{align*}
\]

The general time solutions for the force levels become:

1) If \( a_{Al} > 0 \) and \( a_{Bl} > 0 \)

\[
\begin{align*}
\eta_{A_1}(t) &= \left[ \eta_{A_1}(t_0) - \frac{s}{a_{Al}} \right] \cosh \left( \sqrt{\frac{a_{Bl}}{a_{Al}}} (t-t_0) \right) \\
&\quad + \frac{s}{a_{Al}} \\
\frac{\eta_{B_1}(t)}{\sqrt{a_{Bl}}} &= \left[ \eta_{B_1}(t_0) - \frac{r}{a_{Bl}} \right] \sinh \left( \sqrt{\frac{a_{Bl}}{a_{Al}}} (t-t_0) \right)
\end{align*}
\]
2) If \( a_{A1} \) or \( a_{B1} = 0 \)

\[
n_{A1}(t) = -\frac{a_{B1}}{2} s(t-t_0) + a_{B1} n_{B1}(t_0) - r (t-t_0) + n_{A1}(t_0)
\]

The equations for \( n_{B1}(t) \) are identical in form to equations (13) and (14) above, with the subscripts, of course, being reversed.

For the purpose of determining the "time to elimination", the inverse solution of the above equations are needed. These are:

1) If \( a_{B1} = 0 \) and \( r < 0 \):

\[
t = t_0 + \frac{n_{A1} - n_{A1}(t_0)}{r}
\]

2) If \( a_{B1} = 0 \) and \( r \geq 0 \):

no solution exists; i.e., \( t = \)

3) If \( a_{A1} = 0 \) and \( r < a_{B1} n_{B1}(t_0) \):

\[
t = t_0 + \frac{n_{A1} - n_{A1}(t_0)}{a_{B1} n_{B1}(t_0) - r}
\]

4) If \( a_{A1} = 0 \) and \( r \geq a_{B1} n_{B1}(t_0) \):

no solution exists; i.e., \( t = \)

5) If \( a_{A1} = 0 \) and \( s \neq 0 \):

\[
t = t_0 + \left( \frac{[r-a_{B1} n_{B1}(t_0)] + \sqrt{(a_{B1} n_{B1}(t_0)) - r}^2 + 2 a_{B1} s (a_{A1}(t_0) - n_{A1})}{a_{B1} s} \right)
\]
6) If $a_{A1} = 0$ and $s < 0$ and $r > \sqrt[n_{BL}(t_0)]{\frac{x_{BL} n_{BL}(t_0)}{s_{BL}(n_{A1}(t_0) - n_{A1})}}$

no solution exists; i.e., $t = -\infty$

7) If $a_{A1} = 0$ and $s < 0$ and $r = \sqrt[n_{BL}(t_0)]{\frac{x_{BL} n_{BL}(t_0)}{s_{BL}(n_{A1}(t_0) - n_{A1})}}$

\[ t = t_0 + \frac{[r-a_{BL} n_{BL}(t_0)] + \sqrt{(n_{BL}(t_0)-r)^2 + 2 a_{BL} s_{BL}(n_{A1}(t_0) - n_{A1})}}{n_{BL} d} \]

8) If $\sqrt[n_{BL}(t_0)]{\frac{x_{BL}}{a_{BL} - n_{BL}(t_0)}} + \sqrt[n_{A1}]{n_{A1}(t_0) - \frac{s}{a_{A1}}} = 0 \text{ and } \frac{s}{a_{A1}} < n_{A1}$

\[ t = t_0 + \frac{1}{\sqrt[n_{A1}]{a_{BL}}} \ln \left| \frac{n_{A1} - \frac{s}{a_{A1}}}{\frac{s}{a_{A1}} - n_{A1}} \right| \]

9) If $\sqrt[n_{BL}(t_0)]{\frac{x_{BL}}{a_{BL} - n_{BL}(t_0)}} + \sqrt[n_{A1}]{n_{A1}(t_0) - \frac{s}{a_{A1}}} < 0$

\[ t = t_0 + \frac{1}{\sqrt[n_{A1}]{a_{BL}}} \ln \left( \frac{n_{A1} - \frac{s}{a_{A1}}}{{a_{BL}} \frac{s}{a_{A1}} + \sqrt[n_{A1}]{\frac{x_{BL}}{a_{BL}} - n_{BL}(t_0)}} + \left[ n_{A1} - \frac{s}{a_{A1}} - n_{A1}(t_0) \right] \right) \]

10) If $\sqrt[n_{BL}(t_0)]{\frac{x_{BL}}{a_{BL} - n_{BL}(t_0)}} + \sqrt[n_{A1}]{n_{A1}(t_0) - \frac{s}{a_{A1}}} > 0 \text{ and } \left[ \frac{x_{BL}}{a_{BL} - n_{BL}(t_0)} \right]$

no solution exists; i.e., $t = -\infty$
1.21) If \( \sqrt{a_{BL}} \left( \frac{x}{a_{B1}} - n_{B1}(t_o) \right) + \sqrt{a_{A1}} \left[ n_{A1}(t_o) - \frac{s}{a_{A1}} \right] > 0 \) and \( \left( \frac{x}{a_{B1}} - n_{B1}(t_o) \right) < 0 \), and \( \sqrt{a_{A1}} \left[ n_{A1}(t_o) - \frac{s}{a_{A1}} \right] > n_{A1} - \frac{s}{a_{A1}} \), no solution exists; i.e., \( t = \)

1.22) If \( \sqrt{a_{BL}} \left( \frac{x}{a_{B1}} - n_{B1}(t_o) \right) + \sqrt{a_{A1}} \left[ n_{A1}(t_o) - \frac{s}{a_{A1}} \right] > 0 \) and \( \left( \frac{x}{a_{B1}} - n_{B1}(t_o) \right) < 0 \), and \( \sqrt{a_{A1}} \left[ n_{A1}(t_o) - \frac{s}{a_{A1}} \right] \leq n_{A1} - \frac{s}{a_{A1}} \),

\[
t = t_o + \frac{1}{\sqrt{a_{A1}a_{B1}}} \left\{ \frac{n_{A1} - \frac{s}{a_{A1}}}{\sqrt{a_{A1}}} \left[ n_{A1} + n_{A1}(t_o) - \frac{2s}{a_{A1}} + \frac{a_{B1}}{a_{A1}} \right] \right. \]
\[
+ \left. \frac{\sqrt{a_{B1}} \left\{ n_{A1} + n_{A1}(t_o) - \frac{2s}{a_{A1}} + \frac{a_{B1}}{a_{A1}} \right\}}{\sqrt{a_{A1}}} + \left[ n_{A1}(t_o) - \frac{s}{a_{A1}} \right] \right\}.
\]

(81)
D. PROBLEM I

1. Introduction

To illustrate the methodological techniques for handling an amphibious landing situation, a specific problem will be investigated (see Figure 6). An amphibious landing force $B$, at sea, is to assault a limited area defended by a force $R$. The landing force is to be split up into three surface mobile elements $B_1$, $B_2$, $B_3$ and a mobile element $B_4$, each of which is to initiate action sequentially in time. These elements are projected shore sequentially in time with the first wave consisting of infantry battalion type units, designated $B_1$, then a short time later more infantry plus close support artillery type units, designated $B_2$, and finally after another increment later some more infantry plus a tank section designated $B_3$. The air mobile elements consisting of a vertical envelopment team of infantry and close support artillery units, designated as $B_4$, which are projected instantaneously inland distance $d$ from the beach (see Figure 6) at the same time $B_1$ arrives across the beach. The relative fire power of these four elements of blue are assumed to be in the following ratio: $B_1 : B_2 : B_3 : B_4 = 1.0 : 2.0 : 4.0 : 1.5$. The red defending force, in turn, commits its forces either to the beach (designated $R_1$), to the airmobile (designated $R_2$), or to both at the same time $B_1$ and $B_4$ are deployed. The relative firepower of these two elements of red are assumed to be equal to blue's maximum $B_3 : R_1 : R_2 = 1.0 : 1.0 : 1.0$. Note (from Figure 6) that each element of blue has a predetermined battle commitment time which reflects the amphibious landing force logistic constraints (Reference 4). This time of commitment is symbolized by the clock next to each element in Figure 6. Also, take note that the ensuing battle takes place at two different locations, at the beach and at some inland point. A model developed for this problem reflects this spatial characteristic of the battle by defining a set of time delays $\{t_d\}$ which applies to each force element's battle indicating the amount of time required for each element to traverse this distance if required to do so during the course of the battle. Whether an element advances this distance depends upon whether the force element has an opponent to fight during the initial allocation of forces and successfully destroys the opponent; that any new fighting opponents can only be reached by traversing the distance $d$ shown in Figure 6. If a force element does not have an opponent initially at a particular point on the battlefield then the time delay $t_d$ will denote the time necessary to meet an opponent located at the other point on the battlefield. This time delay reflects the mobility characteristics of the forces involved and hence an important trade-off parameter.
Figure 6. THE AMPHIDIOUS LANDING MODEL
Another aspect of problem I not indicated in Figure 6 is the fact that fighting elements have a priority list of opponents. This means that every fighting element knows its first priority opponent and when the action is successfully completed, it can then attack a predefined second priority opponent. In some cases fighting elements are placed in position to protect the flanks of an operation, no priority is utilized in which case fighting initiates only when the opposing priority initiates the battle action against the flank.

The following questions are pertinent to the above problem description:

a) What is the mathematical structure of the tactical decisions made by both sides in selecting the action variables such as number of forces, time delays, battle time commitments, priorities, etc.?

b) How does this type of an analysis relate back to the real world?

c) What allocation of forces should each side use in initiating the battle? E.g. Blue’s allocation is that of deploying its total units to the beach and/or inland locations and red’s total unit allocation is that of defending the beach and/or staying in the rear to engage the vertical envelopment attack. Both sides must make such allocations quantitatively.

d) What time sequence should the amphibious landing force adhere to in initiating its elements into combat?

The significant input variables to the model based upon the above areas are:

a) Order of battle actions — the total number of troops each side has available for the operation.

b) Firepower — each fighting element’s rate of kill per man per hour.

c) Time sequencing of units into the battle

d) Initial priorities

e) Unit mobility factors

The significant output variables are:

a) The number of survivors at the end of the battle.

b) The duration of the battle for any threshold of defeat of one side.

c) The optimal allocation strategies available to each side, e.g., what fraction of the total forces goes across the beach. Deployed fire units is opposed to the defender’s allocation of what fraction of his forces (his force in defending the beach as opposed to the Fijian kept in reserve to defend against the air mobile attack.

BET AVAIL COPY
A. The natural advantage is represented by the smallest or zero amount of the real-world advantage. In the context of the natural advantage of the natural advantage, it is always the real-world advantage of the natural advantage of the natural advantage. However, it should be noted that the real-world advantage of the natural advantage is the real-world advantage of the natural advantage of the natural advantage. Hence, an alternative method of determining the real-world advantage of the natural advantage is the real-world advantage of the natural advantage of the natural advantage. Consequently, the real-world advantage of the natural advantage is the real-world advantage of the natural advantage of the natural advantage.
criteria that will be used to determine the optimality of the strategies will be min max payoff for red and max min payoff for blue. When min max equals max min a pure solution exists and the payoff corresponding to this pure solution is called the value of the game. If no such equality exists, then it can be shown that min min and the only way to get equality is to redefine the payoff function as anova value of survivors with the players picking their strategic variables (i.e., total forces allocated to air mobile and defending against air mobile) accounts for probability distribution. Decisions made in this manner are called mixed strategies. Usually represent marginal strategies for the side having to employ them. For tactical system design or requirements point of view, one would never know an amphibious operation against a defending force which depended upon a mixed strategy to gain the objective of the operation. This would be tantamount to having the objective of the operation. Whereas the logical plan for an amphibious operation to land with overwhelming superiority and allow any advantage accrued by the element of secrecy in initiating the operation to compensate for faulty threat intelligence estimates, acts of God (e.g., bad weather that is unpredictable), etc. To determine just what constitutes overwhelming superiority in planning an amphibious operation, just how much can one degrade this superiority and still achieve objectives with uncertainty is one of the objectives of this study. The techniques employed to these types of questions will be the subject matter of this section.

2. Discussion

To obtain the game-theoretic solution to the amphibious operation described above (see Figure 1) we first compute the mathematical surface representing it of all possible allocations for both sides (Figure 2). Then we tag the minimum row and the maximum of each column shown respectively as rectangles and ellipses (Figure 3). Blue, the attacker, will select the maximum of the minimums tagged rows while red, the defender, will select the minimum of the maximums tagged columns. The point on the grid (see Figure 4) in which the maximum of the minimum payoff and the minimum of the maximum payoff occurs is the same point and the value of the game is defined as the number of survivors of blue/red (positive for blue, negative for red) located at this point on the surface. The strategies associated with this point are pure. In most cases the min max = max min solution occurs at corners of the matrix.
### P Matrix

**Pend Against Air Mortar**

| 10 | 60 | 70 | 80 | 90 | 100 |

**Payoff:**

- **No. of Blue Survivors at Time t (B.O.)**

...
0 DEFEND AGAINST AIR MOBILE

<p>| | | | | |</p>
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<tr>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
<td>100</td>
</tr>
</tbody>
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MAX-MIN
Once the value of the game and associated decision variables are known with constraints, the solutions determined from natural discontinuities of the surface generated. This is accomplished by placing a constraint raster first on blue's set line starting from the 100% allocation decision level and moving the raster toward the 0% allocation. For each placement of the raster the matrix is solved for the min max = max min solution in the same manner as outlined above on the partial matrix (see Figure 9) consisting of blue's strategy line from 0% allocation to the raster and red's unrestricted strategy line. This sub-optimum solution to the game is then related to the placement of the raster by noting when the solution changes abruptly as the raster moves from 100% to 0% allocation. For example, if blue's unrestricted strategy is to allocate 100% of the fighting elements to the air mobile or vertical envelopment decision the raster is then placed under the 90% level raising blue's strategy from 0% to 90% and the solution to this restricted game is not if the optimum solution is the 90% allocation of the fighting elements via the air mobile mode decision, then this is considered to be no change in the basic strategy. That is, the restricted game still demands that blue send all his fighting units air mobile even though blue is restricted by the 90% allocation level. After this determination, the raster is then moved to the 80% allocation level and the restricted game again solved. If the solution yields the 80% allocation level decision the is moved to the 70% level, etc. In most cases during this process of methodically constraining blue's decision level the game theoretic solution will abruptly change to yield an optimum decision other than the maximum possible fighting units going air mobile. The position of blue's constraint raster when such an abrupt decision occurs is then recorded and the associated sub-optimum strategy restricted within this partitioned matrix is called a natural strategic discontinuity level. The physical meaning of such sub-optimum strategies is that if for any physical reason blue cannot send 100% of his forces air mobile (the optimal policy), at what level of decision constraint must blue change his tactics completely concerning a given side of attack. Therefore, the natural strategic discontinuity level represents a threshold in blue's strategic thinking, above which blue will attack in the vertical envelopment mode with all the fighting elements he physically can get air mobile, and below which he will use the sub-optimum strategy based upon the solution of the partitioned matrix. An example of this threshold from the real world would be the decision of a commander not to send an air mobile strike in support of an across- sea operation if he felt that the air mobile forces would not be able to act a fighting unit during the time necessary for the main body of forces from
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**MAG.DESCRIPTIVITY LEVELS**

**METHOD AGAINST AIR MOBILE**

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**FIGURE 10. SENSITIVITY ANALYSIS**
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The major inputs to the model are represented by the columns. Also, in this analysis is the fact that each fighting element has a second level opponent that the surviving element after an engagement on the battlefield is considered as a new opponent as opposed to staying put waiting for an enemy element from another section of the battlefield. In this way mixed (or asymmetric) opponent is not be an influencing factor. In other words, the payoff function, i.e., the number of survivors, will reflect active combat on both sides.

b. The Standard Case

The standard case with inputs shown in Figure 10 was analyzed by Computer Program No. AMO4B titled, "Assault Tactics." A complete description of the computer program is contained in Appendix A of this report. The results of these runs are displayed as a matrix of survivors for blue, red and the time length of warfare as defined by the particular allocation for each side. Examples of these outputs are shown in Appendix A. The game-theoretic analyses of these outputs were done by hand and displayed in the form of a series of graphs placed in columns, each column representing a different natural constraint of the problem.

Figure 11 gives the game-theoretic analysis of the standard case. The first column represents the unrestricted play of the game in that the strategic available to both sides encompasses the full payoff matrix or decision surface. An independent variable is the order of battle ratio (B/R) which reflects the degree of superiority the attacking force initiates the amphibious operation. In the first in the upper left hand corner the attacker's optimum payoff (B-R) in terms of the number of survivors (positive for blue or attacking survivors, negative for red or defending survivors) is plotted. The defending (red) force is kept constant at 5000 men and the attacking (blue) force is allowed to vary from 0 to 20,000 men. Note that as both sides are reduced to zero at the same time for the optimal allocation (min max min criteria) of forces on the battlefield. This order of battle ratio is the attacker's threshold for successfully completing the operation, i.e., the attack on the defender to no survivors. However, an inspection of the actual decision surfaces (Figures 12 to 16) reveals that this threshold has no meaning as a pay factor because many of the non-optimum choices for both sides will result in the attacker not achieving his objectives. In other words, the optimum allocation of point is an unconservative criteria for use in determining the level of superior...
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**STANDARD CASE**  B/R = 3.0
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16 - STANDARD CASE  B/R = 4.0
the attacker must plan for in order to guarantee his objective, i.e., a positive value for the attacker. This unconservatism is due to the fact that in the world situation intelligence for both sides would be less than perfect and operational decisions would have to be judged from a probabilistic basis. The model used in this study assumes both sides have perfect intelligence which is certainly not the world case. However, it is possible to use the results of this run to determine a proper order of battle ratio for the attacker which is independent of the actual decision made by either side, thus removing the intelligence aspect of the problem (i.e., attacker's advance knowledge as to the defenders allocation of forces) can be done by determining the order of battle ratio above which the payoff decision surface always stays positive. This occurs approximately at \( \frac{B}{R} = 1.4 \) which represents a measure of the amount of intelligence indeterminacy the particular operation scenario (in this case the model of Figure 6) contains. The intelligence aspect of the attacker's advance knowledge as to the defenders allocation of forces can be done by determining the order of battle ratio above which the payoff decision surface always stays positive. This occurs approximately at \( \frac{B}{R} = 1.4 \) to 2.3 representing the increment of forces necessary to guarantee success of the operation with imperfect intelligence. One example of imperfect intelligence would be for the commander of the amphibious operation to err, based upon poor surveillance information in his estimate of the defense's allocation of forces to the beach and/or reserves.

The next graph in the left hand column of Figure I1 represents the duration of the battle \( t^* \) based upon both sides optimizing their decision in the sense defined in the graph above, e.g. \( (B/R)^* = f(B/R) \). Two time solutions exist because red, the defender, has two different force allocations which yield the same optimum payoff. The following two charts in this column represent blue's and red's optimum strategies as a function of \( (B/R) \) for the unrestricted game. Blue, the attacker, sends all his assault elements in the air mode \( (R^* = 1.0) \) and red, the defender, either defends the beach with all his forces at the time the battle is initiated, or stays in the rear with all his forces attacks all of blue's air mobile units deployed. If no blue air mobile units are deployed (a non-optimum play on blue's part), red traverses the distance to the beach in time \( t_d \) after the battle is initiated and attacks blue's units landing on the beach. If red defends the beach and blue goes 100% air mobile (optimum) then battle is initiated \( t_d \) hours after vertical envelopment touchdown. Depending red's optimum choice of strategy, the length of time of the battle will reflect not only the rate of attrition of both sides, but also the possible transport time between both points of the battlefield, i.e., between the beach and the line. Note that when both sides reduce each other to zero force strength at the same time of the battle \( t^* \) goes to infinity (see Figure 11). This is a consequence of the deterministic assumptions which represents a mathematical discontinuity for this
cular solution. It can be avoided by having a finite number or percentage of survivors determine the pay off surface.

Summarizing the above, we can say the left-hand column of Figure 11 represents the results of the unrestricted optimal play of the game between blue, the attacker, and red, the defender (defined in Figure 6) as a function of the order of battle ratio parameter $(B/R)$. The payoff function $(B-R)^* = f(B/R)$ indicates that an order of battle ratio at least $(B/R) > 1.4$ is necessary for blue to achieve its objectives (reducing red to zero survivors while blue survives) under optimal decisions and perfect intelligence for both sides. It was also implied (Figures 12 to 16) that a $(B/R) > 2.3$ was necessary to guarantee the objective independent of the strategic decisions made by either side (defined as perfect intelligence). It should be noted that both sides had complete freedom in decision choice. That is, blue could send any portion of its forces air mobile to the fire power constraints of the air mobile mode. While red could defend the and/or inland area without constraint. This is called the unrestricted play of the game.

We are now in a position to discuss the natural discontinuity levels of the strategic decision surface for both sides. These are indicated in the next three columns in Figure 11.

c. The Natural Strategic Discontinuity Levels

Based upon the above analysis, we would like to pose the following question:

To what extent does possible real world constraints affect the output recorded by the standard case solution for the unrestricted play of the game?

From the mathematical model point of view, real world constraints such as logistic of the amphibious operation have not been directly programmed. However, inherent payoff matrices of Figures 12 to 16 lie in mathematical constraints which indicate sensitivity of optimum strategic decisions made by both sides to the value of the function. For example, in the second column of Figure 11, blue’s strategic discontinuity levels are recorded as a function of the order of battle $(B/R)$.

The third graph in this second column of Figure 11 illustrates the mathematical constraint we are talking about. This chart indicates blue’s per cent air mobile allocation threshold above which blue must deploy to the air mobile mode for the air mobile mode to remain the optimum force allocation or decision for the unrestricted play of the game (the first column of Figure 11). If blue in attempts to play optimally, cannot deploy a greater percent of his forces to the air mobile mode this threshold percent indicates, then a radically different strategy is called...
this case, blue, the attacker, is forced to go 100% across the beach if his to remain optimum vis-a-vis red, the defender. This abrupt change in strategy threshold level indicated is caused by a discontinuity of the optimum decision when solving for the min max = max min payoff point on the decision surface—limiting blue's capacity to fully allocate forces in the air mobile mode.

This threshold parameter is a significant output reflecting the of the mathematical model exercised which was not obvious to the analyst during the construction phase of this study. Also, the air mobile threshold represent a significant output of a command decision made many times in the past by commanding officers, planning and tactical phase of great battles. The question as to whether to deploy in the rear via the air mobile mode and in what percentage of the total force to deploy in this mode is usually answered at the highest levels of command where judgment and experience are without peer. For the amphibious model to be useful, this parameter quantitatively is a significant test of the model in that the other inputs of the amphibious operation model such as fire power, mobility, priority levels, etc. of both the attacker and defender should be considered to know the knowledge and experience necessary to develop requirements, plan, and comprehend the amphibious environment. Before continuing the analysis in this direction, columns of Figure 11 will be defined.

The third column represents red's, the defenders, threshold in a forces to the beach and/or remain inland to attack the air mobile force if deployed. This same type of decision was made in the Battle of Okinawa by the Japanese in the battle of Okinawa by the Japanese. They elected to dig in and let the United States amphibious landing opposed. Based upon the results of this battle, one can say the Japanese played optimally if not successfully. Columns two and three of Figure 11 are the payoff values (B-R)³ and the time t⁴ based upon the constrained strategies resulting from the natural discontinuity effect inherent in the strategic surface, column four of Figure 11 illustrates the results of the play of the game sides are forced into strategies constrained within their thresholds.

Summarizing we can say that the unrestricted play of the game re allocating completely to the air mobile mode while red defends completely either beach or inland reserve (Figure 11, first column). If blue is physically co
hand, \( \frac{B}{A-4} \) is called the superiority case in that it represents the greater relative to the defender, of blue forces used in the standard case. The time parameter \( t_\theta \) is allowed to run from \( 0 \leq t_\theta \leq 4 \) hours and there does not see any visible effect on blue's air mobile threshold for the parity case (see third graph, second column). However, in the superiority case shown in Figure 14, for zero time delay the air mobile threshold drops suddenly to zero. In other words if each fighting element traverses the two points on the battlefield at \( t_\theta \) (a physical impossibility) no air mobile threshold exists for the superior and blue will send as many of his force elements via the air mobile mode as possible, while still keeping within the optimality criteria for both the restricted and restricted play of the game. It should be noted that only discrete \( t_\theta \) values were computed and the curves shown in all these game runs are only valid at the specific points where computations were made. In other words \( t_\theta \) for zero and one were not handled in a continuous manner, it is not known when the threshold value jumps from zero to one. However, for sake of simplicity, straight lines were drawn between computed points on the next graph.

The next variable that was perturbed was the attacker's (blue) Figure 10 indicates the range of values given to all four of blue's firing elements. The firing power of each element was increased by a factor of half in both parity and superiority case (see Figures 19 and 20). There was no effect on blue's decision threshold level which remained constant compared to the standard case. Apparently increasing the relative strength of blue over red using addition power indicates that the surface attack mode becomes more desirable vis-a-vis the air mobile mode, thus keeping the commander's decision threshold relatively high.

The next perturbation would be to weaken blue relative to red's use of fire power. This was done for the parity and superiority cases in Figures 11 and 12 by raising red's fire power relative to blue (see Figure 10). Red's fire power was raised in increments of 50\% and a marked decrease in the commander's decision level is observed in the parity case (Figure 21, third graph, second column). Unfortunately blue, the attacker, does not achieve the objective of the operation just before the threshold starts to decrease the payoff \((B-R)\) to blue negative (that is, red wins). This is demonstrated by the first graph in the column of Figure 21. This parity case could have represented too marginal order of battle ratio superiority for blue. Possibly if blue used a \( \frac{B}{A-4} \) Case (Figure 22), this would not only reduce the commander's decision threshold.
Figure 21. REN'S LINE POWER (p, p') PERIODICITY, PASTI CASE
but also achieve for blue the objective of the operation success. Figur
us by achieving only half the conjecture of the last sentence. That is,
operation objective is truly achieved, but the commander's decision three
right back to the standard case level. It is only when red's fire power
causi blue to lose the battle $R > 1.2$ that the commander's decision t
urns down. At this point one might ask whether the commander's low thr
consistent with optimally gaining the objective of the amphibious opera'
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landing operation in an optimum sense, why does the equivalent real wor
this amphibious operation contain an air mobile mode of troop deployment
magnitude relative to the surface landing represented by the abstract ma
model seems to be saying is that the air mobile mode of deployment of th
in this case represents instantaneous deployment is not as effective as
ing (that is slower deployment) of the surface elements $\{B_1, B_2, B_3\}$
red greater fire power. If this be the case, then let us increase the
air mobile elements only and see what happens to the commander's decisi

Figure 23 gives blue's air mobile element fire power pert
parity case. As expected for $B_4 = 0.3$ the commander's decision threshold
reduced to 70% which is still in the upper range of possible values and cur
current doctrine. It is interesting to note that for $B_4 = 0.225$ not only
optimal strategy reduce to 100% allocation across the beach for the rest
the game, but also red, the defender, optimal strategy reduces to defend
only as opposed to the multiple choice available to red for the unrestri
(Figure 23, fourth graph, column one). This mild reduction in threshold
to be caused by increasing the desirability of the air mobile deployment
cressing its fire power. What happens if the order of battle ratio ($\frac{B}{P}$
increased?

Figure 23 gives these results for the superiority case and
the commander's decision threshold level goes back to the 90% level of t
case. This again illustrates that the commander's air mobile decision t
but also achieve for blue the objective of the operation success. Figure by achieving only half the conjecture of the last sentence. That is, operation objective is truly achieved, but the commander's decision three right back to the standard case level. It is only when red's fire power causing blue to lose the battle \( R > 1.2 \) that the commander's decision turns down. At this point one might ask whether the commander's low threshold consistent with optimally gaining the objective of the amphibious operation threshold level is not consistent with the attacker's achieving a successful landing operation in an optimum sense. Why does the equivalent real world amphibious operation contain an air mobile mode of troop deployment magnitude relative to the surface landing represented by the abstract model seems to be saying is that the air mobile mode of deployment of troops in this case represents instantaneous deployment is not as effective (that is slower deployment) of the surface elements \( \{ B_1, B_2, B_3 \} \) have greater fire power. If this be the case, then let us increase the air mobile elements only and see what happens to the commander's decision.

Figure 33 gives blue's air mobile element fire power parity case. As expected for \( B_4 \geq 0.3 \) the commander's decision threshold reduced to 70% which is still in the upper range of possible values and current doctrine. It is interesting to note that for \( B_4 \geq 0.225 \) not only optimal strategy reduces to 100% allocation across the beach for the rest the game, but also red, the defender, optimal strategy reduces to defend only as oppose to the multiple choice available to red for the unrestricted (Figure 25, fourth graph, column one). This mild reduction in threshold to be caused by increasing the desirability of the air mobile deployment increasing its fire power. What happens if the order of battle ratio \( (B_3) \) increased?

Figure 33 gives these results for the superiority case and the commander's decision threshold level goes back to the 90% level of the case. This again illustrates that the commander's air mobile decision
Figure 24: AIR MOBILE FIRE POWER / b, PENETRATION, INITIATIVITY CASE
is directly related to the success of the amphibious operation. That is, the commander has overwhelming superiority of forces and fire power, optimum in the air mobile mode infers most (greater than 90%) of his forces land via air mobile mode or air mobile deployment mode. If the magnitude of such deployment is for some reason (technically or economically) not feasible, then the optimum allocation of forces is made all forces across the beach.

e. Summary

Justifying these model results to current doctrine is impossible; amphibious operation as presently conceived is designed to overwhelm the defending force such that a toe-hold can be achieved on the defender's territory to support future major objectives. The planning of an amphibious operation assumes the fact that it represents an operation based upon vis a vis the defender. Then why does current doctrine also require a vertical or air mobile deployment mode of forces of insufficient magnitude to assure results? Even if one assumes that the payoff is insensitive to the optimum for the unrestricted or restricted play of the game (which may be true in the economic and/or technical feasibility of the air mobile deployment mode is more severe than the surface mode of deployment of forces. Thus if one performs an amphibious operation using the cost-effectiveness criteria instead of a military payoff criteria, the justification for the air mobile deployment even more difficult.

Still another possible interpretation of the results is that the mathematical model used in the completed study was too abstract a representation amphibious operation and therefore did not adequately reflect all the important parameters affecting the outcome of the battle. This presumes that a more realistic model of the amphibious operation, taking into consideration parameters omitted from the completed study because they were considered of second order would yield results more compatible with reality. Such a study certainly would yield results more compatible with reality. Such a study certainly would yield results more compatible with reality. Such a study certainly would yield results more compatible with reality. Such a study certainly would yield results more compatible with reality. Such a study certainly would yield results more compatible with reality. Such a study certainly would yield results more compatible with reality.

4. Iwo Jima

J. H. Engel in his paper titled, "A Verification of Lanchester's Law the applicability of the Square Law in an actual combat situation where United States forces captured the island of Iwo Jima. In this analysis of the capture of Iwo Jima, the Lanchester's equations, as defined by the Square Law, were found to be

The fire powers accredited to both sides as measured by the author were one the magnitude of the standard case used in this study. This factor of one...
assumes fighting occurred predominately in the day time. This fire power as an input to the amphibious model of this study to see if any useful would result. In Figure 5, the U.S. loss ratio as a function of ord ratio \((I/R)\) is plotted against actual World War II data. The mother results for this case fell well within actual data. Figure 25 gives the computer run in the same form as the previous; sensitivity analysis delay \(t_d\) used was ten hours. The first column, third graph, indicates optimum strategy of allocating all blue's forces across the beach. Fire power strength available during the Iwo Jima Campaign does not upon the assumptions of the model) the sophistication of air mobile mode. Of interest to the analyst would be the strategy crossover point 26 gives the results of the standard case for one-fifth the fire power unrestricted blue strategy remains across the beach. Figure 27 'ad: strategy crossover for blue only at \(I/R\) and \(b\). Column two, third the commander's decision threshold of \(20\%\), the same as the standard.
E. THE VALUE OF MOBILITY, DISPERSION AND SURVEILLANCE IN ACHIEVING EFFECTIVE UTILIZATION OF FIRE POWER

The requirements consistent with limited and tactical warfare have placed stress on the characteristics of mobility and dispersion of units on the battlefield. The problem area in the successful employment of tactical units and associated systems against such targets. In order for such units to be effective, an integrated tactical force comprised of aircraft (both reconnaissance and log
supporting personnel, and equipment must be deployed. Since the enemy presume also attempting to use its own tactical units and weapon systems in an effecti the situation reverts to a 'red' and 'blue' type analysis because each opposit unit represents a potential target to each other. The question then arises as type of mobility and dispersion characteristics each side (red relative to blu versa) would need to perform effectively. If either side was immobile, then only be a question of time when the opposing sides would detect and destroy each The net effect of red versus blue would be a function of the relative number of aircraft, weapon systems and supporting equipment opposing each other and how dispersed. Such a situation could effectively be described by the Lanchester To increase survivability, each side could either disperse into smaller units periodically move about the battle area, or incorporate into the tactical sys active defense. The latter is objectionable because it makes the tactical itcumbersome and there is serious technological doubt as to whether a defense ever be effective against the all out use of offensive force in a battlefie...tain defense. The latter is objectionable because it makes the tactical fu cumbersome and there is serious technological doubt as to whether a defens....beration (e.g. "The best defense is a good offense."). It seems clear that permo bility and unit dispersion are the only alternatives when attempting to sustai...itions on the battlefield. This is true for a wide variety of combat situations, guerrilla warfare to tactical nuclear war. The key factor determining the e...m of fire power utilized against the tactical system. When the attention is the threat of overwhelming fire power is present, then mobility/dispersion considerat ions are much more important that fire power/reaction time considerations. Only when the latter predominates do we have a case for Lanchester's Model. If are interested in the successful employment of tactical systems under all cond...ing. When the attention is on the threat of a nuclear environment, we can assume that the importanc mobility and dispersion supersede fire power considerations and the model des...that follows will have these assumptions in mind. Also please note that Appe contains a different approach to the same model which will be developed below.

b. The Effect of Intermittent Mobility on Detection

Let us suppose two opposing tactical forces (referred to as the red or blues) operate in a battle area. Let capital 'R' be the total area red contra searches to acquire the blue tactical force and small 'r' the rate of search a area per unit time red searches looking for blue. Define capital 'B' and smal...
in a similar manner for the blue tactical force. If both sides search for each systematically covering their controlled battle areas, and if the probability being in an area of the battlefield of size 'r' equals r/R the question arises is the probability \( P_{rb} \) red detects blue first, assuming red moves periodically every \( m \) time units and blue moves periodically every \( m \) time units, each move being independent. This question can be similarly phrased for \( P_{br} \). The probability red detects blue first is important because this indicates which side has the benefit of action against the other. Also if one assumes that the threat of fire power is sufficient to overwhelm the tactical force unit, then the side that is able to preempt its attack suffers negligible attrition. For this reason the determination of which side detects first is all important with firepower attrition factors secondary and a probability of kill conditional on the probability of detecting first. Figure 28 indicates in a time sequence one cycle of the periodic mobility considered to red and blue. \( P_{rb} \) will be computed for one cycle, i.e., the time period \( m \) that blue is stationary.

Figure 28. RED'S AND BLUE'S PERIODIC MOBILITY AS A FUNCTION OF ONE OF EIGHT CONFLICT CYCLES
Using Blue's move as the location of the origin of the time scale (Figure 2) and the random move of Red starting at time \( t = 0 \), we compute the probability of red detecting blue first at time \( t \), \( P_{r,b}(t) \).

Let time run from \( 1 \leq i \leq m \), then \( P_{r,b}(i) \) for \( m \leq k \) can be developed inductively:

\[
P_{r,b}\left\{1 \leq i \leq j\right\} = (1 - \frac{b}{B})^k R_{r} \quad \text{for } i = 0
\]

\[
(1 - \frac{b}{B}) (1 - \frac{b}{B}) \ldots (1 - \frac{b}{B}) (1 - \frac{r}{R}) (1 - \frac{r}{R})
\]

\[
(1 - \frac{r}{R-\left[1-2\frac{R}{B}\right]} R_{r} \quad \text{for } i
\]

The \( i^{th} \) expression can be reduced to

\[
P_{r,b}\left\{1 \leq i \leq j\right\} = \left(\frac{B-1}{B}\right)^k R_{r}
\]

The first term on the right indicates the probability blue does not detect red during the \((0, 1)\) time period and the second indicates probability that red does not detect blue during the \((0, i - 1)\) time period but does detect at the \( i^{th} \) time period. Another way of looking at this probability model would be to assume that red and blue, each sampling sequentially in time and in unison, an urn placed before each, containing white balls except one which is black. The number of white balls in each is \((B/r) - 1\) for red and \((B/b) - 1\) for blue. Therefore the above equation probability red pulls the black ball out of the urn before blue does so or of his urn at the \( i^{th} \) time period conditional to all previous balls pulled from the urn not being replaced (i.e., sampling without replacement).