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Report R-9

PART II

CHARACTERISTICS OF SYSTEMS FEASIBLE FOR INERTIAL NAVIGATION OF SUBMARINES

August 1951

INSTRUMENTATION LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Cambridge 39, Mass.

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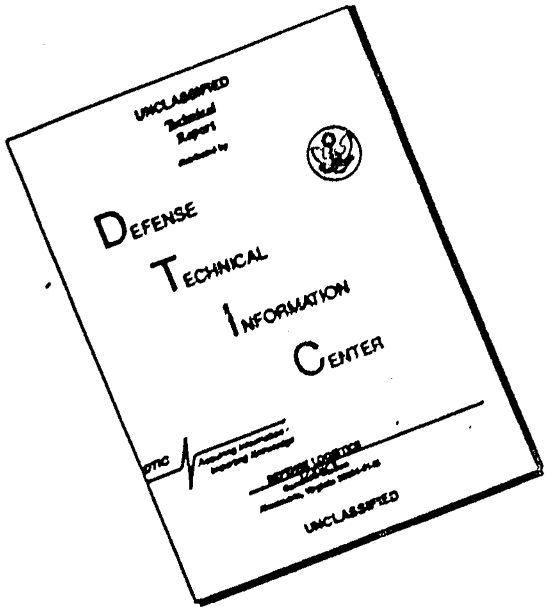

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Report R-9 –

PART II

CHARACTERISTICS OF SYSTEMS FEASIBLE FOR INERTIAL NAVIGATION OF SUBMARINES

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INSTRUMENTATION LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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PREFACE

This report is the second of a sequence of two reports dealing with the study of possible inertial navigation systems for submarine use. The first report, entitled "Theoretical Background of Inertial Navigation for Submarines", was issued in March 1951 and dealt with the fundamental problem of inertial navigation with the discussion restricted to idealized systems. The present report, the second of this sequence, entitled "Characteristics of Systems Feasible for Inertial Navigation of Submarines" considers characteristics of nonideal equipment that must be used in any practical system. On this basis, one system out of the many theoretically possible types has been chosen as most nearly meeting the requirements for submarine operation.

This work has been carried out under Contract N5Ori-07850 for the Office of Naval Research. The principal responsibility and supervision of the report has been carried out by Mr. Forrest E. Houston, Assistant Director of the Instrumentation Laboratory, with the assistance of Mr. John Hovorka. Acknowledgment is also due the Technical Publications Division of Jackson & Moreland for the preparation of the report.

Walter Wrigley

Cambridge, Massachusetts
October, 1951

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ABSTRACT

This report is the result of a study of possible submarine inertial navigation methods. Prior investigation has narrowed the field of inquiry to gravity-field navigation, and pointed to a system design based on a stable vertical element which is essentially an equivalent pendulum with eighty-four-minute-period characteristics, a so-called Schuler-tuned system. The equivalent pendulum is represented by a controlled member which is oriented by accelerometer-monitored gyro units to indicate the vertical. The accelerometer units furnish data, which, after processing, are used to precess the gyro units and drive the controlled member so that it responds to changes in the vertical direction as the ship moves along its course over the surface of the Earth. The gyro units also, in conjunction with the same controlled member drives, simultaneously isolate motions of the base from the controlled member. This kind of vertical indicator is described in detail in the report, with the discussion including such problems as the effects of damping, uncertainties in component outputs, and inaccuracies in inputs from external sources. With this system, Schuler tuning eliminates the requirement for precise external compensation. The optimization of system parameters is discussed, and numerical values are given for optimum parameters.

When the indicated vertical has been established by a controlled member, the problem of position indication can be solved by two general classes of systems. In one class, accelerometer data, before or after processing, are integrated in an open-chain configuration to indicate position change from a known point of origin. The accelerometer itself may be doubly-integrating in character, in which case position change is indicated by utilizing its output directly. This class of position indicators, deriving position information from integrators appended to the vertical-indication loops or by direct double integration of acceleration, is characterized by simplicity of concept and geometry, i.e., only three concentric gimbals are required, no more than are required for vertical indication alone. However, the integrators, being open chain in nature, also integrate false ground-speed components associated particularly with gyro-unit drift and with inaccuracy in Earth-rate compensation. This leads to errors in indicated position which are cumulative, depending on, among other things, the time of operation of the system.

In view of the long operating intervals probably required of a submarine navigation system, methods for avoiding such cumulative errors are important. In

this report, it is shown that position-indication systems of this class, including those using a doubly-integrating accelerometer, have intrinsic cumulative errors in their output, which are reducible only by instrumentation sufficiently precise to keep these errors within preassigned limits during the long operating intervals.

The second class of position indicators discussed requires the use of two additional gimbals, a disadvantage that is at once offset by the absence of cumulative error in latitude indication, and by efficient minimization of cumulative error in longitude, using a longitude indicator which operates in conjunction with the latitude indicator. (Longitude, because of its arbitrary nature, has an inherent possibility for cumulative error in its indication; this is not true of latitude.) This class of systems consists of two general types: the prealigned system and the self-erecting system. Prealignment of inertial gyro unit axes, so that they serve as "star-lines," presents some problems in a submarine installation, particularly in the requirement for an azimuth reference. A self-erecting system avoids these difficulties.

The self-erecting latitude indicator discussed in this report operates in conjunction with an azimuth indicator (an inherent gyrocompass) and with the vertical indicator, as a closed-loop system which causes a reference direction on a fourth gimbal, mounted concentrically with the controlled member, to track the Earth polar axis. A fifth, and innermost gimbal, mounted within the latitude gimbal, carries an integrating gyro unit which serves, with the innermost gimbal drive, to integrate celestial longitude rate. Indicated longitude is derived from this integrated rate, while latitude is indicated by the orientation of the polar-axis-tracking gimbal relative to the controlled member. This position-indication method is examined from the point of view of practical instrumentation and accuracy of indication. The problem of ground-speed indication with this system is also discussed, and it is shown that ground speed can be obtained with few additional components.

A balancing of the advantages and disadvantages of these systems, while keeping in view the practicalities of instrument design and the demands of shipboard operation, permits the recommendation of the last-mentioned position indicator as the most promising. The study of this position indicator has been sufficiently exhaustive to permit the assignment of numerical values to the important system parameters. These are given in the report. In addition, potentially profitable further research concerning certain specified aspects of the problem is suggested.

INTRODUCTION

Inertial guidance of submarines is a method for deriving essential navigation information -- latitude, longitude, heading and speed -- from a system that obtains its data entirely within the submarine. Once initial conditions have been inserted into the system, it must operate as nearly independently of external information as possible, indicating the above quantities with sufficient precision and continuity for the successful navigation of the submarine.

The preceding report in this group, Report R-9, Part I(1)*, contains a discussion of fundamental approaches to this problem. The basic method discussed there involves a stabilized platform as an essential feature. A stabilization servo system receives specific force data from accelerometers or pendulums, and delivers these data to gyro units and platform drives, which cause the platform to indicate the vertical. Two basic methods of utilizing the indicated vertical are then available to obtain the submarine's essentially geocentric angle of travel. One method, referred to as analytical integration in Part I, involves the integration of accelerometer outputs or of gyro inputs, with the accelerometers and gyro units mounted on the stable platform. The other method, called geometric integration, involves the direct comparison of the indicated vertical with a reference vertical. The conclusion arrived at in Part I was that, within the submarine, a servo-driven platform on which were mounted two accelerometers (or pendulums) and two gyroscopes could be used to indicate the vertical, and, with the addition of a third gyroscope, to indicate heading. The vertical indicator serves to show the direction of the Earth's gravity field, and the azimuth indicator shows the projection of the Earth's polar axis in the horizontal plane. This three-axis stabilized platform is the basis for various methods of position indication described in the preceding report, and in addition inherently provides data on heading, roll, pitch and speed.

It is the function of this report, first, to examine the practicalities of instrumentation of a platform stabilized about the vertical and in azimuth, and second, to re-examine the position-indication methods in the light of necessarily imperfect instrumentation and imperfect external compensations. This discussion serves for the rejection of systems which, while they appeared theoretically

* Report R-9, Part I, will hereinafter be referred to as Part I. Superscript numbers in parentheses refer to the bibliography.

feasible as described in Part I, present engineering difficulties in execution.

It is thus possible to apply certain principles of selection and compromise, which are discussed in the following, to arrive at what appears to be the most practical submarine inertial navigation system. The discussion of this system makes up the largest part of the report. Appended to the report is a series of derivations which render the textual discussion in mathematical terms.

GENERAL CONSIDERATIONS CONCERNING SUBMARINE INERTIAL NAVIGATION

The essential problem to be discussed is the determination of the present position of the submarine on the Earth, from data obtained within the submarine. The position is defined in terms of astronomical latitude and longitude, and these are in turn defined as two geocentric angles. Astronomical latitude is defined as the angle, in the meridian plane, between the local vertical and the vertical at the equator. Astronomical longitude is defined as the angle between the meridian plane containing the local vertical and a reference meridian plane. It is to be noted that astronomical position and map data are not necessarily identical.* Position determination thus reduces to

1) determination of the local vertical
2) determination of the angle between the local vertical and two Earth reference directions.

The two Earth reference directions mentioned above may be, respectively, a line parallel to the Earth polar axis as indicated by the navigation system, and a longitude reference direction set into the system (latitude and longitude are then derived directly from angles between the indicated directions); or, alternatively, the original vertical and azimuth at the start of the navigation problem may serve as references (as in cases where rate or acceleration integration methods are used to give the geocentric angle of travel of the submarine from the starting point).

In Part I, ideal components formed the navigation systems considered. These components could of themselves introduce no errors into a system. Practical engineering will require the consideration of nonideal behavior of components. This nonideal behavior can be characterized in various ways: for example, the over-all sensitivity of the component can be regarded as uncertain, i.e., there is an uncertainty in the relation of the output to the input. Or, as will be assumed in this report, the component sensitivities may be regarded as pre-set with negligible error, although some uncertainty may nevertheless occur in the output of a component.

* Astronomical position data are derived from a knowledge of the local vertical, while map data, which are in effect smoothed astronomical data, are derived from geodetic triangulation techniques.[(2),(3),(4)] The corrections required to bring the two sets of position data into coincidence are in general quite small; but if desired, they may be applied in a manner similar to that used to make variation corrections to a magnetic compass reading.

In any case, the nonideal character of a component will be displayed ultimately (in the performance equation) by an uncertainty term appropriate to the nature of the component and to its role in the system. The word uncertainty here refers specifically to the indicated output of the component, and the indicated quantity uncertainty is defined as the indicated quantity minus some statistical average of this quantity. Similarly, the indicated quantity error is defined as the average indicated quantity minus the correct value of the quantity. The negative of the error is defined as the correction to the indicated quantity. Finally, the indicated quantity inaccuracy is defined as the indicated quantity minus the correct value of the quantity. Therefore, at any instant, the inaccuracy is equal to the error plus the uncertainty as here defined. These distinctions are profitable because, in the case of a component within the system, during the "transient" stage of the motion the component output error due to the transient regime is ultimately correctible by the system itself, and only the uncertainty in the component output is important. But, in the case of quantities brought into the system in open-chain configurations from external sources, the system can influence neither the error nor the uncertainty, and the sum of these, the inaccuracy, becomes the important determinant of system behavior.

DETERMINATION OF THE LOCAL VERTICAL

The definition of astronomical position given above requires that the vertical at the present position of the submarine be indicated by the navigation system. On a stationary base the vertical is, by definition, the direction along which a plumb line settles. On an accelerating base, a plumb line no longer indicates the true vertical; instead, it tends to indicate the direction of the apparent vertical, which is a function of the acceleration of the base. It was pointed out in Part I that a useful vertical indicator on a moving base can be conceived of as an accelerometer-monitored servo-controlled platform which acts as a long-period equivalent pendulum. The dynamics of this system would be adjusted by Schuler tuning. This tuning consists in selecting an undamped natural resonant frequency for the equivalent pendulum, consisting of the platform with its control systems, such that the error in the indication of the vertical by the platform is substantially independent of horizontal accelerations of the base.

There is a subtle, basic difference between this kind of vertical indicator and conventional stable verticals and artificial horizons. The latter two navigational aids represent a philosophy in which gravity and accelerations are considered as essentially different physical factors. They require precise compensation inputs, that is, ship-motion data must be available independently and of the same order of precision as that desired in the indication of the vertical direction. This is due to the fact that the vertical is associated with gravity alone, and accelerations are considered as interferences. However, the Schuler-tuned vertical indicator discussed above does not require precise compensation inputs to assure small steady-state errors. Fundamentally, this is because in a Schuler-tuned system the indicated vertical is not associated with gravity alone, nor are accelerations of the base to be regarded as interferences; instead, the vertical and accelerations of the base are interrelated through the geometry and kinematics of motion over the essentially spherical earth, with the resultant minimization of the importance of external compensation data. It is true that compensation inputs will be required by a practical vertical indicator of the Schuler-tuned type, but their required precision is such that complete failure of compensations should not seriously impair the system performance; they act as a factor of safety. On the other hand, systems without Schuler tuning cannot achieve required performance unless the compensation is of a high order of precision.

The essential operating components of the vertical indicator under discussion are shown in Fig. 1. Two single-axis accelerometers (or pendulum units) and three single-axis* integrating gyro units are shown. This system is described on p. 22 of Part I. In terms of the component functions, the system is described there from the standpoint of three subsystems:

1) a system for obtaining data on the resultant specific force on the platform (controlled member) by means of either accelerometers or pendulums;
2) a system for orienting the controlled member in response to data from (1), so that the controlled member indicates the vertical; this involves gyro units and servo drives;
3) a system for modifying data from (1) before they are applied to (2), to control the dynamic performance of the whole system.

A gyro unit is characterized by an ability to maintain a fixed direction in inertial space, when no torques are applied to it. In this system, torques applied by the roll and pitch of the ship are removed by fast gyro-monitored servo drives, which orient the controlled member and the gyro units mounted on it with respect to the outer gimbals; and at the same time, torques applied by the processed accelerometer data, acting to precess the gyro units more slowly, cause the controlled member to rotate approximately geocentrically at the angular velocity of the indicated vertical. A second kind of system is conceivable, in which the gyro units are mounted on a gimbal separate from the controlled member carrying the accelerometer units, in which case the controlled member can be oriented with respect to the gimbal carrying the gyro units. This gimbal then remains fixed in inertial space, serving ultimately as a source of reference directions, or "star lines",

* The choice of single-axis gyro and accelerometer units, as compared with two-axis units, has been made on the basis of earlier studies[5].

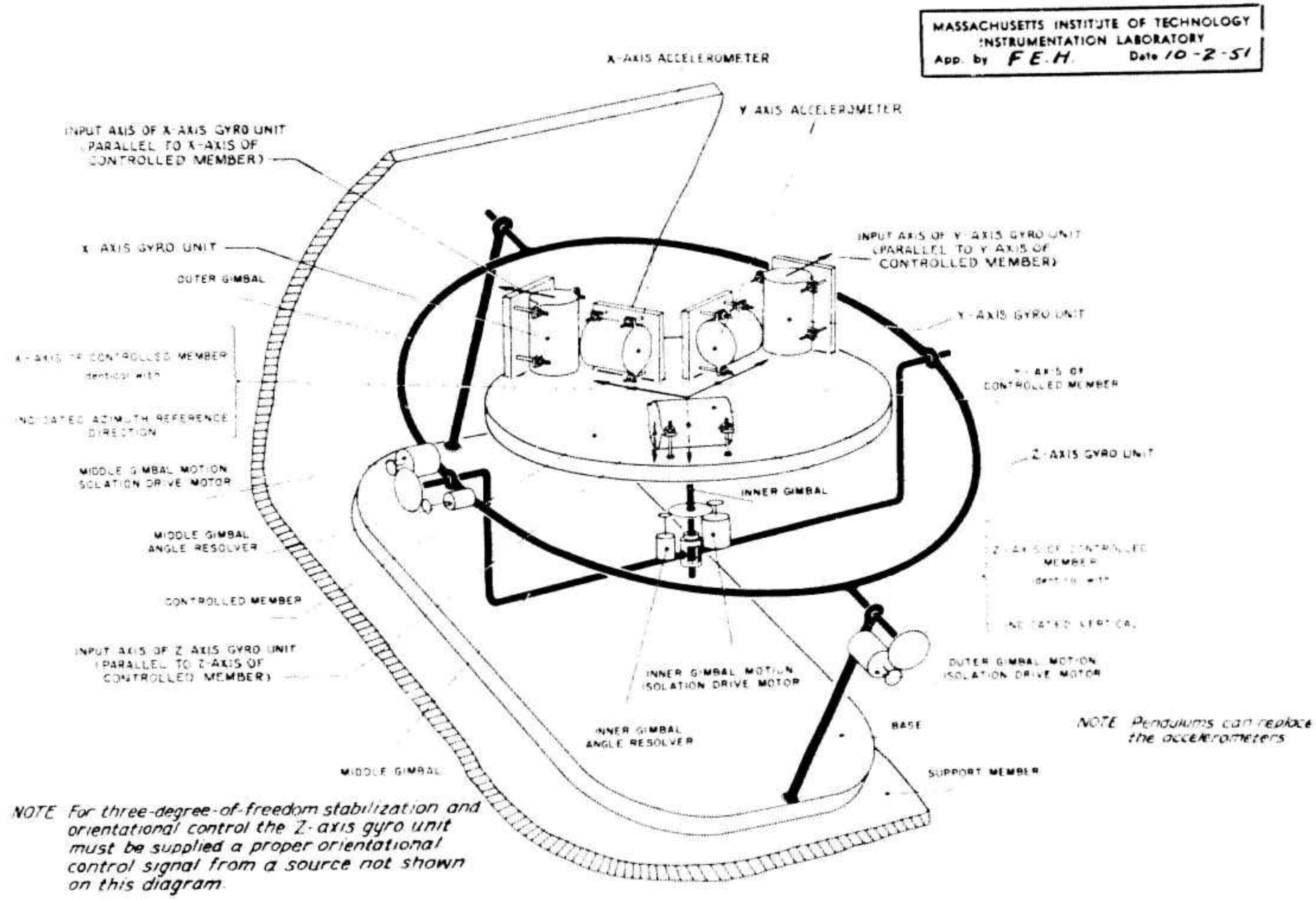

Fig. 1. Line schematic diagram showing essential geometrical parts of a complete vertical indicating system using two single-axis accelerometers and three single-degree-of-freedom gyro units.

carried within the system, for position indication. Interference isolation against deck motion is then obtained by the gyro-monitored gimbal drives, and the controlled member drives need no gyro units. The practicability of this scheme in submarine navigation will be discussed later in this report. In both this system and that shown in Fig. 1, the principal function of the gyro units is to obtain base motion isolation, and the angles between the gimbals external to the controlled member indicate roll and pitch of the base, and therefore of the submarine, incidental to the indication of the vertical by the controlled member.

DETERMINATION OF POSITION

Once the problem of the indication of the local vertical has been solved, an attempt can be made to indicate position with the data implicit in the vertical indicator. For example, suppose it is desired to indicate latitude by integrating, with respect to time, the input signal to the y-gyro unit,i.e., the unit controlling motion about the east-west axis. This signal is proportional to the y-component of the angular velocity of the indicated vertical, and its first integral is approximately proportional to the angular displacement of the indicated vertical in the meridian plane, from some starting position which serves as a reference. The approximation, however, depends strongly on the instrumentation, and can lead to serious difficulties in maintaining correct position indication at all points on the course.

It was mentioned before that two classes of torques act to precess the gyro units; those due to base motion, i.e., roll and pitch of the ship, and those due to processed accelerometer data. A third source of torques on a gyro unit lies in such phenomena as mass-unbalance and friction within the gyro unit itself. These will be referred to as uncertainty torques. They give rise to an uncertainty in the gyro unit gimbal angular velocity called gyro unit drift. This drift is indistinguishable in the gyro unit output from the true angular velocity of the indicated vertical; i.e., gyro drift produces the effect of a false ground speed component.

There is a similar additional source of false ground speed which comes about in the following way: Suppose that, by means of a gyro compass operating in conjunction with the z-gyro in Fig. I, the controlled member is oriented with the x-axis tracking true north, so that the y-axis is pointed approximately east. Then the x-axis gyro unit will sense the horizontal projection of the sidereal rotation of the Earth (hereinafter referred to as horizontal Earth rate), while the y-axis gyro unit will not. The x-axis gyro unit, if the controlled member is to be kept in the horizontal plane, must be compensated for horizontal Earth rate, i.e., this particular input angular velocity component must be cancelled in some way. One possibility is to precess the gyro unit with a signal proportional to horizontal Earth rate. This results in a compensation torque on the gyro unit which exists in addition to the torques discussed previously. Lack of precision in this Earth-rate compensation input will result in a residual gyro unit gimbal angular velocity inaccuracy. This inaccuracy is functionally identical with the aforementioned gyro unit drift in its effect; combined, these constitute a false ground speed component.

Note that, since these errors are essentially dynamic, they leave the steady-state indicated vertical unaffected. This is so because the system is accelerometer-monitored, and the integrator between the accelerometer and the gyro automatically acquires the proper bias to null this false ground speed component; the indicated vertical is unaltered. These phenomena are common to all systems which use accelerometers or pendulums to track the apparent vertical and which use precessed gyro units to effect base motion isolation.

The aforementioned false ground speed components are necessarily included in the integration of rate or acceleration data to indicate position, and lead to an error in position which increases with time. Further examination will show that such cumulative errors are unavoidable with this method of indicating either latitude or longitude.

Similarly, the direct double integration of the angular acceleration of the indicated vertical, by whatever method — e.g., a doubly-integrating accelerometer — involves again a cumulative error in the indicated position. A false acceleration component, corresponding to the false ground speed component in another part of the vertical indicator loop, is responsible for the cumulative error in this case. Thus, these "open-chain integration" and direct double-integration methods of indicating position, while having the virtue of relative simplicity, simultaneously present serious drawbacks in practice.

A further difficulty enters this problem when the possibility of integrator drift, a common characteristic of such devices, is considered. In the accelerometer data processing, drift in the integrators used will affect the indicated ground speed, while in open-chain integrators used to determine position, integrator drift will lead to cumulative errors in the indicated position. The entire problem of cumulative errors is traceable to the fact that only indicated, not true, motions are available as data from the vertical indicator; more generally, aboard the ship, motions of the various parts of the system with respect to each other and to the ship are the only motions from which the position of the ship can be deduced.

It is therefore of interest to consider the possibility of establishing an inertial reference, fixed in space, and relatively independent of the accelerometer-monitored controlled member. As was mentioned in the discussion of vertical indication, the accelerometers of Fig. 1 might be mounted on the controlled member as shown, but with the three gyro units mounted on a separate platform, which remains fixed in inertial space; then the controlled member can be rotated with respect to the platform by a sidereal time drive, about a "polar axis" established on the gyro unit platform. Evidently, latitude and longitude can, with suitable initial gyro unit alignment, be indicated directly as angles between gimbals. Only two cumulative errors are then possible: that due to gyro unit drift, and that due to inaccuracy in the sidereal time drive. At the present time accuracies of the order of one part in one hundred thousand appear attainable in matching sidereal time drive angular velocity to Earth rate. It may, therefore, be expected that the gyro unit drift will be the major source of error in a system of this type.

This situation appears to be unavoidable in the case of longitude indication. Astronomical longitude is not a unique quantity; only longitude difference with respect to a present reference can be indicated. The indicated longitude will in no case be more precise than this reference setting. Furthermore, it must be obtained by some form of integration (as in the case just described, where the gyros and drives act as integrators), with the concomitant cumulative error.

Latitude indication is not necessarily bound by these restrictions. A self-erecting latitude system is conceivable, with a gimbal which moves with respect to the controlled member until the gimbal is parallel to the Earth polar axis. Thereafter the gimbal moves at the indicated latitude rate, about the east-west axis, with respect to the controlled member; it is monitored by the azimuth stabilizer, and ultimately by the y-axis accelerometer. The angle between this gimbal and the indicated horizontal is then the indicated latitude.

These matters will each be discussed in more detail on the pages following.

THE DAMPED VERTICAL INDICATOR

An undamped vertical indicator of the type considered in Part 1, pp. 28 ff., is necessarily subject to a fixed-amplitude continuous oscillation with an 84-minute period. Such a system might be practical under the following conditions:

1) The time of operation is sufficiently short, i.e., of the order of one period.
2) The initial conditions are set into the system with sufficient precision, i.e., such that the amplitude of oscillation plus the steady-state error, where it occurs, will not exceed the preassigned error in the indication of the vertical.

In submarine operation, an operation time of the order of at least eight Schuler periods must be allowed for, and in addition the precision of the initial settings is likely, for practical reasons, to be somewhat poorer than the ultimate accuracy desired in vertical indication. The long operating time allows a large probability for the system to be disturbed during a run, making it possible to incur errors larger than those associated with uncaging of the instrument. Insufficient precision in initial-condition settings requires that the system reduce its amplitude of oscillation after the submarine is set in motion.

Evidently, an undamped system satisfies neither requirement. On the other hand, a damped system suffers from two serious limitations:

1) Insofar as damping suppresses high input frequencies, it lengthens the solution time.* This effect is important both when the system is initially uncaged, and when it suffers transient disturbances while the submarine is in motion.
2) The delay in achieving a solution is operative at other times also; effectively, a damped system responds to values of the input averaged over a time of the order of the solution time, rather than to instantaneous input values. The result is manifested as errors which depend on the rapidity of the inputs. These are called <u>forced dynamic errors</u> (cf. Part 1, p. 17).

* Solution time is defined for these reports as the time required for a system to achieve ninety-five percent of the change a[illegible]ated with a given input change.

This unavoidable compromise between short solution time and small forced dynamic errors confronts the designer of any practical system.

Consider the vertical indicator of Fig. 1. The simplest damping method might involve putting a direct by-pass around the integrator, between the accelerometer and the gyro unit. However, this corresponds to a high-pass filter, and would result in a system that is excessively responsive to high* frequencies, some of which, in view of the practicalities of instrumentation, can be expected to be spurious. The resultant system is analogous to a pendulum with frictional damping, and is describable by a second-order differential equation.

High-frequency suppression can be obtained by combining the integrator and by-pass with a low-pass filter, which can be represented as an integrator with direct feedback. This kind of system is shown in Fig. 2. Its performance equation is derived in Derivation 1A. (Derivations are included in the Appendix, beginning on page 52). The differential performance equation, Eq. (1-16), is of third order in time derivatives of the correction to the indicated vertical. The tuning of the system to an approximate 84-minute period (Schuler tuning) is therefore possible in alternative ways. In terms of the physical inputs to the system, Schuler tuning may be accomplished by:

1) minimizing the acceleration error, or
2) minimizing the jerk (acceleration derivative) error.

The result of each method is displayed in the two amplitude-ratio vs frequency-ratio logarithmic plots, Fig. 3. It should be noted that, were the damping sufficiently reduced in either system, the system would approach an equivalent Schuler pendulum with a natural frequency corresponding to a frequency ratio of unity (since the reference frequency is the Schuler frequency). But the frequency ratio for maximum amplitude ratio is higher for acceleration-error minimization and lower for jerk-error minimization, the departure of the frequency ratio from unity being determined in both cases by the damping.** The relative heights of the peaks in the curve are also determined by the damping.

The choice between the two response functions can then be based on whether it is desirable, relatively, to suppress low frequencies or high frequencies; i.e., whether the expected forced errors due to necessarily imperfect compensation inputs will be primarily of a high- or low-frequency character. (Instrumentation error will presumably be sufficiently reducible to give the desired over-all accuracy). The assumption is here made that these uncontrolled forced errors will be short-period, compared with 84 minutes, and in Derivation 1A, the jerk error is minimized

* It is assumed that the system in this discussion will be Schuler-tuned, so that the reference frequency corresponds to a period of 84 minutes.

** Specifically, by the *damping ratio*, which is defined as the ratio of the actual damping to critical damping.

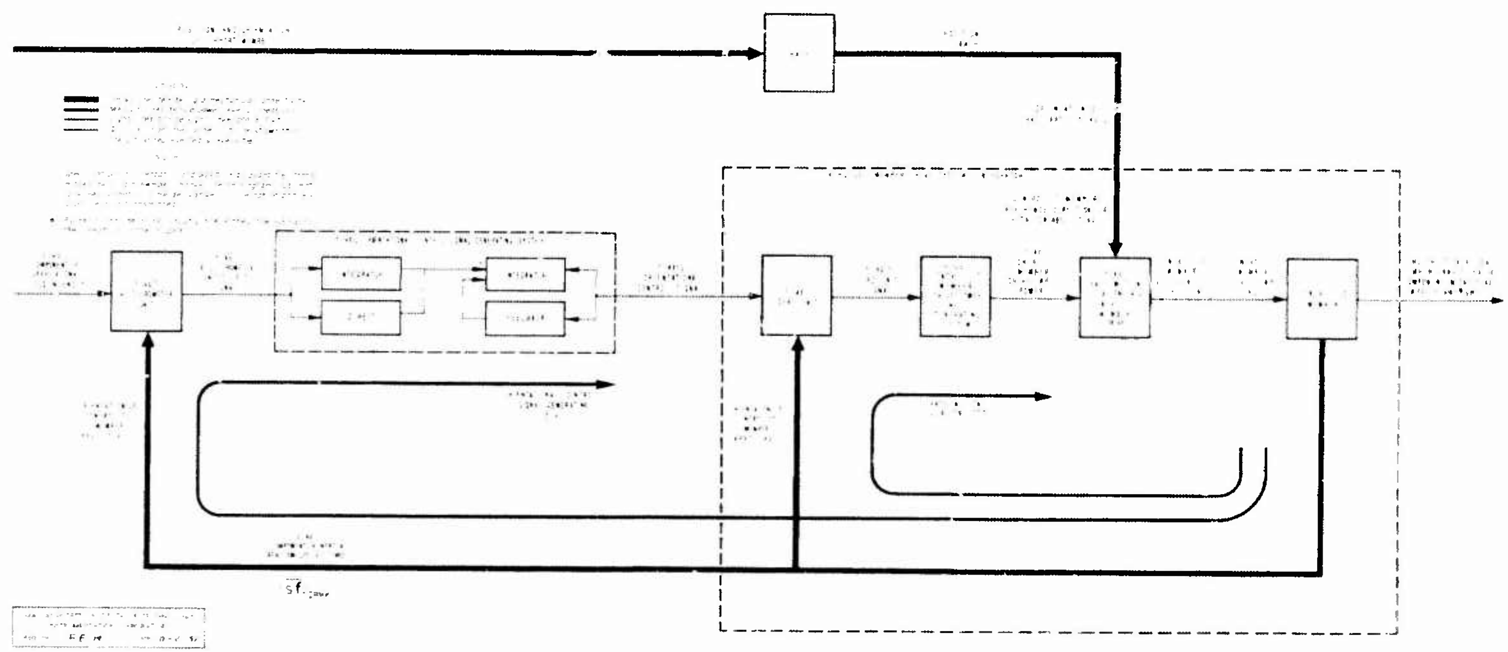

Fig. 2. Functional diagram showing interrelations among essential operating components of a damped single-degree-of-freedom system for indicating the vertical when the gravity field vector lies in the plane normal to the y-axis.

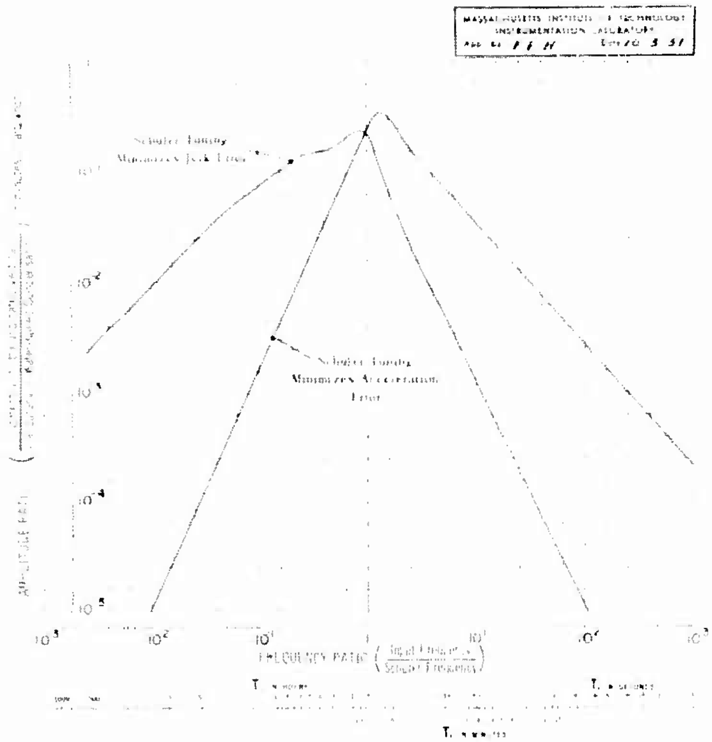

Fig. 3. Amplitude ratio (correction to indicated vertical/inaccuracy in water-speed compensation) vs. frequency ratio (input frequency/Schuler frequency) for two vertical indicators: (1) with jerk error minimized by Schuler tuning, acceleration error minimized by choice of undamped natural frequency, and (2) with acceleration error minimized by Schuler tuning, jerk error minimized by choice of undamped natural frequency. Damping ratio is 0.3 in both cases.

by an adjustment of the system parameters, which corresponds to Schuler tuning (Eq. (1-13)).

The acceleration error, however, remains. It is proposed to minimize this with a signal representing the appropriate ship velocity component. In the case of the x-system, this component is the longitude rate; in the y-system, the latitude rate. This velocity component, hereinafter referred to as the water-speed compensation, is fed from the ship pitometer log, through the azimuth resolver, to the high-frequency integrator; the effect of this treatment of the ship velocity signal is to minimize the acceleration error, if the signal is suitably adjusted (Eqs. (1-14) and (1-15)). Actually, the error could be made more nearly zero if the ship velocity with respect to the Earth were available as a datum; even a perfect pitometer log can give only the water speed. Therefore, ocean current gradients (although not the ocean current itself) are included along with uncertainties in the water-speed indication in the velocity-compensation inaccuracy (see Eq. (1-14)).

The differential performance equation, Eq. (1-16), is, as observed, third order, with the following principal forcing terms, shown on the right-hand side of the equation in the order given:

1) A term proportional to the rate of change of the inaccuracy of the water-speed compensation with respect to true ground speed.
2) A term proportional to the rate of change, and second rate of change, of the inaccuracy in the Earth-rate compensation current, and a gyro drift error term proportional to the rate of change, and second rate of change, of the gyro gimbal angular velocity uncertainty.
3) A term proportional to the accelerometer output uncertainty and to its rate of change, and a term proportional to the difference between the rate of change of the uncertainty in the output of the low-frequency integrator by-pass and the rate of change of the uncertainty in the feedback channel output. These constitute an acceleration error.

The left-hand side of Eq. (1-16) is concerned with the system dynamic characteristics. Derivation 1B deals with the assignment of numerical values to these characteristics. The dynamic performance can be described in terms of the damping ratio and the undamped natural frequency of the quadratic term, and the characteristic time of the first-order term. The numerical values of these three quantities ultimately determine the actual values of component sensitivities (Eq. (1-17)). These three parameters are chosen for convenience in the system design, since they are universally applicable to linear systems of this general nature. In Derivation 1B, the numerical values are assigned according to the following procedures:

1) The damping ratio, as has been observed, has an optimum value of zero, but the resulting undamped system would be objectionable for the reasons given. The chosen damping ratio is therefore a compromise. The smallest damping ratio consistent with a reasonable solution time is considered to be about 0.3.
2) The resonant frequency for the undamped system is selected to minimize the acceleration error in Eq. (1-12), before the water-speed compensation is applied. It is desirable to do this because of the aforementioned expected difficulty in providing a perfect velocity-input in making the compensation. The resonant frequency thus found is determined by the chosen damping ratio, and is 95.4 percent of the Schuler frequency.
3) Given the damping ratio and the resonant frequency for the undamped system, the characteristic time is determined by Eq. (1-24) to be 1.43 hours.

The transient behavior of the system is studied in Derivation 1C by considering the homogeneous equation, Eq. (1-27), derived from Eq. (1-16). As in Derivation 1B, it is assumed that the left-hand side of Eq. (1-16) is factorable into a first-order term, containing the characteristic time, and a second-order term, containing the damping ratio and the natural frequency. Laplace transform methods then furnish the solution (Eq. (1-30)). The correction to the indicated vertical contains the following two important groups of terms:

1) A damped exponential, controlled by the characteristic time.
2) A damped sinusoid, with a maximum-amplitude frequency dependent on the damping, as has been observed; the frequency ratio at the frequency of maximum amplitude is equal to unity minus the square of the damping ratio. The damping of the sinusoidal term is controlled by the damping-ratio-natural-frequency product.

Refer again to Eq. (1-16) of Derivation 1. On the right-hand side of the equation are several forced-error terms, which will be discussed in terms of a method to be outlined below. The most important forced errors are caused by the following:

1) velocity compensation inaccuracy,
2) gyro unit drift,
3) accelerometer output uncertainty.

The functional form of these errors is not precisely known; this obviates the possibility of obtaining a particular solution to Eq. (1-16) by such means as Laplace transform methods. However, it is possible to exclude certain regions from the frequency spectrum of each of these forced errors as being highly improbable. This fact, coupled with the frequency response of the system to sinusoidal variations in each of the forced errors taken individually, permits some statements to be made concerning the effect of these errors on the over-all accuracy in the indication of the vertical.

Thus, since the exact frequency spectrum for each error is unknown, it can first be assumed that all frequencies occur with equal amplitudes, i.e., that the input to the system consists of "white noise". A harmonic analysis of the output then gives the normalized frequency response. Figure 4 shows the result of such an analysis applied to the velocity-compensation inaccuracy. The curve exhibits a peak near the Schuler frequency. At high and low frequencies the response is much attenuated. The conclusion is that velocity-compensation-inaccuracy oscillations having frequency components near the Schuler frequency will be most important, and that other frequencies will be less so. To evaluate the net effect of the velocity-compensation inaccuracy on the over-all error, then, it must be known whether or not frequency components in this forced error are near the Schuler frequency. Little data are available on this subject, but the presence of 84-minute period forcing functions for an appreciable length of time does not appear likely.

Similar considerations apply to Figures 5 and 6, which apply to the gyro drift and accelerometer uncertainty errors, respectively.

Since this is a single-axis system, and two such systems, geometrically orthogonal, are required for vertical indication, the coupling between two similar systems was investigated, as follows: An equivalent complete vertical indicator, represented by Eqs. (1-39) and (1-40) in Derivation 1D, was set up on a Reeves Electronic Analogue Computer (REAC) in the Instrumentation Laboratory. The simulated vertical indicator was effectively uncaged with equal errors in the vertical about both the x- and y-axes. The system came to rest in the manner of a damped Foucault pendulum(6) (the damping ratio chosen was 0.3). Fig-

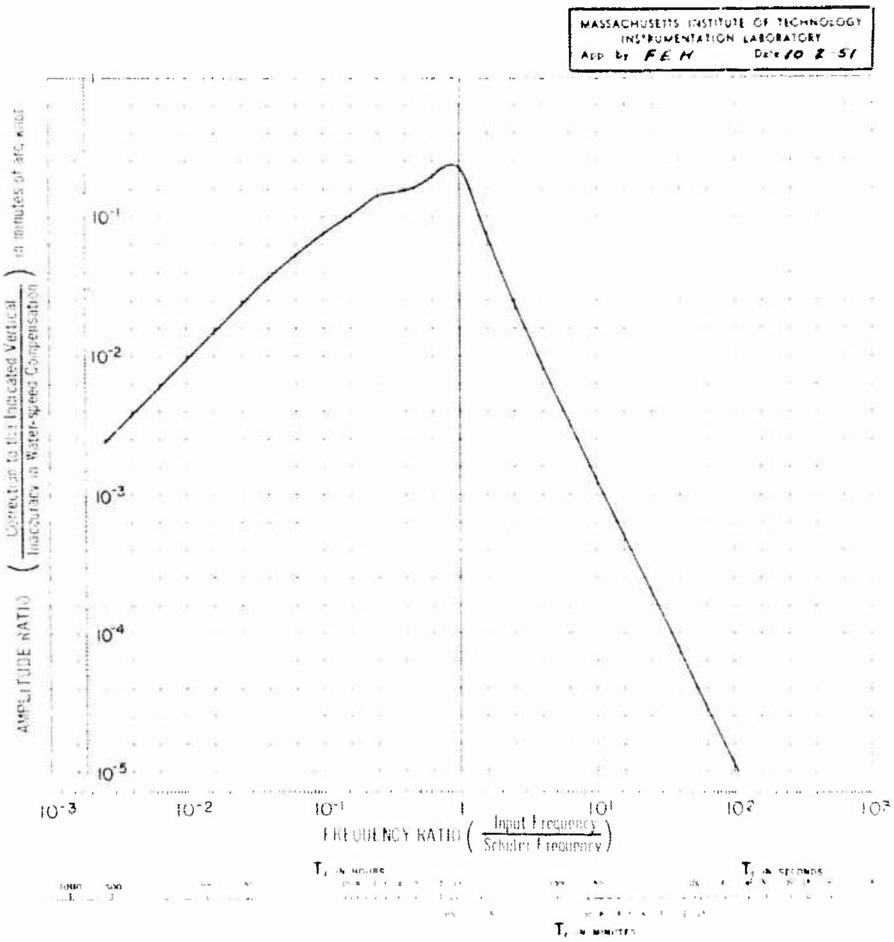

Fig. 4. Effect of water-speed compensation inaccuracy on vertical indication, as a function of frequency.

ure 7b shows a plot of x- vs y- displacements from the true vertical as a solution was reached. It is seen that the projections of the two-dimensional oscillations on the x- and y- input planes are practically the same, a situation that would occur for decoupled x- and y- systems (Fig. 7a). The conclusion is that this two-dimensional pendulum effect can be neglected, and single-axis results are immediately extensible to the three-dimensional case. These remarks are important because in subsequent discussions it will be assumed that as a good approximation, the x- and y-systems are independent.

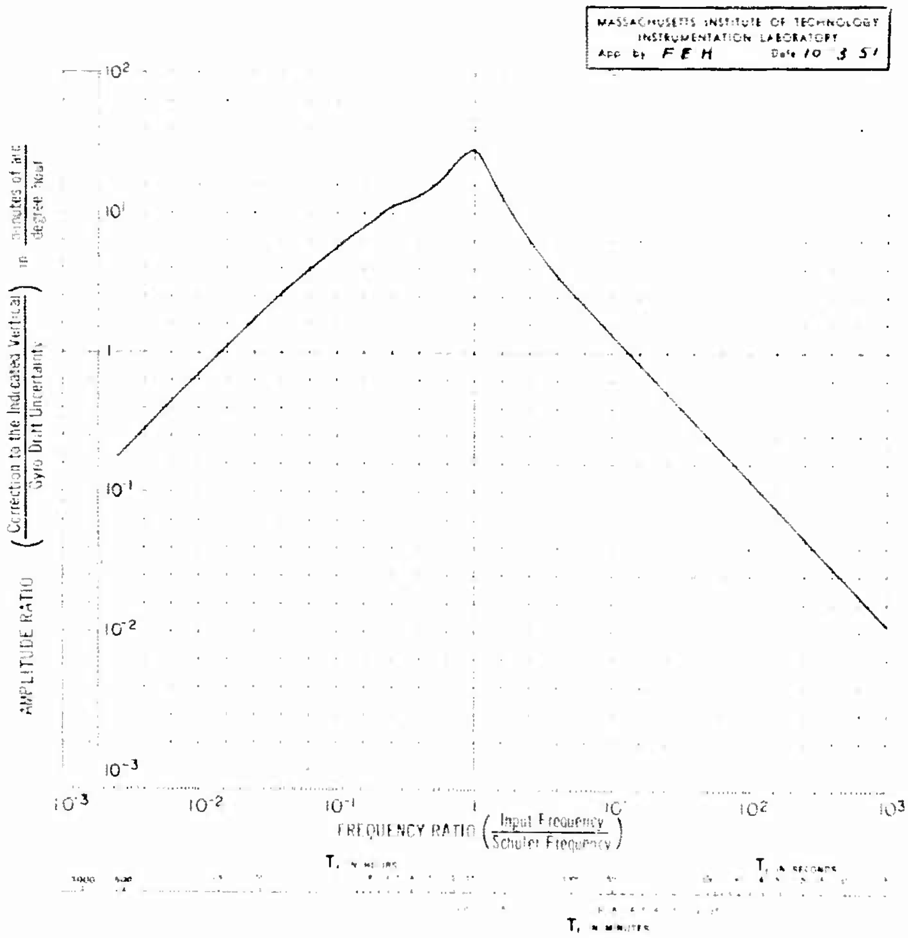

Fig. 5. Effect of gyro unit drift on vertical indication, as a function of frequency

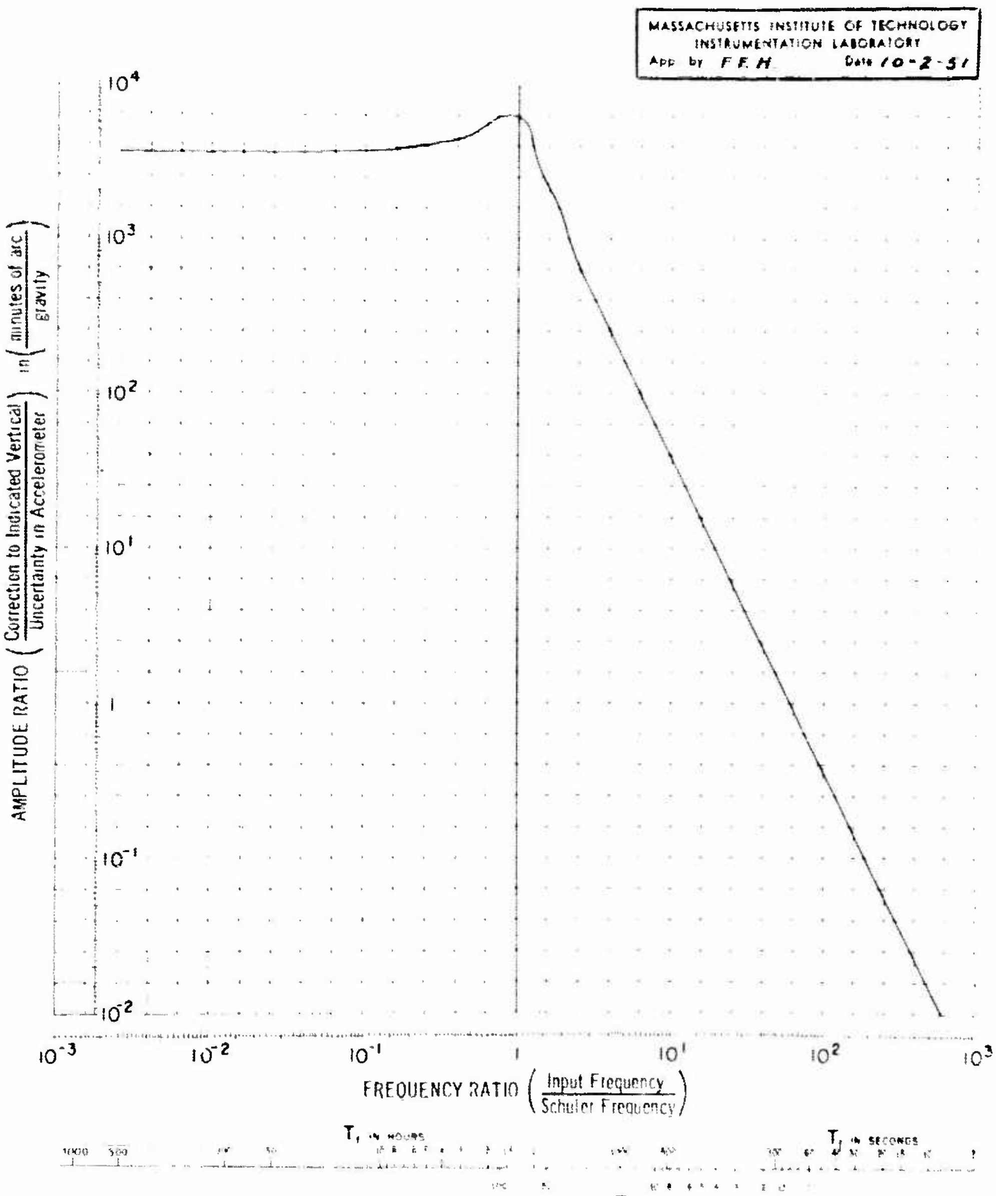

Fig. 6. Effect of accelerometer uncertainty on vertical indication, as a function of frequency.

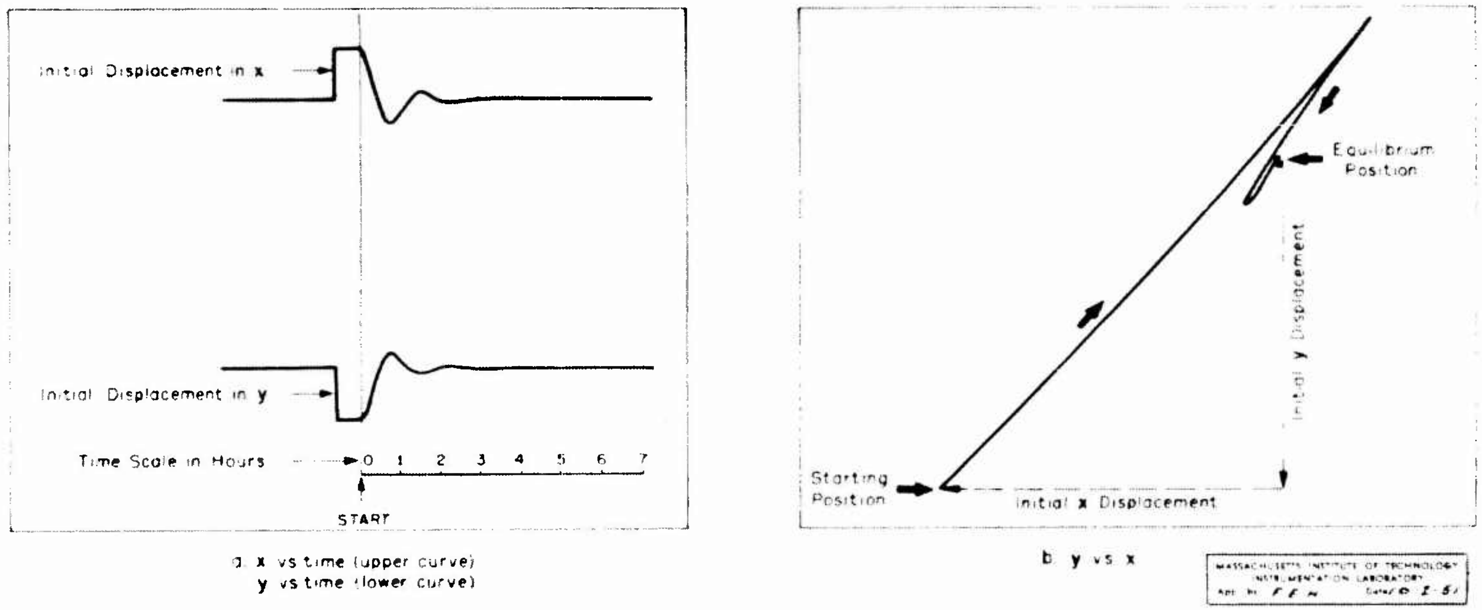

Fig. 7. Vertical indicator component systems simulated on the REAC (Reeves Electronic Analogue Computer) to display x and y coupling through earth rate.

POSITION INDICATION BY ANALYTICAL INTEGRATION OF RATES

A distinction was made in Part 1 (p. 27) between two general methods of position indication: analytic integration of the accelerometer output data, and geometric integration of the accelerometer output data. These methods will be defined again here in the same way. Analytical integration is characterized by a modification of the accelerometer output data by means of two integrators, so that the output of the second integrator is a signal proportional to the essentially geocentric angular displacement. Geometric integration, on the other hand, is characterized by a physical comparison of the indicated vertical with a reference vertical; the angle between these two verticals gives the indicated position.

First, analytical-integrating systems will be discussed. They will be treated in terms of single-axis damped systems, with vertical indication provided in each case by a system similar to that of Fig. 2.

Open-chain integration of the angular velocity of the indicated vertical

This system is shown in Fig. 8. A performance equation, Eq. (2-9), is derived in Derivation 2A, to assess the effect of component uncertainties and input inaccuracies (which are treated as in Derivation 1). The correction to the indicated position as given by Eq. (2-9) is equal to the following sum:

1) the initial position correction,

2) the change in the correction to the indicated vertical,

3) an indicated ground speed integral with the time of operation as its limits.

The most serious source of error is the last term, in which Earth-rate compensation inaccuracy, gyro drift, and the position-integrator drift, which appear as false ground-speed components, are integrated with respect to time. This cumulative error, unlimited except by the time of operation, is the direct result of the open-chain nature of the last integration giving the position. Note that a one-nautical-mile-per-hour false ground speed component gives rise to a minute of arc error in the indicated position for each hour of operation time. A false ground-speed component of this magnitude would correspond to a gyro drift of about 0.015 degree per hour, or to an Earth-rate compensation signal which failed to be constant to within less than 0.1 percent or to a position-integrator drift of one minute of arc per hour. Several hours' operation time can thus be expected to give rise to

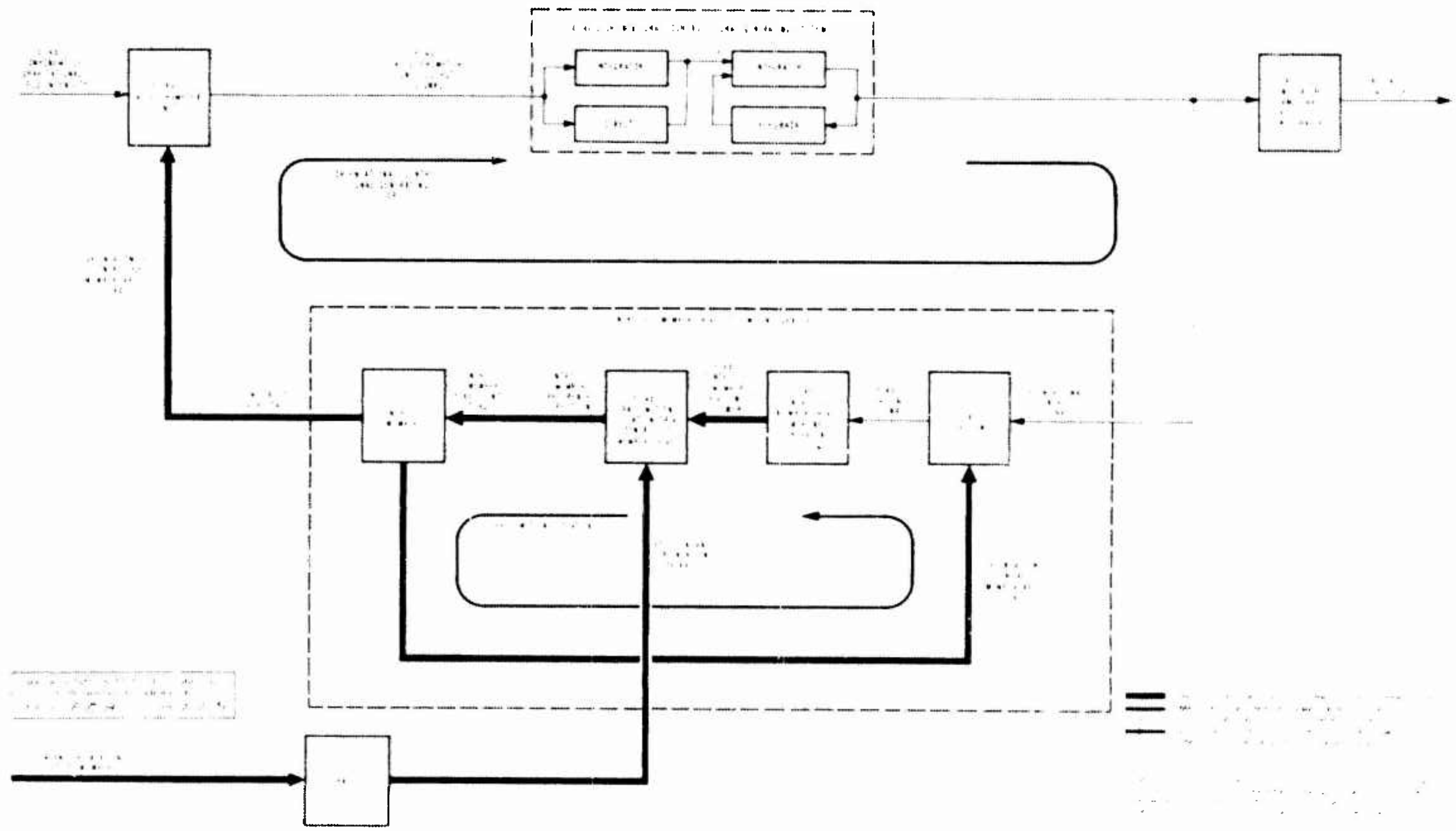

Fig. 8. Functional diagram showing interrelations among essential operating components of a damped latitude indicator, using open-chain single integration of the angular velocity of the indicated vertical.

excessive errors, unless this false ground speed component can be kept sufficiently small. The relative simplicity of conception of this method of position indication is thus offset by a disadvantage in the high precision of the instrumentation required to minimize the false ground speed component. The magnitude of the operation time is a strong determining factor in judging the practicability of this system, and is to be balanced against possible instrumentation problems, which must include stability and maintenance considerations in the initial design procedure. In submarine navigation, operation time intervals of the order of several hours are required; moreover, the operation time is necessarily somewhat indefinite in any case, a situation to be contrasted, for example, with the guidance of a vehicle over a preselected course between a given starting point and destination. Systems of this type, i.e., those subject to cumulative errors, are therefore to be regarded with disfavor as solutions to the submarine navigation problem.

Open-chain double integration of the angular acceleration of the indicated vertical

This system is shown in Fig. 9, and a performance equation, Eq. (2-37), is derived in Derivation 2B. The most important effect of component uncertainties and input inaccuracies is to create two cumulative error terms. In view of the previous discussion, these terms will now be examined. One of the terms is a single integration, with respect to the time of operation, of a false ground speed component; the other is a double integration of a false acceleration component. The presence of a term of this form is traceable to the open-chain double integration peculiar to this method of position indication. In connection with the false ground speed component, note that there are four principal sources of error:

1) a term proportional to the initial value of the correction to the indicated vertical,
2) gyro drift,
3) Earth-rate compensation inaccuracy,
4) velocity compensation inaccuracy.

A false ground speed component of one nautical mile per hour, giving rise to an error of one minute of arc in the indicated position for each hour of operation time, corresponds to an initial correction to the indicated vertical of 0.03 minute of arc, or to a gyro drift of 0.003 degree per hour. Evidently, the difficulties in accurate position indication associated with cumulative error in the previous case discussed are here even more serious. Further examination of the errors incurred in using an open-chain double-integration method confirms this. Therefore, the considerations of the effect of cumulative errors on a submarine navigation system given in the previous case warrant rejection of the present method as infeasible.

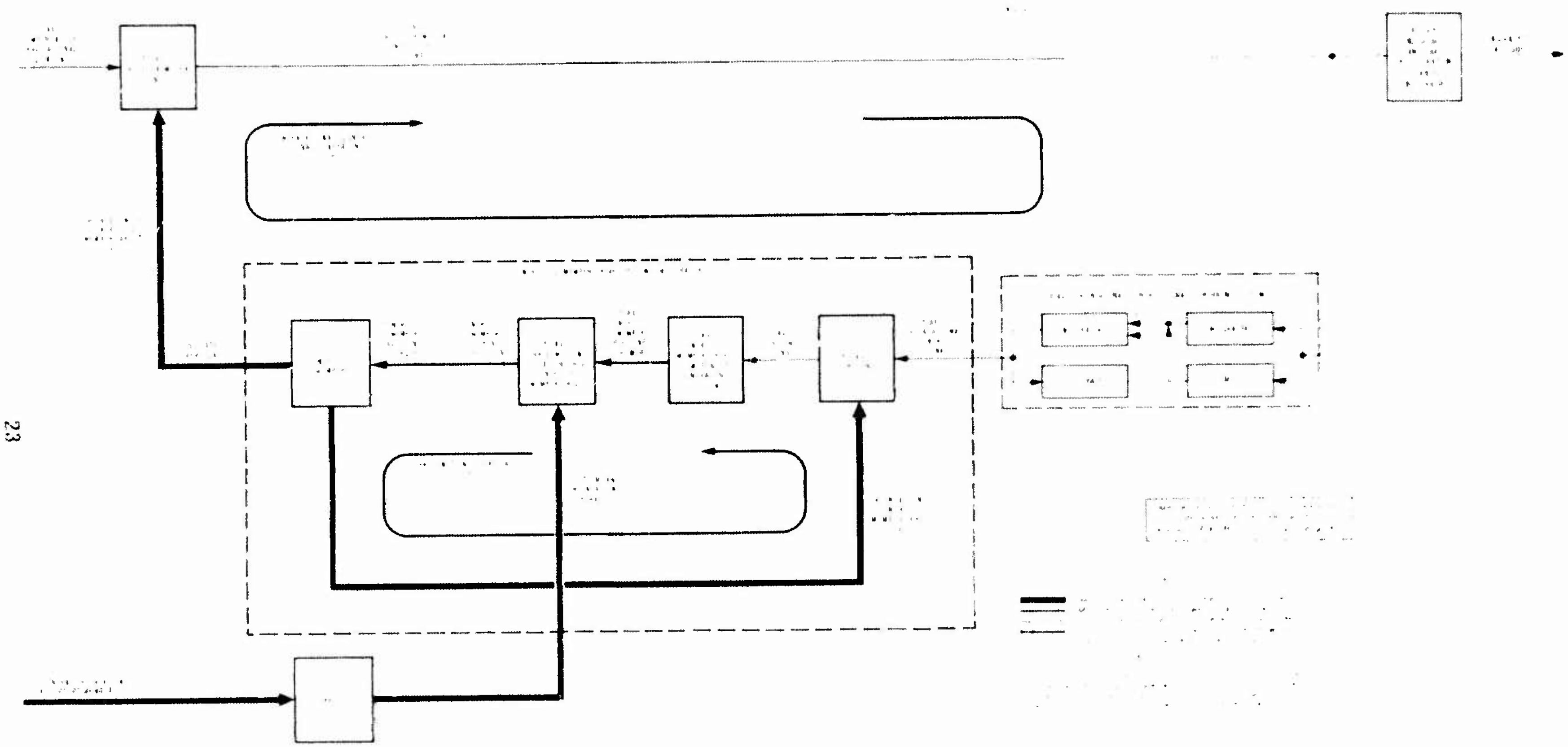

Fig. 9. Functional diagram showing interrelations among essential operating components of a damped latitude indicator, using open-chain double integration of the angular acceleration of the indicated vertical.

Direct double integration of acceleration

This system is shown in Fig. 10. The accelerometer here is inherently doubly-integrating (Part I, p. 34), and therefore is an immediate source of position data. A modification of the vertical indicator of Fig. 2 is used; this is discussed at the beginning of Derivation 2C. In Derivation 2C a performance equation (Eq. (2-67)) for position indication is also derived. The most significant errors are cumulative, as in the two previous cases, i.e., proportional to a false ground speed component integrated over the time of operation. The integrand contains the following two principal sources of error:

1) gyro drift,
2) Earth-rate compensation inaccuracy.

The same conclusions regarding indicated position errors apply to this case as apply to the case of the single open-chain integration method already discussed. That is, a false ground speed component of one nautical mile, creating a minute of arc error in the indicated position for each hour of operating time, corresponds to a gyro drift of about 0.015 degree per hour, or to an Earth-rate compensation error of 0.1 percent.

Again, although the method appears superficially simple and therefore of potential applicability, the long time of operation encountered in submarine navigation would probably make instrumentation problems difficult.

Note that there is a duplication of effort, so to speak, involved in differentiating the already-integrated position data to obtain the velocity signal required to precess the integrating gyro unit. This situation can be avoided by using a rate-gyro unit and omitting the differentiating network, but this would involve poorer drift characteristics in the gyro unit than can be obtained with an integrating gyro. Retention of the differentiating network, on the other hand, involves accentuated system response to high frequencies. It is possible that this problem might be resolved by further study. However, the cumulative errors would still be present, and appear to be unavoidable. For this reason this system is rejected as not being applicable to the submarine problem.

AZIMUTH STABILIZATION

All of the foregoing systems require some form of azimuth stabilization, i.e., orientation of the controlled member in the horizontal plane about the z-axis, so that the x-axis indicates north. This problem is discussed in Part I, pp. 43 ff. and 83 ff. The basic stabilization method will be reviewed briefly here, after which the problem will be considered in detail, since the self-erecting latitude indicator to be discussed below, operates in conjunction with the azimuth stabilizer.

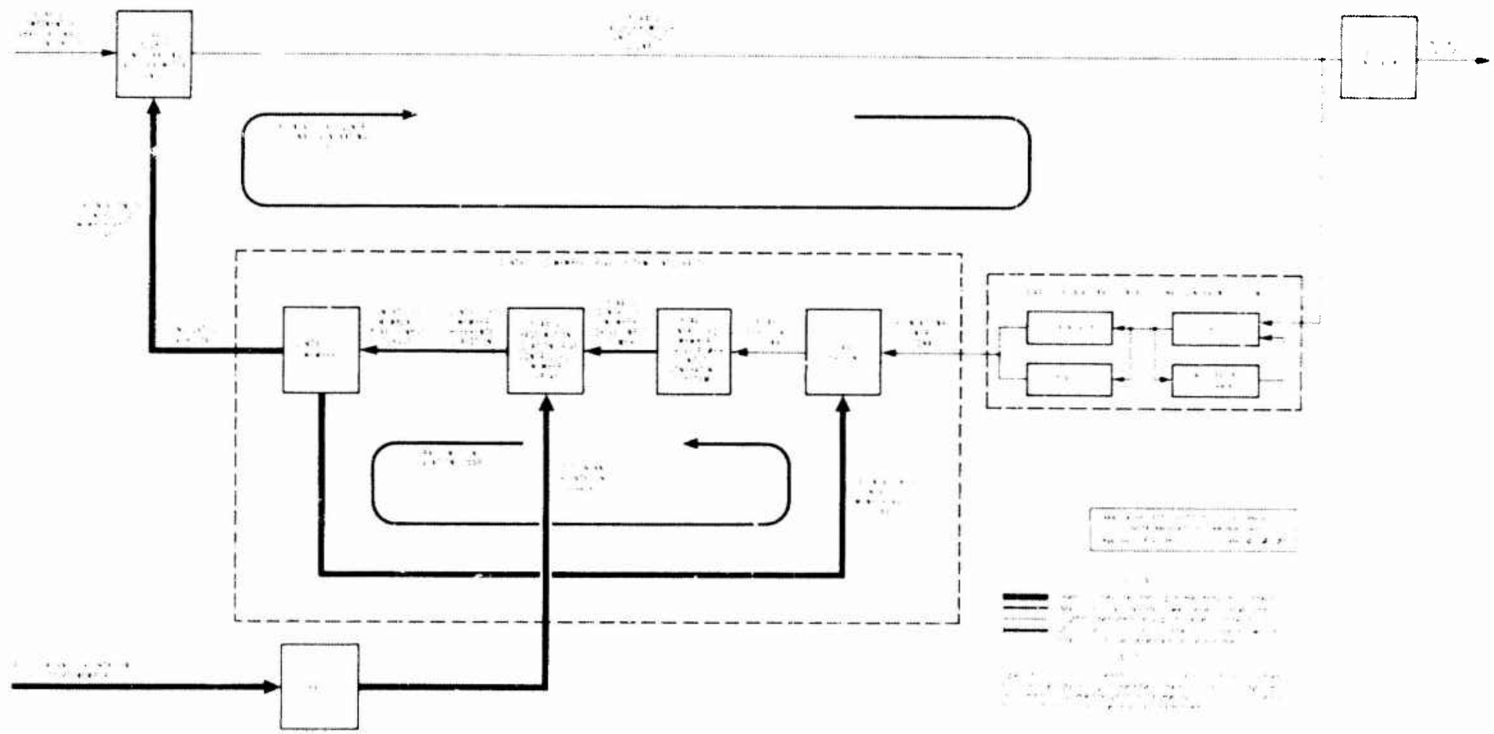

Fig. 10. Functional diagram showing interrelations among essential operating components of a damped latitude indicator, using a doubly-integrating accelerometer.

The y-gyro unit in the vertical indicator, that is, the one whose input axis coincides with the y-axis, senses two angular velocity components: the y-axis component of the angular velocity of the indicated vertical (normal to the indicated meridian plane), and the component of Earth-rate projected on the y-axis (see Fig. 11). The y-axis Earth-rate component is a measure of azimuth misalignment, i.e., the departure of the indicated east axis (the y-axis) from true east (the Y-axis). Specifically, this angular velocity component is proportional to the sine of the correction to indicated north, defined as the angle between true north (the X-axis) and indicated north (the x-axis). (Since the correction is kept small by the system, the sine of the correction is taken as equal to the correction itself in the azimuth stabilizer synthesis given in Derivation 3.) The y-axis Earth-rate angular velocity component can be nulled by rotating the controlled member in the indicated horizontal plane about the z-axis until the y-axis points east and the x-axis points north. This is accomplished in the manner described in Part I. Consider the special case where the angular velocity of the indicated vertical is zero, and an

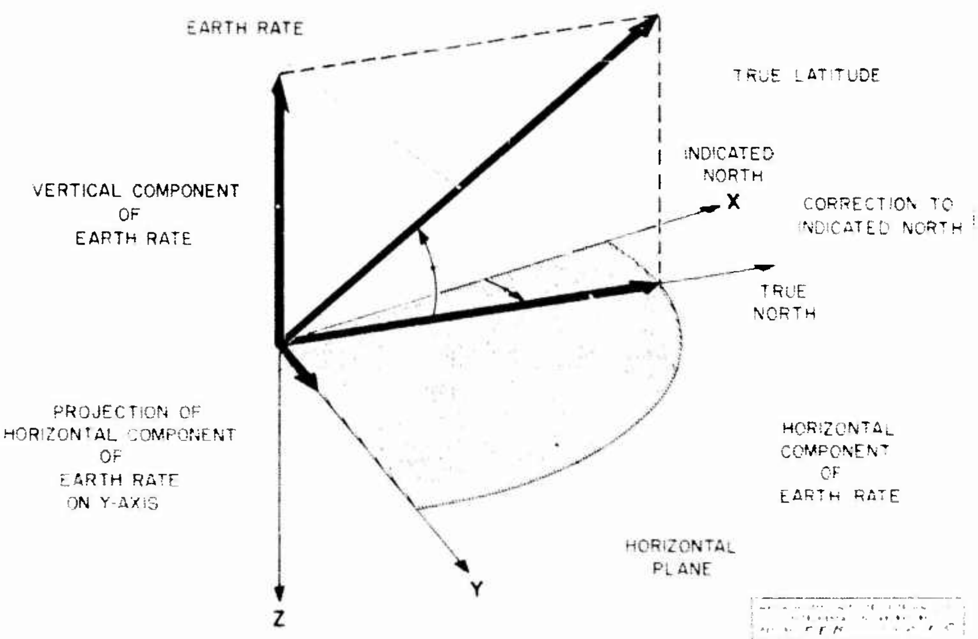

Fig. 11. Horizontal Earth-rate component projection on y-axis due to azimuth misalignment.

azimuth error is present, with the y-axis pointed somewhat north of east. As stated, the y-gyro will sense the component of Earth-rate projected on the y-axis. The y-gyro unit then sends a signal to the controlled member drive which rotates the controlled member about the y-axis, i.e., causes the x-axis to dip. This in turn will cause the y-axis accelerometer unit to send an orientational control signal to both the y- and z-gyro units. The signal to the y-gyro and its associated drive will tend to level the controlled member, while the signal to the z-gyro unit will produce a simultaneous rotation of the controlled member about the indicated vertical until the angular velocity sensed by the y-gyro unit is substantially zero, with the y-axis pointed east (i.e., perpendicular to the vector representing the horizontal component of Earth rate). If, removing the restriction above, the base has a velocity component in a northward direction, the y-gyro will be nulled when it is pointed slightly north of east (and south of east for southward velocity). That is, the y-axis will tend to align itself in the direction of a vector perpendicular to the vector resultant of the horizontal component of the angular velocity of the Earth (in the direction of true north) and the northward angular velocity of the base (in the direction of true west). The steady-state indication of east (and north) will be in error by an angle whose sine is the ratio of the northward angular velocity of the base to the horizontal component of Earth rate. This error will be discussed later in more detail.

An additional angular velocity component that is sensed by the z-gyro unit is the vertical component of Earth rate. To prevent the controlled member from being rotated about the z-axis at this angular velocity, an Earth-rate compensation signal must be provided. The effect of the compensation is to maintain the gyro-unit axes in a coordinate system fixed with respect to the Earth rather than with respect to inertial space. The compensation signal which precesses the z-gyro is proportional to Earth-rate multiplied by the sine of true latitude. In the practical case, as given in Derivation 3, this compensation signal is derived from the x-orientational control signal which contains Earth rate multiplied by the cosine of true latitude. This signal is multiplied by the tangent of indicated latitude, derived from the latitude indicator, to furnish the z-gyro unit Earth-rate compensation.

The y-gyro unit is thus the signal source for the azimuth drive as well as one of two signal sources for the y-vertical indicator drive, the second source in the latter case being the y-accelerometer. In this sense the y-gyro unit may be referred to as the east-seeking gyro unit, since, with the aid of the y-accelerometer and y- and z-controlled member drives, its input axis tracks true east.

As indicated above, the y- and z-systems are coupled, and their dynamic characteristics are interrelated. Part I which treats undamped systems exclusively, shows (pp. 83 ff.) the nature of the interdependence when the z-system is a second-order system (i.e., representable by a second-order differential performance equation. When the y-vertical indicator is damped, in the manner shown in Fig. 2, the coupling becomes more complex. It has been found feasible, in the latter case, to represent the z-system by a first-order equation, Eq. (3-21), in Derivation 3. This equation is applicable to the z-system shown in Fig. 12.

The characteristic time of the z-system is an important adjustable parameter. Stability considerations* place a lower limit on its value when the y- and z-systems are combined into a single fourth-order system.** Figure 13 shows a series of transient-response curves for the coupled y- and z-systems, taken with the REAC. The following assumptions were used in simulating the problem on the analogue computer:

1) The y-system is Schuler-tuned and has a damping ratio of 0.3, when decoupled from the z-system. (It is assumed, incidentally, that in the practical case the x-system would have similar characteristics.)
2) Component uncertainties and input inaccuracies are ignored, since they do not primarily affect stability.
3) The system is supposed to be on a stationary base, so that the oscillations shown in Fig. 13 occur about the fixed directions of the true vertical and true north, respectively.

It will be seen from Fig. 13 that the general effect of decreasing the characteristic time is to decrease the system damping. The general effect of increasing the characteristic time is to increase the solution time. The entire system response is in addition a function of the true latitude. Neither latitude variations from zero to 80 degrees, nor z-channel characteristic-time variations of comparable effect, have much influence on over-all stability for a characteristic time of about 1.3 hours; and this value allows a solution time in azimuth of about four hours, or three Schuler periods. The value 1.3 hours is therefore chosen tentatively as an optimum z-channel characteristic time.

The effects of errors due to imperfect components and external inputs to the system deserve examination. Equation (3-25), the last equation in Derivation 3, displays these sources of error when the y-system, z-system, and the self-erecting-latitude-system (to be discussed in the pages following) are regarded as a single system having a fifth-order performance equation in time-derivatives. As far as errors are concerned, those associated with the steady-state behavior of the system are of greatest significance, since the other errors are zero except under transient conditions. There are two such sources of steady-state error:

* Specifically, if Routh's stability criterion is applied to the y- and z-combined systems, and the y-system is Schuler-tuned and has a damping ratio of 0.3 when decoupled from the z-system, the characteristic time of the z-system has a lower limit of approximately 0.42 hour.

** The x-system will not enter these considerations, because of the aforementioned Foucault pendulum study.

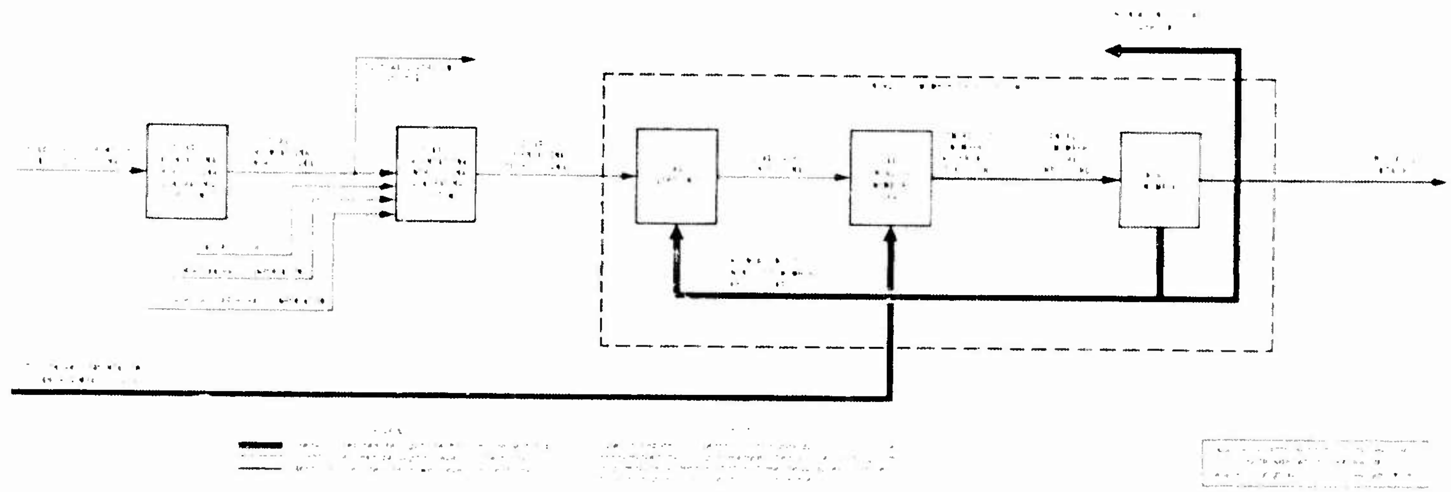

Fig. 12. Partial functional schematic diagram of an azimuth stabilization system applicable to the z-axis orientation of the controlled member of Fig. 1

Note: All curves are from REAC (Reeves Electronic Analogue Computer) traces. Displacements are all on one relative amplitude scale.
The y-system parameters are taken from Derivation 1B

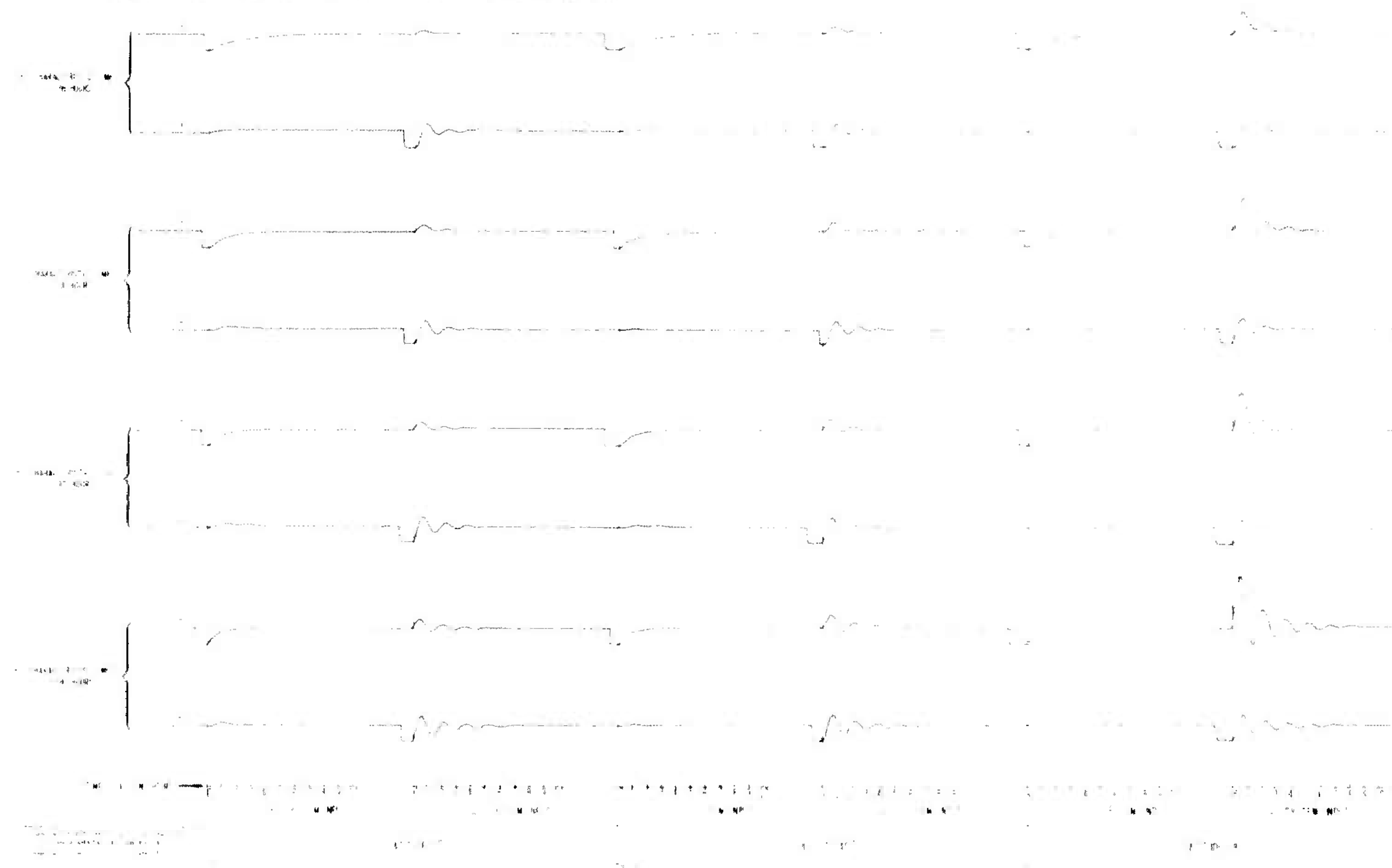

Fig. 13. Transient response of z-system (azimuth) and y-vertical indicator for successive z and y step-inputs with a range of azimuth system characteristic time settings from 1.9 to 0.54 hour at latitudes 0,45, and 80 degrees.

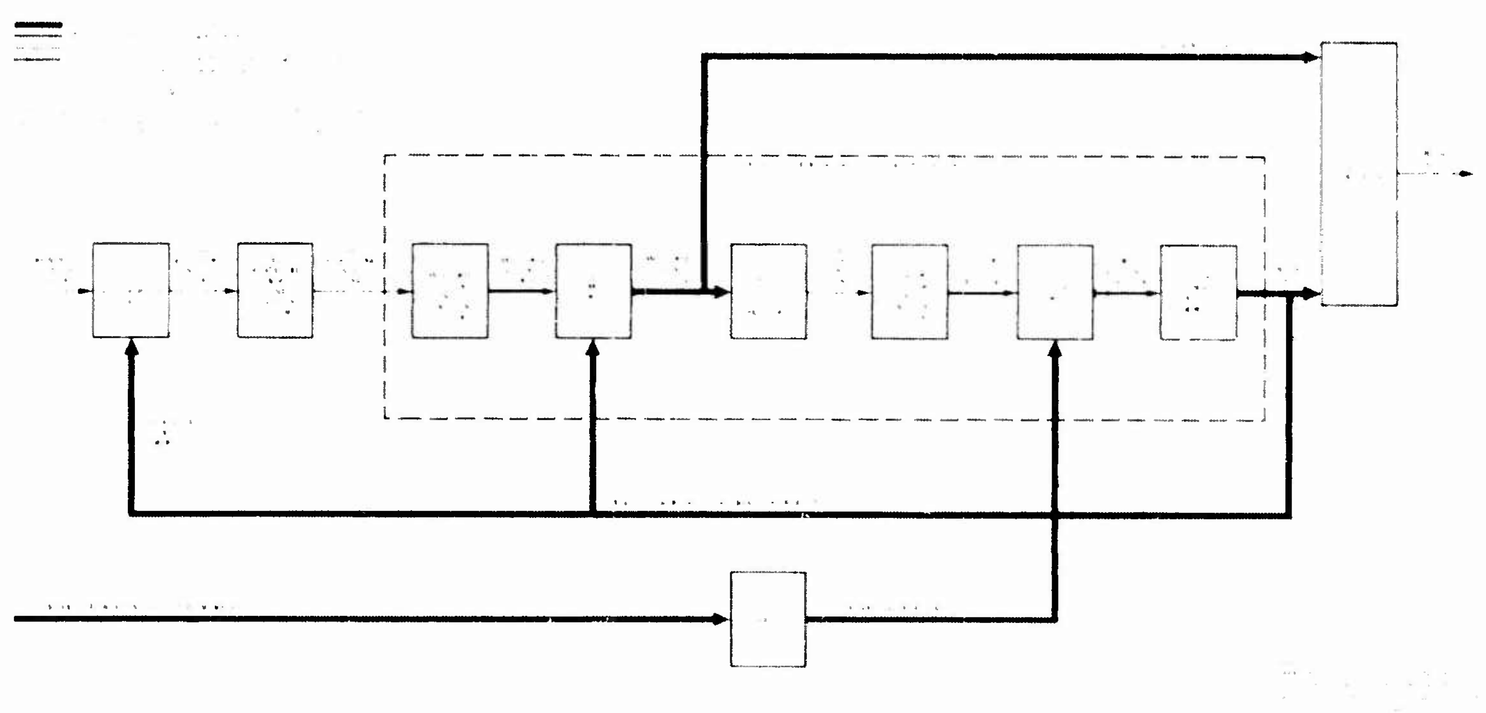

Fig. 14. Functional diagram showing interrelations among essential operating components of a latitude indicator which compares the indicated vertical with a reference vertical maintained by an inertial gyro unit.

1) A term proportional to the inaccuracy in the latitude-rate (water-speed) compensation. This term varies as the secant of latitude. At latitude 45 degrees an inaccuracy in the latitude-rate compensation of 0.7 knot will create an error of one mil* in indicated azimuth.

2) A term proportional to y-gyro unit drift. This term also varies as the secant of latitude. At latitude 45 degrees, a drift of 0.01 degree per hour will create an error of one mil in indicated azimuth.

The first of these two errors would involve the aforementioned ratio of the latitude rate to the horizontal component of Earth rate. The use of a northerly water-speed compensation signal reduces this error to the residual one given above, which is due to the inaccuracy in the water-speed compensation. Note that no cumulative errors are incurred in azimuth indication.

POSITION INDICATION BY GEOMETRIC INTEGRATION USING A PREALIGNED INERTIAL GYRO UNIT

The position indication system is shown in Fig. 14, and is discussed in Part I, pp. 38 ff. and 101 ff. The major objection to this system is the requirement for prealignment of the gyro axes with the directions of astronomical star-lines, data which are not readily available with sufficient accuracy on a system installed in a submarine. In particular, the required azimuth reference would be difficult to establish, a situation to be contrasted with, for example, the problem of aircraft guidance, where the reference can be established on the ground before take-off. In submarine navigation systems, in general, self-erecting systems are to be regarded as preferable to pre-aligned systems. Therefore, prealigned systems are not discussed further here. However, it will be shown subsequently that if a self-erecting system is used to orient an inertial gyro unit so that it is sensitive only to angular velocity components effectively parallel to the Earth polar axis, the gyro unit will be a source of data on longitude.

A SELF-ERECTING GEOMETRIC LATITUDE INDICATOR

The indication of latitude by geometric integration was discussed in Part I, pp. 37 ff. The integration is accomplished by mounting a gyro unit on a latitude gimbal, which causes a line on the gimbal to track a line parallel to the Earth polar axis, thereby furnishing a reference vertical; i.e., the perpendicular to the Earth polar axis is a vertical at the equator.

The indicated latitude is required in the system itself, e.g., for vector component resolution of Earth-rate compensation in the x-system, for resolution of the x-current when used for compensation in the z-channel, and for sensitivity

* The *mil* will be defined for this report as it is commonly used for fire-control applications: a mil means one milliradian, or 1/1000 of a radian.

control through the secant of latitude in the z-channel. This suggests an inevitable feedback from the latitude gimbal to the vertical and azimuth indicators. This feedback effect may be utilized to erect the latitude gimbal as is subsequently shown, thereby making the latitude gyro unit mentioned in Part I unnecessary. The mechanization of such a device will now be described. The latitude system performance equation, Eq. (4-11), is derived in Derivation 4. The essential operating components of the latitude indicator are shown in Fig. 15.

The kind of signal input to the latitude gimbal drive is determinable from the principal requirement for indicated latitude imposed by the rest of the system: namely, the multiplication of the x-orientational-control signal by the tangent of indicated latitude, so that the product can be used to compensate the z-gyro unit for the vertical component of Earth-rate plus longitude change. As indicated in Derivation 4, the latitude drive signal chosen consists of the y- orientational-control signal, with true latitude rate partly compensated by water speed, and, in addition, the water-speed compensation itself.

Consider first the case where the base is stationary with respect to the Earth, and assume that originally no errors are present in the vertical and azimuth systems. Then the input axis of the z-gyro unit is vertical and the input axis of the y-gyro unit points east. Assume that the latitude gimbal is not positioned correctly. Under these conditions the latitude resolver (see Fig. 16) will not receive the correct angle as an input. This in turn will produce an incorrect resolution of the x-gyro unit current, which is used to compensate the z-gyro unit for the vertical component of Earth rate. The z-gyro unit will then sense the uncompensated portion of vertical Earth rate, and, through the azimuth drive, will produce a rotation of the controlled member about the indicated vertical. When this rotation occurs, the input axis of the y-gyro unit will be moved from its eastward position, and the unit will sense a component of horizontal Earth rate. From this point the action is similar to that described previously in connection with azimuth stabilization except that there are now involved three coupled systems instead of two. The y-gyro unit, by means of its drive, will produce a rotation of the controlled member about the y-axis. This in turn will cause the y-axis pendulum to send simultaneous orientational control signals to the y-gyro, the z-gyro and the latitude drive. The signal to the y-gyro unit will tend to level the controlled member, the signal to the z-gyro unit will produce a rotation of the controlled member about the indicated vertical until the angular velocity sensed by the y-gyro unit is substantially zero, and the signal to the latitude drive will rotate the latitude gimbal (see Fig. 16) until the uncompensated angular velocity sensed by the z-gyro unit is zero. The last condition will exist only when the latitude gimbal is correctly positioned and the latitude resolver receives the particular input angle necessary to produce accurate compensation for vertical Earth rate in the z-gyro unit.

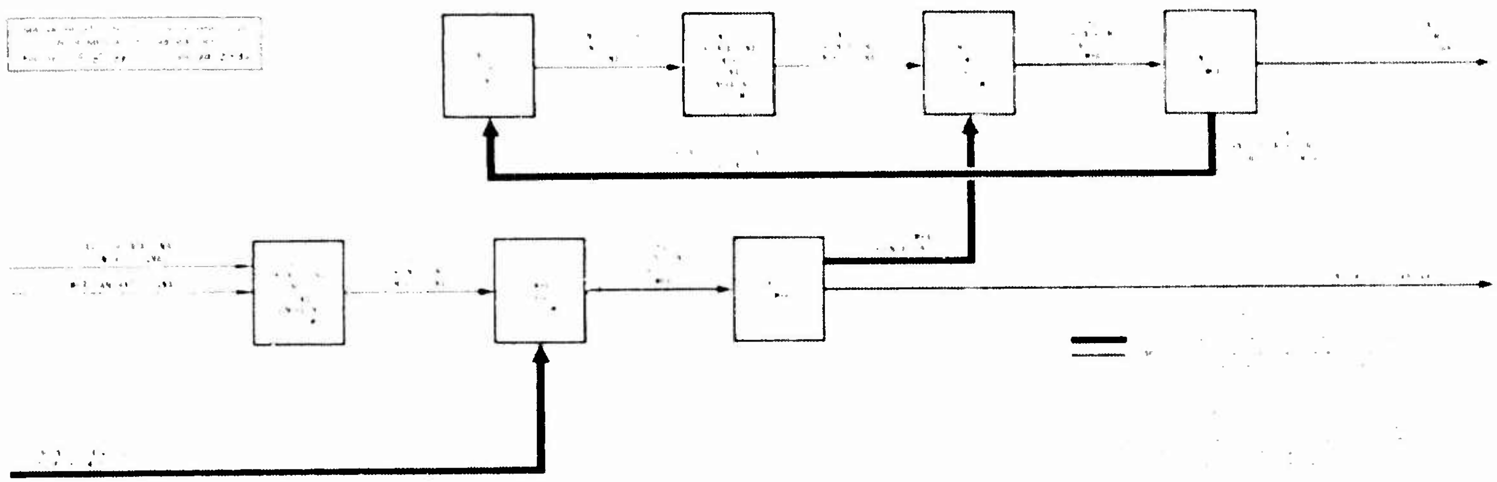

Fig. 15. Partial functional schematic diagram of a three-axis system for position indication using detection of angular velocity components perpendicular and parallel to the Earth's polar axis, using the system of Fig. 12 to furnish three-axis stabilization in azimuth.

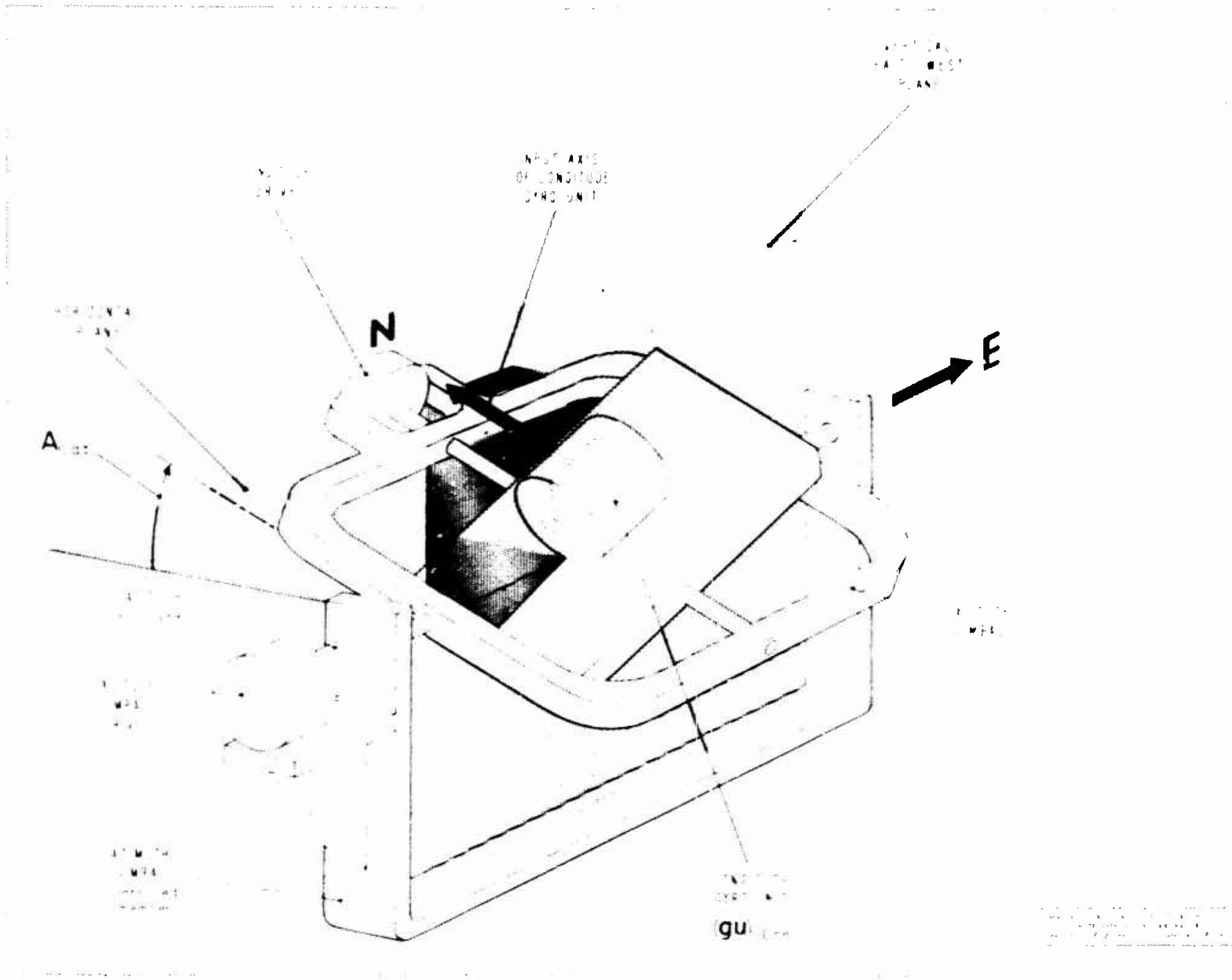

Fig. 16. Latitude and longitude position indicator using geometric integration of the indicated vertical.

The operation of the system on a base with a non-zero latitude rate is essentially the same except for the effects on the dynamic performance of the azimuth stabilization system (noted previously) and the latitude system. The dynamics of the latitude system under discussion, when it is not compensated for latitude rate, are such as to make the system subject to error if there is a finite velocity input. The disadvantages of such an arrangement are apparent. Two possible solutions appear to be feasible: viz., to change the system dynamics and thereby produce a latitude system with zero velocity error, or to use an external means for latitude-rate compensation so that the latitude gimbal is driven at the approximate latitude rate at all times. The first method is superior for the submarine problem, since it minimizes the effect of compensation inaccuracy and is the one which offers most promise for further development. However, it does introduce a stability problem which, because of the coupling between the latitude, azimuth, and y-systems, is difficult to solve. This problem is being studied. For the purpose of the present report, it will be shown to what extent a system utilizing water-speed compensation is practical. Derivation 4 is therefore based on the latter approach.

In accordance with the foregoing discussion, the latitude drive current is made up of the following component signals (see Eq. (4-11)):

1) The azimuth orientational control signal, minus Earth-rate compensation. This signal consists of the y-orientational control signal plus latitude-rate compensation.
2) The rate of change of latitude, based on pitometer-log information.

These signals are sent to the latitude gimbal drive system, mounted on the controlled member. Equation (4-11), the performance equation for the latitude loop, is of first order in time derivatives of the correction to indicated latitude; the equation for the three systems mentioned, taken as a single system, is therefore a quintic (Eq. (4-12)). Routh's criterion, when applied to the latter equation, with y- and z-parameters chosen as already indicated, shows that for stability the lower limit for the characteristic time in the latitude system is about 0.6 hour. Figure 17 shows the response to step-inputs in y-, z-, and latitude-channels for a simulated* system. It is assumed in the simulation that the system base is stationary with respect to the Earth and that the inaccuracies and uncertainties in the quantities upon which the system operates are zero; this simplification is consistent with stability determination as an isolated problem. A practical lower limit on the latitude-characteristic time is seen from Fig. 17 to be about 1.3 hours, and this is the recommended value.

The latitude system is subject to steady-state errors; this is shown by Eq. (4-12) when the time derivatives of correction terms are equated to zero. The

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* The system was simulated on the REAC.

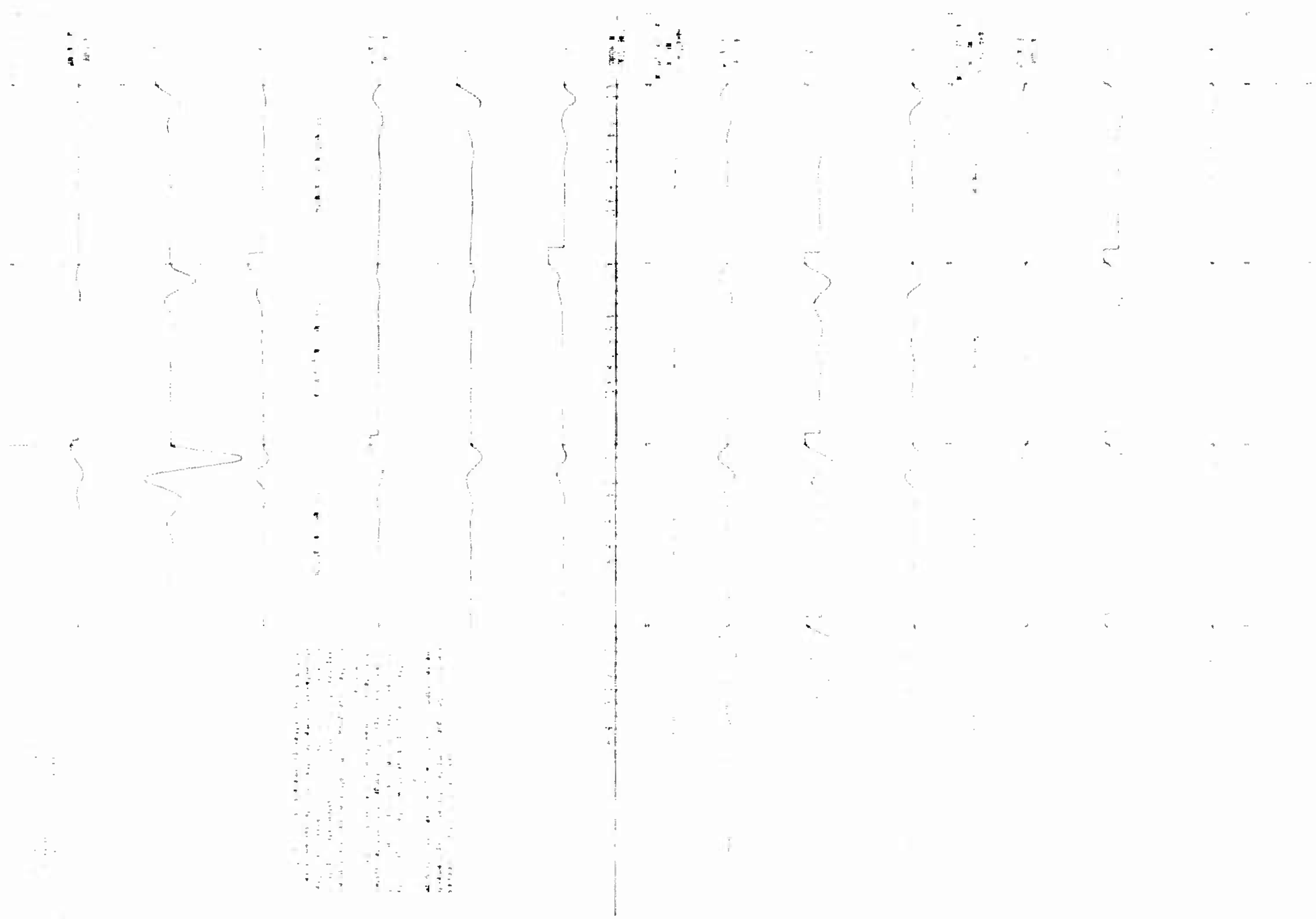

Fig. 17. Transient response of y-vertical indicator, z (azimuth) system, and latitude system (polar-axis indicator), for various step inputs and latitude-system characteristic time settings.

magnitude of these errors can be estimated for typical operating conditions by considering the submarine to be moving at a speed of 20 knots on a meridian course at latitude 45 degrees. The steady-state error in indicated latitude under these conditions is made up principally of the following terms:

1) Resolver uncertainty: about 1.5 minutes of arc error in indicated latitude, if the resolver is accurate to 0.05 percent.
2) Gyro unit drift: about 1.6 minutes of arc error in indicated latitude for either x-gyro unit drift or z-gyro unit drift of 0.01 degree per hour.
3) Latitude rate compensation inaccuracy: about 1.3 minutes of arc error in indicated latitude for each knot of inaccuracy in latitude rate compensation.
4) Accelerometer uncertainty: about 0.25 minute of arc error in indicated latitude.
5) Orientational control signal uncertainty: about 0.2 minute of arc error in indicated latitude for a noise level in the z-orientational control signal generator of 0.3 percent.

Generally speaking, in terms of the performance of these components as they are used in current applications, the principal serious sources of error can be expected to be the resolver, the latitude rate compensation, and the gyro units, in that order.

LONGITUDE INDICATION BY CELESTIAL-LONGITUDE-RATE TRACKING

The indication of longitude necessarily involves the integration of longitude rate. A straightforward method of accomplishing this involves the use of a gyro unit, coupled to a servo drive, as the integrator; this at once minimizes the effect of integrator drift (in this case, drift effects are confined to the gyro) and allows the use of the latitude gimbal just described to orient the longitude gyro input axis. The mounting of the longitude gyro on the latitude gimbal is shown schematically in Fig. 15. The input axis of the gyro unit is oriented by the latitude gimbal in such a way that the gyro unit is sensitive only to angular velocity components that are effectively parallel to the Earth polar axis (see Part I, pp. 38 and 97 ff.).

These components include the Earth rate, and since the gyro unit is sensitive to motion with respect to inertial space, the signal from the gyro unit to the longitude gimbal drive will cause the drive to operate at the celestial longitude rate. A separate time drive can be used to subtract Earth rate from this motion, so that the longitude gimbal rotates at the indicated longitude rate, and the angle between the longitude gimbal and a preset reference direction on the latitude gimbal is indicated longitude. It appears preferable to mount the time drive in a console off the gimbal and carry out the above subtraction in the console. This permits a reduction in the size of the gimbal and more convenient system packaging.

This method is discussed in detail in Derivation 5. The following conclusions may be drawn from Derivation 5, Eq. (5-24):

1) The transient behavior of the longitude system is somewhat influenced by the rest of the indication system, particularly the x- and z-systems. However, there is no feedback from the longitude loop to the rest of the system.

2) The steady-state accuracy of the system is dependent on the longitude velocity, unless the longitude system characteristic time is sufficiently small.

3) The longitude gyro unit contributes a drift term to the performance equation (Eq. (5-24). The integral of this drift term with respect to time appears; this means that a cumulative error can be expected, determined by the drift rate and time of operation of the system.

4) To a lesser extent, the inaccuracy in the sidereal time drive contributes to the above cumulative error in the same manner as gyro unit drift.

5) The error in setting reference longitude on the indicator dial is not cumulative and appears in the same form in each longitude reading.

To assess the effect of these errors on longitude indication, consider Eq. (5-24) in Derivation 5 under steady-state conditions, i.e., after transients have settled out, so that the time derivatives of correction terms are zero. Assume that the characteristic time for the longitude system is 10 seconds, and that the submarine is at latitude 45 degrees, traveling northward at 30 knots. Under these conditions a one-mil error in azimuth, or a one-mil error in the x-component of the correction to the indicated vertical, will produce approximately 0.02 minute of arc error in indicated longitude for each hour of operating time. A longitude gyro unit drift of 0.015 degree per hour will be a source of about one minute of arc error in indicated longitude for each hour of operating time. If the sidereal time drive is in error by 0.01 percent there will be an error from this source of about 0.1 minute of arc in indicated longitude for each hour of operating time.

As stated initially, the indication of longitude requires that longitude rate be integrated. The consequence appears to be an unavoidable cumulative error associated with the integration. The inevitability of the integration requirement stems ultimately from the non-unique quality of longitude and the symmetry of the Earth about its polar axis. Therefore, the engineering procedure in longitude indicator design must be directed at the reduction of cumulative errors, through efficient disposition of components and sufficiently precise instrumentation.

All of the position-indicating systems described have outputs in the form of angular, rather than linear, displacements of the submarine. It is to be expected, therefore, that ground speed obtained from this type of navigation system will be derived from the essentially geocentric angular velocity of the indicated vertical with respect to the Earth. The product of this quantity and the radius of the Earth then gives the ground speed. Thus the chief problem in indicating ground speed is the extraction of the angular velocity of the indicated vertical with respect to the Earth from the navigation system.

This angular velocity may be obtained either from 1) the vertical indicator itself or 2) the latitude and longitude indicators. These methods are described in Derivation 6. Ground-speed indication by each of these two methods will now be discussed.

THE VERTICAL INDICATOR AS A SOURCE OF GROUND-SPEED DATA

In method (1) where the vertical indicator is the initial source of ground-speed data, the quantities of importance are the x- and y-axis orientational control signals. These signals are respectively proportional to the x- and y-components of the angular velocity of the indicated vertical with respect to inertial space. The y-component of the angular velocity of the indicated vertical with respect to the Earth is equal to that with respect to inertial space except for small correction terms. It is therefore at once usable to indicate the northerly ground-speed component. The x-component, however, contains an Earth-rate term, which must be removed before east-west ground speed is available from this source. Specifically, a signal proportional to Earth-rate compensation multiplied by the cosine of indicated latitude (the latter derived from the latitude indicator) is subtracted from the x-axis orientational control signal. The resultant signal is approximately proportional to the x-component of the angular velocity of the indicated vertical with respect to the Earth. The indicated-latitude function and Earth-rate compensation term cause the x-component to be influenced to some extent by Earth-rate compensation inaccuracy and latitude-rate compensation inaccuracy. Similarly, the y-component of the angular velocity of the indicated vertical with respect to the Earth is affected by the correction to indicated north. These errors are subsequently discussed; for the present, note that two signals can be divided from the vertical indicator, each essentially proportional to a component of ground speed.

It is necessary that these components be added in quadrature by a resolver set at an angle of 90 degrees, so that the resolver output will be proportional to the magnitude of the vector angular velocity of the indicated vertical with respect to the Earth. The result of such addition is given for the case of a system in equilibrium by Eq. (6-13) of Derivation 6. The indicated ground speed is equal to the true ground speed plus an inaccuracy which depends on the direction of motion over the Earth, i.e., on the cosine or sine of true azimuth. The important influences on the inaccuracy term are due to:

1) The x-gyro and y-gyro drift terms: for each 0.01 degree per hour of drift, the contributory error in the eastward and northward indicated ground-speed components respectively will be approximately 0.6 knot.
2) The term containing the correction to indicated north; at latitude 45 degrees, each mil of error in indicated azimuth contributes approximately 0.6 knot of error to the eastward indicated ground-speed component. (The contribution to the northward ground-speed component error from this source is much smaller, being only about 0.02 knot for each mil of azimuth error, at a ship speed of 20 knots.)
3) The Earth-rate compensation inaccuracy term; if the compensation signal is constant to within an accuracy of 0.01 percent, the contributory error in the eastward indicated ground-speed component will be about 0.09 knot.
4) The term containing the correction to indicated latitude; at latitude 45 degrees, each minute of arc error in latitude indication contributes about 0.2 knot of error to the northward component of indicated ground speed. The y-axis correction to the indicated vertical contributes the same relative error to this ground-speed component and the x-axis correction to the indicated vertical contributes the same relative error to the eastward component of indicated ground speed.

THE LATITUDE AND LONGITUDE INDICATORS AS SOURCES OF GROUND-SPEED DATA

If ground-speed information is obtained from the latitude and longitude indicators (method (2) above), it is necessary to add, in quadrature, signals proportional to the indicated longitude rate, multiplied by the cosine of indicated latitude, and the indicated latitude rate. One of the signals suitable for this purpose is that furnished to the latitude drive; this signal is proportional to indicated latitude rate. The indicated longitude can be obtained from a tachometer attached to a shaft whose rotation rate is proportional to indicated longitude rate. The indicated latitude is obtained as before from the latitude system. The indicated ground speed is then given by Eq. (6-20) of Derivation 6. The angular velocity of the indicated vertical is equal to the angular velocity of the true vertical plus an inaccuracy term as before, the importance influences on the inaccuracy in this case being as follows:

1) The longitude gyro drift term: at latitude 45 degrees, for each 0.01 degree per hour of drift, the contributory error in the northward ground-speed component will be approximately 0.4 knot.

2) The longitude-rate tachometer uncertainty term: this contributes an error to the northward indicated ground-speed component, at latitude 45 degrees and with a component of 20 knots magnitude, of about 0.1 knot for a one percent uncertainty in the tachometer.

3) The Earth-rate compensation (sidereal time drive) inaccuracy term: at latitude 45 degrees, when the error in the time-drive rotational rate is 0.01 percent, the contributory error in the northward indicated ground-speed component will be about 0.06 knot. Note that the sidereal time-drive inaccuracy appears in the inaccuracy in the indicated ground speed, as compared with the Earth-rate compensation signal inaccuracy in the previous case. At present, a time-drive can be engineered, in general, as an inherently more accurate device than a constant-level signal generator.

4) Terms containing the correction to indicated latitude, the correction to indicated north, and the x-axis correction to the indicated vertical: at latitude 45 degrees, and northward and eastward ship velocity components of 20 knots each, one mil of error in azimuth places the northward ground-speed component in error by about 0.01 knot, while one minute of arc error in latitude or x-vertical indication contributes approximately 0.005 knot to the indicated ground-speed error northward component. Note that at equilibrium, i.e., when the system is not suffering a transient shock, the eastward ground-speed component is without errors of the type discussed above.

One advantage of this method over the first one discussed is that the gyro drift is here multiplied by the cosine of latitude. Furthermore, the azimuth error, which is finite in the steady state, makes a less serious contribution here. It may be concluded that the method of ground-speed indication discussed immediately above is preferable to the first method. The principal limitation would derive from the requirement for a precise tachometer for the measurement of longitude rate.

GENERAL CONSIDERATIONS

Report R-9, Parts I and II, represents the result of a study of several methods of vertical- and position-indication, of which the most promising have been discussed. The basic theory has been presented, along with suggestions as to how presently available components could be assembled into a workable system.

The nature of the instrumentation art as reflected in the precision of components currently under development, rather than any basic theoretical reason, determines the choice of the best navigation system from the engineering point of view. Four classes of components of particular significance in their effect on the performance of an inertial position indicating system are:

1) gyro units,

2) accelerometer units,

3) latitude resolvers,

4) sidereal time drives.

The present precision of these units may be specified as follows: a gyro drift of 0.01 degree per hour is within the goal of current system development, accelerometers, used in a vertical indicator of the type discussed in this report, can be expected to contribute about 0.25 minute of arc error in indication of the vertical; a resolver accuracy of 0.05 percent is attainable; and a sidereal time drive, such as is required in a longitude indicator using a gyro unit as an integrator, can be constructed as a crystal-controlled clock, with a shaft rotation that is proportional to Earth rate to within one part in 100,000.

It has been shown in this report that in certain respects, some position indicators are superior to others. The selection of the best position-indication method necessarily involves a compromise, because no single method is entirely free from faults, and even the best systems described here are susceptible to improvement through further study.

VERTICAL INDICATION, Roll and Pitch

As a result of this study it appears practical to make each component subsystem of the vertical indicator similar to that shown in Fig. 2. The optimum system parameters are given on page 14, and the theory is based on Derivation 1 in the Appendix. The controlled member, or azimuth gimbal, is to be mounted inside a pitch gimbal, and this in turn inside a roll gimbal. Such an assembly is sketched in detail in Fig. 18.

The dynamics of this vertical indicator are adjusted so that it is Schuler-tuned, i.e., the eighty-four-minute period characteristics are incorporated. Its performance depends on this property rather than on precise external compensation inputs. Errors will be of an essentially dynamic nature, and do not accumulate with time. The chief source of error, due to the accelerometer, is expected to produce about 0.25 minute of arc error in vertical indication after the transient stage is over. The solution time (the time required to settle out) associated with the transient stage is about three hours.

By means of the gimbal system, the vertical, i.e., the azimuth gimbal orientation, is compared with the roll- and pitch-gimbal orientations to indicate respectively the roll and pitch of the ship.

AZIMUTH STABILIZATION, Heading

The system of Fig. 12 is recommended for azimuth stabilization. The fundamental operation involved is the nulling of the angular velocity input to the east-seeking gyro unit (used by the vertical indicator) on the azimuth gimbal. The operation is analogous to that of a standard gyrocompass, orienting the azimuth gimbal so that a reference line on the gimbal, perpendicular to the input axis of the east-seeking gyro unit, will track true north when latitude-rate compensation is provided from an external source. The inaccuracy in this compensation is a source of steady-state error, to the extent of about one-mil error in indicated azimuth for a 0.7-knot error in latitude-rate compensation, at latitude 45 degrees. The drift of the east-seeking gyro unit under similar conditions will result in an error of one mil in indicated azimuth for a gyro unit drift of 0.01 degree per hour. The solution time of the azimuth system is about four hours. The optimum system parameters are given on page 28, and the theoretical considerations are given in Derivation 3. Figure 18 shows this azimuth system added to the azimuth gimbal. The angle between the north-reference line on the azimuth gimbal and the fore-aft line of the ship is indicated ship heading.

LATITUDE INDICATION

The self-erecting latitude indicator described on page 32 is recommended. The theory is given in Derivation 4. It involves the rotation of a resolver whose output is fed back into the azimuth system. Since the azimuth and vertical indicators are coupled, as mentioned above, latitude indication is ultimately monitored by the accelerometer unit associated with the east-west axis on the vertical indicator. The latitude, azimuth, and east-west vertical indicators make up a

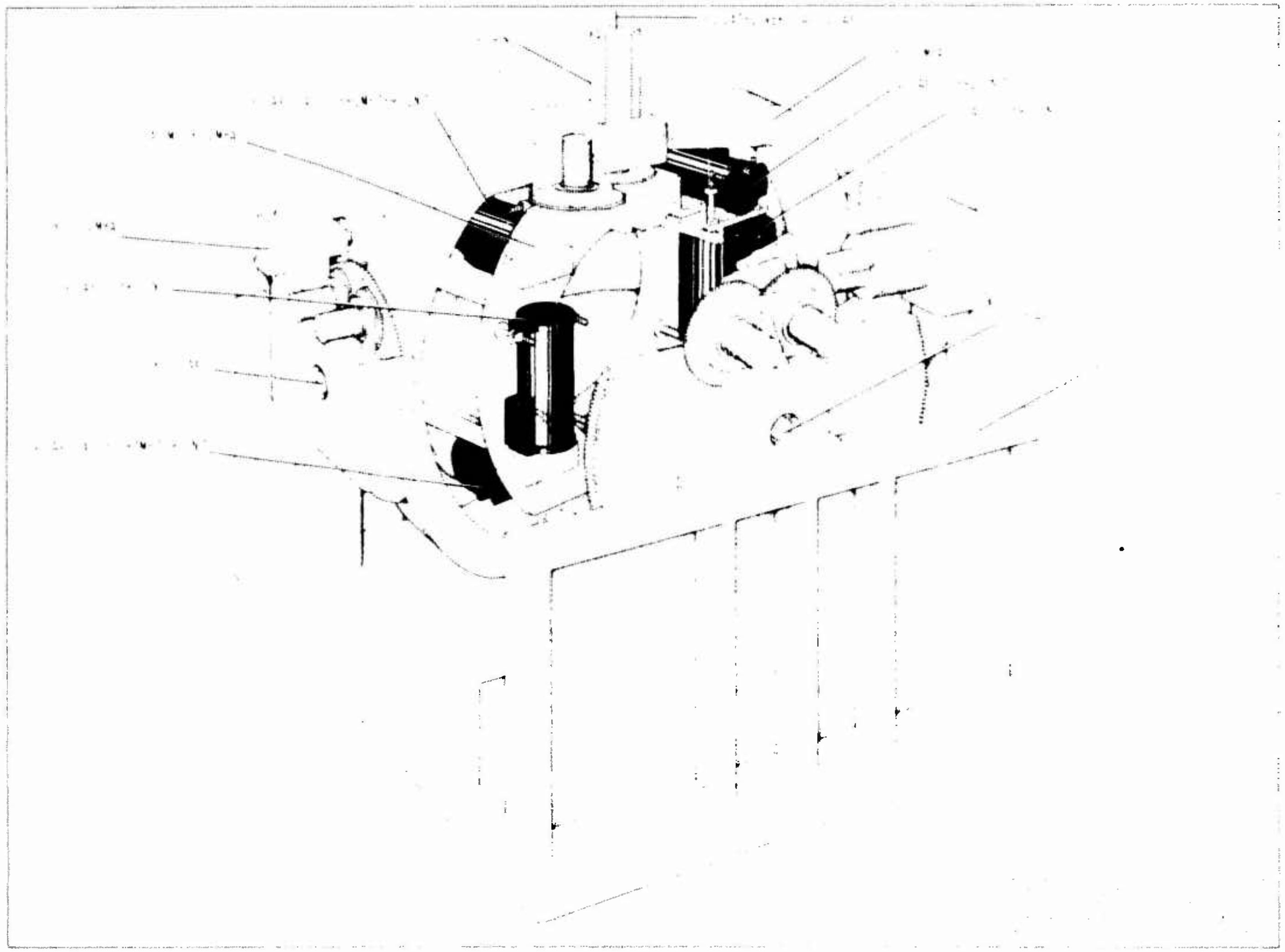

Fig. 18. Pictorial cutaway of gimbal configuration for vertical and azimuth systems.

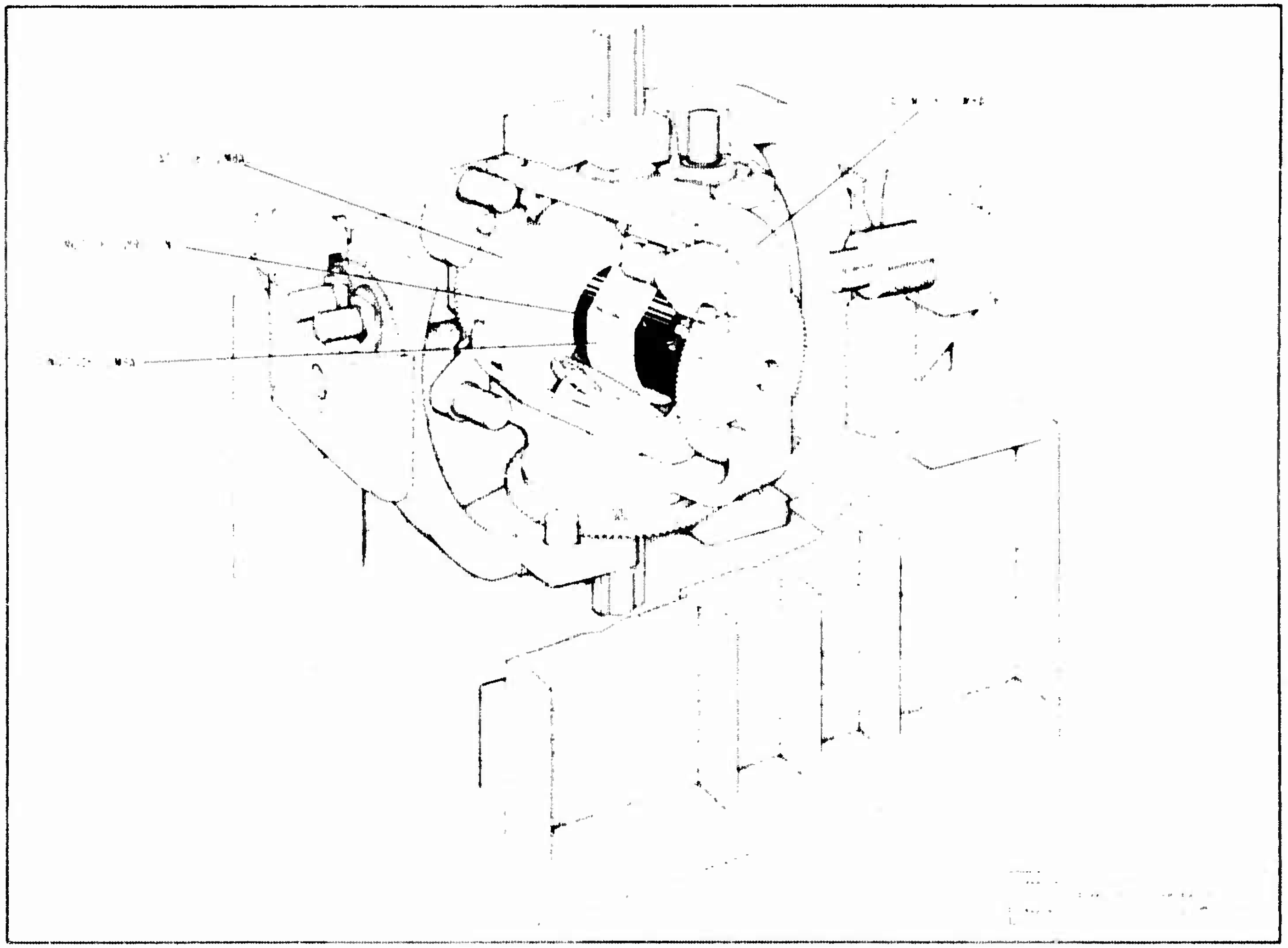

Fig. 19. Pictorial cutaway of latitude and longitude gimbal configuration.

closed-loop system, and latitude indication, along with vertical and azimuth indication, is not subject to cumulative errors. The rotation of the latitude resolver is associated with the rotation of the latitude gimbal, which effectively tracks the Earth polar axis, and is sketched in Fig. 19. This system is such that the latitude gimbal is self-erecting, and rotates until a reference line on the gimbal indicates the polar axis without initial setting or prealignment. Latitude-rate compensation from an external source is required to minimize a servo velocity error. Indicated latitude will be in error by approximately one minute of arc for each knot of error in the latitude-rate compensation, and by approximately 1.5 minutes of arc for a drift of 0.01 degree per hour of the two gyro units whose input axes are in the indicated meridian plane. A 0.05 percent uncertainty in the output of the latitude resolver will also create an error of about 1.5 minutes of arc in indicated latitude. These are steady-state errors, after the transient stage has passed. The solution time is about four hours. The optimum latitude system parameters are given on page36. The angle between the indicated horizontal (indicated by the azimuth gimbal) and the indicated polar axis (furnished by the latitude gimbal) gives the indicated latitude.

LONGITUDE INDICATION

The most practical method for indicating longitude is to track celestial longitude rate with an integrating gyro unit mounted to rotate within the latitude gimbal, as shown in Fig. 20. This gyro unit responds to the angular velocity parallel to the indicated Earth polar axis. The gimbal on which the longitude gyro unit is mounted, rotating about the indicated polar axis within the latitude gimbal, forms the innermost of the five concentric gimbals of the navigation system. The gimbal rotates at the indicated celestial longitude rate. This information is carried to a mechanical differential remote from the gimbal system, where Earth rate, generated as a shaft rotation by a sidereal clock, is subtracted from indicated celestial longitude rate. The differential output gives indicated longitude difference. This output is read as indicated longitude relative to a longitude reference index setting on the indicator. The chief errors in longitude indication are cumulative. Under typical operating conditions, a longitude gyro unit drift of 0.015 degree per hour will be a source of about one minute of arc error in indicated longitude for each hour of operating time. The sidereal time drive may be expected to contribute an error in indicated longitude of about 1.5 minutes of arc in a week of operating time. The longitude system is discussed on page 38 and in Derivation 5.

Fig. 20. Photograph of wooden mock–up of complete gimbal assembly.

GROUND-SPEED INDICATION

It is recommended that ground speed be computed by utilizing signals proportional to indicated longitude rate (from a tachometer geared to the longitude indicator) and to indicated latitude rate (the latitude gimbal drive input signal). The indicated longitude-rate signal is then multiplied by the cosine of the indicated latitude, obtained from a resolver on the latitude gimbal. This quantity is then added in quadrature to the indicated latitude-rate signal, by means of another resolver, to obtain the magnitude of the essentially geocentric angular velocity of the indicated vertical. This last quantity, multiplied by the Earth radius, is the <u>indicated ground speed.</u> The chief error in the indicated ground speed is in the northward component, to which a 0.01 degree per hour longitude gyro unit drift will contribute an error of about 0.4 knot. This method of ground-speed indication is discussed on page 41 and in Derivation 6.

GENERAL DATA

The following tabulation summarizes the over-all characteristics of the proposed system:

Approximate outer dimensions of gimbal system, including cover	42 in. high, 42 in. wide, and 46 in. long
Radius of largest gimbal (to go through a standard 20 in. by 39 in. hatch, during installation)	36 in.
Number of self-contained single-axis integrating gyro units required	4 (3 inertial gyros, 1 used as an integrator)
Number of accelerometers required	2
Number of integrators required*	4
Estimated input power required from ship power supplies	5 kilowatts
Estimated dimensions of console mounting electronic components	5 ft high, 5 ft long, and 3 ft deep

* The number of integrators might be reduced to three by using the z-system in its entirety as one of the y-system integrators. This line of attack is being investigated.

The submarine navigation system discussed in the preceding section and pictured in the sketches, Figs. 18 and 19, is shown as a wooden mock-up in the photograph, Fig. 20. While this system is regarded at present as most suitable for the application, its recommendation is also described as tentative because it is based on a time-limited study. Nevertheless, it is possible to suggest the direction in which subsequent work might best proceed in order to improve on the proposed system.

Four gyro units are required in the proposed system; three of these are inertial gyro units, used to set up a three-axis coordinate system, while the fourth (the longitude gyro unit) is used as an integrator. The first three gyro units represent an irreducible minimum number if single-axis gyro units are used. It may be possible to combine the function of the longitude gyro unit with that of the other units, and reduce the total number of single-axis gyro units to three. This would require a somewhat different gimbal configuration from that of the system proposed in this report.

Another field for future work based on the present study is in the problem of the adequate use of external compensations. Particular reference is made to the water-speed compensation, a signal derivable, for example, from the ship pitometer-log. The eastward and northward components of the water speed are used by the two component systems of the vertical indicator. The specific application here is the minimization of acceleration, not velocity, error; the vertical-indicating system, in effect, serves to differentiate external water-speed information in using it to compensate for dynamic errors. The result of this particular application of water-speed data is to leave residual errors in the vertical indication that are proportional to the rate of change of the inaccuracy in the water-speed compensation signal rather than to the inaccuracy itself, i.e., no steady-state velocity error stems from this source. Furthermore, since the compensation is dynamic in character, its effects are strongly dependent upon frequency; only oscillations near the Schuler frequency are of consequence, and for frequencies outside this region, the compensation is not important.

In the proposed navigation system, these considerations do not hold for the azimuth and latitude indicators. Water-speed compensation signals are used in these cases essentially to eliminate effectively geocentric angular velocity components of the indicated vertical in the process of determining the direction of the Earth-rate vector. The result of this application of water-speed compensation is steady-state errors in indicated azimuth and latitude that are proportional to the inaccuracy in the water-speed compensation, not to its rate of change. It is therefore desirable to investigate other means to remove this source of steady-state error, and thereby relieve the present dependence of the azimuth and latitude systems on data external to the navigation system. A possible approach is to feed indicated latitude rate from the latitude system back into the azimuth indicator. This involves additional complexity in the concept of the system as a whole, and possibly raises stability problems. This approach is a subject of present studies, and appears promising.

APPENDIX

The mathematical derivations on the following pages are referred to by number in the foregoing text. The self-defining notation used in the equations is based on a formulation given by Draper[7].

DERIVATION 1. DAMPED SINGLE-AXIS VERTICAL INDICATOR.

A. DERIVATION OF DIFFERENTIAL PERFORMANCE EQUATION.

If $[(C)V]_{(t,i)}$ is defined as the correction to the indicated vertical,

$$p\,[(C)V]_{(t,i)} = W_{(EV)t} - W_{(EV)i} \tag{1-1}$$

$$W_{(EV)i} = S_{(cmds)(i,\mathbb{W})}\,i_{(hfi)} + (I)W_{(IE)(cp)} + (U)W_{(gu)} \tag{1-2}$$

Refer to Fig. I-1.

$$i_{(hfi)} = S_{(hfi)(e,\dot{i})}\,\frac{1}{p}\,[e_{(dir)} + e_{(lfi)} + e_{(cp)} - e_{(fb)}] + \frac{1}{p}(U)\dot{i}_{(hfi)(df)} + i_{(hfi)o} \tag{1-3}$$

$$e_{(dir)} = S_{(dir)(e,e)}\,e_{(au)} + (U)e_{(dir)} \tag{1-4}$$

$$e_{(lfi)} = S_{(lfi)(e,\dot{e})}\,\frac{1}{p}\,e_{(au)} + \frac{1}{p}(U)\dot{e}_{(lfi)(df)} + e_{(lfi)o} \tag{1-5}$$

$$e_{(cp)} = (PF)_{(cp)(\mathbb{W},e)}\,W_{(EV)(cp)} \tag{1-6}$$

where $W_{(EV)(cp)}$ is the equivalent angular velocity of the true vertical derived from pitometer-log data, and $(PF)_{(cp)(\mathbb{W},e)}$ is to be determined.

$$e_{(fb)} = S_{(fb)(i,e)}\,i_{(hfi)} + (U)e_{(fb)} \tag{1-7}$$

$$e_{(au)} = S_{(au)(a,e)}\,\{g\,[(C)V]_{(t,i)} + RpW_{(EV)t}\} + (U)e_{(au)} \tag{1-8}$$

Substitute the foregoing in Eq. (1-2). The result is

$$\begin{aligned} W_{(EV)i} = {} & S_{(cmds)(i,\mathbb{W})}\left[\frac{S_{(hfi)(e,\dot{i})}}{p + S_{(hfi)(e,\dot{i})}S_{(fb)(i,e)}}\right]\{S_{(dir)(e,e)}S_{(au)(a,e)}\,[g\,[(C)V]_{(t,i)} \\ & + RpW_{(EV)t}] + S_{(dir)(e,e)}(U)e_{(au)} + (U)e_{(dir)} + S_{(lfi)(e,\dot{e})}S_{(au)(a,e)}\frac{1}{p}[g\,[(C)V]_{(t,i)} \\ & + RpW_{(EV)t}] + S_{(lfi)(e,\dot{e})}\frac{1}{p}(U)e_{(au)} + \frac{1}{p}(U)e_{(lfi)(df)} + e_{(lfi)o} + (PF)_{(cp)(\mathbb{W},e)}W_{(EV)(cp)} \\ & - (U)e_{(fb)}\} + \frac{S_{(cmds)(i,\mathbb{W})}}{p + S_{(hfi)(e,\dot{i})}S_{(fb)(\dot{i},e)}}(U)\dot{i}_{(hfi)(df)} + (I)W_{(IE)(cp)} + (U)W_{(gu)} \end{aligned} \tag{1-9}$$

NOTE. $(SR)_{(gu)(i,W)}$ *is defined as the sensitivity ratio of the gyro unit, i.e., the ratio of the sensitivity for a current input-torque output to the sensitivity for an angular velocity input-torque output, these torques being balanced to null by the gyro unit operation. The controlled member drive is sufficiently fast to make* $S_{(cmds)(i,W)} = (SR)_{(gu)(i,W)}$, *as in any high-gain feedback loop.*

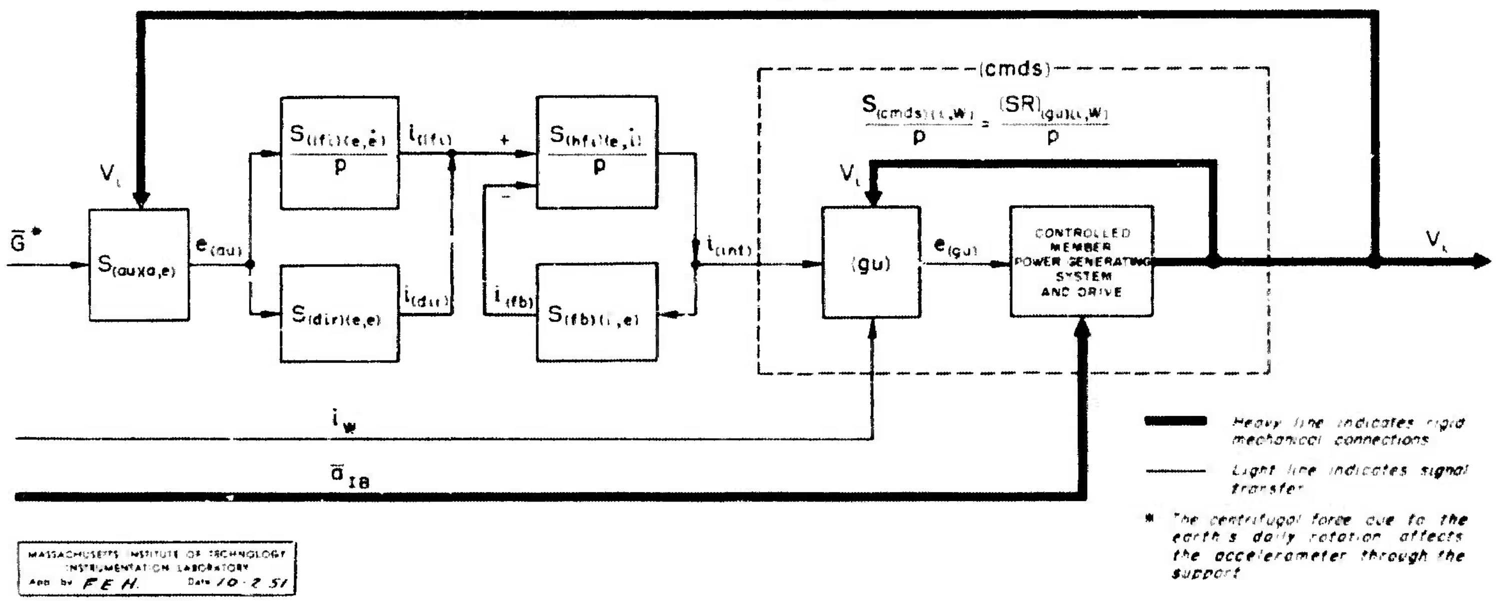

Fig. 1–1. Mathematical functional diagram of a damped single–axis vertical indicating system using the components shown in Fig. 1.

Define

$$\left.\begin{aligned}
&S_{(cmds)(i,\Psi)}S_{(hfi)(e,\dot{i})}S_{(dir)(e,e)}S_{(au)(a,e)} = S_{(Vi)(a,\dot{\Psi})}\\
&\frac{S_{(dir)(e,e)}}{S_{(lfi)(e,\dot{e})}} = (CT)_{(ld)}\\
&\frac{1}{S_{(hfi)(e,\dot{i})}S_{(fb)(i,e)}} = (CT)_{(lg)}
\end{aligned}\right\} \qquad (1-10)$$

Substitute these definitions and Eq. (1–9) into Eq. (1–1). There results

$$p\,[(C)V]_{(t,i)} = W_{(EV)t} - \left(\frac{1}{p + \frac{1}{(CT)_{(lg)}}}\right)\Big\{S_{(Vi)(a,\dot{\Psi})}\,g\,[(C)V]_{(t,i)} + S_{(Vi)(a,\dot{\Psi})}RpW_{(EV)t}$$

$$+ \frac{S_{(Vi)(a,\dot{\Psi})}}{S_{(au)(a,e)}}(U)e_{(au)} + \frac{S_{(Vi)(a,\dot{\Psi})}}{(CT)_{(ld)}}\frac{1}{p}\,g\,[(C)V]_{(t,i)}$$

$$+ \frac{S_{(Vi)(a,\dot{\Psi})}}{(CT)_{(ld)}}\frac{1}{p}RpW_{(EV)t} + \frac{S_{(Vi)(a,\dot{\Psi})}}{(CT)_{(ld)}S_{(au)(a,e)}}\frac{1}{p}(U)e_{(au)}$$

$$+ S_{(cmds)(i,\Psi)}S_{(hfi)(e,\dot{i})}\left[(U)e_{(dir)} - (U)e_{(fb)}\right]$$

$$+ S_{(cmds)(i,\Psi)}S_{(hfi)(e,\dot{i})}\frac{1}{p}(U)\dot{e}_{(lfi)(df)} + S_{(cmds)(i,\Psi)}S_{(hfi)(e,\dot{i})}e_{(lfi)o}$$

$$+ S_{(cmds)(i,\Psi)}S_{(hfi)(e,\dot{i})}(PF)_{(cp)(\Psi,e)}W_{(EV)(cp)}\Big\}$$

$$- \frac{S_{(cmds)(i,\Psi)}}{\left(p + \frac{1}{(CT)_{(lg)}}\right)}(U)\dot{i}_{(hfi)(df)} - (I)W_{(IE)(cp)} - (U)W_{(gu)} \qquad (1-11)$$

Operate on Eq. (1–11) with $\left[p\left(p + \frac{1}{(CT)_{(lg)}}\right)\right]$, collect terms in $[(C)V]_{(t,i)}$, $pW_{(EV)t}$ and $p^2W_{(EV)t}$; there results

$$\left\{p^3 + \frac{1}{(CT)_{(lg)}}p^2 + S_{(Vi)(a,\dot{\Psi})}\,g\,p + \frac{gS_{(Vi)(a,\dot{\Psi})}}{(CT)_{(ld)}}\right\}[(C)V]_{(t,i)}$$

$$= \left\{\left[1 - S_{(Vi)(a,\dot{\Psi})}R\right]p + \left[\frac{1}{(CT)_{(lg)}} - \frac{S_{(vi)(a,\dot{\Psi})}R}{(CT)_{(ld)}}\right]\right\}pW_{(EV)t}$$

$$
- S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)} (PF)_{(cp)(\Psi,e)}\, p W_{(EV)(cp)}
$$

$$
- \frac{S_{(Vi)(a,\Psi)}}{S_{(au)(a,e)}} \left(p + \frac{1}{(CT)_{(ld)}}\right) (U)e_{(au)} - S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)}\, p \left[(U)e_{(dir)}\right.
$$

$$
\left. - (U)e_{(fb)}\right] - S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)} (U)\dot{e}_{(lfi)(df)}
$$

$$
- S_{(cmds)(i,\Psi)}\, p (U)\dot{i}_{(hfi)(df)} - \left(p^2 + \frac{1}{(CT)_{(lg)}}\, p\right) \left[(I)W_{(IE)(cp)} + (U)W_{(gu)}\right] \quad , \qquad (1-12)
$$

The jerk error is minimized through Schuler tuning, i.e., the coefficient of $p^2 W_{(EV)t}$ is equated to zero. There results

$$
S_{(Vi)(a,\dot{\Psi})} = \frac{1}{R_s} \qquad (1-13)
$$

where R_s is a set value for the Earth radius.

Assume $R_s \equiv R$ in the following. The acceleration error is minimized, in part, by using the pitometer-log data, as follows. Define

$$
(I)W_{(EV)(cp)} \equiv W_{(EV)(cp)} - W_{(EV)t} \qquad (1-14)
$$

and set

$$
(PF)_{(cp)(\Psi,e)} = \frac{\left[\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\right]}{S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)}} \qquad (1-15)
$$

The differential performance equation is then

$$
\left(p^3 + \frac{1}{(CT)_{(lg)}} p^2 + \frac{g}{R_s} p + \frac{1}{(CT)_{(ld)}} \frac{g}{R_s} \right) \left[(C)V\right]_{(t,i)}
$$

$$
= - \left[\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\right] p(I)W_{(EV)(cp)} - \left[p^2 + \frac{1}{(CT)_{(lg)}} p\right] \left\{(I)W_{(IE)(cp)}\right.
$$

$$
\left. + (U)W_{(gu)}\right\} - \frac{1}{R_s S_{(au)(a,e)}} \left(p + \frac{1}{(CT)_{(ld)}}\right) (U)e_{(au)}
$$

$$
- S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)}\, p \left[(U)e_{(dir)} - (U)e_{(fb)}\right]
$$

$$
- S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)} (U)\dot{e}_{(lfi)(df)}
$$

$$
- S_{(cmds)(i,\Psi)}\, p(U)\dot{i}_{(hfi)(df)} \qquad (1-16)
$$

B. OPTIMIZATION OF UNDAMPED RESONANT FREQUENCY

Considering only the water-speed compensation error term in Eq. (1–16), the equation becomes

$$\left(p^3 + \frac{1}{(CT)_{(1g)}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s} \frac{1}{(CT)_{(1d)}}\right) [(C)V']_{(t,i)}$$

$$= \left(\frac{1}{(CT)_{(1g)}} - \frac{1}{(CT)_{(1d)}}\right) p(I)W_{(FV)(cp)} \qquad (1-17)$$

The problem is to select that value for the resonant frequency associated with a selected damping ratio of the system for which the right-hand side of the equation is a minimum. To do this, the following notation is defined

W_n = undamped resonant frequency of the quadratic term

(DR) = damping ratio of the quadratic term

(CT) = characteristic time of first-order term

$W_{ns} = \frac{g}{R_s}$ = the Schuler natural frequency

Then in operator form

$$p^3 + \frac{1}{(CT)_{(1g)}} p^2 + W_{ns}^2 p + \frac{W_{ns}^2}{(CT)_{(1d)}} = \left[p + \frac{1}{(CT)}\right] \left(p^2 + [2(DR)W_n] p + W_n^2\right)$$

$$= p^3 + \left[2(DR)W_n + \frac{1}{(CT)}\right] p^2 + \left[\frac{2(DR)W_n}{(CT)} + W_n^2\right] p + \frac{W_n^2}{(CT)} \qquad (1-18)$$

Define

$$F = \frac{1}{(CT)_{(1g)}} - \frac{1}{(CT)_{(1d)}}$$

From Eq. (1–18)

$$F = 2(DR)W_n + \frac{1}{(CT)} - \frac{1}{(CT)} \frac{W_n^2}{W_{ns}^2} \qquad (1-19)$$

and

$$\frac{2(DR)W_n}{(CT)} + W_n^2 = W_{ns}^2 \qquad (1-20)$$

Eliminating (CT) between Eq. (1–19) and (1–20), there results

$$\xi = \frac{1}{2(DR)W_{ns}^2}\left\{\frac{W_n^4 + 2W_{ns}^2\,[2(DR)^2 - 1]\,W_n^2 + W_{ns}^4}{W_n}\right\} \tag{1-21}$$

Differentiate with respect to W_n, and set the result equal to zero. The solution is

$$W_n^2 = W_{ns}^2\left\{\frac{1 - 2(DR)^2 \pm 2\sqrt{(DR)^4 - (DR)^2 + 1}}{3}\right\} \tag{1-22}$$

Expansion of the square root by the binomial theorem gives as the positive root, to first approximation,

$$W_n^2 = [1 - (DR)^2]\,W_{ns}^2 \tag{1-23}$$

For $(DR) = 0.3$

$$g = 32.2 \frac{\text{Ft}}{\text{sec}^2}$$

$$R_s = 3437.7 \text{ nautical miles}$$

the value of W_n is $0.954\, W_{ns}$ or $0.00118 \frac{\text{Rad}}{\text{Sec}}$

From Eq. (1–18),

$$\frac{2(DR)W_n}{(CT)} + W_n^2 = W_{ns}^2 \tag{1-24}$$

Therefore, utilizing Eq. (1-23),

$$(CT) = \frac{2[1 - (DR)^2]}{W_n(DR)} \tag{1-25}$$

(DR), W_n and (CT) are therefore chosen as follows:

$$(DR) = 0.3$$

$$W_n = 0.00118 \frac{\text{Rad}}{\text{Sec}} \tag{1-26}$$

$$(CT) = 5140 \text{ sec} = 1.43 \text{ hrs.}$$

C. TRANSIENT RESPONSE.

The homogeneous equation derivable from Eq. (1–16) is

$$\left\{p^3 + \frac{1}{(CT)_{(1g)}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s} \frac{1}{(CT)_{(1d)}}\right\} [(C)V]_{(t,i)} = 0 \qquad (1\text{–}27)$$

This is of the form

$$\left[p^3 + \left[2(DR)W_n + \frac{1}{CT}\right] p^2 + \left[W_n^2 + \frac{2(DR)W_n}{CT}\right] p + \frac{W_n^2}{CT}\right] [(C)V]_{(t,i)} = 0 \qquad (1\text{–}28)$$

The solution is

$$[(C)V]_{(t,i)} = \left([(C)V]_{(t,i)}\right)_o \left\{\left(\frac{W_n^2}{\frac{1}{(CT)^2} - \frac{2(DR)W_n}{(CT)} + W_n^2}\right) e^{-\frac{t}{(CT)}}\right.$$

$$+ \frac{1}{CT\sqrt{1-(DR)^2}} \left[\frac{1}{\left(\frac{1}{(CT)^2} - \frac{2(DR)W_n}{(CT)} + W_n^2\right.}\right]$$

$$\left.\times e^{-(DR)W_n t} \sin\left(W_n\sqrt{1-(DR)^2}\, t + A_{(ph)1}\right)\right\}$$

$$+ \left([(C\dot{)}V]_{(t,i)}\right)_o \left\{\left(\frac{2(DR)W_n}{\frac{1}{(CT)^2} - \frac{2(DR)W_n}{(CT)} + W_n^2}\right) e^{-\frac{t}{(CT)}}\right.$$

$$+ \frac{1}{W_n\sqrt{1-(DR)^2}} \left[\left(\frac{\frac{1}{(CT)^2} + \frac{2(DR)W_n}{(CT)} + W_n^2}{\frac{1}{(CT)^2} - \frac{2(DR)W_n}{(CT)} + W_n^2}\right)\right]^{\frac{1}{2}}$$

$$\left.\times e^{-(DR)W_n t} \sin\left(W_n\sqrt{1-(DR)^2}\, t + A_{(ph)2}\right)\right\}$$

$$+ \left([(\dot{C})V]_{(t,i)} \right)_o \left\{ \left(\frac{1}{\frac{1}{(CT)^2} - \frac{2(DR)W_n}{(CT)} + W_n^2} \right) e^{-\frac{t}{(CT)}} \right.$$

$$+ \frac{1}{W_n(1 - (DR)^2} \left(\frac{1}{\frac{1}{(CT)^2} - \frac{2(DR)W_n}{(CT)} + W_n^2} \right)^{\frac{1}{2}}$$

$$\left. \times e^{-(DR)W_n t} \sin W_n \sqrt{1 - (DR)^2}\, t - A_{(ph)3} \right\} \qquad (1\text{–}29)$$

where

$$A_{(ph)1} = \tan^{-1} \frac{\sqrt{1 - (DR)^2}}{(DR)} - \tan^{-1} \frac{W_n \sqrt{1 - (DR)^2}}{\frac{1}{CT} - (DR)W_n}$$

$$A_{(ph)2} = \tan^{-1} \frac{W_n \sqrt{1 - (DR)^2}}{(DR)W_n + \frac{1}{CT}} - \tan^{-1} \frac{W_n \sqrt{1 - (DR)^2}}{\frac{1}{CT} - (DR)W_n}$$

$$A_{(ph)3} = \tan^{-1} \frac{W_n \sqrt{1 - (DR)^2}}{\frac{1}{CT} - (DR)W_n}$$

Practical values for (DR), W_n, and (CT) are given in part B, Eq. (1–26). If these are substituted in Eq. (1–29), the result gives the transient response:

$$[(C)V]_{(t,i)} = \left([(C)V]_{(t,i)} \right)_o \{ 1.078\ e^{-1.950 \times 10^{-4}\, t} + 0.18\ e^{-3.54 \times 10^{-4}\, t} \sin(.001126t$$

$$- 0.437) \} + \left([(C)\dot{V}]_{(t,i)} \right)_o \{ 549\ e^{-1.950 \times 10^{-4}\, t}$$

$$+ 976\ e^{-3.54 \times 10^{-4}\, t} \sin(.001126t - 0.594) \}$$

$$+ \left([(C)\ddot{V}]_{(\cdot,i)} \right)_o \{ 775000\ e^{-1.950 \times 10^{-4}\, t}$$

$$+ 78100\ e^{-3.54 \times 10^{-4}\, t} \sin(.001126t - 1.71) \} \qquad (1\text{–}30)$$

D. EFFECT OF x AND y COUPLING.

In part A if it is assumed that $i_{(int)} = i_{(int)x}$ and $(U)W_{(gu)} = 0$,

$$i_{(int)x} = \frac{(W_{(IV)i})_x}{S_{(cmds)(i,W)x}} = \frac{1}{S_{(cmds)(i,W)x}} \{W_{(IV)x} - p\,[(C)V]_{(t,i)x}\}$$

$$= \frac{1}{S_{(cmds)(i,W)x}} \{[W_{IE} + (\dot{Lon})_t]\cos(Lat)_t - p\,[(C)V]_{(t,i)x}\} \tag{1-31}$$

This applies to a single-axis system. In a two-axis system, the Earth rate couples the vertical indicators as follows. Define

$$i_{(int)x} \equiv i_x \tag{1-32}$$

Then

$$i_x = \frac{1}{S_{(cmds)(i,W)x}} \{[W_{IE} + (\dot{Lon})_t]\cos(Lat)_t - [W_{IE} + (\dot{Lon})_t]\,[(C)V]_{(t,i)y}\sin(Lat)_t$$

$$- p\,[(C)V]_{(t,i)x}\} \tag{1-33}$$

where

$$(\dot{Lon})_t\cos(Lat)_t - (\dot{Lon})_t\,[(C)V]_{(t,i)y}\sin(Lat)_t = W_{(EV)t\,x} \tag{1-34}$$

An analogous situation exists for the y-system

$$i_y = \frac{1}{S_{(cmds)(i,W)y}} \{-(\dot{Lat})_t + [W_{IE} + (\dot{Lon})]_t\,[(C)V]_{(t,i)x}\sin(Lat)_t - p\,[(C)V]_{(t,i)y}\} \tag{1-35}$$

where

$$-(\dot{Lat})_t + (\dot{Lon})_t\,[(C)V]_{(t,i)x}\sin(Lat)_t = W_{(EV)ty} \tag{1-36}$$

The x- and y-systems are thus coupled through two terms, each proportional to $[W_{IE} + (\dot{Lon})_t]\sin(Lat)_t$. for purposes of the coupling effect analysis given in the text, assume that

$$\left.\begin{aligned}(\dot{Lon})_t &= 0 \quad (1\text{-}37)\\ (\dot{Lat})_t &= 0 \quad (1\text{-}38)\end{aligned}\right\}\ \text{i.e., the system base is stationary on the Earth.}$$

$$S_{(cmds)(i,W)x} = S_{(cmds)(i,W)y} \tag{1-39}$$

Inserting these values of i_x and i_y into the development in part A, the two performance equations in x and y (ignoring uncertainties in component outputs as well as inaccuracies in system inputs) are

$$(p + \frac{1}{(CT)})(p^2 + 2(DR)W_n p + W_n^2)[(C)V]_{(t,i)y} = \{p^2 + [2(DR)W_n + \frac{1}{(CT)}]p\}W_{IE} \sin(Lat)_i [(C)V]_{(t,i)x} \quad (1-40)$$

$$(p + \frac{1}{(CT)})(p^2 + 2(DR)W_n p + W_n^2)[(C)V]_{(t,i)x} = - \{p^2 + [2(DR)W_n + \frac{1}{(CT)}]p\} W_{IE} \sin(Lat)_i [(C)V]_{(t,i)y} \quad (1-41)$$

It is on these two equations that Fig. 7 in the text is based.

DERIVATION 2. SINGLE-AXIS POSITION INDICATORS

A. OPEN-CHAIN INTEGRATION OF THE ANGULAR VELOCITY OF THE INDICATED VERTICAL

Refer to Figure 2-1. The indicated position is given by:

$$P_{(i)} = S_{(Pi)(i\dot{P})} \frac{1}{p} i_{(hfi)} + \frac{1}{p} (U) \dot{P}_{(i)(df)} + (P_i)_o \tag{2-1}$$

From Eq. (1-8) in Der. 1A,

$$i_{(int)} = \frac{1}{S_{(cmds)(i,W)}} [W_{(EV)i} - (U) W_{(gu)} - (I) W_{(IE)(cp)}] \tag{2-2}$$

$$W_{(EV)i} = W_{(EV)t} - p \, [(C)V]_{(t,i)} \tag*{[1-9]}$$

set

$$S_{(Pi)(i,\dot{P})} = S_{(cmds)(i,W)} \tag{2-3}$$

(This constitutes an arbitrary adjustment of system parameters.)

Note that

$$\frac{1}{p} W_{(EV)t} = P_t - (P_t)_o \tag{2-4}$$

and

$$\frac{1}{p} p \, [(C)V]_{(t,i)} = [(C)V]_{(t,i)} - [(C)V]_{(t,i)o} \tag{2-5}$$

From the foregoing, Eq. (2-1) may be expressed as follows:

$$\begin{aligned} P_i = P_t - (P_t)_o - [(C)V]_{(t,i)} + [(C)V]_{(t,i)o} \\ - \frac{1}{p} [(U) W_{(gu)} + (I) W_{(IE)(cp)}] + (P_i)_o + \frac{1}{p} (U) \dot{P}_{(i)(df)} \end{aligned} \tag{2-6}$$

Define the correction to the indicated position as follows:

$$[(C)P]_{(t,i)} \equiv P_t - P_i \tag{2-7}$$

At the start of the navigation problem, this becomes

$$[(C)P]_{(t,i)o} = (P_t)_o - (P_i)_o \tag{2-8}$$

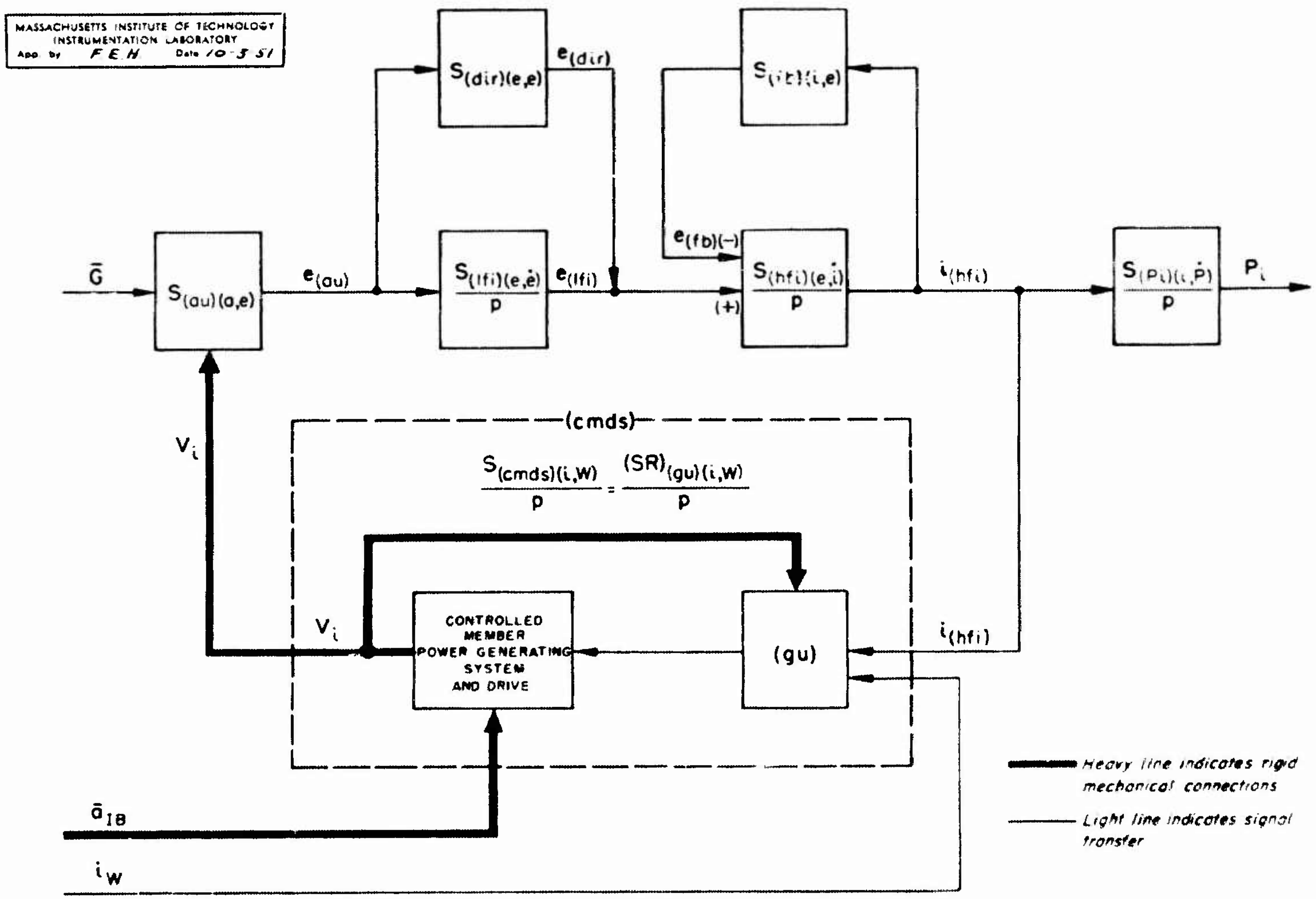

Fig. 2—1. Mathematical functional diagram of a damped system for indicating position by means of open—chain integration of the angular velocity of the indicated vertical.

Substitute Eq. (2-7) and (2-8) in (2-6). The result is

$$[(C)P]_{(t,i)} = [(C)P]_{(t,i)o} + [(C)V]_{(t,i)} - [(C)V]_{(t,i)o}$$

$$+ \frac{1}{p}\,[(U)W_{(gu)} + (I)W_{(IE)(cp)} - (U)\dot{P}_{(i)(df)}] \tag{2-9}$$

B. OPEN-CHAIN INTEGRATION OF THE ANGULAR ACCELERATION OF THE INDICATED VERTICAL

Refer to Fig. 2-2. The vertical-indicating function of this system has been analyzed in Derivation 1. The position-indicating function is analyzed as follows. From Fig. 2-2, (as in Derivation 1, Eq. (1-3))

$$i_{(hfi)} = S_{(hfi)(e,\dot{i})}\frac{1}{p}[e_{(dir)} + e_{(lfi)} + e_{(cp)} - e_{(fb)}] + \frac{1}{p}(U)\dot{i}_{(hfi)(df)} + i_{(hfi)o} \tag{2-10}$$

$$i_{(hfi)o} = \frac{1}{S_{(cmds)(i,\dot{W})}}\,W_{(EV)s} \tag{2-11}$$

where $W_{(EV)s}$ is the initial setting, in the second integrator in the vertical indicator, of the angular velocity of the indicated vertical with respect to the earth. Also, from the figure,

$$e_{(fb)} = S_{(fb)(i,e)}i_{(hfi)} + (U)e_{(fb)} \tag{2-12}$$

therefore

$$[1 + S_{(hfi)(e,\dot{i})}S_{(fb)(i,e)}\frac{1}{p}]\,i_{(hfi)} = S_{(hfi)(e,\dot{i})}\frac{1}{p}[e_{(dir)} + e_{(lfi)}$$

$$+ e_{(cp)} - (U)e_{(fb)}] + \frac{1}{p}(U)\dot{i}_{(hfi)(df)} + \frac{1}{S_{(cmds)(i,\dot{W})}}\,W_{(EV)s} \tag{2-13}$$

Also, from Fig. 3-2,

$$e_{(dir)} = S_{(dir)(e,e)}e_{(au)} + (U)e_{(dir)} \tag{2-14}$$

$$e_{(lfi)} = S_{(lfi)(e,\dot{e})}\frac{1}{p}e_{(au)} + \frac{1}{p}(U)\dot{e}_{(lfi)(df)} + e_{(lfi)o} \tag{2-15}$$

The constant $e_{(lfi)o}$ can be evaluated from Eq. (2-10), with initial conditions inserted such that the input to hfi (high frequency integrator) is nulled:

$$e_{(lfi)o} = \frac{S_{(fb)(i,e)}}{S_{(cmds)(i,\dot{W})}}\,W_{(EV)s} - S_{(dir)(e,e)}e_{(au)o} - (U)e_{(dir)o} - e_{(cp)o} + (U)e_{(fb)o} \tag{2-16}$$

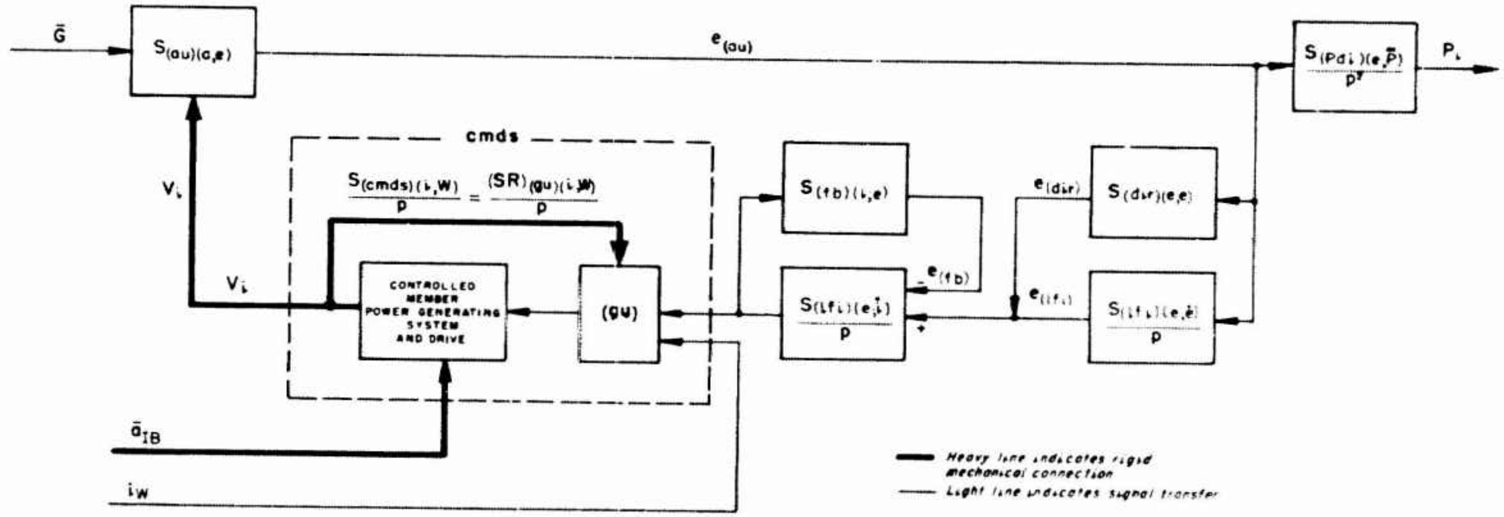

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Fig. 2–2. Mathematical functional diagram of damped system for indicating position by means of open–chain double integration of the angular acceleration of the indicated vertical.

Substituting Eq. (2-14) and (2-15) into (2-13) there results

$$
\begin{aligned}
&[1 + S_{(hfi)(e,\dot{i})} S_{(fb)(i,e)} \frac{1}{p}]\, i_{(hfi)} = S_{(hfi)(e,\dot{i})} \frac{1}{p} [S_{(dir)(e,e)} \\
&+ S_{(lfi)(e,\dot{e})} \frac{1}{p}]\, e_{(au)} + S_{(hfi)(e,\dot{i})} \frac{1}{p} [e_{(cp)} - (U)\, e_{(fb)}] \\
&+ S_{(hfi)(e,\dot{i})} \frac{1}{p} \{(U)\, e_{(dir)} + \frac{1}{p} (U)\, \dot{e}_{(lfi)(df)} \\
&+ \frac{S_{(fb)(i,e)}}{S_{(cmds)(i,W)}} W_{(FV)s} - S_{(dir)(e,e)}\, e_{(au)o} - (U) e_{(dir)o} - e_{(cp)o} \\
&+ [(U) e_{(fb)}]_o \} + \frac{1}{p} (U) \dot{i}_{(hfi)(df)} + \frac{1}{S_{(cmds)(i,W)}} W_{(EV)s}
\end{aligned}
\qquad (2-17)
$$

Define

$$\frac{1}{(CT)_{(lg)}} = S_{(hfi)(e,\dot{i})}\, S_{(fb)(i,e)} \qquad (2-18)$$

$$\frac{1}{(CT)_{(ld)}} = \frac{S_{(lfi)(e,\dot{e})}}{S_{(dir)(e,e)}} \qquad (2-19)$$

The gyro input is (cf. Eq. (1-2) and (1-3))

$$i_{(hfi)} = \frac{1}{S_{(cmds)(i,W)}} \{W_{(EV)t} - p\,[(C)\,V]_{(t,i)} - (U)\, W_{(gu)} - (I) W_{(lE)(cp)}\} \qquad (2-20)$$

Solving for $e_{(au)}$, Eq. (2-17) becomes

$$
\begin{aligned}
&S_{(hfi)(e,\dot{i})}\, S_{(dir)(e,e)} [\frac{1}{p} + \frac{1}{(CT)_{(ld)}} \frac{1}{p^2}]\, e_{(au)} \\
&= \frac{1}{S_{(cmds)(i,W)}} [1 + \frac{1}{(CT)_{(lg)}} \frac{1}{p}]\, [W_{(EV)t} - p\,[(C)\,V]_{(t,i)} - (U)\, W_{(gu)} \\
&-(I) W_{(lE)(cp)}] - S_{(hfi)(e,\dot{i})} \frac{1}{p} \{e_{(cp)} - e_{(cp)o} - (U)\, e_{(fb)} \\
&+ (U) e_{(fb)o} + (U)\, e_{(dir)} - (U) e_{(dir)o} - S_{(dir)(e,e)}\, e_{(au)o} \\
&+ \frac{1}{p} (U) \dot{e}_{(lfi)(df)} \} - \frac{1}{(CT)_{(lg)}\, S_{(cmds)(i,W)}} \frac{1}{p} W_{(EV)s} \\
&- \frac{1}{p} (U) \dot{i}_{(hfi)(df)} - \frac{1}{S_{(cmds)(i,W)}} W_{(EV)s}
\end{aligned}
\qquad (2-21)
$$

From Fig. (2-2), the indicated position is

$$P_i = S_{(Pdi)(e,\ddot{P})}\frac{1}{p^2}e_{(au)} + (\dot{P}_i)_s t + (P_i)_o + \frac{1}{p^2}(U)\ddot{P}_{(di)(df)} + \frac{1}{p}(U)\dot{P}_{(di)(df)} \tag{2-22}$$

where $(\dot{P}_i)_s$ is the initial setting in the position integrator of the rate of change of position.

$$\frac{1}{p^2}e_{(au)} = \frac{1}{S_{(Pdi)(e,\ddot{P})}}\{P_i - (P_i)_o - (\dot{P}_i)_s t + \frac{1}{p^2}(U)\ddot{P}_{(di)(df)} - \frac{1}{p}(U)\dot{P}_{(di)(df)}\} \tag{2-23}$$

$$\frac{1}{p}e_{(au)} = \frac{1}{S_{(Pdi)(e,\ddot{P})}}\{p(P_i) - (\dot{P}_i)_s - \frac{1}{p}(U)\ddot{P}_{(di)(df)} - (U)\dot{P}_{(di)(df)}\} \tag{2-24}$$

$$S_{(hfi)(e,i)}S_{(dir)(e,e)}[\frac{1}{p} + \frac{1}{(CT)_{(ld)}}\frac{1}{p^2}]e_{(au)}$$

$$= \frac{S_{(hfi)(e,i)}S_{(dir)(e,e)}}{S_{(Pdi)(e,\ddot{P})}}\{(p + \frac{1}{(CT)_{(ld)}})P_i - \frac{1}{(CT)_{(ld)}}(P_i)_o$$

$$-(\dot{P}_i)_s - \frac{1}{(CT)_{(ld)}}(\dot{P}_i)_s t - (p + \frac{1}{(CT)_{(ld)}})[\frac{1}{p^2}(U)\ddot{P}_{(di)(dt)}$$

$$+ \frac{1}{p}(U)\dot{P}_{(di)(df)}]\} \tag{2-25}$$

From Eq. (2-21),

$$(p + \frac{1}{(CT)_{(ld)}})P_i - \frac{1}{(CT)_{(ld)}}(P_i)_o - (p + \frac{1}{(CT)_{(ld)}})(\dot{P}_i)_s t$$

$$-(p + \frac{1}{(CT)_{(ld)}})[\frac{1}{p^2}(U)\ddot{P}_{(di)(df)} + \frac{1}{p}(U)\dot{P}_{(di)(df)}]$$

$$= \frac{S_{(Pdi)(e,\ddot{P})}}{S_{(hfi)(e,i)}S_{(dir)(e,e)}S_{(cmds)(i,W)}}\{W_{(EV)t} + \frac{1}{(CT)_{(lg)}}(P_t - (P_t)_o)$$

$$- p[(C)V]_{(t,i)} - \frac{1}{(CT)_{(lg)}}([(C)V]_{(t,i)} - [(C)V]_{(t,i)o})$$

$$-(1 + \frac{1}{(CT)_{(lg)}}\frac{1}{p})[(U)W_{(gu)} + (I)W_{(IE)(cp)}] - W_{(EV)s}$$

$$-\frac{1}{(CT)_{(lg)}}\frac{1}{p}W_{(EV)s}\} - \frac{S_{(Pdi)(e,\ddot{P})}}{S_{(dir)(e,e)}}\frac{1}{p}\{e_{(cp)} - e_{(cp)o}$$

$$-(U)e_{(fb)} + (U)e_{(fb)o} + (U)e_{(dir)} - (U)e_{(dir)o} - S_{(dir)(e,e)}e_{(au)o}$$

$$+\frac{1}{p}(U)\dot{e}_{(lfi)(df)}) - \frac{S_{(Pdi)(e,\dot{P})}}{S_{(hfi)(e,\dot{i})}\,S_{(dir)(e,e)}}\,\frac{1}{p}[(U)\dot{i}_{(hfi)(df)}] \tag{2-26}$$

It will be assumed that the system be initially aligned under conditions of zero acceleration; then

$$e_{(au)o} = S_{(au)(a,e)}\,g\,[(C)V]_{(t,i)o} + (U)e_{(au)o} \tag{2-27}$$

Let

$$S_{(cmds)(i,W)}\,S_{(hfi)(e,\dot{i})}\,e_{(cp)} = \left(\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\right)W_{(EV)(cp)} \tag{2-28}$$

The position indicator is calibrated by setting

$$S_{(Pdi)(e,\dot{P})} = S_{(hfi)(e,\dot{i})}\,S_{(dir)(e,e)}\,S_{(cmds)(i,W)} \tag{2-29}$$

Substituting the above into Eq. (2–26), there results

$$\begin{aligned}
&\left(p + \frac{1}{(CT)_{(ld)}}\right)(P_i - P_t) - \frac{1}{(CT)_{(ld)}}[(P_i)_o - (P_t)_o] = \left[\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\right](P_t - (P_t)_o) \\
&\quad - W_{(EV)s} - \frac{1}{(CT)_{(lg)}} W_{(EV)s}t + (\dot{P}_i)_s + \frac{1}{(CT)_{(ld)}}(\dot{P}_i)_s t \\
&\quad - \left(\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\right)\frac{1}{p}[W_{(EV)(cp)} - W_{(EV)(cp)o}] - \left(p + \frac{1}{(CT)_{(lg)}}\right)[(C)V]_{(t,i)} \\
&\quad + \left(\frac{1}{(CT)_{(lg)}} + S_{(Pdi)(e,\ddot{P})}\,S_{(au)(a,e)}\,g\,\frac{1}{p}[(C)V]_{(t,i)o}\right. \\
&\quad - \left(1 + \frac{1}{(CT)_{(lg)}}\,\frac{1}{p}\right)[(U)W_{(gu)} + (I)W_{(IE)(cp)}] \\
&\quad - S_{(hfi)(e,\dot{i})}\,S_{(cmds)(i,W)}\,\frac{1}{p^2}\,p[(U)e_{(dir)} - (U)e_{(fb)}] \\
&\quad + S_{(Pdi)(e,\ddot{P})}\,\frac{1}{p}(U)e_{(au)o} - S_{(hfi)(e,\dot{i})}\,S_{(cmds)(i,W)}\,\frac{1}{p^2}(U)\dot{e}_{(lfi)(df)} \\
&\quad - S_{(cmds)(i,W)}\,\frac{1}{p}(U)\dot{i}_{(hfi)(df)} + \left(p + \frac{1}{(CT)_{(ld)}}\right)\left[\frac{1}{p^2}(U)\ddot{P}_{(di)(df)}\right. \\
&\quad \left. + \frac{1}{p}(U)\dot{P}_{(di)(df)}\right]
\end{aligned} \tag{2-30}$$

Define

$$[(C)P]_{(t,i)} \equiv P_t - P_i \tag{2-31}$$

$$
\begin{aligned}
(p + \frac{1}{(CT)_{(1d)}})[(C)P]_{(t,i)} &= \frac{1}{(CT)_{(1d)}}[(C)P]_{(t,i)o} - [\frac{1}{(CT)_{(1g)}} - \frac{1}{(CT)_{(1d)}}]\frac{1}{p}(W_{(EV)t} \\
&- W_{(EV)(cp)}) + \frac{1}{(CT)_{(1g)}}(W_{(EV)s}t - W_{(EV)(cp)o}t) \\
&- \frac{1}{(CT)_{(1d)}}((\dot{P}_i)_s t - W_{(EV)(cp)o}t) + (p + \frac{1}{(CT)_{(1g)}})[(C)V]_{(t,i)} \\
&- (\frac{1}{(CT)_{(1g)}} + S_{(au)(a,e)}S_{(hfi)(e,\dot{i})}S_{(dir)(e,e)}S_{(cmds)(i,W)}\,g\,\frac{1}{p})[(C)V]_{(t,i)o} \\
&+ (1 + \frac{1}{(CT)_{(1g)}}\frac{1}{p})[(U)W_{(gu)} + (I)W_{(IE)(cp)}] \\
&+ S_{(hfi)(e,\dot{i})}S_{(cmds)(i,W)}\frac{1}{p}[(U)e_{(dir)} - (U)e_{(dir)o} - (U)e_{(fb)} \\
&+ (U)e_{(fb)o} + \frac{1}{S_{(hfi)(e,\dot{i})}}(U)\dot{i}_{(hfi)(df)} - S_{(Pdi)(e,\ddot{P})}\frac{1}{p}(U)e_{(au)o} \\
&+ S_{(hfi)(e,\dot{i})}S_{(cmds)(i,W)}\frac{1}{p^2}(U)\dot{e}_{(1fi)(df)} \\
&- (p + \frac{1}{(CT)_{(1d)}})[\frac{1}{p^2}(U)\ddot{P}_{(di)(df)} + \frac{1}{p}(U)\dot{P}_{(di)(df)}]
\end{aligned} \tag{2-32}
$$

Since $W_{(EV)s}$ and $(\dot{P}_i)_s$ both represent the best information on the initial velocity of the base at zero time,

$$W_{(EV)s} = (\dot{P}_i)_s \tag{2-33}$$

Define the inaccuracy in the initial position rate setting as follows :

$$(I)W_{(EV)s} \equiv W_{(EV)s} - (W_{(EV)t})_o \tag{2-34}$$

and the inaccuracy in the velocity compensation as

$$(I)W_{(EV)(cp)} \equiv W_{(EV)(cp)} - W_{(EV)t} \tag{2-35}$$

From Eq. (1-9) and (1-13), note that

$$S_{(au)(a,e)}S_{(hfi)(e,\dot{i})}S_{(dir)(e,e)}S_{(cmds)(i,W)} = \frac{1}{R_s} \tag{2-36}$$

$$
\begin{aligned}
&\left(p + \frac{1}{(CT)_{(ld)}}\right)[(C)P]_{(t,i)} = \frac{1}{(CT)_{(ld)}}[(C)P]_{(t,i)o} - \frac{1}{(CT)_{(lg)}}[(C)V]_{(t,i)o} \\
&\quad + \left(p + \frac{1}{(CT)_{(lg)}}\right)[(C)V]_{(t,i)} + (U)W_{(gu)} + (I)W_{(IE)(cp)} \\
&\quad - (U)\dot{P}_{(di)(df)} + \frac{1}{p}\{\left(\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\right)[(I)W_{(EV)(cp)} \\
&\quad - (I)W_{(EV)s}] - \frac{g}{R_s}[(C)V]_{(t,i)o} + \frac{1}{(CT)_{(lg)}}[(U)W_{(gu)} \\
&\quad + (I)W_{(IE)(cp)}] + S_{(hfi)(e,i)}S_{(cmds)(i,W)}[(U)e_{(dir)} \\
&\quad - (U)e_{(dir)o} - (U)e_{(fb)} + (U)e_{(fb)o}] + S_{(cmds)(i,W)}(U)\dot{i}_{(hfi)(df)} \\
&\quad - \frac{1}{R_s S_{(au)(a,e)}}(U)e_{(au)a} - (U)\ddot{P}_{(di)(df)} \\
&\quad - \frac{1}{(CT)_{(ld)}}(U)\dot{P}_{(di)(df)}\} + \frac{1}{p^2}\{S_{(hfi)(e,i)}S_{(cmds)(i,W)}(U)\dot{e}_{(lfi)(df)} \\
&\quad - \frac{1}{(CT)_{(ld)}}(U)\ddot{P}_{(di)(df)} \qquad (2\text{-}37)
\end{aligned}
$$

C. APPLICATION OF DOUBLY-INTEGRATING ACCELEROMETER TO POSITION INDICATION.

Refer to Fig. 2-3. The vertical-indicating function of this system can be analyzed by a method similar to that of Derivation 1 to arrive at a differential performance equation in $[(C)V]_{(t,i)}$ when Schuler tuning is used to minimize the jerk error. The position-indicating function of this system is analyzed as follows. From Fig. 2-3,

$$i_{(dir)2} = S_{(dir)2(e,i)}\, e_{(dir)1} + (U)_{(dir)2} \tag{2-38}$$

$$i_{(diff)} = S_{(diff)(\dot{e},i)}\, p\, e_{(dir)1} + (U)i_{(diff)} \tag{2-39}$$

$$e_{(int)} = S_{(int)(e,\dot{e})} \frac{1}{p} e_{(dir)1} + \frac{1}{p}(U)\dot{e}_{(int)(df)} + e_{(int)o} \tag{2-40}$$

$$\begin{aligned} e_{(dir)1} &= S_{(dir)1(e,e)} [e_{(dia)} - e_{(int)}] + (U)e_{(dir)1} \\ &= S_{(dir)1(e,e)}\, e_{(dia)} - S_{(dir)1(e,e)} S_{(int)(e,\dot{e})} \frac{1}{p} e_{(dir)1} \\ &\quad - S_{(dir)1(e,e)} \frac{1}{p}(U)\dot{e}_{(int)(df)} - S_{(dir)1(e,e)}\, e_{(int)o} + (U)e_{(dir)1} \end{aligned} \tag{2-41}$$

$$i_{(cp)} = (PF)_{(cp)(W,i)}\, W_{(EV)(cp)} \tag{2-42}$$

$$\begin{aligned} e_{(dia)} &= S_{(int)(e,\dot{e})} \left[\frac{1}{S_{(dir)1(e,e)} S_{(int)(e,\dot{e})}} + \frac{1}{p}\right] e_{(dir)1} \\ &\quad + \frac{1}{p}(U)\dot{e}_{(int)(df)} + e_{(int)o} - \frac{1}{S_{(dir)1(e,e)}}(U)e_{(dir)1} \end{aligned} \tag{2-43}$$

Define

$$(CT)_{(lg)} \equiv \frac{1}{S_{(dir)1(e,e)} S_{(int)(e,\dot{e})}} \tag{2-44}$$

$$(CT)_{(ld)} \equiv \frac{S_{(diff)(\dot{e},i)}}{S_{(dir)2(e,i)}} \tag{2-45}$$

Operating on Eq. (2-43) with $(1 + (CT)_{(ld)}p)$ and substituting Eq. (2-44) and (2-45), the result is

$$\begin{aligned} [1 + (CT)_{(ld)}p]e_{(dia)} &= S_{(int)(e,\dot{e})} [(CT)_{(lg)} + \frac{1}{p}] \{\frac{1}{S_{(dir)2(e,i)}} [i_{(dir)2} \\ &\quad + i_{(diff)} - (U)i_{(dir)2} - (U)i_{(diff)}]\} + S_{(int)(e,\dot{e})} (CT)_{(ld)} [e_{(dir)1}]_o \\ &\quad + [1 + (CT)_{(ld)}p] \frac{1}{p}(U)\dot{e}_{(int)(df)} + [1 + (CT)_{(ld)}p] \{e_{(int)o} \\ &\quad - \frac{1}{S_{(dir)1(e,e)}}(U)e_{(dir)}\} \end{aligned} \tag{2-46}$$

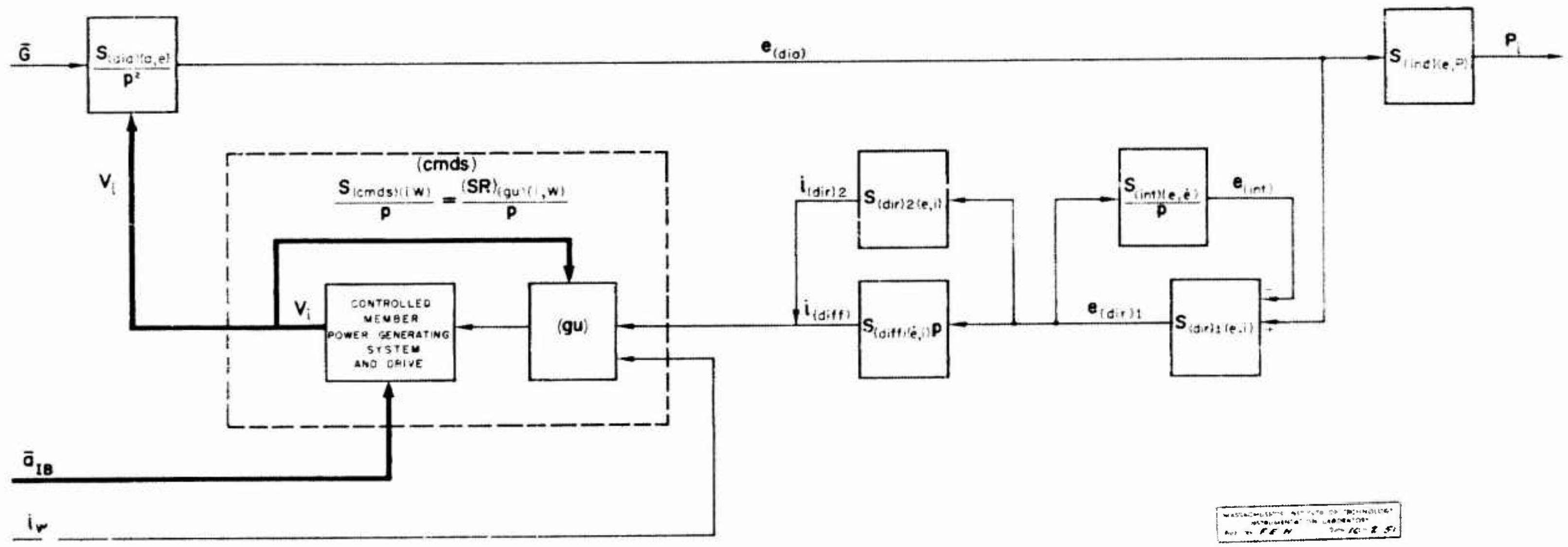

Fig. 2–3. Mathematical functional diagram of a damped system for indicating position by means of a doubly–integrating accelerometer unit.

From Eq. (2–41)

$$e_{(dia)o} = \frac{1}{S_{(dir)1(e,e)}} \left([e_{(dir)1}]_o - [(U)\,e_{(dir)1}]_o \right) + e_{(int)o} \tag{2-47}$$

Multiplying Eq. (2–46) by $\frac{1}{(CT)_{(ld)}}$ and substituting Eq. (2–47), there results

$$\begin{aligned}(p + \frac{1}{(CT)_{(ld)}})\, e_{(dia)} &= \frac{1}{(CT)_{(lg)}} e_{(dia)o} + (\frac{1}{(CT)_{(ld)}} - \frac{1}{(CT)_{(lg)}})\, e_{(int)o} \\ &+ \frac{1}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}} (1 + \frac{1}{(CT)_{(lg)}} \frac{1}{p}) [i_{(dir)2} \\ &+ i_{(diff)} - (U)i_{(dir)2} - (U)i_{(diff)}] + (p + \frac{1}{(CT)_{(ld)}})(\frac{1}{p}(U)\,\dot{e}_{(int)(df)}) \\ &- \frac{1}{S_{(dir)1(e,e)}} (U)\,e_{(dir)1} + S_{(int)(e,\dot{e})} [(U)\,e_{(dir)1}]_o\end{aligned} \tag{2-48}$$

At $t = 0$,

$$\begin{aligned}\dot{e}_{(dia)o} + [\frac{1}{(CT)_{(ld)}} - \frac{1}{(CT)_{(lg)}}]\, e_{(dia)o} &= [\frac{1}{(CT)_{(ld)}} - \frac{1}{(CT)_{(lg)}}]\, e_{(int)o} \\ &+ \frac{1}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}} \{ [i_{(dir)2}]_o + i_{(diff)o} - [(U)i_{(dir)2}]_o \\ &- (U)\,i_{(diff)o}\} + (U)\,\dot{e}_{(int)(df)o} - \frac{1}{S_{(dir)1(e,e)}} ((U)\,\dot{e}_{(dir)1})_o + (S_{(int)(e,\dot{e})} \\ &- \frac{1}{(CT)_{(ld)}\, S_{(dir)1(e,e)}}) [(U)\,e_{(dir)1}]_o\end{aligned} \tag{2-49}$$

The gyro input is

$$i_{(gu)} = i_{(dir)2} + i_{(diff)} + i_{(cp)} \tag{2-50}$$

$$i_{(gu)} = \frac{1}{S_{(cmds)(i,W)}} (W_{(EV)t} - p\,[(C)\,V]_{(t,i)} - (U)\,W_{(gu)} - (I)\,W_{(IE)(cp)}) \tag{2-51}$$

At zero time, it is desired that

$$i_{(gu)o} = \frac{W_{(EV)s}}{S_{(cmds)(i,W)}} \tag{2-52}$$

and

$$i_{(cp)} = 0 \tag{2-53}$$

Here, $W_{(EV)s}$ is the best estimate of the velocity of the base with respect to the Earth at that instant. Eq. (2–49) now becomes

$$\left[\frac{1}{(CT)_{(ld)}} - \frac{1}{(CT)_{(lg)}}\right] e_{(int)o} = \left[\frac{1}{(CT)_{(ld)}} - \frac{1}{(CT)_{(lg)}}\right] e_{(dia)o} + \dot{e}_{(dia)o}$$

$$- \frac{W_{(EV)s}}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}\, S_{(cmds)(i,W)}} + \left[\frac{1}{(CT)_{(ld)}}\right.$$

$$\left. - \frac{1}{(CT)_{(lg)}}\right] \frac{1}{S_{(dir)1(e,e)}} [(U) e_{(dir)1}]_o + \frac{1}{S_{(dir)1(e,e)}} ((U)\dot{e}_{(dir)1})_o$$

$$- (U)\dot{e}_{(int)(df)o} + \frac{1}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}} ([(U)i_{(dir)2}]_0 + (U)i_{(diff)o}) \qquad (2\text{–}54)$$

Substituting the above into Eq. (2–48), there results,

$$(p + \frac{1}{(CT)_{(ld)}}) e_{(dia)} = \frac{1}{(CT)_{(ld)}} e_{(dia)o} + \dot{e}_{(dia)o}$$

$$+ \frac{1}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}} (1 + \frac{1}{(CT)_{(ld)}} \frac{1}{p}) (i_{(gu)} - i_{(cp)})$$

$$- \frac{W_{(EV)s}}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}\, S_{(cmds)(i,W)}} - \frac{1}{S_{(dir)1(e,e)}} \{ p(U) e_{(dir)1}$$

$$- [(U)\dot{e}_{(dir)1}]_o + \frac{1}{(CT)_{(ld)}} [(U) e_{(dir)1} - [(U) e_{(dir)1}]_o \}$$

$$+ \frac{1}{(CT)_{(ld)}} \frac{1}{p} [(U)\dot{e}_{(int)(df)}] + (U)\dot{e}_{(int)(df)} - (U)\dot{e}_{(int)(df)o}$$

$$- \frac{1}{S_{(dir)1(e,e)}\, S_{(diff)(\dot{e},i)}} \left[\frac{1}{(CT)_{(lg)}} \frac{1}{p} \{(U)i_{(dir)2} + (U)i_{(diff)}\} + (U)i_{(dir)2}\right.$$

$$\left. - [(U) i_{(dir)2}]_o + (U) i_{(diff)} - (U)i_{(diff)o}\right] \qquad (2\text{–}55)$$

From Fig. 2–3, the indicated position is

$$P_i = S_{(ind)(e,P)}\, e_{(dia)} + (U)P_i \qquad (2\text{–}56)$$

$$[p + \frac{1}{(CT)_{(ld)}}] P_i = \frac{1}{(CT)_{(ld)}} (P_i)_o + (\dot{P}_i)_o$$

$$+\frac{S_{(ind)(e,P)}}{S_{(dir)1(e,e)}S_{(diff)(\dot{e},i)}S_{(cmds)(i,W)}}\left[\left(1+\frac{1}{(CT)_{(1g)}}\frac{1}{p}\right)(W_{(EV)t}-p\,[(C)V]_{(t,i)}\right.$$

$$\left.-(U)W_{(gu)}-(I)W_{(1E)(cp)}-S_{(cmds)}\,i_{(cp)}-W_{(EV)s}\right]-\frac{S_{(ind)(e,P)}}{S_{(dir)1(e,e)}}\{p(U)e_{(dir)1}$$

$$-[(U)\dot{e}_{(dir)1}]_o+\frac{1}{(CT)_{(1d)}}[(U)e_{(dir)1}]-[(U)e_{(dir)1}]_o\}$$

$$+S_{(ind)(e,P)}\left\{\frac{1}{(CT)_{(1d)}}\frac{1}{p}(U)\dot{e}_{(int)(df)}+(U)\dot{e}_{(int)(df)}-(U)\dot{e}_{(int)(df)o}\right\}$$

$$-\frac{S_{(ind)(e,P)}}{S_{(dir)1(e,e)}S_{(diff)(\dot{e},i)}}\left\{\frac{1}{(CT)_{(1g)}}\frac{1}{p}[(U)i_{(dir)2}+(U)i_{(diff)}]+(U)i_{(dir)2}\right.$$

$$\left.-[(U)i_{(dir)2}]_o+(U)i_{(diff)}-(U)i_{(diff)o}\right\} \tag{2-57}$$

The best available estimate of the velocity of the base with respect to the Earth is $W_{(EV)s}$. Therefore set

$$(\dot{P}_i)_o = W_{(EV)s} + (\dot{P}_i)_{(dt)o} \tag{2-58}$$

set

$$\left(1+\frac{1}{(CT)_{(1g)}}\frac{1}{p}\right)S_{(cmds)(i,W)}\,i_{(cp)} = \left(\frac{1}{(CT)_{(1g)}}-\frac{1}{(CT)_{(1d)}}\right)\frac{1}{p}W_{(EV)(cp)} \tag{2-59}$$

The position indicating system is calibrated by setting

$$S_{(ind)(e,P)} = S_{(dir)1(e,e)}\,S_{(diff)(\dot{e},i)}\,S_{(cmds)(i,W)} \tag{2-60}$$

then

$$\left(p+\frac{1}{(CT)_{(1d)}}\right)P_i = \frac{1}{(CT)_{(1d)}}(P_i)_o + W_{(EV)s} + (\dot{P}_i)_{(df)o} + W_{(EV)t}$$

$$+\frac{1}{(CT)_{(1g)}}\frac{1}{p}W_{(EV)t}-\left(1+\frac{1}{(CT)_{(1g)}}\frac{1}{p}\right)(p\,[(C)V]_{(t,i)}+(U)W_{(gu)}$$

$$+(I)W_{(1E)(cp)})-\left(\frac{1}{(CT)_{(1g)}}-\frac{1}{(CT)_{(1d)}}\right)\frac{1}{p}W_{(EV)(cp)}-W_{(EV)s}$$

$$-S_{(diff)(\dot{e},i)}S_{(cmds)(i,W)}\{p(U)e_{(dir)1}-[(U)\dot{e}_{(dir)1}]_o$$

$$+\frac{1}{(CT)_{(1d)}}[(U)e_{(dir)1}-[(U)e_{(dir)1}]_o]\}+S_{(ind)(e,P)}\left\{\frac{1}{(CT)_{(1d)}}\frac{1}{p}(U)e_{(int)(df)}\right.$$

$$+ (U)\dot{e}_{(int)(df)} - (U)\dot{e}_{(int)(df)o} \} - S_{(cmds)(i,W)} \{ \frac{1}{(CT)_{(lg)}} [(U)i_{(dir)2}$$

$$+ (U)i_{(diff)}] + (U)i_{(dir)2} - [(U)i_{(dir)2}]_o + (U)i_{(diff)} - (U)i_{(diff)o} \} \qquad (2\text{-}61)$$

Define

$$[(C)P]_{(t,i)} = P_t - P_i \qquad (2\text{-}62)$$

Eq. (2–61) may then be expressed in terms of $[(C)P]_{(t,i)}$:

$$(p + \frac{1}{(CT)_{(ld)}}) [(C)P]_{(t,i)} = (p + \frac{1}{(CT)_{(ld)}}) P_t - \frac{1}{(CT)_{(ld)}} (P_i)_o - W_{(EV)s} - (\dot{P}_i)_{(df)o}$$

$$- W_{(EV)t} - \frac{1}{(CT)_{(lg)}} \frac{1}{p} W_{(EV)t} + (1 + \frac{1}{(CT)_{(lg)}} \frac{1}{p}) (p [(C)V]_{(t,i)}$$

$$+ (U)W_{(gu)} + (I)W_{(IE)(cp)}) + (\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}) \frac{1}{p} W_{(EV)(cp)} + W_{(EV)s}$$

$$+ S_{(diff)(\dot{e},i)} S_{(cmds)(i,W)} \{p (U)e_{(dir)1} - [(U)\dot{e}_{(dir)1}]_o + \frac{1}{(CT)_{(ld)}} [(U)e_{(dir)1}$$

$$- [(U)e_{(dir)1}]_o] \} - S_{(ind)(e,P)} \{\frac{1}{(CT)_{(ld)}} \frac{1}{p} (U)e_{(int)(df)} + (U)\dot{e}_{(int)(df)}$$

$$- (U)\dot{e}_{(int)(df)o} \} + S_{(cmds)(i,W)} \{\frac{1}{(CT)_{(lg)}} \frac{1}{p} [(U)i_{(dir)2} + (U)i_{(diff)}]$$

$$+ (U)i_{(dir)2} - [(U)i_{(dir)2}]_o + (U)i_{(diff)} - (U)i_{(diff)o} \} \qquad (2\text{-}63)$$

In the equation above,

$$\frac{1}{p} W_{(EV)t} = P_t - (P_t)_o \qquad (2\text{-}64)$$

$$\frac{1}{p} W_{(EV)(cp)} = P_i - (P_i)_o \qquad (2\text{-}65)$$

$$p P_t = W_{(EV)t} \qquad (2\text{-}66)$$

Thus Eq. (2–63) may be written:

$$(p + \frac{1}{(CT)_{(lg)}}) [(C)P]_{(t,i)} = (\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_{(lg)}}) ([(C)P]_{(t,i)})_o - (\dot{P}_i)_{(df)o}$$

$$+ (1 + \frac{1}{(CT)_{(lg)}} \frac{1}{p}) (p\, [(C)V]_{(t,i)} + (U)W_{(gu)} + (I)W_{(IE)(cp)})$$

$$+ S_{(diff)(\dot{e},i)}\, S_{(cmds)(i,\Psi)} \{ p\,(U)e_{(dir)1} - [(U)\dot{e}_{(dir)1}]_o$$

$$+ \frac{1}{(CT)_{(ld)}} [(U)e_{(dir)1} - [(U)e_{(dir)1}]_o] \} - S_{(ind)(e,P)} \frac{1}{(CT)_{(ld)}} \frac{1}{p} (U)e_{(int)(df)}$$

$$+ (U)\dot{e}_{(int)(df)} - (U)\dot{e}_{(int)(df)o} \} + S_{(cmds)(i,\Psi)} \{ \frac{1}{(CT)_{(lg)}} \frac{1}{p} [(U)i_{(dir)2}$$

$$+ (U)i_{(diff)}] + (U)i_{(dir)2} - [(U)i_{(dir)2}]_o + (U)i_{(diff)} - (U)i_{(diff)o} \} \qquad (2\text{–}67)$$

DERIVATION 3. AZIMUTH STABILIZATION SYSTEM

Refer to Fig. 3–1. The angular velocity of indicated north with respect to inertial space, about the z-axis, is:

$$W_{(IN)i} = S_{(cmds)(i,W)z}\, i_z + (U)W_{(gu)z} \tag{3-1}$$

This is related to the angular velocity of indicated north with respect to the Earth as follows:

$$W_{(EN)i} = W_{(IN)i} + W_{(IE)} \sin(Lat)_t + [(C)V]_{(t,i)y} W_{IE} \cos(Lat)_t \tag{3-2}$$

From these equations,

$$W_{(EN)i} = S_{(cmds)(i,W)z}\, i_z + (U)W_{(gu)z} + W_{IE} \sin(Lat)_t + [(C)V]_{(t,i)y} W_{IE} \cos(Lat)_t \tag{3-3}$$

Define $[(C)N]_{(t,i)}$ as the correction to indicated north. Then

$$p\overline{[(C)N]}_{(t,i)} = \overline{W}_{(EN)t} - \overline{W}_{(EN)i} - \overline{W}_{(EB)} \times \overline{[(C)N]}_{(t,i)} \tag{3-4}$$

At marine speeds, $[\overline{W}_{EB} \times \overline{[(C)N]}_{(t,i)}] = 0$. (See Part I, pp. 85-86). Therefore,

$$p[(C)N]_{(t,i)} = W_{(EN)t} - W_{(EN)i} \tag{3-5}$$

From the foregoing, this may be written,

$$p[(C)N]_{(t,i)} = W_{(EN)t} - S_{(cmds)(i,W)z}\, i_z - (U)W_{(gu)z}$$

$$- W_{IE} \sin(Lat)_t - [(C)V]_{(t,i)y} W_{IE} \cos(Lat)_t \tag{3-6}$$

From Fig. 3–1,

$$i_z = \{S_{(dir)(i,i)z}(i_y + i_{(cp)z}) + (U)i_{(dir)z}$$

$$- i_x (\sin(Lat)_i + (U)Res)\}(\sec(Lat)_i + (U)Res) \tag{3-7}$$

where $(U)Res$ is the uncertainty in the resolver generating $\sin(Lat)_i$ and $\sec(Lat)_i$. From geometrical considerations, i_y and i_x are given by:

$$i_y = \frac{1}{S_{(cmds)(i,W)y}} \{-(\dot{Lat})_t + [W_{IE} + (\dot{Lon})_t] [(C)V]_{(t,i)x} \sin(Lat)_t$$

$$+ [W_{IE} + (\dot{Lon})_t] [(C)N]_{(t,i)} \cos(Lat)_t - p[(C)V]_{(t,i)y} - (U)W_{(gu)y}\} \tag{3-8}$$

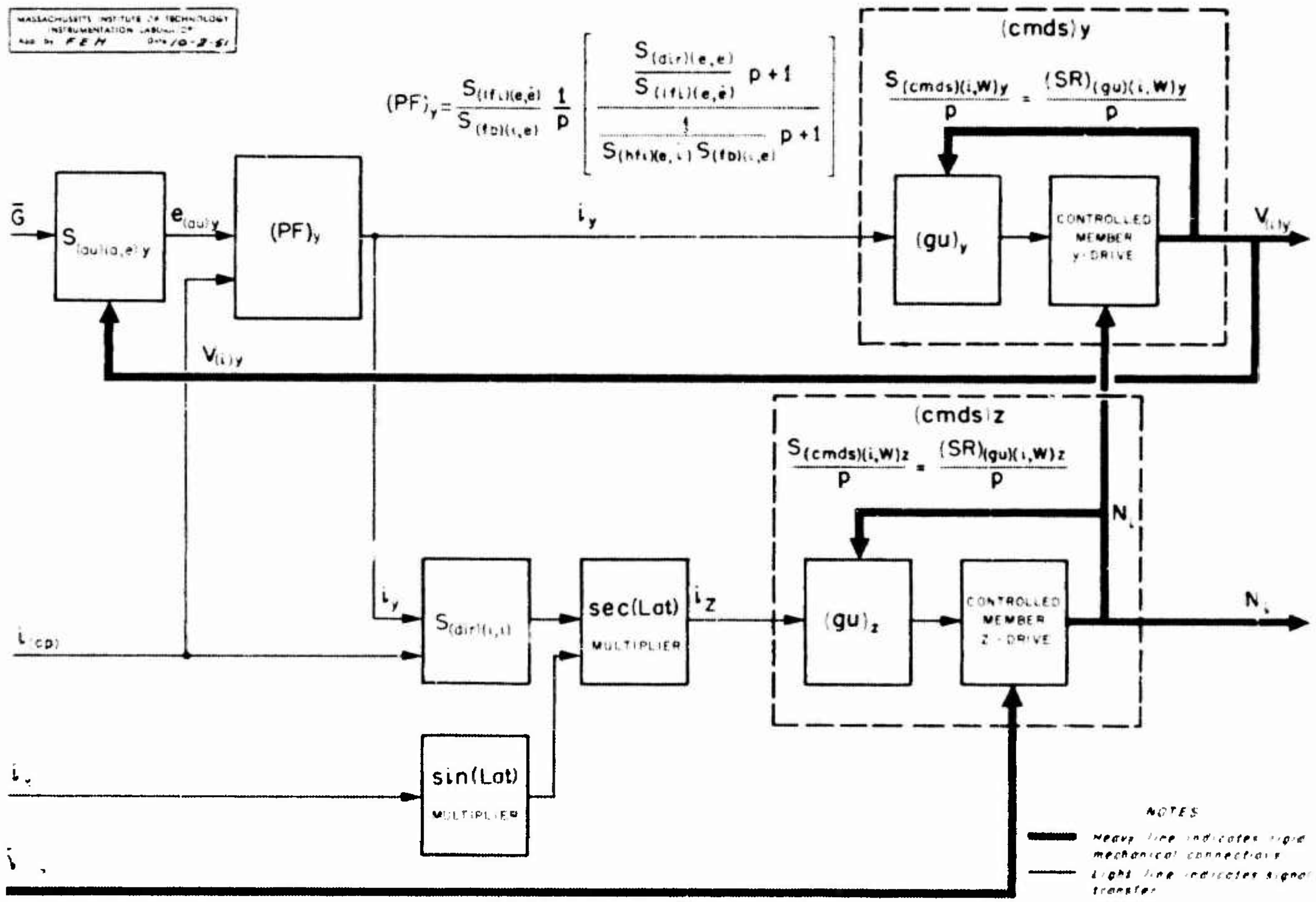

Fig. 3–1. Mathematical functional diagram of combined damped y–axis vertical indicator and z–axis stabilizer.

$$i_x = \frac{1}{S_{(cmds)(i,\Psi)x}} \{ [W_{IE} + (\dot{Lon})_t] \cos(Lat)_t$$

$$- [W_{IE} + (\dot{Lon})_t] \; [(C)V]_{(t,i)y} \sin(Lat)_t$$

$$+ (\dot{Lat})_t \; [(C)N]_{(t,i)} - p\,[(C)V]_{(t,i)x} - (U)W_{(gu)x} \} \tag{3-9}$$

Set

$$S_{(cmds)(i,\Psi)x} = S_{(cmds)(i,\Psi)y} = S_{(cmds)(i,\Psi)z} \tag{3-10}$$

Combination of Eqs. (3–7), (3–8), (3–9) and (3–10) with (3–6) gives the following:

$$p\,[(C)N]_{(t,i)} = W_{(EN)t} - S_{(dir)(i,i)z} \, [-(\dot{Lat})_t + S_{(cmds)(i\Psi)} i_{(cp)z}] \, [\sec(Lat)_i + (U)Res]$$

$$- S_{(dir)(i,i)z} \, [W_{IE} + (\dot{Lon})_t] \; [(C)V]_{(t,i)x} [\sin(Lat)_t] \; [\sec(Lat)_i + (U)Res]$$

$$- S_{(dir)(i,i)z} [W_{IE} + (\dot{Lon})_t] \; [(C)N]_{(t,i)} [\cos(Lat)_t] \; [\sec(Lat)_i + (U)Res]$$

$$+ S_{(dir)(i,i)z} [\sec(Lat)_i + (U)Res]\, p \; [(C)V]_{(t,i)y}$$

$$+ S_{(dir)(i,i)z} \, [\sec(Lat)_i + (U)Res] (U)W_{(gu)y}$$

$$+ [W_{IE} + (\dot{Lon})_t] \; [\cos(Lat)_t] \; [\sin(Lat)_i + (U)Res] \; [\sec(Lat)_i + (U)Res]$$

$$- W_{IE} \sin(Lat)_t - [W_{IE} + (\dot{Lon})_t] \; [(C)V]_{(t,i)y} [\sin(Lat)_t] \; [\sin(Lat)_i$$

$$+ (U)Res] \; [\sec(Lat)_i + (U)Res] + \{(\dot{Lat})_t \; [(C)N]_{(t,i)} - p\,[(C)V]_{(t,i)x}$$

$$- (U)W_{(gu)x}\} \; [\sin(Lat)_i + (U)Res] \; [\sec(Lat)_i + (U)Res]$$

$$- [(C)V]_{(t,i)y} W_{IE} \cos(Lat)_t - (U)W_{(gu)z} - S_{(cmds)(i,\Psi)} (U)i_{(dir)z} \tag{3-11}$$

Define the latitude rate compensation inaccuracy as follows:

$$(I)(\dot{Lat})_{cp} = S_{(cmds)(i,\Psi)} \, i_{(cp)z} - (\dot{Lat})_t \tag{3-12}$$

Define the correction to indicated latitude as follows:

$$[(C)Lat]_{(t,i)} = (Lat)_t - (Lat)_i \tag{3-13}$$

Then

$$\sin(Lat)_i = \sin(Lat)_t - [(C)\,Lat]_{(t,i)} \cos(Lat)_t \tag{3-14}$$

$$\sec(Lat)_i = \frac{1}{\cos(Lat)_i}$$

$$\cong \sec(Lat)_t - [(C)\,Lat]_{(t,i)} \tan(Lat)_t \sec(Lat)_t \tag{3-15}$$

Since the time variations in $(\dot{Lon})_t$ and $(\dot{Lat})_t$ are small when compared with the variations in $[(C)\,V]_{(t,i)x}$, $[(C)\,V]_{(t,i)y}$ and $[(C)\,N]_{(t,i)}$, it will be assumed that $(\dot{Lon})_t$ and $(\dot{Lat})_t$ are quasi-static. This simplifies the subsequent analysis. Note that

$$W_{(EN)t} = -(\dot{Lon})_t \sin(Lat)_t - (\dot{Lat})_t [(C)\,V]_{(t,i)x}$$
$$-(\dot{Lon})_t \cos(Lat)_t [(C)\,V]_{(t,i)y} \tag{3-16}$$

From the foregoing, the performance equation is, ignoring second-order terms,

$$[p + S_{(dir)(i,i)z} [W_{IE} + (\dot{Lon})_t] - (\dot{Lat})_t \tan(Lat)_t] [(C)\,N]_{(t,i)}$$

$$= \{S_{(dir)(i,i)z} \sec(Lat)_t p - [W_{IE} + (\dot{Lon})_t] [\tan(Lat)_t \sin(Lat)_t$$

$$+ \cos(Lat)_t]\} [(C)\,V]_{(t,i)y} - \{\tan(Lat)_t p + (\dot{Lat})_t$$

$$+ S_{(dir)(i,i)z} [W_{IE} + (\dot{Lon})_t] \tan(Lat)_t\} [(C)\,V]_{(t,i)x}$$

$$+ S_{(dir)(i,i)z} \sec(Lat)_t (U)\,W_{(gu)y} - (U)\,W_{(gu)x} \tan(lat)_t$$

$$-(U)\,W_{(gu)z} - S_{(dir)(i,i)z} (I)(\dot{Lat})_{(cp)} \sec(Lat)_t$$

$$- S_{(cmds)(i,W)} [\sec(Lat)_t] (U)\,i_{(dir)z}$$

$$- [W_{IE} + (\dot{Lon})_t] [\sec(Lat)_t] [(C)\,Lat]_{(t,i)}$$

$$+ [(U)\,Res] (1 + \sin(Lat)_t \cos(Lat)_t) [W_{IE} + (\dot{Lon})_t] \tag{3-17}$$

The following assumptions are warranted if the maximum ship speed is taken as 40 knots:

$$(\dot{Lon})_t << W_{IE} \tag{3-18}$$

$$(\dot{Lat})_t << W_{IE} \tag{3-19}$$

Define

$$S_{(dir)(i,i)z} W_{IE} = \frac{1}{(CT)_z} \tag{3-20}$$

The performance equation then simplifies to

$$[p + \frac{1}{(CT)_z}] [(C)N]_{(t,i)} - \frac{1}{(CT)_z W_{IE}} [\sec(Lat)_t] p [(C)V]_{(t,i)y}$$

$$- W_{IE} \sec(Lat)_t \{ [(C)V]_{(t,i)y} + [(C)Lat]_{(t,i)} \}$$

$$- \{ p + \frac{1}{(CT)_z} \} [\tan(Lat)_t] [(C)V]_{(t,i)x}$$

$$+ \frac{1}{(CT)_z W_{IE}} [\sec(Lat)_t] (U) W_{(gu)y}$$

$$- \tan(Lat)_t (U) W_{(gu)x} - (U) W_{(gu)z}$$

$$- \frac{1}{(CT)_z W_{IE}} [(I)(\dot{Lat})_{(cp)}] \sec(Lat)_t$$

$$- S_{(cmds)(i,\Psi)} [\sec(Lat)_t] (U)i_{(dir)z}$$

$$+ [1 + \sin(Lat)_t \cos(Lat)_t] W_{IE} [(U)Res] \qquad (3-21)$$

Ignoring x – coupling, the y – current is

$$i_y = \frac{1}{S_{(cmds)(i,\Psi)}} \{ - (\dot{Lat})_t + [(C)N]_{(t,i)} W_{IE} \cos(Lat)_t - p [(C)V]_{(t,i)y} - (U) W_{(gu)y} \} \qquad (3-22)$$

By a procedure similar to that of Derivation 1, Part A, the y – performance equation including z – coupling can be shown to be

$$\{ p^3 + \frac{1}{(CT)_{(lg)y}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s} \frac{1}{(CT)_{(ld)y}} \} [(C)V]_{(t,i)y}$$

$$= [p^2 + \frac{1}{(CT)_{(lg)y}} p] W_{IE} [(C)N]_{(t,i)} \cos(Lat)_t$$

$$- [\frac{1}{(CT)_{(lg)y}} - \frac{1}{(CT)_{(ld)y}}] p(I)(\dot{Lat})_{(cp)}$$

$$- [p^2 + \frac{1}{(CT)_{(lg)y}} p] (U) W_{(gu)y} - \frac{1}{R_s S_{(au)(a,e)}} \{ p + \frac{1}{(CT)_{(ld)y}} \} (U)e_{(au)}$$

$$- S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)} [(U)e_{(dir)y} - (U)e_{(fb)}]$$

$$- S_{(cmds)(i,\Psi)} S_{(hfi)(e,i)} (U)\dot{e}_{(lfi)(df)}$$

$$- S_{(cmds)(i,\Psi)} p(U)i_{(hfi)(df)} \qquad (3-23)$$

Operate on Eq. (3-21) with $[p^3 + \frac{1}{(CT)_{(1g)y}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s} \frac{1}{(CT)_{(1d)y}}]$, utilizing Eq. (3-23) to eliminate $[(C)V]_{(t,i)y}$. The result is, ignoring x – coupling except for x- gyro unit drift,

$$\{p^4 + \frac{1}{(CT)_{(1g)y}} p^3 + [\frac{g}{R_s} + W^2_{IE}]p^2$$

$$+ [\frac{g}{R_s}(\frac{1}{(CT)_{(1d)y}} + \frac{1}{(CT)_z}) + \frac{W^2_{IE}}{(CT)_{(1g)y}}]p + \frac{g}{R_s(CT)_{(1d)y}(CT)_z}\}[(C)N]_{(t,i)}$$

$$= -W_{(IE)}\sec(Lat)_t[p^3 + \frac{1}{(CT)_{(1g)y}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s}\frac{1}{(CT)_{(1d)y}}][(C)Lat]_{(t,i)}$$

$$- \frac{\sec(Lat)_t}{(CT)_z W_{IE}}[p^3 + (\frac{2}{(CT)_{(1g)y}} - \frac{1}{(CT)_{(1d)y}})p^2$$

$$+ [\frac{g}{R_s} + (CT)_z W^2_{IE}(\frac{1}{(CT)_{(1d)y}} - \frac{1}{(CT)_{(1g)y}})]p$$

$$+ \frac{g}{R_s}\frac{1}{(CT)_{(1d)y}}](I)(\dot{Lat})_{(cp)} - \tan(Lat)_t(p^3 + \frac{1}{(CT)_{(1g)y}} p^2$$

$$+ \frac{g}{R_s} p + \frac{g}{R_s}\frac{1}{(CT)_{(1d)y}})(U)W_{(gu)x} + \sec(Lat)_t[W_{IE}p^2$$

$$+ (\frac{W_{IE}}{(CT)_{(1g)y}} + \frac{1}{R_s(CT)_z W_{IE}})p + \frac{g}{R_s}\frac{1}{(CT)_{(1d)y}(CT)_z W_{IE}}](U)W_{(gu)y}$$

$$- (p^3 + \frac{1}{(CT)_{(1g)y}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s}\frac{1}{(CT)_{(1d)y}})(U)W_{(gu)z}$$

$$- \frac{\sec(Lat)_t}{R_s S_{(au)(a,e)}(CT)_z W_{IE}}[p^2 - ((CT)_z W^2_{IE} + \frac{1}{(CT)_{(1d)y}})p$$

$$- \frac{(CT)_z}{(CT)_{(1d)y}} W^2_{IE}](U)e_{au} - \frac{S_{(cmds)(i,W)} S_{(hfi)(e,i)} \sec(Lat)_t}{(CT)_z W_{IE}}[p$$

$$-(CT)_z W^2_{IE}]p[(U)e_{(dit)} - (U)e_{(fb)}]$$

$$- \frac{S_{(cmds)(i,W)} S_{(hfi)(e,i)} \sec(Lat)_t}{(CT)_z W_{IE}}[p - (CT)_z W^2_{IE}](U)\dot{e}_{(lfi)(df)}$$

$$- \frac{S_{(cmds)(i,W)} \sec(Lat)_t}{(CT)_z W_{IE}} (p - (CT)_z W_{IE}^2) \, p \, (U) \dot{i}_{(hfi)(df)}$$

$$- S_{(cmds)(i,W)} \sec(Lat)_t \left(p^3 + \frac{1}{(CT)_{(lg)y}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s} \frac{1}{(CT)_{(ld)y}}\right) (U) i_{(dir)z}$$

$$+ (1 + \sin(Lat)_t \cos(Lat)_t) W_{IE} \left(p^3 + \frac{1}{(CT)_{(lg)y}} p^2 + \frac{g}{R_s} p + \frac{g}{R_s} \frac{1}{(CT)_{(ld)y}}\right) (U) Res \tag{3-24}$$

In Derivation 4, which follows, an expression for $[(C)Lat]_{(t,i)}$ in terms of $[(C)V]_{(t,i)y}$ and $[(C)N]_{(t,i)}$ is derived. Solving Eqs. (4-11), (3-23), and (3-21) simultaneously and ignoring small terms, there results:

$$p^5 + \frac{1}{(CT)_{(lg)}} p^4 + \frac{g}{R_s} p^3 + \frac{g}{R_s} \left(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\right) p^2 + \frac{g}{R_s} \frac{1}{(CT)_z} \left(\frac{1}{(CT)_{(ld)}}\right.$$

$$\left. + \frac{1}{(CT)_L}\right) p + \frac{g}{R_s} \left(\frac{1}{(CT)_{(ld)}(CT)_L(CT)_z}\right) [(C)N]_{(t,i)} \cong - \left[\frac{1}{(CT)_z W_{IE}} p^4 \right.$$

$$+ \frac{1}{(CT)_z W_{IE}} \left(\frac{2}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_L}\right) p^3 + \frac{1}{(CT)_z W_{IE}} \left(\frac{g}{R_s}\right.$$

$$\left. + \frac{2}{(CT)_L (CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}(CT)_L}\right) p^2 + \frac{g}{R_s} \left(\frac{1}{(CT)_z (CT)_L W_{IE}}\right.$$

$$\left. + \frac{1}{(CT)_{(ld)}(CT)_z W_{IE}} - W_{IE}\right) p + \frac{g}{R_s} \frac{1}{(CT)_{(ld)}} \left(\frac{1}{(CT)_{(ld)}(CT)_z W_{IE}}\right.$$

$$\left. \left. - W_{IE}\right) \right] \sec(Lat)_t \, [(I)\dot{Lat}]_{(cp)} - \tan(Lat)_t \left[p^4 + \frac{1}{(CT)_{(lg)}} p^3 + \frac{g}{R_s} p^2\right.$$

$$\left. + \frac{g}{R_s} \frac{1}{(CT)_{(ld)}} p\right] (U) W_{(gu)x} + \sec(Lat)_t \left[W_{IE} p^3 + \frac{g}{R_s} \frac{1}{(CT)_z W_{IE}} p^2\right.$$

$$\left. + \frac{g}{R_s} \frac{1}{(CT)_z W_{IE}} \left(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\right) p + \frac{g}{R_s} \frac{1}{(CT)_{(ld)}(CT)_L(CT)_z W_{IE}}\right] (U) W_{(gu)x}$$

$$- \left[p^4 + \frac{1}{(CT)_{(lg)}} p^3 + \frac{g}{R_s} p^2 + \frac{g}{R_s (CT)_{(ld)}} p\right] (U) W_{(gu)z}$$

$$
\begin{aligned}
& - \frac{\sec(Lat)_t}{R_s S_{(au)(a,e)}} \left[\frac{1}{(CT)_z W_{IE}} p^3 + \left(\frac{1}{(CT)_L (CT)_z W_{IE}} + \frac{1}{(CT)_{(1d)} (CT)_z W_{IE}} \right.\right. \\
& \left.\left. - W_{IE} \right) p^2 + \left(\frac{1}{(CT)_{(1d)} (CT)_L (CT)_z W_{IE}} - \frac{W_{IE}}{(CT)_{(1d)}} \right) p \right] (U) e_{(au)} \\
& - S_{(cmds)(i,w)} S_{(hfi)(e,\dot{i})} \sec(Lat)_t \left[\frac{1}{(CT)_z W_{IE}} p^2 + \left(\frac{1}{(CT)_L (CT)_z W_{IE}} \right.\right. \\
& \left.\left. - W_{IE} \right) p \right] (U) \dot{e}_{(1fi)(df)} - S_{(cmds)(i,w)} \sec(Lat)_t \left[\frac{1}{(CT)_z W_{IE}} p^3 \right. \\
& \left. + \left(\frac{1}{(CT)_L (CT)_z W_{IE}} \quad W_{IE} \right) p^2 \right] (U) i_{(hfi)(df)} \\
& - S_{(cmds)(i,w)} S_{(hfi)(e,\dot{i})} \sec(Lat)_t \left[\frac{1}{(CT)_z W_{IE}} p^3 + \left(\frac{1}{(CT)_L (CT)_z W_{IE}} \right.\right. \\
& \left.\left. - W_{IE} \right) p^2 \right] \left[(U) e_{(dir)y} - (U) e_{(fb)} \right] - S_{(cmds)(i,w)} \sec(Lat)_t \left[p^4 + \frac{1}{(CT)_{(1g)}} p^3 \right. \\
& \left. + \frac{g}{R_s} p^2 + \frac{g}{R_s} \frac{1}{(CT)_{(1d)}} p \right] (U) i_{(dir)z} + \left[1 + \sin(Lat)_t \cos(Lat)_t \right] W_{IE} \left[p^4 \right. \\
& \left. + \frac{1}{(CT)_{(1g)}} p^3 + \frac{g}{R_s} p^2 + \frac{g}{R_s} \frac{1}{(CT)_{(1d)}} p \right] (U) Res \qquad (3\text{-}25)
\end{aligned}
$$

DERIVATION 4. LATITUDE INDICATION

Refer to Fig. 5-1. The latitude drive signal is

$$i_L = S_{(dir)(i,i)L}\,(i_y + i_{(cp)z}) - i_{(cp)z}$$

$$= S_{(dir)(i,i)L}\, i_y + (S_{(dir)(i,i)L} - 1)\, i_{(cp)z} \qquad (4-1)$$

Since the drive rotates at the indicated latitude rate, but in the opposite direction,

$$p(Lat)_i = - S_{(Lds)(i,W)}\, i_L \qquad (4-2)$$

From Der. 3, Eq. (3-6)

$$i_y = \frac{1}{S_{(cmds)(i,W)}} \{ - (\dot{Lat})_t$$

$$+ [W_{IE} + (\dot{Lon})_t]\, [(C)V]_{(t,i)x} \sin (Lat)_t$$

$$+ [W_{IE} + (\dot{Lon})_t]\, [(C)N]_{(t,i)} \cos (Lat)_t$$

$$- p\,[(C)V]_{(t,i)y} - (U)W_{(gu)y} \} \qquad (4-3)$$

Assume as in Der. 3 that $W_{IE} >> (\dot{Lon})_t$, and that x-coupling may be neglected. Substitute Eq. (4-3) and Eq. (4-1) into Eq. (4-2). The result is

$$p(Lat)_i = - S_{(dir)(i,i)L} \frac{S_{(Lds)(i,W)}}{S_{(cmds)(i,W)}} \{ - (\dot{Lat})_t$$

$$+ S_{(cmds)(i,W)}\, i_{(cp)z} + W_{IE}\, [(C)N]_{(t,i)} \cos (Lat)_t$$

$$- p\,[(C)V]_{(t,i)y} - (U)W_{(gu)y} \} + S_{(Lds)(i,W)}\, i_{(cp)z} \qquad (4-4)$$

Define

$$(I)(Lat)_{(cp)} = S_{(cmds)(i,W)}\, i_{(cp)z} - (Lat)_t \qquad (4-5)$$

and

$$[(C)Lat]_{(t,i)} = (Lat)_t - (Lat)_i \qquad (4-6)$$

set

$$S_{(Lds)(i,W)} = S_{(cmds)(i,W)} \qquad (4-7)$$

Then

$$p\,[(C)Lat]_{(t,i)} = S_{(dir)(i,i)L}\,\{(I)(\dot{Lat})_{(cp)} + W_{IE}\,[(C)N]_{(t,i)}\cos(Lat)_t - p\,[(C)V]_{(t,i)y} - (U)W_{(gu)y}\} - (I)(\dot{Lat})_{(cp)} \qquad (4\text{-}8)$$

Transposing terms in Eq. (3-21) (ignoring x-coupling, except for x-gyro drift), solving for $[(C)N]_{(t,i)}$ in terms of $p[(C)N]_{(t,i)}$, and multiplying through by $(CT)_z W_{IE}$, gives the following:

$$\begin{aligned}
W_{IE}\,[(C)N]_{(t,i)} = & - W_{IE}\,(CT)_z\,p\,[(C)N]_{(t,i)} + \sec(Lat)_t\,p\,[(C)V]_{(t,i)y} \\
& - W_{IE}^2\,(CT)_z\,\sec(Lat)_t\,\{[(C)V]_{(t,i)y} + [(C)Lat]_{(t,i)}\} \\
& + \sec(Lat)_t\,(U)W_{(gu)y} - W_{IE}\,(CT)_z\,[\tan(Lat)_t](U)W_{(gu)x} \\
& - W_{IE}(CT)_z(U)W_{(gu)z} - (I)(\dot{Lat})_{(cp)}\sec(Lat)_t \\
& - W_{IE}(CT)_z\,S_{(cmds)(i,\Psi)}\,[\sec(Lat)_t](U)i_{(dir)z} \\
& + [1 + \sin(Lat)_t\cos(Lat)_t]\,W_{IE}^2\,(CT)_z\,[(U)Res]
\end{aligned} \qquad (4\text{-}9)$$

Define

$$\frac{S_{(dir)(i,i)L}}{S_{(dir)(i,i)z}}\,W_{IE} = S_{(dir)(i,i)L}(CT)_z\,W_{IE}^2 = \frac{1}{(CT)_L} \qquad (4\text{-}10)$$

Substitute Eq. (4-7) in Eq. (4-6), utilizing Eq. (4-8). There results,

$$\begin{aligned}
[p + \frac{1}{(CT)_L}]\,[(C)Lat]_{(t,i)} = & -\frac{1}{(CT)_L}\,[(C)V]_{(t,i)y} \\
& - \frac{1}{W_{IE}\,(CT)_L}\,p\,[(C)N]_{(t,i)}\cos(Lat)_t \\
& - (I)(\dot{Lat})_{(cp)} - \frac{1}{(CT)_L W_{IE}}\sin(Lat)_t(U)W_{(gu)x} \\
& - \frac{S_{(cmds)(i,\Psi)}}{(CT)_L W_{IE}}\,(U)\,i_{(dir)z} \\
& + [1 + \sin(Lat)_t\cos(Lat)_t]\,[(U)Res]\,\frac{1}{(CT)_L}\cos(Lat)_t
\end{aligned}$$

$$-\frac{1}{(CT)_L\, W_{IE}}\,[\cos(Lat)_t]\,(U)W_{(gu)z} \qquad (4\text{-}11)$$

To eliminate $[(C)V]_{(t,t)y}$ and $[(C)N]_{(t,t)}$, Eq. (3-21) and (3-23) are utilized. Their simultaneous solution is (omitting small quantities):

$$\Big[p^5 + \frac{1}{(CT)_{(lg)}}\,p^4 + \frac{g}{R_s}\,p^3 + \frac{g}{R_s}\Big(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\Big)\,p^2$$

$$+ \frac{g}{R_s}\Big(\frac{1}{(CT)_{(ld)}(CT)_z} + \frac{1}{(CT)_L(CT)_z}\Big)\,p$$

$$+ \frac{g}{R_s}\,\frac{1}{(CT)_{(ld)}(CT)_L(CT)_z}\Big]\,[(C)Lat]_{(t,t)} \cong -\frac{1}{(CT)_L(CT)_z(W_{IE})^2}\,p^4$$

$$+ \frac{1}{(CT)_L(CT)_z(W_{IE})^2}\Big(\frac{1}{(CT)_{(lg)}} - \frac{1}{(CT)_{(ld)}}\Big)\,p^3$$

$$+ \frac{g}{R_s}\,\frac{1}{(CT)_L(CT)_z(W_{IE})^2}\,p^2 + \Big[\frac{2}{(CT)_{(lg)}(CT)_L(CT)_z}$$

$$- \frac{g}{R_s}\Big(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\Big)\Big]p + \frac{g}{R_s}\Big(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\Big)\Big\}\,(I)(L\dot{a}t)_{(cp)}$$

$$- \frac{g}{R_s}\,\frac{1}{(CT)_L(CT)_z W_{IE}}\Big(p + \frac{1}{(CT)_{(ld)}}\Big)\sin(Lat)_t\,(U)W_{(gu)x}$$

$$- \frac{g}{R_s}\,\frac{1}{(CT)_L(CT)_z(W_{IE})^2}\,p\Big(p + \frac{1}{(CT)_{(ld)}}\Big)(U)W_{(gu)y}$$

$$- \frac{g}{R_s}\,\frac{1}{(CT)_L(CT)_z W_{IE}}\Big(p + \frac{1}{(CT)_{(ld)}}\Big)\cos(Lat)_t\,(U)W_{(gu)z}$$

$$+ \frac{1}{(CT)_L(CT)_z}\Big(\frac{1}{(W_{IE})^2}\,p^3 + \frac{1}{(CT)_{(ld)}(W_{IE})^2}\,p^2 + p$$

$$+ \frac{1}{(CT)_{(ld)}}\Big)\frac{1}{R_s S_{(au)(a,c)}}\,(U)e_{(au)}$$

$$+ S_{(cmds)(i,w)}\,S_{(hfi)(c,i)}\Big(\frac{1}{(CT)_L(CT)_z}\Big)\Big(\frac{1}{(W_{IE})^2}\,p^3 + p\Big)\Big[(U)e_{(dir)y} - (U)e_{(fb)}\Big]$$

$$+ S_{(cmds)(i,W)} S_{(hfi)(e,i)} \frac{1}{(CT)_L (CT)_z} \left(\frac{1}{(W_{IE})^2} p^2 + i\right) (U) \dot{e}_{(lfi)(ldf)}$$

$$+ S_{(cmds)(i,W)} \frac{1}{(CT)_L (CT)_z} \left(\frac{1}{(W_{IE})^2} p^3 + p\right) (U) \dot{i}_{(hfi)(df)}$$

$$+ S_{(cmds)(i,W)} \frac{1}{(CT)_L W_{IE}} \left[\left(\frac{1}{(CT)_z} - \frac{1}{(CT)_L}\right) p^3 + \frac{g}{R_s} \frac{1}{(CT)_z} p\right.$$

$$\left. + \frac{g}{R_s} \frac{1}{(CT)_{(ld)} (CT)_z}\right] (U) i_{(dir)z}$$

$$+ \frac{g}{R_s} \frac{\cos(Lat)_t}{(CT)_L (CT)_z} \left[1 + \sin(Lat)_t \cos(Lat)_t\right] \left(p + \frac{1}{(CT)_{(ld)}}\right) (J) Res \qquad (4-12)$$

In Derivation 9 of Part I it was shown that

$$[(C)Lat]_{(t,i)} = [(C)V]_{(t,i)y} - [(C)PA]_{(t,i)} \qquad (4-13)$$

where $[(C)PA]_{(t,i)}$ = correction to the indicated polar axis. $[(C)V]_{(t,i)y}$ becomes of interest in the discussion of roll and may also be obtained through the simultaneous solution of Eq. (3-21), (3-23), and (4-11), as follows:

$$\left\{p^5 + \frac{1}{(CT)_{(lg)}} p^4 + \frac{g}{R_s} p^3 + \frac{g}{R_s} \left(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\right) p^2 + \frac{g}{R_s} \left(\frac{1}{(CT)_{(ld)} (CT)_z}\right.\right.$$

$$\left.\left. + \frac{1}{(CT)_L (CT)_z}\right) p + \frac{g}{R_s} \frac{1}{(CT)_{(ld)} (CT)_L (CT)_z}\right\} [(C)V]_{(t,i)y}$$

$$= \left\{\left(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z} - \frac{1}{(CT)_{(lg)}}\right) p^3 + \frac{1}{(CT)_z} \left(\frac{1}{(CT)_L} - \frac{1}{(CT)_{(ld)}}\right) p^2\right.$$

$$\left. + \frac{1}{(CT)_{(ld)} (CT)_L (CT)_z}\right\} (I)(\dot{L}at)_{(cp)} - W_{IE} \sin(Lat)_t \left(p^3 + \left(\frac{2}{(CT)_L}\right.\right.$$

$$\left.\left. + \frac{1}{(CT)_{(lg)}}\right) p^2 + \frac{2}{(CT)_L (CT)_{(lg)}} p\right) (U) W_{(gu)x} - \left(p^4 + \left(\frac{2}{(CT)_z} + \frac{1}{(CT)_{(lg)}}\right) p^3\right.$$

$$\left. + \frac{2}{(CT)_z} \left(\frac{1}{(CT)_L} + \frac{1}{(CT)_{(lg)}}\right) p^2 + \frac{1}{(CT)_{(lg)} (CT)_L (CT)_z} p\right) (U) W_{(gu)y}$$

$$+ W_{IE}\cos(Lat)_t\left[p^3 + \left(\frac{2}{(CT)_L} + \frac{1}{(CT)_{(lg)}}\right)p^2 + \frac{2}{(CT)_L(CT)_{(lg)}}p\right](U)W_{(gu)}$$

$$- \frac{1}{R_s S_{(au)(a,e)}}\left[p^3 + \left(\frac{1}{(CT)_{(ld)}} + \frac{1}{(CT)_z}\right)p^2 + \frac{1}{(CT)_z}\left(\frac{1}{(CT)_L} + \frac{1}{(CT)_{(ld)}}\right)p\right.$$

$$\left. + \frac{1}{(CT)_{(ld)}(CT)_L(CT)_z}\right](U)e_{(au)} - S_{(cmds)(i,w)}S_{(hfi)(e,i)}\left[p^2 + \frac{1}{(CT)_z}p\right.$$

$$\left. + \frac{1}{(CT)_L(CT)_z}\right](U)\dot{e}_{(hfi)(df)} + S_{(cmds)(i,w)}\left(p^3 + \frac{1}{(CT)_z}p\right.$$

$$\left. + \frac{1}{(CT)_L(CT)_z}p\right)(U)\dot{i}_{(hfi)(df)} + S_{(cmds)(i,w)}S_{(hfi)(e,i)}\left(p^3 + \frac{1}{(CT)_z}p^2\right.$$

$$\left. + \frac{1}{(CT)_L(CT)_z}p\right)\left[(U)e_{(dir)y} - (U)e_{(fb)}\right] + S_{(cmds)(i,w)}W_{IE}\left[p^3\right.$$

$$\left. + \left(\frac{1}{(CT)_{(lg)}} + \frac{1}{(CT)_L}\right)p^2 + \frac{1}{(CT)_{(lg)}(CT)_L}p\right](U)i_{(dir)z}$$

$$- W_{IE}^2\cos(Lat)_t\left[1 + \sin(Lat)_t\cos(Lat)_t\right]\left[p^3 + \left(\frac{2}{(CT)_L} + \frac{1}{(CT)_{(lg)}}\right)p^2\right.$$

$$\left. + \frac{2}{(CT)_{(lg)}(CT)_L}p\right](U)Res \qquad (4\text{-}14)$$

DERIVATION 5. LONGITUDE INDICATION

Refer to Fig. 5–1. The longitude drive (upper right-hand corner of the figure) rotates at the indicated celestial longitude rate:

$$p(\mathrm{Lon})_{(cel)i} = S_{(lds)(e,\Psi)} e_{(sg)} \tag{5-1}$$

where $(\mathrm{Lon})_{(cel)i}$ is the indicated celestial longitude and $e_{(sg)}$ is the output of the integrating gyro unit mounted on the longitude gimbal.

$$(\mathrm{Lon})_{(cel)i} = W_{(IE)i} + (\mathrm{Lon})_i \tag{5-2}$$

$$e_{(sg)} = S_{(sg)(A,e)} A_g \tag{5-3}$$

where A_g is the gyro gimbal angle. Define $W_{(PA)i}$ as the angular velocity with respect to inertial space sensed by the longitudinal gyro. Then the torque balance for the gyro unit is

$$H[W_{(PA)i} + (U)W_{(gu)I} - p(\mathrm{Lon})_{(cel)i}] = cpA_g \tag{5-4}$$

From the foregoing

$$p^2(\mathrm{Lon})_{(cel)i} = S_{(lds)(e,\Psi)} S_{(sg)(A,e)} \left(\frac{H}{c}\right) [W_{(PA)i} + (U)W_{(gu)I} - p(\mathrm{Lon})_{(cel)i}] \tag{5-5}$$

$W_{(PA)i}$ may be evaluated as follows. From geometrical considerations

$$W_{(PA)i} = (W_{(IV)i})_x \cos(\mathrm{Lat})_i - W_{(IN)i} \sin(\mathrm{Lat})_i \tag{5-6}$$

From Derivation 3, Eq. (3–1),

$$W_{(IN)i} = S_{(cmds)(i,\Psi)} i_z + (U)W_{(gu)z} \tag{3-1}$$

Substitution of Eq. (3–16) into (3–6) and solving for $S_{(cmds)(i,\Psi)} i_z$ results in the following

$$W_{(IN)i} = -(\dot{\mathrm{Lon}})_t \sin(\mathrm{Lat})_t - (\dot{\mathrm{Lat}})_t [(C)V]_{(t,i)x} - (\dot{\mathrm{Lon}})_t [\cos(\mathrm{Lat})_t] [(C)V]_{(t,i)y}$$

$$- p[(C)N]_{(t,i)} - W_{IE} \sin(\mathrm{Lat})_t - [(C)V]_{(t,i)y} W_{IF} \cos(\mathrm{Lat})_t \tag{5-7}$$

$$(W_{(IV)i})_x = S_{(cmds)(i,\Psi)} i_x + (U)W_{(gu)z} \tag{5-8}$$

Solving Eq. (3–9) for $S_{(cmds)(i,\Psi)} i_x + (U)W_{(gu)z}$, there results

$$(W_{(IV)i})_x = [W_{IE} + (\dot{\mathrm{Lon}})_t] \cos(\mathrm{Lat})_t - [W_{IE} + (\dot{\mathrm{Lon}})_t] [(C)V]_{(t,i)y} \sin(\mathrm{Lat})_t$$

$$+ (\dot{\mathrm{Lat}})_t [(C)N]_{(t,i)} - p[(C)V]_{(t,i)x} \tag{5-9}$$

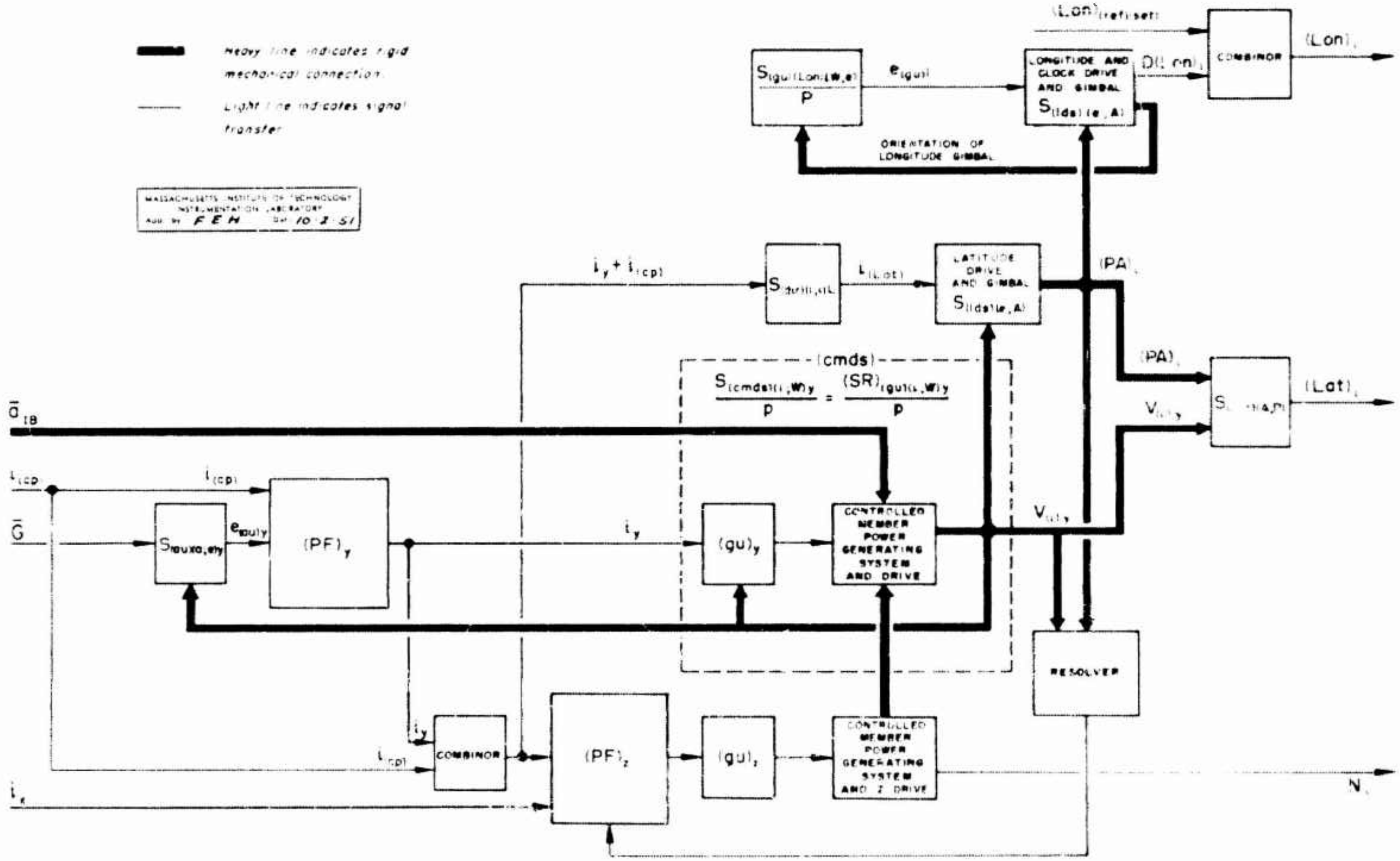

Fig. 5–1. Mathematical functional diagram of a position indicator using (1) the detection of earth rate from a stable platform, for latitude indication, and (2) celestial longitude rate tracking for longitude indication. Coupling with y–component vertical indicator and with the azimuth indicator is shown on the lower half of the diagram.

From Eqs. (5–7) and (5–9), (5–6) becomes

$$W_{(PA)i} = [W_{IE} + (\dot{Lon})_t] [\cos(Lat)_t \cos(Lat)_i + \sin(Lat)_t \sin(Lat)_i]$$

$$+ (\dot{Lat})_t [(C)N]_{(t,i)} \cos(Lat)_i + \{(\dot{Lat})_t \sin(Lat)_i - [\cos(Lat)_i] p\} [(C)V]_{(t,i)x}$$

$$+ [\sin(Lat)_i] p [(C)N]_{(t,i)} \tag{5-10}$$

Define

$$[(C)Lat]_{(t,i)} \equiv (Lat)_t - (Lat)_i \tag{5-11}$$

Then in the first term on the right in Eq. (5–10), assuming $[(C)Lat]_{(t,i)}$ is a small angle,

$$\cos(Lat)_t \cos(Lat)_i + \sin(Lat)_t \sin(Lat)_i$$

$$= \cos[(Lat)_t - (Lat)_i] = \cos[(C)(Lat)]_{(t,i)} \cong 1 \tag{5-12}$$

ignoring second-order terms, Eq. (5–10) now becomes

$$W_{(PA)i} = W_{IE} + (\dot{Lon})_t + (\dot{Lat})_t [(C)N]_{(t,i)} \cos(Lat)_i$$

$$+ [(\dot{Lat})_t \sin(Lat)_t - [\cos(Lat)_t] p] [(C)V]_{(t,i)x}$$

$$+ [\sin(Lat)_t] p [(C)N]_{(t,i)} \tag{5-13}$$

Eq. (5–5) may now be written

$$[p + S_{(lds)(e,W)} S_{(sg)(A,e)} (\frac{H}{c})] p(Lon)_{(cel)i}$$

$$= S_{(lds)(e,W)} S_{(sg)(A,e)} (\frac{H}{c}) \Big\{ W_{IE} + (\dot{Lon})_t + (\dot{Lat})_t [(C)N]_{(t,i)} \cos(Lat)_t$$

$$+ \{(\dot{Lat})_t \sin(Lat)_t - [\cos(Lat)_t] p\} [(C)V]_{(t,i)x}$$

$$+ [\sin(Lat)_t] p [(C)N]_{(t,i)} \Big\} + S_{(lds)(e,W)} S_{(sg)(a,e)} (\frac{H}{c}) (U) W_{(gu)l} \tag{5-14}$$

Define

$$\frac{1}{(CT)_e} \equiv S_{(lds)(e,W)} S_{(sg)(A,e)} (\frac{H}{c}) \tag{5-15}$$

Note that

$$(Lon)_{(cel)t} = W_{IE} + (\dot{Lon})_t \tag{5-16}$$

$$[p + \frac{1}{(CT)_1}]\, p(Lon)_{(cel)i} = \frac{1}{(CT)_1} p(Lon)_{(cel)t} + \frac{1}{(CT)_1} \{(\dot{Lat})_t \cos(\dot{Lat})_t$$

$$+ [\sin(Lat)_t]\, p\}\, [(C)N]_{(t,i)} + \frac{1}{(CT)_1} \{(\dot{Lat})_t \sin(Lat)_t$$

$$- [\cos(Lat)_t]\, p\}\, [(C)V]_{(t,i)x} + \frac{1}{(CT)_1} (U) W_{(gu)1} \tag{5-17}$$

Define the sidereal time drive angular velocity $W_{(tds)}$ as follows

$$W_{(tds)} \equiv p(Lon)_{(cel)i} - p(Lon)_i = W_{(IE)i} \tag{5-18}$$

Substitute Eq. (5–18) into Eq. (5–17) and multiply by $(CT)_1$

$$[1 + (CT)_1 p]\, p(Lon)_i + (1 + (CT)_1 p) W_{(tds)} = p(Lon)_t + W_{IE} + (U) W_{(gu)1}$$

$$+ [(\dot{Lat})_t \cos(Lat)_t + (\sin(Lat)_t)\, p]\; [(C)N]_{(t,i)} + [(\dot{Lat})_t \sin(Lat)_t$$

$$- (\cos(Lat)_t)\, p]\; [(C)V]_{(t,i)x} \tag{5-19}$$

Define the inaccuracy in the time drive angular velocity

$$(I) W_{(tds)} \equiv W_{(tds)} - W_{IE} \tag{5-20}$$

Define the correction to indicated longitude:

$$[(C)Lon]_{(t,i)} \equiv (Lon)_t - (Lon)_i \tag{5-21}$$

then

$$[1 + (CT)_1 p]\; p[(C)Lon]_{(t,i)} = (CT)_1\, p^2 (Lon)_t + (1 + (CT)_1\, p)(I) W_{(tds)}$$

$$+ (CT)_1 W_{IE} - (U) W_{(gu)1}\; [(\dot{Lat})_t \cos(Lat)_t + (\sin(Lat)_t)\, p]\; [(C)N]_{(t,i)}$$

$$- [(\dot{Lat})_t \sin(Lat)_t - (\cos(Lat)_t)\, p]\; [(C)V]_{(t,i)x} \tag{5-22}$$

The variation in Earth rate is very small; therefore,

$$p W_{IE} = 0 \tag{5-23}$$

Integrating Eq. (5–22), there results

$$[1 + (CT)_1 p]\; [(C)Lon]_{(t,i)} = [(C)Lon]_{(t,i)o} + (CT)_1\; [(C)\dot{Lon}]_{(t,i)o}$$

$$
\begin{aligned}
&+ (CT)_1 \left[p(Lon)_t - (p(Lon)_t)_o \right] + (CT)_1 \left[(I) W_{(tds)o} \right] \\
&+ \frac{1}{p} \left[(I) W_{(tds)} - (U) W_{(gu)1} \right] - \frac{1}{p} \left\{ \left[(\dot{Lat})_t \cos(Lat)_t + (\sin(Lat)_t)\, p \right] \left[(C) N \right]_{(t,i)} \right. \\
&\left. - \left[(\dot{Lat})_t \sin(Lat)_t - (\cos(Lat)_t)\, p \right] \left[(C) V \right]_{(t,i)x} \right\}
\end{aligned}
\tag{5-24}
$$

DERIVATION 6. GROUND-SPEED INDICATION

The true ground speed is $R_s W_{(EV)t}$, where R_s is a set value of the Earth-radius and $W_{(EV)t}$ is given in terms of its components by:

$$W_{(EV)t} = \sqrt{[(\dot{Lon})_t \cos(Lat)_t]^2 + (\dot{Lat})_t^2} \qquad (6-1)$$

From the above, the true ground speed, which is the magnitude of the true velocity of the base with respect to the Earth, is:

$$v_{(EB)t} = R_s \sqrt{[(\dot{Lon})_t \cos(Lat)_t]^2 + (\dot{Lat})_t^2} \qquad (6-2)$$

Case I. The vertical indicator as a source of ground-speed data.

The quantities in Eq. (6–2) are, in part, implicit in the x-and y-gyro units' orientational control signals, as follows:

$$i_x = \frac{W_{(IV)(i)x}}{S_{(cmds)(i,W)}} = \frac{1}{S_{(cmds)(i,W)}} \{ [(W_{IE} + (\dot{Lon})_t] \cos(Lat)_t$$

$$- [W_{IE} + (\dot{Lon})_t] [\sin(Lat)_t] [(C)V]_{(t,i)y} + (\dot{Lat})_t [(C)N]_{(t,i)}$$

$$- p[(C)V]_{(t,i)x} - (U) W_{(gu)x} \} \qquad (6-3)$$

$$i_y = \frac{W_{(IV)(i)y}}{S_{(cmds)(i,W)}} = \frac{1}{S_{(cmds)(i,W)}} \{ - (\dot{Lat})_t$$

$$+ [W_{IE} + (\dot{Lon})_t] [\sin(Lat)_t] [(C)V]_{(t,i)x}$$

$$+ [W_{IE} + (\dot{Lon})_t] [\cos(Lat)_t] [(C)N]_{(t,i)} - p[(C)V]_{(t,i)y} - (U) W_{(gu)y} \} \qquad (6-4)$$

Define the gyro-unit signal compensating for the horizontal projection of the Earth's daily rotation with respect to inertial space as follows:

$$i_W = \frac{W_{(IE)(cp)}}{S_{(cmds)(i,W)}} \cos(Lat)_i \qquad (6-5)$$

The quantities i_x, i_y and i_W may be combined as follows to obtain the indicated ground speed, in a way analogous to Eq. (6–2):

$$v_{(EV)i} = S_{(cmds)(i,W)} R_s \sqrt{(i_x - i_W)^2 + i_y^2} \qquad (6-6)$$

From Eq. (6–3), (6–4) and (6–6),

$$V_{(EB)i} = R_s \{ [(\dot{Lon})_t \cos(Lat)_t + W_{IE} \cos(Lat)_t - W_{(IE)(cp)} \cos(Lat)_i$$

$$- [W_{IE} + (\dot{Lon})_t] [\sin(Lat)_t] [(C)V]_{(t,i)y} + (\dot{Lat})_t [(C)N]_{(t,i)}$$

$$- p[(C)V]_{(t,i)x} - (U)W_{(gu)x}]^2 + [- (\dot{Lat})_t + [W_{IE}$$

$$+ (\dot{Lon})_t] [\sin(Lat)_t] [(C)V]_{(t,i)x} + [W_{IE} + (\dot{Lon})_t] [\cos(Lat)_t] [(C)N]_{(t,i)}$$

$$- p[(C)V]_{(t,i)y} - (U)W_{(gu)y}]^2 \}^{\frac{1}{2}} \tag{6-7}$$

Define

$$(I)W_{(IE)(cp)} = W_{(IE)(cp)} - W_{IE} \tag{6-8}$$

and

$$[(C)Lat]_{(t,i)} = (Lat)_t - (Lat)_i \tag{6-9}$$

From Eq. (6–9),

$$\cos(Lat)_i \cong \cos(Lat)_t + [(C)Lat]_{(t,i)} \sin(Lat)_t \tag{6-10}$$

Substitute the last three relationships into Eq. (6–7), and expand the square root by the binominal theorem, utilizing Eq. (6–2) for $V_{(EB)t}$. Note also th. ·

$$\frac{R_s(\dot{Lon})_t \cos(Lat)_t}{V_{(EB)t}} = \cos(Az)_t \tag{6-11}$$

and

$$\frac{-R_s(\dot{Lat})_t}{V_{(EB)t}} = \sin(Az)_t \tag{6-12}$$

The expansion of the square root, ignoring second-order quantities, then results in the following:

$$V_{(EB)i} = V_{(EB)t} + R_s \{[- (I)W_{(IE)(cp)} - W_{IE} [(C)Lat]_{(t,i)} \sin(Lat)_t$$

$$- [W_{IE} + (\dot{Lon})_t] [\sin(Lat)_t] [(C)V]_{(t,i)y} + (\dot{Lat})_t [(C)N]_{(t,i)}$$

$$- p[(C)V]_{(t,i)x} - (U)W_{(gu)x}] \cos(Az)_t + ([W_{IE} + (\dot{Lon})_t] [\sin(Lat)_t] [(C)V]_{(t,i)x}$$

$$+ \left[W_{1t} + (\dot{Lon})_t \right] \left[\cos(Lat)_t \right] \left[(C)N \right]_{(t,i)} - p\left[(C)V \right]_{(t,i)x}$$

$$- (U)W_{(gu)v} \Big) \sin(Az)_t \Big\} \qquad (6-13)$$

Case II. Self-erecting latitude indicator and celestial-Earth-rate-tracking longitude indicator as sources of ground-speed data.

These indicators are discussed in Derivations 4 and 5, respectively. By differentiation of Eq. (4-6) in Derivation 4, the indicated latitude rate can be given by:

$$p(Lat)_i = (\dot{Lat})_t - p\left[(C)Lat \right]_{(t,i)} = S_{(lds)(i,\Psi)} i_L \qquad (6-14)$$

From equations (5-19) and (5-20) of Derivation 5, if $(CT)_1$ is sufficiently small,

$$p(Lon)_i \approx (\dot{Lon})_t - (I)W_{tds} + (U)W_{(gu)l} + (\dot{Lat})_t \left[(C)N \right]_{(t,i)} \cos(Lat)_t$$

$$+ \left[\sin(Lat)_t \right] p\left[(C)N \right]_{(t,i)} + (\dot{Lat})_t \left[(C)V \right]_{(t,i)x} \sin(Lat)_t$$

$$- \left[\cos(Lat)_t \right] p\left[(C)V \right]_{(t,i)x} = S_{(lds)(i,\Psi)} i_1 \qquad (6-15)$$

For ground speed indication using the foregoing, consider the following expression for $v_{(EB)i}$

$$v_{(EB)i} = R_s \sqrt{\left\{ \left[p(Lon)_i \right] \cos(Lat)_i \right\}^2 + \left[p(Lat)_i \right]^2} \qquad (6-16)$$

These quantities are obtained from the system as follows. Longitude rate is obtained as a current derived from the output of a tachometer geared to the longitude indicator. If the sensitivity of the tachometer and its associated circuits is $S_{T(\dot{l},i)}$ and the output is i_T,

$$p(Lon)_i = \frac{1}{S_{T(\dot{l},i)}} i_T \qquad (6-17)$$

Indicated latitude is proportional to the latitude gimbal drive signal, from Eq. (4-2) of Derivation 4:

$$p(Lat)_i = - S_{(lds)(i,\Psi)} i_L \qquad (4-2)$$

From the foregoing,

$$v_{(EB)i} = R_s \sqrt{\left[\frac{1}{S_{T(\dot{l},i)}} i_T \cos(Lat)_i \right]^2 + \left[S_{(lds)(i,\Psi)} i_L \right]^2} \qquad (6-18)$$

The square root may be evaluated as follows. From Eq. (6-10):

$$\cos(Lat)_i \cong \cos(Lat)_t + \left[(C)Lat \right]_{(t,i)} \sin(Lat)_t \qquad (6-10)$$

so that, from eq. (6–15):

$$[p(\mathrm{Lon})_i]\cos(\mathrm{Lat})_i = (\dot{\mathrm{Lon}})_t\cos(\mathrm{Lat})_t + (\dot{\mathrm{Lon}})_t[(C)\mathrm{Lat}]_{(t,i)}\sin(\mathrm{Lat})_t$$

$$- (I)W_{tds}\cos(\mathrm{Lat})_t + (U)W_{(gu)I}\cos(\mathrm{Lat})_t + (\dot{\mathrm{Lat}})_t[(C)N]_{(t,i)}\cos^2(\mathrm{Lat})_t$$

$$+ [\sin(\mathrm{Lat})_t\cos(\mathrm{Lat})_t]\,p[(C)N]_{(t,i)} + (\dot{\mathrm{Lat}})_t[(C)V]_{(t,i)x}\sin(\mathrm{Lat})_t\cos(\mathrm{Lat})_t \qquad (6\text{–}19)$$

From Eq. (6–2),

$$v_{(EB)t} = R_s\sqrt{[(\dot{\mathrm{Lon}})_t\cos(\mathrm{Lat})_t]^2 + (\dot{\mathrm{Lat}})_t^2} \qquad (6\text{–}2)$$

Therefore, ignoring small quantities in the binominal expansion of the square root,

$$v_{(EB)i} = v_{(EB)t}$$

$$+ R_s\Big\{\Big[(\dot{\mathrm{Lon}})_t[(C)\mathrm{Lat}]_{(t,i)}\sin(\mathrm{Lat})_t - (I)W_{tds}\cos(\mathrm{Lat})_t$$

$$+ (U)W_{(gu)I}\cos(\mathrm{Lat})_t + (\dot{\mathrm{Lat}})_t[(C)N]_{(t,i)}\cos^2(\mathrm{Lat})_t$$

$$+ [\sin(\mathrm{Lat})_t\cos(\mathrm{Lat})_t]\,p[(C)N]_{(t,i)}$$

$$+ (\dot{\mathrm{Lat}})_t[(C)V]_{(t,i)x}\sin(\mathrm{Lat})_t\cos(\mathrm{Lat})_t\Big]\cos(\mathrm{Az})_t$$

$$- [p[(C)\mathrm{Lat}]_{(t,i)}]\sin(\mathrm{Az})_t\Big\} \qquad (6\text{–}20)$$

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