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CORG-SP-128

HISTORICAL DATA
AND
LANCHESTER'S THEORY OF COMBAT

by

R. L. Helmbold

This work was sponsored by the Combat Developments Section, Headquarters, United States Continental Army Command, Fort Monroe, Virginia under Department of the Army Contract No. DA-19-020-ORD-5351.

1 July 1961

This paper represents the individual views of the author and does not necessarily represent the opinion or approval of the Combat Operations Research Group, the United States Continental Army Command, or of the Department of the Army.

ABSTRACT

The purpose of this paper is to search for empirical relations among various quantitative aspects of ground combat suggested by Lanchester's square law model. The motivation for considering this problem is discussed at some length in the introductory section. Theoretical and methodological issues bearing on the problem are discussed. Data on 92 historical battles are studied for information bearing on the problem. The principal results are contained in several tables and figures and are exhaustively evaluated. Areas where these results might be applied are indicated.

FOREWORD

"War is a science replete with shadows in whose obscurity one cannot move with an assured step. Routine and prejudice, the natural result of ignorance, are its foundation and support.

All sciences have principles and rules. War has none. The great captains who have written of it give us none. Extreme cleverness is required even to understand them. And it is impossible to base any judgment on the relations of the historians, for they only speak of war as their imaginations paint it. As for the great captains who have written of it, they have attempted rather to be interesting than instructive, since the mechanics of war is dry and tedious."

(Marshal Maurice De Saxe, Reveries on the Art of War, 1757)

"The principles of war do exist; but our present concepts of them are inadequate, misleading, and of exaggerated practical use . . . .

The study of history is only of use when we can convert the hindsight of past analysis into foresight for future action; otherwise this study is merely pedantic. To develop this foresight the chain of causes and effects must be discovered. This is exactly what scientific principles enunciate . . . .

As the principles of war stand they do not show the relationship between cause and effect, between their application and victory in battle. For instance, we cannot say that if offensive action is applied victory will result. We cannot even say that victory will be probable, because offensive action may not be the right principle to emphasize under the circumstances. Further it may even be that the application of this principle will contribute to defeat. So much for the practical value of the principles of war as they stand today . . .

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The (scientific) principles of war can be discovered but have not yet been discovered."

(Major J. Nazareth, Two Views on the Principles of War, Military Review, February 1961, published by the United States Army command and General Staff College)

" 'Weapons change, but the principles of war and our geographical situation remain constant.' One has seen a statement similar to the above many times. Such dogmatic statements are common in military journals, and before being accepted at their face value should be examined very critically.

In the first place, it is difficult to discover what these principles are.

A glance at the accompanying chart shows that at various times in the last hundred years or so, at least 24 principles of war have been put forward; and of the lists shown on the chart, no two are the same. Further research would no doubt reveal many more lists, all different. The exponents of most of these lists have stated quite categorically that 'this is the list of the principles of war.' One wishes at least that they had the modesty of the writers of the Holy Gospel, who did not state that theirs was The Gospel but only the Gospel according to the writer."

(Major M. J. W. Wright, Two Views on the Principles of War, Military Review, February 1961, published by the United States Army Command and General Staff College)

"The growing complexity of modern warfare has led some students to take a fresh look at the principles that have traditionally guided military strategists in war . . . .

In number, the stated principles have varied from writer to writer, from period to period, and from nation to nation . . . .

The individual authors of the lists have almost uniformly claimed the principles to be immutable . . . .

Other authorities have argued that the claim of immutability cannot be accepted literally; that there is little agreement as to what the principles are and mean; that they overlap; that they are fluid and require constant re-examination; that they are not comparable with scientific laws since no two military situations are ever completely alike; that the so-called principles are not really principles at all, but merely methods and common-sense procedures adopted by great commanders of the past; and that changes in the conditions of war alter their relative importance.

The debate over principles has been renewed with the coming of the atomic era. Some theorists argue that the new weapons have destroyed whatever infallibility still remain; others contend that the principles are as valid as ever, even more so. To some extent this is a debate over semantics. Defenders point out that each age must make its own applications of the 'fundamental truths' of strategy. Opponents argue that there can be no set rules for the art; the so-called principles must by no means be interpreted as pat formulas for victory to be followed blindly and rigidly; the only sound guide in war and strategy is flexibility."

(Maurice Matloff, Article on Strategy, Encyclopedia Brittanica)

"To the statistically-minded these scraps of evidence are provocative rather than convincing. What is needed is a comprehensive survey by counting, for counting is an antiseptic against prejudice . . . .

A fundamental rule of scientific method is Ockham's so-called 'Razor,' to the effect that: 'Entities are not to be postulated without necessity.' For shaving off the superabundant growth of mathematical uncertainties and difficulties I have made frequent appeal to an analogous rule: 'Formulae are not to be complicated without good evidence.' This is a difficult and groping empiricism . . . .

Whoever says that these results are rough should compare them with our previous blank ignorance."

(Lewis Fry Richardson, Statistics of Deadly Quarrels)

CONTENTS

HISTORICAL DATA
AND
LANCHESTER'S THEORY OF COMBAT

INTRODUCTION

Purpose

To search for empirical relations among various quantitative aspects of ground combat.

Background

Military Aspects

Through the ages much has been written concerning the nature of war and the manner in which it should be conducted. Nevertheless, among modern practitioners of the military art, there is a division of opinion regarding the interpretation and relative importance of the various so-called "principles of war." (See Ref. 1.) This division of opinion and the resultant uncertainty concerning the fundamental elements of ground combat can hardly fail to have an adverse effect on many U. S. Army activities, including (and perhaps, especially) those of concern to the Combat Developments system. Therefore, it is considered important to try to elucidate the relationships among various aspects of ground combat.

Technical Aspects

The past few years have seen the growth of operations research and similar activities devoted to the application of formal or "scientific" techniques to a wide variety of military problems. At least one facet of this work represents a significant departure from classical methods -- namely, the introduction and use of relatively simple mathematical equations which purport to describe the salient features of ground combat.[1] Many of these combat models consist of modifications or extensions of Lanchester's model,[2] a detailed discussion of which, emphasizing those features of interest to this study, is contained in the sequel. Here it suffices to note that most of these models are based on the so-called "square law," or its analogues. In effect this states that the combat power of a military organization is proportional to the square of the number of combatants in that organization.

Despite the frequency with which such "square law" models are proposed, only one detailed comparison of theoretical predictions with observed combat data is known to this writer.[3] This comparison showed that the theory originally given by Lanchester (save for a straight-forward modification to represent the landing rate of fresh troops) is in good agreement with data available on the World War II battle of Iwo Jima (Ref. 9). Therefore, the relation of such models to observed combat phenomena is an important area of investigation.[4]

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[1] Here and in the following we refer to such mathematical theories as "models," in accord with prevailing operations research jargon. The reader is cautioned that in other contexts the term "model" may include any or all of a wide variety of representations of combat or other phenomena.

[2] See Reference 2. [illegible] parts of Reference 2 containing the development of Lanchester' [illegible] del are reproduced in Reference 3. Reference 4 also discusses [illegible] hester's theory. For exampl[illegible] of theories inspired by Lanche[illegible] work, see References 4, 5, 6, [illegible] 9.

[3] Referenc[illegible] A less detailed comparison of the square law model with combat data [illegible] Reference 5, where it is shown that the data lends s[illegible] tion to the square law.

[4] I[illegible] that data is the body, and theory the soul, of scientific endeavor. It then follows that without adequate data, though the spirit is willing, the flesh is weak.

Scope

Presumably, difficulties in collecting the necessary detailed data have hampered further attempts to validate the square law,[5] although Engel (Ref. 9) clearly recognizes the potential value of additional effort along these lines. In particular, Engel notes that when a sufficiently large number of specific combat situations have been investigated, it may be possible to determine the parameter values characteristic of various classes of combat situations and to discover relationships between these parameters and other factors associated with the various classes of situations. Engel pointedly remarks that "such relationships will be of particular value if it is possible to measure these other factors prior to the inception of a particular engagement." The

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[5]The amount of detail required for model validation is indicated by the following list of information used in Reference 9:

(a) The total number of friendly troops put ashore each day (no friendly troops ashore prior to the beginning of the engagement)

(b) The total number of friendly casualties each day and, separately, those killed in action

(c) The number of enemy troops ashore at the beginning of the engagement

(d) The time the island was declared secure (after this time, although the battle continued, it may have done so at a different rate)

(e) The time the engagement ended (after the island was declared secure)

(f) The number of enemy troops at the end of the engagement (zero if all were destroyed)

(g) To which we add: The number of enemy troops put ashore each day (in this case, none).

The recurring phrase "each day" in the above list indicates the degree of detailed required for a test of square-law validity. (If enemy reinforcements and friendly or enemy non-battle losses are non-negligible, then the time history of enemy casualties and, separately, non-battle losses must also be known in order to test model validity.)

crucial first step in carrying out this program has also been clearly identified by Engel -- a determination of the range of validity of alternative forms of Lanchester-type models. Unfortunately, as indicated earlier, successful execution of this important first step is directly dependent on the availability of a large body of sufficiently detailed data which do not seem to be readily available.

Although not emphasized by Engel, a certain amount of progress on other phases of the indicated program may be possible, even in the absence of detailed data, provided we tentatively assume, as a working hypothesis, the validity of some form of Lanchester-type model. It is the purpose of this study to attempt just such a program: adopting the validity of the Lanchester square law as a working hypothesis and point of departure. Reasons for adopting the square law hypothesis rather than some other are as follows:

i. It is relatively easy to deal with mathematically. (Considerations of analytical simplicity are not viewed as final. In exploratory studies, such as this, easily handled models are used to expedite the work.)

ii. At least one battle (Iwo Jima, mentioned above) has been shown to follow the square law predictions.

iii. The square law might be viewed as an approximation, obtained by replacing in the correct model, discrete by continuous variables (or random variables by their average values), and linearizing the result. If there is any truth in this view, then the square law represents at least a first approximation to the correct model.

iv. The complexity of the square law, as reflected in the number of parameters, does not exceed the amount of detail obtainable from more-or-less readily available sources of data. That is, the number of unknown constants is small enough that we can determine all of them from such data as is available.

v. Even if the square law is not a valid model, if the features of combat represented by the square law are in fact related to other factors which can be determined prior to the initiation of combat, then we may be able to at least define predictors which permit a forecast of those features once the appropriate factors are known. In other words we may be able to make assertions such as, "If you are going to use the square law for predictions, then by measuring certain factors and performing certain operations you will obtain a prediction of such-and-such an aspect of combat." (Note that we may not understand why such predictors work unless and until additional theoretical insight becomes available.) Discussions of other factors influencing the scope of the study are contained in pp. 10 - 17 below.

METHOD

Exposition of Method

Theory

The available data consists, for the most part, of information on a number of historical battles. The sources from which these data were assembled usually confined discussion to the following:

a. Date and place of occurrence of a battle.

b. Forces present on or in the immediate vicinity of the battlefield during the engagement.

c. A narrative account of the salient features of the action, normally including:

(1) An identification of one side as attacker and the other as defender.

(2) Duration of the engagement.

d. A statement of the outcome, including:

(1) Identification of one side as victorious and the other as defeated.

(2) Losses to each side.

For the most part, no detailed information regarding the rate of reinforcement is available. Since most of the data assembled by the author to date involve battles of short duration, the process of reinforcement will be neglected. Operational attrition will also be neglected.

It is then possible to write Lanchester's equations as:

$$\begin{aligned} dx/dt &= \dot{x} = -Dy \\ dy/dt &= \dot{y} = -Ax \end{aligned} \qquad (1)$$

which is the form adopted for study in this paper. These equations assert that the instantaneous rate of attrition to attacking troops ($\dot{x}$) is proportional to the number of defending (y) troops, and the instantaneous rate of attrition to defending troops ($\dot{y}$) is proportional to the number of attacking troops (x). The proportionality parameter D and A respectively represent the rate of attacker (defender) attrition per defender (attacker) per unit time. For convenience in the following discussion, these and other important parameters are given (more or less whimsical) names. Thus, D will be known as "defender's activity" and A will be termed "attacker's activity." With this terminology, for example, the first member of Equation (1) may be read as "The instantaneous rate of attrition to the attacker is jointly proportional to defender activity and the number of defending troops."

Let x_0 and y_0 be (respectively) the attacker's and defender's pre-battle or "initial" strengths. Then Equations (1) may be rewritten as:

$$\dot{a} = -\delta d$$
$$\dot{d} = -\alpha a \tag{2}$$

where we have set:

$$x/x_0 = a \text{ (surviving fraction of attacker)}$$

$$y/y_0 = d \text{ (surviving fraction of defender)}$$

then,

$$D\, y_0/x_0 = \delta$$

$$A\, x_0/y_0 = \alpha\ .$$

In addition to the parameter names indicated above, x_0/y_0 will be called the (initial) "force ratio" of attacker to defender. Since, at the initiation of combat, $a = d = 1$, the parameters α and δ represent the initial fractional attrition rates to defender and attacker respectively.

Dividing the first member of Equation (2) by the second, separating variables, and integrating, we obtain:

$$\mu^2 = \frac{1 - a^2}{1 - d^2}\ , \tag{3}$$

where we have set $\mu^2 = \frac{\delta}{\alpha}$. In this form we see that μ is related to the relative advantage of the two sides. In particular, if $\mu > 1$, then the surviving fraction of attackers goes to zero before the surviving fraction of defenders; if $\mu < 1$, then the surviving defender fraction is the first to reach zero. This suggests that we define the (defender's) "advantage parameter", V, by:

$$V = \ln \mu \tag{4}$$

where "ln" denotes "natural logarithm of."

Then it can be shown[6] that a positive defender advantage means that the attacker's fractional attrition rate is greater than the defender's; a negative defender advantage means that the defender's fractional attrition rate is greater than the attacker's; a zero advantage denotes equality of attacker's and defender's fractional attrition rates.

By referring to the definitions of α, δ, and μ, we may write Equation (3) as:

$$D/A = \frac{1 - a^2}{1 - d^2} (x_0 / y_0)^2 \tag{5}$$

thus determining D/A, the "activity ratio," in terms of the surviving fractions and the force ratio.

The complete solution of Equations (2) may be written as:

$$\begin{aligned} a &= \cosh \epsilon - \mu \sinh \epsilon \\ d &= \cosh \epsilon - \mu^{-1} \sinh \epsilon \end{aligned} \tag{6}$$

where μ is defined by Equation (3), and $\epsilon = \lambda t$; where $\lambda = \sqrt{\alpha \delta}$, and t is the duration of the engagement. Referring to the definition of α and δ, we see that

$$\lambda = \sqrt{AD}$$

i.e., λ is the geometric mean of defender and attacker activity. Since it represents an average activity, it seems reasonable to call λ the "intensity" parameter and to interpret it as, in some sense, a measure of the average intensity of battle. With this interpretation of λ, ϵ may be interpreted as the product of battle duration and average battle intensity. As such, it seems appropriate to call ϵ the "bitterness" parameter and interpret it as, in some sense, a measure of the bitterness of battle.

E.g., by inspecting the time derivatives of a and d calculated from Equations (6).

By writing the hyperbolic functions in Equations (6) as exponentials, it is possible[7] to derive the following expression for the bitterness parameter in terms of surviving fractions and μ :

$$\epsilon = \ln \left(\frac{1 + \mu}{a + d\mu} \right) \qquad (7)$$

Summarizing the results obtained, we see that if the historical data includes items b, c1, and d2 of those given earlier, then it is possible to use this data to estimate the following parameters:

1. Force ratio (x_0 / y_0)
2. Surviving fractions (a and d)
3. Advantage (V or μ)
4. Activity ratio (D/A)
5. Bitterness (ϵ).

If the data also give the duration of the engagement, then the following additional parameters may be estimated:[8]

6. Intensity (λ)
7. Initial fractional attrition rates (α and δ)
8. Attacker and defender activities (A and D respectively).

---

[7] See Appendix A.

[8] This comes about as follows:

$\lambda = \epsilon / t$ by definition of ϵ. By definition of λ and μ, we then can estimate $\alpha = \lambda\mu$ and $\delta = \lambda\mu$. D and A may then be estimated, since by definition of α and δ we must have:

$$D = \delta\, x_0 / y_0 \text{ and } A = \alpha\, y_0 / x_0 .$$

Methodological Principles

The general features of the method are easy to state. We collect a sizable body of data dealing with historic battles, use this data to estimate the parameter associated with each battle, and search the results for regularities. Unfortunately, this program, in the form given above, ignores several important methodological issues.

First, practical considerations limit the search for data to battles in a restricted area of space and time, and, even in a restricted area of space and time, data on some battles may be either incomplete or in error. As a result, it may be that the detected regularities are spurious, i.e., could be due to accidental circumstances influencing the selection of and errors in the data. In other words, the regularities supposedly "detected" might disappear or be altered in some way if the data included all battles and/or if the data contained no errors or other random fluctuations.

Second, even if we detect <u>some</u> regularities, we may not have found <u>all</u> of the regularities lurking in the data.

Third, information on past battles may not be applicable to future battles, in which case extrapolation on the basis of detected regularities would be improper.

The author is incapable of resolving these difficulties to the satisfaction of all possible readers. In fact, considerable scientific and philosophical debate still revolves around the methodological issues raised above. The author will state his views on these problems as concisely as possible, but the reader will have to determine for himself whether or not these views are acceptable to him. We will take up these issues in reverse order.

If information on past battles is not applicable to future battles, then extrapolation of detected regularities is improper. This is admitted. Nevertheless, we have no way to judge the future save by our present understanding of the past. While opinions may differ with regard to the meaning assigned to the term "understanding," there is general agreement that increased understanding of the past has been and will continue to be useful.

Whether the data and our understanding of it are sufficiently comprehensive to justify extrapolation into the future is largely a matter of individual judgment as to the nature and degree of change in future conditions as compared to past conditions. Nevertheless, a relation which holds for the past, over a wide range of conditions, or which shows a constant trend for an extended period of time, is more likely to hold in the future than one which is unpredictably dependent on circumstances.

If we can detect <u>some</u> regularities in the data, then we have probably achieved a certain amount of understanding of the past. If we could detect <u>all</u> regularities, then our understanding of the data would leave nothing more to be discovered, and would, in that sense, be perfect. The work presented here is not to be viewed as the acme of perfection or completeness. Far from being finished, the work is hardly begun. If it is of value for some aspects of practical problems or if it smooths the way for a deeper understanding of some of the problems of warfare, then it has achieved all that could reasonably be expected of it. The position taken in the above discussion is hardly distinguishable from that adopted by Clausewitz (Ref. 12).

"If theory investigates the things that make up war, if it separates more distinctly that which at first sight seems confused, if it explains fully the properties of the means, if it shows their probable effects, if it clearly defines the nature of the ends in view, if it sheds the light of a deliberate, critical observation over the whole field of war -- then it has achieved the main object of its task. It then becomes a guide to whoever wishes to become familiar with war from books; it everywhere lights up for him the road, facilitates his progress, educates his judgment, and keeps him from going astray . . . .

So it turns to experience and directs its attention to those precedents which military history already has to show. In this way it will certainly be a limited theory, which only fits the circumstances as military history present them. But this limitation is from the first inevitable, because, in every case, what theory says of things, it must either have abstracted from military history, or at all events compared with that history. Besides, such limitation is in any case more theoretical than real.

One great advantage of this method is that theory cannot lose itself in subleties, hair-splittings and chimeras, but must remain practical . . . .

Examples from history make everything clear, and in addition, they afford the most convincing kind of proof in the empirical fields of knowledge. This applies to the art of war more than to anything else."

With regard to the possibility of erroneous or spurious regularities, we will make use of Richardson's principle of historical data samples (Ref. 10).

"Let us assume, as a working hypothesis, that every finite set of historical events is only a sample of what might have happened. Any quantitative theory of history is therefore not required to agree precisely with actual historical events, but to agree only within the range of uncertainty ascribable to sampling."

Adoption of this principle, in effect, arms us with all the tools and techniques of modern statistical theory. Use of these techniques will not, to be sure, eliminate <u>all</u> possibility of spurious regularities, but their application does provide a framework within which these problems can be discussed, together with methods for estimating the probability that a given regularity is spurious. Further testing of possible regularities by independent investigators and the study of further data in the form of additional independent historical samples will also help to decrease the likelihood of spurious regularities.

To summarize our discussion of methodological principles, we may say that the proposed procedure amounts to neither more nor less than an application of the so-called "scientific method," as the author understands it, using military history as a source of data and aided by modern statistical techniques. As such, the proposed procedure carries with it the same advantages and disadvantages as any other application of the "scientific method."

Limitations of the Method

Several limitations have been discussed in the Introduction (pp. 3 - 5) and in the paragraphs immediately above (pp. 10 - 13). These include the limitations imposed by the inability to consider more than a limited sample of battles, the inability to detect all regularities contained in even a limited sample, and the uncertainties involved in extrapolating to the future such regularities as may be discovered. Also included is the self-imposed restriction to discussing the data in terms of the classical Lanchester square-law model, including the neglect of detailed consideration of reinforcements and operational attrition.

The reader is cautioned that this study in no sense provides a test of the validity of Lanchester's square-law model of combat. As noted in the Introduction (pp. 3 - 5), such a validity test requires highly detailed information on the casualty experience of each combatant. Other than the data used in Reference 9, the author is not aware that any such data exists in the unclassified literature.[9]

The author considers that the quality of the data used is a possible limitation on the results obtainable by the method, and therefore will briefly discuss some of the difficulties encountered in the collection and

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[9] For some data available from classified sources, see Reference 14

use of data.[10] Four categories of data appear to merit discussion. These are (1) identification of attacker and defender, (2) initial strength, (3) casualty estimates, and (4) duration of engagement.

Identification of Attacker and Defender

In some battles (e.g., Bunker Hill) it seems to be quite clear which side is the attacker and which the defender. In others, it is considerably less clear. In reviewing narrative accounts of action, the author found that in some instances a force attacked the enemy but, finding themselves out numbered, "were immediately thrown on the defensive." Also, for engagements over extended intervals of time, a pattern of attack, counter-attack, counter-counter-attack, etc., is common. In cases like these, identification of one side or the other as attacker must be nominal at best. The identifications given in Table I are based either on the identifications given in the sources consulted or on the author's impression, based on the narrative accounts, of which side behaved most aggressively. When allied nations participate on the attacking or on the defending side, an attempt was made to present identifications which correspond either to the nationality of the largest body of troops or to the nationality of the commander of the allied forces.

Initial Strength

To some extent, an attempt was made to include in Table I only the number of personnel "engaged" on each side, although not all of the sources consulted unambiguously indicate how many were engaged as opposed to the number present somewhere in the general battle area. The narrative accounts of some battles indicate that sizable bodies of troops, though not greatly distant from the battlefield, were not engaged for one reason or another. In addition, it is sometimes difficult to decide how to count the influence of reserves and maneuvering elements.

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[10] Discussion of data in relation to specific sources is contained in Appendix B.

For example, in the battle of Chancellorsville, Jackson took a large force (about 26,000 men) and marched westward on a 15-mile detour to envelop the Federal right flank. In this, as in similar cases, maneuvering elements and an unknown fraction of reserve troops are counted as part of the initial force strength. (See Table I.)

As might be expected, the trustworthiness of much of the data is questionable. For example, Reference 11 states that in the battle of Bunker Hill "about 1,200 men" were defending and were attacked by "between 2,000 and 3,000" British. In the battle of Palo Alto, Taylor estimated his opponent's strength as 6,000 (Ref. 15). Reference 11 merely states that Arista's strength has been "variously estimated at between 4,000 and 6,000." Reference 15 says that Arista gave his strength as about 3,000 and adds that this is "probably about right." With regard to the data on American Civil War battles, Reference 15 repeatedly remarks that the number engaged is difficult to determine.

A third possible source of error in estimated initial strengths arises from the relatively superficial treatment of reinforcement. As mentioned before, we have included the total number of troops on or near the battlefield at any time with the initial strength figures, which is tantamount to assuming that the reinforcements were engaged from the beginning. However, in nearly all cases, examination of the influence of reinforcements, detachments, maneuver elements, reserves, etc., in greater detail than is undertaken here would require more precise data than can be found in readily accessible historical treatments of battles.

Casualty Estimates

In addition to participating in some of the sources of error described above for initial strength data, other errors or confusions are possible here.

Since, as we have seen, initial strength values are often in error, and since, in some instances, casualties may have been estimated by subtracting "strength after battle" from "strength before battle," the casualty values may in some cases represent the result of subtracting one large inaccurate number from another large inaccurate (but approximately equal) number. It is well-known that this sort of procedure can result in extremely large errors.

Although perhaps not a source of error as such, confusion can readily be created (and the difficulty of data interpretation increased) by variations in the criteria adopted for a "casualty." The following criteria, or combinations thereof, seem to be generally used:

a. Operational attrition -- including illnesses, accidents, administrative transfers, desertion, confusion, etc.

b. Missing in action -- MIA

c. Prisoner of war -- POW

d. Wounded in action -- WIA (including died of wounds -- DOW)

e. Killed in action -- KIA

Although an attempt has been made to include in the casualty data only the estimated value corresponding to the combined category WIA + KIA, in some instances only the MIA + WIA + KIA figure was available from the data sources consulted. In some cases the number MIA may be small so that lumping it with KIA + WIA does not appreciably alter the estimated casualty value. In some instances the references gave only a figure labelled as "losses," and it would seem from the context that "losses" were to be interpreted as MIA + POW + WIA + KIA, although exactly what the author had in mind is often unclear. Webster's New Collegiate Dictionary defines "Loss, (mil.); The losing of soldiers in battle or by surrender; also, chiefly in plural, killed, wounded, or captured soldiers." The term "losses" is not defined in AR-320-5

Dictionary of U.S. Army Terms, although "battle casualty" is, and includes those "killed, wounded, missing, captured, or interned."

Duration of Engagement

In general, the duration of an engagement is difficult to determine. For example, should the time occupied by the skirmishes and minor fights preceding the main battle of Gettysburg be counted as part of the duration of the battle or not? In rare instances (e.g., the account of the battle of Austerlitz in Reference 11), the reference asserts that the stated number of casualties occured in a certain period of time. Usually, only vague indications are given, such as "the battle opened on the morning of . . ." and was "over before nightfall on . . . ." In some cases (e.g., the Pacific Island amphibious operations) the beginning of the land battle is quite clear-cut, although the effects of the preceding Naval and Air operations may not be adequately represented if the occasion of landing troops is taken as the initiation of the battle. Moreover, it is notoriously difficult to select a time 'ich represents the end of these island campaigns.

The estimates of battle duration used here represent, for the most part, this author's best guesses, on the basis of limited study of the narrative accounts, as to the initial and terminal times. The reader is cautioned that, for some of the older battles, the duration has been adjusted under the assumption of no night fighting. Thus, for example, the Seven Day's Battles are assigned a duration of 70 hours (7 days with 10 hours of usable daylight each day). A like procedure was not adopted for the World War II data, nor in other cases when a reading of the narrative account did not seem to clearly justify the assumption of negligible night fighting.

RESULTS

Data

Data on the historical battles treated in this report were assembled from References 9, 11, 15, 16, 17, 18, and 19.[11] The principal quantitative components of this data, together with the identifications of attacker and of victor, are presented in Table I. (Except for the data used in the comparison of sources, for which see Appendix B. Further discussion of data and treatment refers to that in Table I, unless a specific assertion to the contrary is made.)

1. Population and Sample[12]

In accord with Richardson's "Principle of Historical Data Samples," we consider this collection of data as a sample from some population. As often happens, the exact nature of the population is virtually impossible to determine. We will describe certain features of the sample which might be of assistance in judging the nature of the population.

a. Sample Size

Table I contains data on a total of 92 distinct historical battles. For each of these battles we list the name or designation of the battle, the year in which it occured, the source of data used, the

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[11] Reference 17 does not provide either narrative accounts or an identification of attacker and defender. Where necessary, this lack is supplied from either Reference 11 or 18. In this report, the only use made of Reference 18 is in supplying identifications of attacker and defender to be used with other battle data obtained from Reference 17, and as a source of data for the comparison of data sources given in Appendix B. Also in this report, Reference 19 is used only for the comparison of sources in Appendix B.

[12] See Reference 13 for a discussion of the concepts involved in the terms "population" and "sample."

identifications of attacker and defender, the estimated initial troop strength on each side, the estimated casualties suffered by each side (together with a notation approximately identifying the casualty criterion involved), and an identification of the victor. For 82 of these battles, an estimate of the time duration is also supplied.

b. **Distribution of Sample Data in Time**

Although battle dates vary from 1741 A. D. to 1945 A. D., all but 5 of the 92 battles occured between 1757 and 1877, inclusive Figure 1 shows the distribution of battles in time by twenty-year intervals. The high degree of clustering reflects the occurrence of certain periods of general military activity, such as the Napoleonic Wars, the Wars of Frederick the Great, etc.

c. **Distribution of Sample Data in Space**

Of the 92 sample battles, 68 were fought in the Eurasian area, 20 in North America, and 4 on Pacific Ocean islands. These battles took place on terrain of various types, including the open Plains of Abraham near Quebec, the wooded thickets of the Wilderness battles, the hills and caves of Iwo Jima, and the cultivated fields near Waterloo.

d. **Distribution Among Countries**

The governments which participated in these battles and the approximate frequency of participation[13] in numbers of battles, were:

France	40	battles
Prussia or Germany	36	"
Austria	30	"
United States of America	23	"

[13] It is difficult to know how much weight to give to participation with allies under a United Command.

Russia	14	battles
Confederate States of America	12	"
Britain	7	"
Turkey	5	"
Mexico	4	"
Japan	4	"
Piedmont	3	"
Hungary	1	"
Spain	1	"
Sardinia	1	"
Denmark	1	"
Hanover	1	"

e. **Magnitude of the Sample Battles**

In terms of X, the total number of troops involved ($X = x_0 + y_0$), the smallest of the 92 battles is Cowpens with X = 2,000, and the largest is Leipzig with X = 472,500. The overall total forces (the sum of X's for all 92 battles) amounts to 19,784,873, and the average total force (average of X's over 92 battles) is about 117,227.

In terms of C, the total number of casualties ($C = C_x + C_y$), the smallest of the 92 battles is Cowpens with C = 303, and the largest is Leipzig with 93,000. The overall total casualties (the sum of C's for all 92 battles) amounts to 1,431,898, and the average total casualties (average of C's over 92 battles) is about 15,564.

In terms of F, the total casualty fraction (F = C/X), the smallest of the 92 battles is Temesvar with F = 0.007, and the largest is Iwo Jima with F = 0.448. The average total casualty fraction (average of F's over 92 battles) is about 0.138.

In terms of t, the duration of battle, the shortest of the 82 battles for which this datum is recorded are Rossbach and the break-out at Plevna, each with a duration of 2 hours, and the longest is Iwo Jima with 864 hours (36 days). The overall total battle duration (the sum of t's for all 82 battles) amounts to 3,651.5 hours, and the average battle duration (average of t's over 82 battles) is about 44.5 hours.

In terms of M, the number of battle man-hours (M = Xt), the largest of the 82 battles for which this datum is available is Iwo Jima with M = 81,648,000, and the smallest is Lundy's Lane with M = 25,000. The overall total battle man-hours (the sum of M's for all 82 battles) amounts to 377,308,000, and the average number of battle man-hours (average of M's over 82 battles) is 4,601,000.

f. Numerical Superiority

Of the 92 battles the one with the greatest force ratio (x_0/y_0) is Iwo Jima with $x_0/y_0 = 3.395$, and the one with the least force ratio is Plevna with $x_0/y_0 = 0.250$. The average force ratio (over 92 battles) is 1.323.

g. Victorious Side

In 47 of the 92 battles the attacker is given credit for the victory. In the remaining 45 battles the defending side is counted as victorious.

2. Errors in the Data

In order to obtain an estimate of the variability attributable to differences between sources, data from a few more-or-less independent sources was obtained for each of 7 separate battles. These data and the Lanchester parameters estimated from them are discussed in detail in Appendix B. It is sufficient to note here that the sample data consists of estimates of the combat situation rather than the situation itself. As such, different sources present different estimates.

Although the variability between estimates could presumably be reduced by reference to primary, rather than secondary, sources of information, it seems likely that some variability and uncertainty would always remain. Consequently, it is necessary to estimate the amount of this variability and to consider its implications on various aspects of the program being pursued. This is done in the development and discussion of results.

3. Combat Parameters

Lanchester parameters for each battle were estimated from the data of Table I by the procedures outlined in Chapter II, Method, and are tabulated in Table II.

Other combat parameters are introduced and defined as needed in a manner similar to that adopted above for the discussion of battle magnitude.

Findings

As noted in Chapter II, Method, we neither claim nor expect that we have uncovered all the relationships and regularities concealed in the data. It is appropriate to note here that many of the findings presented in the sequel are negative in nature, i.e., we assert that certain regularities or relationships are <u>not</u> supported by the data under investigation. These negative findings are often as important as the positive findings in contributing to an understanding of combat and constitute an essential part of the findings. In addition, some of our findings will be uncertain or indefinite, i.e., we assert that certain regularities are neither clearly supported or denied by the data under investigation. Results of this nature constitute problems for further study.

1. Correspondence Between Parameters and Phenomena

In Chapter II, Method, names were given to the theoretical Lanchester parameters (e.g., bitterness for ϵ and advantage for $\ln \mu$). The names assigned suggest a correspondence between the theoretical symbols appearing in Lanchester's theory and certain phenomena occuring in the real world. It is important to determine whether the implied correspondences are valid or whether they are merely fictitious and misleading. We will investigate the situation for the bitterness and advantage parameters.

a. Bitterness

We begin by investigating whether the Lanchester parameter, ϵ , can legitimately be considered as an index of battle bitterness. Figure 2 shows a scatter diagram (See Ref. 13 for definition.) of ϵ versus F (in other words, of bitterness versus total casualty fraction), together with a graph of the function[14] $e^F - 1$. The data points closely follow the plotted curve for total casualty fractions less than 0.4, but apparently tend to diverge from it for larger total casualty fractions. Some divergence is expected, since we learn from Appendix A that

$$\epsilon = \cosh^{-1}\left(\frac{1 + a d}{a + d}\right)$$

which implies that when a and d are both small (in which case F must be close to unity) ϵ must be large. In fact, as a and d both approach zero, F approaches unity but ϵ approaches infinity. As F approaches unity, however, $e^F - 1$ approaches $e - 1$, or approximately 1.7, rather than infinity. Hence, on theoretical grounds, we expect the data points to diverge from the curve $e^F - 1$ when F is large.

Since the total casualty fraction may reasonably be taken as a measure of battle bitterness, we conclude that, at least for a wide range of total casualty fraction values, the Lanchester bitterness parameter, ϵ , adequately reflects the more usual, intuitive concept of combat bloodiness, bitterness, and the like. We may then also conclude

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[14]Here and in the following, e will represent the base of the system of natural logarithms and is approximately equal to 2.718281828 . . . .

that the Lanchester intensity parameter, reflects the more usual concept of combat intensity, activity, and the like, since by definition we have $\lambda = \epsilon / t$, where t is the duration of battle.

b. Advantage

Our problem here is to decide whether the so-called "advantage" parameter, $\ln \mu$, can legitimately be considered as an index of defender's relative advantage. Table IIIa shows the results of a tabulation of the number of battles won by side and by the sign of $\ln \mu$ (if $\ln \mu > 0$, then the defending side theoretically has the advantage; if $\ln \mu < 0$, the attacking side theoretically has the advantage). Table IIIb displays the same information expressed as a percentage of the number of victories by the respective sides.

We say that the Lanchester advantage parameter, $\ln \mu$, "follows the victor" in a battle if the sign of $\ln \mu$ is "right" (i.e., positive when defender wins; negative when attacker wins). Thus, Table IIIa shows that the Lanchester advantage parameter follows the victor in 68 (73.9%) of the 92 battles and does not follow the victor in 24 (26.1%) of the battles.

It is important to ask whether it is likely that these results are spurious or not. Table IIIa, especially when put in the form of Table IIIb, appears to indicate that $\ln \mu$ is more likely to be positive than negative when the defending side wins, and more likely to be negative than positive when the attacking side wins, but how sure are we that this interpretation is justified? To answer questions of this nature, statisticians frequently use the so-called Chi-square test. (See Reference 13. Note that we are dealing with what is technically known as a 2 x 2 contingency table, so Yates's correction will be applied.) On the basis of the data in Table IIIa, the value of Chi-square (calculated using Yates's correction) turns out to be 20.32, with 1 degree of freedom. This large a value of Chi-square would occur, if chance were the only factor affecting the data, considerably less than ½ of one percent of the time. We must conclude that, beyond any reasonable doubt,[15] some factor other than or in addition to pure chance has given rise to the data of Table IIIa. Essentially, this amounts to saying that the sign of the Lanchester advantage parameter is indeed somehow associated with the victorious side, so in this case our less formal interpretation of Tables IIIa and IIIb is supported by the statistical computations.

The above does not entirely resolve the issue, however, for if ln μ is to be interpreted as an index of advantage we need to show that the greater victories tend to be associated with extreme values of the advantage parameter. To investigate this question, a list of the advantage parameter, ln μ , values for all 92 battles, ordered from the most extreme negative value to the most extreme positive value, was prepared and the corresponding victorious side was listed beside each advantage parameter value. Table IV exhibits this arrangement. Note that, in reading down the columns in order from the algebraically smallest value of ln μ to the algebraically largest value, the victor is first exclusively the attacking side, then predominantly the attacking side, then neither side predominates; and then gradually, the victor becomes predominately the defending side, and finally becomes exclusively the defending side. From Table IV, we obtain Table Va, which

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[15] The usual statistical convention is to consider that the occurrence of a result which would have, if chance were the only factor affecting the data, an 0.05 or smaller probability of occurrence constitutes "proof beyond reasonable doubt" that chance is not the only factor affecting the data. The occurrence of a result which would occur more than 5% of the time by chance is not usually taken to demonstrate beyond reasonable doubt that some factor other than chance is operating. We will follow these conventions here and in the following discussions, but will usually indicate the probability of occurrence which a result under discussion would have if chance were the only factor affecting the data. By so doing, the reader will be better able to judge for himself whether there are or are not grounds for "reasonable doubt" that chance alone is affecting the data.

exhibits the number of victories by side with Lanchester advantage parameters either greater than 0.3, between + 0.3 and - 0.3, or less than - 0.3. Table Vb presents the same information expressed as a percentage of the number of victories by the respective sides. From Table Va we see that, for values of ln μ numerically greater than or equal to 0.3, the Lanchester advantage parameter follows the victor in 28 (84.8%) of 33 battles, and does not follow the victor in 5 (15.2%) of the 33 battles.

Treating Table Va as a 2 x 3 contingency table, we compute a value for Chi-square of 16.80, with 2 degrees of freedom. This large a value of Chi-square would occur, by chance alone, less than ½ of one percent of the time, so again we must conclude that the association between victory and the Lanchester advantage parameter is not spurious.

In sum, the Lanchester advantage parameter seems to adequately reflect the more usual, intuitive concept of probably victory.

c. Other Lanchester Parameters

The remaining Lanchester parameters have been defined in such a way (See Chapter II, Method.) that they clearly reflect a large part of the more usual concepts indicated by the names assigned to them.

We have presented evidence, which the author believes is highly convincing, in support of the proposition that the names given to the various Lanchester parameters are meaningful and may rightly be taken as numerical indicators of the associated phenomena.

2. Behavior of Individual Parameters

Having identified the various parameters with relevant and important aspects of combat phenomena, we begin the investigation of these parameters. We first consider individual parameters and later consider relations among two or more parameters.

a. Force Ratio

Table VI shows the theoretical frequency of battles for ranges of logarithmic force ratio, ln (x_0/y_0), computed on the basis of a theoretical normal distribution. (With the same mean and standard deviation as the observed values of logarithmic force ratio. See Reference 13 for definitions and discussion of "normal distribution," "mean" and "standard deviation.") The observed frequency of battles is shown for comparison. Figure 3 exhibits the theoretical cumulative normal distribution and the observed cumulative distribution of logarithmic force ratio, and Figure 4 displays the theoretical normal frequency distribution and observed frequency distribution of logarithmic force ratio.

The degree of agreement between the theoretical normal distribution and the observed distribution has been tested using the Chi-square goodness-of-fit test (Ref. 13). Since this test is an approximation whose accuracy is poor if any of the theoretical frequencies are less than 5, frequencies were grouped prior to the Chi-square calculation as indicated in the last two columns of Table VI. (This accords with the procedure recommended in Reference 13.) The value of Chi-square obtained was 10.56 at 10 degrees of freedom. This value of Chi-square indicates that random sampling would produce a poorer fit to the theoretical distribution more than 25% of the time. Hence, the data under discussion do not "prove beyond a reasonable doubt" that the logarithmic force ratio is not normally distributed. In statistical terminology, we say that the data under discussion are consistent with the hypothesis that the logarithmic force ratio is normally distributed.[16]

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[16] This is not the same as asserting that the data proves the hypothesis, since the data may be consistent with other hypotheses. The view normally adopted in this type of situation is to suspend final judgement pending the analysis of additional data, but to use a hypothesis consistent with the data and other available information on a tentative basis until additional data become available.

A hypothesis that, on theoretical grounds, is not expected to hold exactly is that the force ratio, x_0/y_0, (not its logarithm) is approximately normally distributed. The theoretical grounds for rejecting this hypothesis is that it would ascribe a non-zero probability of occurrence to negative values of force ratio, which is known to be false since negative force ratios are impossible. Testing the fit between observed force ratio data and the normal distribution, the author obtained a Chi-square of 17.3 at 5 degrees of freedom. A quality of fit this bad would obtain, in random samples, less than 0.5 percent of the time. In other words, the data is inconsistent with the hypothesis that the force ratio is normally distributed.

Estimated mean and standard deviation (See Ref. 13 for definitions.) for the logarithmic force ratio, $\ln(x_0/y_0)$, are 0.156 and 0.512, respectively.

b. Advantage

Table VII shows the theoretical frequency of battles for ranges of defender advantage, $\ln \mu$, computed on the basis of a theoretical normal distribution. The observed frequency of battles is shown for comparison. Figure 5 exhibits the theoretical cumulative normal distribution of defender advantage.

The Chi-square goodness-of-fit test was applied to the data (grouped as indicated by the last two columns of Table VII), yielding a value for Chi-square of 9.39 at 8 degrees of freedom. This value of Chi-square indicates that random sampling would produce a poorer fit to the theoretical distribution about 33% of the time. Hence, the data are consistent with the hypothesis that defender advantage, $\ln \mu$, is normally distributed. The author has not attempted to fit other theoretical distributions to these data.

Estimated mean and standard deviation for defender advantage are 0.057 and 0.350, respectively.

c. Bitterness

Table VIII shows the theoretical frequency of battles for ranges of logarithmic bitterness, $\ln \epsilon$, computed on the basis of a theoretical normal distribution. The observed frequency of battles is shown for comparison. Figure 6 exhibits the theoretical cumulative normal distribution and the observed cumulative distribution of logarithmic bitterness.

The Chi-square goodness-of-fit test was applied to the data (grouped as indicated in the last two columns of Table VIII), yielding a value for Chi-square of 5.46 at 7 degrees of freedom. This value of Chi-square indicates that a <u>random</u> sampling would produce a poorer fit to the theoretical distribution more than 50% of the time. Hence, the data are consistent with the hypothesis that logarithmic bitterness, $\ln \epsilon$, is normally distributed. No attempt has been made to fit other theoretical distributions to these data.

Estimated mean and standard deviation of logarithmic bitterness, $\ln \epsilon$, are - 2.157 and 0.834, respectively.

d. Surviving Fraction

The average surviving fraction of attacker (averaged over 92 battles) is 0.850; the corresponding value for the defending side is 0.855. As indicated in Table IX, there may be a tendency for the victorious side to have a larger and less variable fraction than the defeated side. The author has not subjected these differences to careful statistical analysis, and will leave this area to future investigations. The statistically-trained reader will note that in addition to the question of the proper form of distributions for surviving fractions on each side, additional difficulties may arise if attacker and defender surviving fractions are not statistically independent.

Table X is presented to provide the reader with a more readily comprehended picture of the distribution of surviving fractions than that afforded by Table II.

3. Relations Between Selected Pairs of Parameters

The principal tools used in this study to examine relations between two or more parameters are scatter diagrams and linear regression analyses of data or transformed data. (See Ref. 13 for definitions.) Other techniques are introduced as necessary or convenient. In the following, some of the seemingly more important parameters are selected and their relation to other parameters is examined. Parameters which appear to be less important are studied in less detail.

a. Victory and Force Ratio

Table XIa gives the number of battles won by side and by numerical superiority or inferiority (i.e., force ratio). Table XIb shows the same information as a percentage of the bottom row. Applying the Chi-square test for independence in contingency tables (using Yates's correction), we find a value for Chi-square of 0.35 at one (1) degree of freedom. This value would be exceeded in random sampling more than 50% of the time, so the data of Table XIa is consistent with the hypothesis that victory is not dependent on numerical superiority.

It might be objected that grouping battles with both a small and a large degree of numerical superiority, as is done in Table XIa, might tend to mask the effect of force ratio on victory. In an attempt to reduce any such masking effect, values of force ratio were chosen so as to divide the 92 battles into three approximately equal groups, as indicated in Table XII. The Chi-square value computed from Table XII is 3.56 at 2 degrees of freedom, which would be exceeded by random samples less than 25%, but more than 10%, of the time. In this case, as formerly, the data are consistent with the hypothesis that victory is independent of numerical superiority.

It may still be objected that the force ratio values are not extreme enough to adequately determine whether the sought-for effect is present or absent. In this connection, a 2-to-1 numerical superiority is often mentioned as critical. Of the 92 battles, 12 have a force ratio of 2.000 or greater. The defender is credited with the victory in 6 of these battles, the attacker is credited with the victory in the remaining 6 battles. Of the 92 battles, 5 have a force ratio of 0.500 or less. The defender is credited with victory in 3, the attacker in 2, of these battles.

These results do not necessarily mean that force ratio and victory are entirely unrelated, but they do indicate that any such supposed relation is too weak and tenuous to be detected in the data under analysis, where by "detected" we mean "distinguished, beyond reasonable doubt, from a chance or accidental effect."

b. Activity Ratio and Force Ratio

Figure 7 shows a linear[17] scatter diagram of activity ratio, D/A, against force ratio, x_0/y_0. Note that the data points appear to scatter more widely for greater values of the force ratio than for small values of force ratio. Before statistical analysis is attempted, it is desirable (for technical reasons) to transform the data (See Ref. 13.) in such a way as to make the amount of scatter more nearly uniform. This can be illustrated by plotting the data points on non-linear graph paper, as shown by Figure 8, which presents a logarithmic scatter diagram of activity ratio against force ratio. An approximate straight-line fit, or trend line, to the data is also indicated, the origin of which will be explained below.

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[17] It will be convenient when discussing figures to indicate the type of graphical scale employed. We do this by prefacing the description of the figure by an appropriate adjective. Thus, we refer to linear figures (linear scale for each coordinate axis), logarithmic figures (logarithmic scale for each axis), and semi-logarithmic figures (linear scale on one axis, logarithmic scale on the other).

Note that the activity ratio (defender relative to attacker) tends to increase with force ratio (attacker relative to defender). In other words, the more the attacking side outnumbers the defending side the lower the attacking side's activity relative to the defending side's activity.

In order to obtain a brief quantitative summary of the activity ratio trend exhibited by Figure 8, we might try a straight-line fit. Algebraically, such an attempt takes the form:

$$\ln (D/A) = b^* + c^* \ln (x_0/y_0)$$

where b^* and c^* are constants to be determined, from the data, to provide a good fit. From Equations (3) and (5) of Section II, Method, however, we may write:

$$D/A = \mu^2 (x_0/y_0)^2$$

Substituting this expression for the activity ratio into the equation expressing the straight-line fit and solving the result for $\ln \mu$, we find:

$$\ln \mu = b + c \ln (x_0/y_0)$$

where

$$b = b^*/2$$

$$c = (c^* - 2)/2$$

In other words, a straight-line fit to the data of Figure 8 can be obtained from a straight-line fit to a logarithmic scatter diagram of μ against force ratio; in fact, the trend line of Figure 8 was obtained in just this way. The details are given in the following paragraph.

c. Defender Relative Advantage

Because of its importance, we give in this section a fairly comprehensive discussion of the results obtained for the relations between the advantage parameter, $\ln \mu$, and other parameters of interest.

Since the issue was raised in the preceding paragraph, we first examine the relation of advantage to force ratio.

(1) Advantage and Force Ratio

Figure 9 exhibits a logarithmic scatter diagram of μ against force ratio, together with the regression line of advantage, $\ln \mu$, on logarithmic force ratio, $\ln (x_0 / y_0)$.[18] Table XIII displays the detailed results of the regression of $\ln \mu$ on $\ln (x_0 / y_0)$. These results show that defender's advantage, beyond reasonable doubt, tends to decrease with increasing (attacker's) force ratio (since, for example, a degree of correlation numerically as large as that in Table XIII would arise, if chance alone were operating, considerably less than one percent of the time).

Note that the values of $\ln (x_0 / y_0)$ used in the regression analysis include a "measurement error" (as shown in Appendix B) as well as an "inherent variability." (See Ref. 13 for definitions.) The regression technique used to analyze the data does not rigorously take into account the influence of these measurement error but assumes that the values of logarithmic force ratio, $\ln (x_0 / y_0)$, reflect only inherent variability. No attempt has been made to examine the amount of error introduced by neglecting the effects of these measurement errors (e.g., by comparing the results given in this report with results obtained using more sophisticated analysis techniques) either here or in the following, partly because of lack of time and partly because the author feels that the approximate analysis methods do not lead to serious errors.

The analysis technique is predicated on the linearity of the regression function and on a normal distribution of the data about

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[18] "Regression line" is the technical term for the expressions "trend line" or "straight-line fit" used earlier. From now on we will use the more explicit technical terminology. (See Ref. 13 for a discussion of regression analysis techniques and interpretation.)

the regression line (as well as precisely known abcissa values for the data points).[19] In order to investigate the degree to which these assumptions of linearity and normality are satisfied, we introduce the concept of residual advantage, which is the deviation of a data point from the estimated, or sample regression line and may be analytically expressed as:

$$\text{Residual Advantage} = \ln \mu - b - c \ln (x_0 / y_0)$$

where b and c are the estimated regression coefficients. (See Table XIII.) In other words, the residual advantage is that portion of the defender's relative advantage which remains after the average effect of force ratio is eliminated. As such, the residual advantage provides an index of the absolute amount of defender's relative advantage, where by "absolute" we mean that the "contaminating" effect of force ratio has been (approximately) eliminated. Thus, residual advantage presumably arises from factors other than those directly dependent on force ratio. It might be supposed that the residual advantage reflects the impact of advancing weapon technology; of superior organization, training, or experience; of relative tactical skill and similar factors in addition to accident and luck. We will attempt to examine some of the factors affecting residual advantage after discussing the linearity and normality assumptions.

Figure 10 shows the observed cumulative distribution of residual advantage together with a theoretical normal distribution. Table XIV shows the theoretical and observed frequency of battles with various residual advantages. Grouping as indicated in Table XIV, we find a value for Chi-square of 9.90 at 5 degrees of freedom. This poor

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[19] It is known, however, that regression techniques do not usually lead to serious errors when the distribution of the data about the regression line exhibits a moderate deviation from normality.

a fit could be expected to arise by chance alone about 8% of the time, so the data is consistent with the hypothesis that residual advantage is normally distributed.

Reference 13 suggests using a run test as a rough check on linearity. This involves, for the case in hand, considering the sequence of signs of the residual advantage values in order of increasing force ratio. If the "true" (i.e., population) regression curve is non-linear, the number of runs in this sequence of signs usually tends to be less than if it were linear. A count shows that 42 of the data points lie above, and 50 below, the sample regression line, and that 46 runs occur in the sample of 92 residual advantage values (in order of increasing force ratio). Chance fluctuations about a linear regression curve would produce fewer than 46 runs much more than 5% of the time, so the data is consistent with the hypothesis of a linear population regression curve.

Despite this test for linearity, however, the semi-logarithmic scatter diagram of residual advantage against force ratio given by Figure 11 appears to show a tendency for residual advantage to increase with increasing force ratio below a force ratio of about 1.3, and to decrease with increasing force ratio above 1.3. On the other hand, a reversal in the sign of residual advantage for 4 or 5 deliberately selected data points could easily change the entire appearance of the scatter diagram and cause the supposed non-linear effect to vanish. It should be noted that a reversal in sign corresponds to interchanging attacker and defender identifications, and that improper identifications may occur.

Considering the facts developed above, the author concludes that neither non-linearity nor non-normality hypotheses are adequately supported by the data considered in this report, and that the linearity and normality hypotheses should be tentatively adopted pending further investigation.

(2) Advantage and Bitterness

Figure 12 shows a logarithmic scatter diagram of bitterness, ϵ, against μ. No trend is discernible by eye. Nevertheless, a regression analysis of the data was performed in order to see whether any trend might be detected by quantitative analysis, the results of which are presented in Table XV. The data are consistent with the hypothesis that logarithmic bitterness is not correlated with advantage (and since both variables are approximately normal, the data is also consistent with the hypothesis that bitterness and advantage are statistically independent of each other), since, for example, a degree of correlation larger than that of Table XV would arise from the action of chance alone much more than 5% of the time.

Since the advantage, $\ln \mu$, is not correlated with logarithmic bitterness, ϵ, it is not likely to be correlated with either the total casualty fraction, F, or the surviving fractions, a and d. If the advantage is also not correlated with battle duration, t, then it is also not likely to be correlated with intensity, λ, since $\lambda = \epsilon / t$.

(3) Advantage and Battle Duration

Figure 13 shows a logarithmic scatter diagram of μ, against battle duration, t. Although no trend is discernible to the eye, a regression analysis of this data was performed, the results of which are given in Table XVI. The analytical results are consistent with the hypothesis that advantage is not correlated with logarithmic battle duration.

As noted above, when taken with other results this suggests that advantage is not correlated with intensity, λ. This suggestion is not investigated in detail.

(4) Residual Advantage and Various Other Parameters

i. Bitterness

Figure 14 gives a semi-logarithmic scatter diagram of residual advantage against bitterness, ϵ . No correlation is apparent to the eye.

ii. Total Force

Figure 15 gives a semi-logarithmic scatter diagram of residual advantage against total force, X. No correlation is apparent to the eye.

iii. Total Casualties

Figure 16 gives a semi-logarithmic scatter diagram of residual advantage against total casualties, C. No correlation is apparent to the eye.

None of the above cases have been subjected to formal statistical analysis, principally because the results seem to be predictable from simple inspection of the scatter diagrams.

(5) Residual Advantage and Battle Date

Figure 17 shows a linear scatter diagram of residual advantage against battle date. Although no consistent trend is apparent, the battles occurring between 1757 and 1760 appear to exhibit a tendency to greater attacker superiority than would be expected. For example, 9 of these 11 battles have a negative residual advantage. If population residual advantage is assumed to be symmetrically distributed about zero residual advantage, then chance alone would produce a result like the one observed (i. e., 2 battles out of 11 with one sign of residual advantage and 9 with the opposite sign) about 5.4% of the time.[20]

A better analysis is obtained if Student's t test (See Ref. 13.) is used to compare the mean residual advantage of the 11

[20] Footnote on bottom of next page.

battles in question with that of the other 81 battles in the sample. The value of Student's t obtained is about 2.22 at 90 degrees of freedom. A more extreme value of Student's t would arise, by chance alone, only about 3% of the time, so the data are not consistent with the hypothesis that mean residual advantage for these 11 battles is the same as that for the other 81 battles.

However, the author is inclined to reject the implication that non-random factors affected the outcome of the 11 battles, even though the consequence of such rejection involves accepting the proposition that between 1757 and 1760 the attacking side enjoyed a run of good luck so remarkable that a greater deviation from average conditions would occur only about 3% of the time. Certainly, the data of Figure 17 give no indication that either the attacking or defending side has been able to consistently turn any supposed non-random factors to its advantage, the period between 1757 and 1760 being excepted. The author finds belief in such a short-lived and unduplicated set of non-random factors favoring one side or the other at least as repugnant as belief in an extraordinary series of accidents.

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[20] One way of demonstrating this is as follows:

The probability that a particular battle will have a given sign of residual advantage is $\frac{1}{2}$. The probability that 9 of 11 battles will have a given sign of residual advantage and the other 2 the opposite sign is:

$$\binom{11}{2}\left(\tfrac{1}{2}\right)^{11}$$

But the given sign for the 9 battles may be independently chosen in two possible ways. Thus, the probability of 9 battles of one (unspecified) sign and 2 of the opposite sign out of 11 battles is:

$$\binom{11}{2}\left(\tfrac{1}{2}\right)^{10} \cong 0.0536$$

and we have obtained the desired result.

(6) Residual Advantage, Data Errors, and Unexplained Variability

Since all attempts to date to find correlations between advantage and other parameters have been unsuccessful, with the exception of that with force ratio and perhaps (in a very limited sense) with battle date, it is necessary to determine whether it is likely that any such correlations exist, and how important they might be. We consider the first of these problems in this section, and the second problem in the next section.

We begin by considering the amount of variability in the advantage parameter and our current understanding of this variability. (Clearly, if we could explain or somehow account for all of the variability in advantage, then our understanding of the advantage parameter would be essentially complete, in the sense that we would have obtained all the information contained in the sample, except for that masked by data inaccuracies.) In order to deal in quantitative terms, the amount of variability will be measured by the variance (i.e., square of the standard deviation -- see Reference 13) of the appropriate parameter.[21]

From Table XIII we find that the total sample variance of advantage is 0.122. The variance remaining after the "contaminating" effects of force ratio are eliminated is the same as the variance of residual advantage (and also the same as $s^2_{\ln \mu \mid \ln (x_0/y_0)}$), or 0.088. The difference between these variances, 0.034, it is the amount of advantage variance which is attributable to force ratio. In other words, the regression of advantage on logarithmic force ratio accounts for a

[21] This is often done in statistical work. The variance is better suited than the standard deviation to problems like those considered in this section (e.g., variances may be combined by simple addition or subtraction while standard deviations cannot).

variance of 0.034 out of a total of 0.122, or about 27.9% of the advantage variance.

A part of the advantage variance is caused by inaccuracies in the strength and casualty data. The amount of advantage variance attributable to data inaccuracies is estimated in Appendix B[22] to be about 0.034.

We now consider whether the hypothesis, that population residual advantage variance is equal to population data inaccuracy variance, is supported by the data. Application of the F test for comparing variances, taking sample residual advantage variance (90 degrees of freedom), divided by sample data inaccuracy variance (19 degrees of freedom) produces an F value of 2.59 at 90 and 19 degrees of freedom. If the hypothesis of equal population variances were correct, this large an F-value would occur by chance alone only a little more than 1% of the time, so the data are not consistent with the hypothesis. In other words, even after removing the "contaminating" effects of force ratio, more variability of advantage remains than can reasonably be ascribed to data inaccuracies. (Note, however, that this procedure assumes that the estimate of data inaccuracy variance obtained in Appendix B is adequate and that it may be validly extrapolated to cover data inaccuracies for other battles and other sources than those specifically considered in Appendix B. It would be desirable to check these assumptions against additional data.)

Subtracting from advantage variance the variability explained by correlation with force ratio and also the estimated

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[22] This estimate uses some of the same data as is used in estimating total sample advantage variance, so the two estimates are not strictly independent in the statistical sense. However, since the author believes that only negligible errors result from treating these estimates as though they were independent, and since so doing considerably simplifies the analysis, they are treated as independent in the following discussion.

variability caused by data inaccuracies leaves an unexplained variance of 0.054, or about 44.2% of the total variance in advantage. Our current understanding of the variability in the advantage parameter may be summarized as follows: about 28% of the variability is accounted for by the correlation of advantage and force ratio, an estimated additional 28% is accounted for by data inaccuracy, and an estimated 44% is at present unexplained. In the following paragraphs, some of the not as yet fully investigated factors which might contribute to this unexplained 44% of total variance in advantage are examined to provide guidance for further study.

i. The fact that the 11 battles discussed in a previous paragraph deviate noticably from the average has an effect on the variance of residual advantage, but the effect is small. By dropping the residual advantage values of these 11 battles from consideration, the author estimated a residual advantage variance (based on the remaining 81 battles) of about 0.083, while the value computed for all 92 battles is 0.088. In principle, new values for the regression coefficients should have been computed and a new residual advantage variance about this regression line should have been obtained, all based on 81 battles. Actually, the simpler procedure of using the old regression coefficients and residual advantages with respect to the old regression line was used, because the resulting error for the situation in hand is negligible.

ii. Presumably a part of the variability in advantage results from the gross treatment of reinforcement and non-battle losses. Even if insufficient historical data are available to support a detailed analysis, some feel for the effects of these factors might be gained by generating artificial or synthetic data from a general Lanchester-type model with a variable reinforcement/non-battle-loss term, and then using this synthetic data to calculate the advantage (and other parameters) on the basis of the square-law model as outlined in Section II, Method. It might be possible to devise (at least for special cases) an analytical expression relating the parameters of the square-law model to those of the general model and thereby reduce the computational effort required. The author has not yet attempted any work along the above lines, but is currently of the opinion that in many cases the effect will be small. (For example, compare the parameters in Table II for the battle of Iwo Jima with those obtained in Reference 9.) If the effects should turn out to be large enough to account for residual advantage variance, less data inaccuracy variance, then use of square-law model should be reconsidered.

iii. Note that a part of the unexplained 44% of total variability in advantage is almost certain to be caused by random and unforseeable events, i.e., by the operation of chance and luck. The author is not able at present to form any clear conception of the amount of variability which might reasonably be interpreted as occasioned by random factors, but is reluctant to assume that all of the unexplained 44% is due to the operation of chance and luck until many more attempts to determine causative factors have been made and found to be unacceptable as explanations.

iv. In an effort to determine whether non-quantitative causative factors might give rise to some of the unexplained variance of advantage, narrative accounts of eight battles with the most extreme values of residual advantage were selected for further study -- four battles for the most extreme residual advantage favoring the attacking side and four favoring the defending side. In order of descending value of residual advantage, these eight battles are Palo Alto (0.662),[23] Lisaine (0.601), Tschernaja (0.577), Plevna (0.514), Cerro Gordo (- 0.569), Mortara (- 0.583), Rossbach (- 0.821), and Contreras (- 1.064). The narrative accounts for these battles are contained in Appendix C, more than one narrative for each battle being included whenever available from the sources consulted. No pretense of completeness

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[23] Numbers in parenthesis give the value of residual advantage.

is made, and time was not available for a wider search for narrative accounts. Unfortunately, little can be gained from such a small number of narratives, some of which are very sketchy (descriptions of events immediately preceeding the battle, to include the grand tactical situation and information bearing on the condition of the troops on each side, are given in some of the sources, although this material is not included in Appendix C). There seems to be a suggestion that (1) extreme negative residual advantage values are associated with success by the attacking side in achieving surprise, accompanied by a successful envelopment, and (2) extreme positive residual advantage values are associated with failure of the attacking side to achieve surprise, accompanied by an unsuccessful enveloping maneuver. It would be interesting to pursue these speculations, but that pursuit will be deferred to another time.

(7) Residual Advantage and Victory

The importance, in terms of its influence on victory in battle, of the unexplained variance in advantage will now be considered. In previous sections the relation between advantage and victory has been investigated (See pp. 24 - 26.) The principal results of that investigation are contained in Tables III, IV, and V. In order to determine the importance of the unexplained variance in advantage, similar tables have been prepared using residual advantage. This results in Tables XVII, XVIII, and XIX.

From Table XVIIa we see that residual advantage follows the victor in 65, or 70.6% of the 92 battles and does not follow the victor in 27, or 29.4% of the battles.[24] To test the observed data against chance, we use the Chi-square test as before, applying Yates's correction and considering Table XVIIa as a 2 x 2 contingency table. This yields a Chi-square of 14.06 of 1 degree of freedom, a value which would be

[24]Advantage follows the victor in 73.9% of the battles.

exceeded less than ½ of one percent of the time if chance were the only factor affecting the data.

Table XVIII gives a list of residual advantage values in order of increasing (algebraic) value, together with the corresponding victorious side.

From Table XIXa we see that, for values of residual advantage numerically greater than 0.2, the residual advantage follows the victor in 32 (78%) of 41 battles and does not follow the victor in 9 (22%) of 41 battles. Treating Table XIXa as a 2 x 3 contingency table, we compute a Chi-square value of 16.06 with 2 degrees of freedom. A larger value for Chi-square would arise by chance alone less than ½ of one percent of the time.

In view of the immediately preceeding results it is evident that the principal determinants of victory are contained in residual advantage. This is consistent with the earlier findings to the effect (1) that the data was consistent with the hypothesis that victory was not correlated with force ratio, and (2) that the advantage parameter is beyond reasonable doubt correlated with force ratio. We conclude that an elucidation of the factors contributing to the variability of residual advantage are critically important to an understanding of the causes of victory in battle. The negative findings in the foregoing Sections now take on an enhanced importance, for they mark off areas which offer little promise of contributing significantly to an understanding of the causes of victory in battle.

d. The Effects of Battle Date

The relation between battle date and residual advantage has been examined in previous paragraphs. The relation between battle date and some of the other parameters will be considered in this section.

(1) Force Ratio and Battle Date

Figure 18 gives a linear scatter diagram of logarithmic force ratio against battle date. No correlation is apparent to the eye.

(2) Duration and Battle Date

Figure 19 gives a semi-logarithmic scatter diagram of duration in hours against battle date. The most remarkable feature of this scatter diagram is the increase in duration of World War II battles over that of earlier battles. There may also be a tendency for duration to gradually increase with battle date between 1741 and 1877.

(3) Bitterness and Battle Date

Figure 20 shows a semi-logarithmic scatter diagram of bitterness ϵ, against battle date. Bitterness does not seem to show any consistent trend, although it may have been somewhat lower than average in the decade 1840 - 1849 and somewhat higher than average during World War II.

In this connection, it is interesting to note one of Richardson's conclusions (Ref. 10) even though it applies to a historical sample of <u>wars</u> rather than of <u>battles</u>:

"In contrast with the enormous increase in deadlines from the sword and arrow of the Middle Ages to the rifle and shell of the nineteenth century, it may be said that the percentage of casualties has remained relatively constant. The explanation presumably is that human endurance to suffering has changed much less than weapons. One side admitted defeat when it could not bear any more suffering. The chief form of suffering was casualties. To a side that was debating within itself whether or not to admit defeat, the weapons were usually of only secondary interest."

(4) Intensity and Battle Date

Figure 21 gives a semi-logarithmic scatter diagram of intensity, λ, against battle date. Despite all efforts to improve weapon design, the average casualty-production rate per man per day expressed by intensity seems to have gradually declined between 1740 and 1880 and to have been markedly lower during World War II than formerly. This decline in intensity may be due in part to an increasing proportion of noncombatant support personnel, but it hardly seems likely that this could cause the order of magnitude decrease found for the World War II battles. Besides, the motive for an increased proportion of support personnel is presumably to increase the average combat power per man per day expressed by the activity parameters D or, as the case may be, A. Yet recalling that intensity is the geometric mean of A and D, we deduce from Figure 21 that the activity of each side must have been much lower for the World War II battles than formerly, a deduction which is confirmed by an inspection of Table II. Moreover, a conjecture that an increased proportion of combat support personnel has led to a decline in intensity appears inconsistent with the finding that bitterness is much less sensitive than intensity to battle date.

e. Bitterness, Intensity, and Duration

The finding that duration, t, and intensity, λ, vary in opposite senses when each is plotted against battle date suggests that there may be an inverse relation between intensity and duration. We will explore this possibility by studying the relation between bitterness, ϵ, and duration, t, since intensity, $\lambda = \epsilon / t$, is determined once bitterness and duration are known.

Figure 22 gives a logarithmic scatter diagram of bitterness, ϵ, against duration, t, together with the regression line of logarithmic bitterness, $\ln \epsilon$, on logarithmic duration, $\ln t$. Table XX gives the results of the regression of $\ln \epsilon$ on $\ln t$.

(1) Residual Logarithmic Bitterness, Data Errors, and Unexplained Variability

From Table XX we find that the total sample variance of logarithmic bitterness, $\ln \epsilon$, is 0.704. The variance remaining after the regression of logarithmic bitterness, $\ln \epsilon$, on logarithmic duration, ln t, is $s^2_{\ln \epsilon \mid \ln t} = 0.557$. The difference between these two variances, 0.147, is the amount of logarithmic bitterness variance which is attributable to the effects of duration. The amount of logarithmic bitterness variance attributable to data inaccuracies is estimated in Appendix B[25] to be about 0.409. We now consider whether the hypothesis, that population residual logarithmic bitterness variance is equal to population data inaccuracy variance, is supported by the data. Application of the F test for comparing variances, taking sample residual logarithmic bitterness variance (80 degrees of freedom) divided by sample data inaccuracy variance (19 degrees of freedom), produces an F value of 1.362 at 80 and 19 degrees of freedom. If the population variances were equal, larger F-values would arise, by the action of chance alone, about 24% of the time, so the data are consistent with the hypothesis of equal population variances. In other words, the variability in logarithmic bitterness which is not accounted for by regression on duration may reasonably be ascribed to inaccuracies in the basic data.

In view of the above, our understanding of the variability in logarithmic bitterness may be summarized as follows: of a total variance of 0.704 a variance of 0.147, or 21%, is accounted for by the correlation of logarithmic bitterness and logarithmic duration; the remaining variance of 0.557, or 79% of total logarithmic bitterness variance, may reasonably be ascribed to data inaccuracies.

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[25] See footnote 22.

Table XXI shows the theoretical frequency of battles for various ranges of residual logarithmic bitterness, computed on the basis of theoretical normal distribution. The observed frequency of battles is shown for comparison. Application of the Chi-square goodness-of-fit test to these data (grouped as indicated in the last two columns of Table XXI) yields a Chi-square value of 14.00 at 7 degrees of freedom, a value which would be exceeded, by chance alone, only 5% of the time. On this basis, the data do not appear to be consistent with the hypothesis that residual logarithmic bitterness is normally distributed. Inspection of Figure 23, which gives the observed and theoretical cumulative distribution of residual logarithmic bitterness, indicates that the departure from normality arises chiefly from a greater frequency of battles with extremely low values of residual logarithmic bitterness than would be expected from the theoretical normal distribution.

In an attempt to isolate the battles with extraordinarily low values of residual logarithmic bitterness, the author has prepared Figure 24, which is a linear scatter diagram of residual logarithmic bitterness against battle date. Inspection of Figure 24 indicates that sample battles with unusually low values of residual logarithmic bitterness tend to be concentrated in the decade 1840 - 1849. Application of Student's t test on the difference between mean residual logarithmic bitterness for the 8 battles in the decade 1840 - 1849 and mean residual logarithmic bitterness for the other 74 battles for which duration data is available yields a t value of about 5.27 at 80 degrees of freedom. This t value would be exceeded by chance alone less than $\frac{1}{20}$ of one percent of the time, so the data are not consistent with the hypothesis that the corresponding population means are equal.

The variance of residual logarithmic bitterness, estimated on the basis of the 8 battles in decade 1840 - 1849, is 1.143. Application of the F test for comparing this variance estimate with the estimated variance in residual logarithmic bitterness attributable to data inaccuracy (See Appendix B.) yields an F value of 1.143/0.409 = 2.79 at 7 and 19 degrees of freedom, which would be exceeded, by chance alone, only about 3% of the time. Thus, the data are not consistent with the hypothesis that the corresponding population variances are equal.[26]

The author is not sure what position should be taken on the basis of the facts outlined above. The possibilities seem to be as follows: Either (1) there is a real difference in residual logarithmic bitterness for the sample battles which occurred in the 1840 - 1849 decade, or (2) data errors for these battles are significantly greater than for the other sample battles. Considering the participants engaged in these 8 battles, the author feels that the latter hypothesis is more likely, and will assume its validity in the following discussions, otherwise withholding final judgment pending the analysis of additional information.

(2) Intensity and Duration

Figure 25 gives a logarithmic scatter diagram of intensity, λ, against duration, t. Although λ clearly tends to decrease with increasing t, presumably the causative factor here is intensity, i.e., a battle which is being conducted at a low level of intensity tends to require a greater battle duration before a decision is reached. Taking this with the fact that bitterness (and hence total casualty fraction) tends to increase with duration suggests that greater casualty fractions tend to be more readily withstood when the casualties occur at a low rate. This

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[26] This suggests that Student's t test, as used in the preceding paragraph, is not strictly applicable (since the test assumes equality of population variances). The author has not estimated the amount of error introduced into this application of Student's t test by the apparent inequality of population variances. Note that the non-normality of residual logarithmic bitterness probably does not introduce serious errors into the regression of logarithmic bitterness on logarithmic duration. (See footnote 19.)

may be due in part to the shock effect of high casualty rates and in part to the ability to rehabilitate and re-commit forces to battle when the time necessary for recovery of effectiveness is available, although these speculations can at present be considered only as suggestions for further investigation.

f. Total Casualties and Total Force

Figure 26 presents a logarithmic scatter diagram of total casualties, C, against total force, X. Also shown for comparison is the line $C = 0.15\ X$, where the constant 0.15 was selected to agree approximately with the average total casualty fraction (averaged over 92 battles).

With the exception of a relatively small number of data points, all representing fewer casualties than average, the line $C = 0.15\ X$ appears to be a reasonable fit to the data. No formal statistical evaluation of the data presented in Figure 26 has as yet been attempted. For an informal comparison with variability attributable to data inaccuracy, compare Figure 26 with the analogous scatter diagram in Appendix B.

g. Force Ratio and Total Force

Figure 27 exhibits a logarithmic scatter diagram of force ratio, x_0/y_0, against total force, X. Also shown for comparison is the line corresponding to the average logarithmic force ratio.

No trend in force ratio with total force is apparent to the eye (this data has not been formally analyzed).

h. Duration, Total Force, and Force Ratio

Figure 28 shows a logarithmic scatter diagram of duration, t, against total force, X, and Figure 29 shows a logarithmic scatter diagram of duration against force ratio, x_0/y_0. These data have not been subjected to formal analysis, and the author has not been able to derive any significant conclusions from an informal inspection of these figures.

(Other scatter diagrams which involve duration and which have been discussed in previous sections are displayed in Figures 13, 19, 22, and 25.)

i. **Participating Nation and Victory**

It has been argued in an earlier section that the data should not be interpreted as indicating a correlation of residual advantage with battle date. The evidence seems to the author to indicate that neither the attacking nor the defending side has been able to consistently maintain an advantage over the other. However, even if the above arguments are accepted, no evidence has as yet been presented which could be used to indicate whether or not a particular nation might consistently maintain an advantage over its opponents. Since a nation may participate in one battle on the attacking side, in another on the defending side, etc., any tactical superiority enjoyed by that nation may not be adequately reflected in arguments based solely on the superiority of the attacking or defending side.

In an attempt to obtain some information relevant to these issues, residual advantage data was arranged to represent the residual advantage favorable to each of the five most frequently participating nations.[27] Figures 30 - 34 present the results for the respective nations in the form of linear scatter diagrams of residual advantage favorable to each nation against battle date. Since residual advantage does not always follow the victor, a table showing victories and defeats for each nation

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[27] The residual advantage favorable to a nation participating in a battle on the defending side is the same as the residual advantage defined in pp. 33 - 35. For a nation participating on the attacking side, it is numerically the same as the residual advantage previously defined but taken with the opposite algebraic sign.

and side was also prepared and is presented as Table XXII. Table XXIII exhibits the number of victories for each participating nation as a fraction of that nation's total participation and gives approximate 95% confidence limits for the probability of victory.

It must be remembered in interpreting these figures and tables that the information given for a particular nation is not independent of that given for other nations, since e.g., a victory by Prussia is simultaneously a defeat for France, Austria, Russia, or some other nation, and a residual advantage favorable (positive) to one participating nation is unfavorable (negative) for another. It must also be remembered that the identification of attacker and defender is subject to error and becómes rather arbitrary when allied forces participate on one or both sides.

Since the circumstances outlined above make rigorous statistical analysis difficult (and, perhaps, of little value for the rather small samples involved), no formal analysis of these data, other than that involved in estimating the approximate confidence intervals for probability of victory given in Table XXIII, has been attempted. Informally, it appears that no one nation has consistently been tactically superior to its opponents. For example, Table XXIII appears to indicate that the data are consistent with the hypothesis that the probability of victory is about 0.50. That one (i.e., a little more than 6%) of the 16 95% confidence intervals listed in Table XXIII does not include the 0.50 probability value may be of no significance, since 5% of such confidence intervals will exclude the true probability value by the action of chance alone. Moreover, Figure 31 does not appear to support the hypothesis of Prussian or German tactical superiority, except possibly during the decade 1860 - 1869.

4. Recapitulation of Findings

a. For the range of total casualty fraction, F, values represented by the sample of 92 battles considered in this study, the Lanchester bitterness parameter, ϵ, is a good index of total casualty fraction.

b. The Lanchester advantage parameter, $\ln \mu$, is closely connected with probability of victory.

c. The other Lanchester parameters are valid indices of the real-world phenomena identified by the names of the respective parameters.

d. Logarithmic force ratio, $\ln (x_0 / y_0)$, is approximately normally distributed with mean 0.156 and standard deviation 0.512.

e. Advantage, $\ln \mu$, is approximately normally distributed with mean 0.057 and standard deviation 0.350.

f. Logarithmic bitterness, $\ln \epsilon$, is approximately normally distributed with mean -2.157 and standard deviation 0.834.

g. There may be a tendency for the victorous side to have a larger and less variable surviving fraction, a or d, than the defeated side.

h. For the range of values represented in the sample battles, victory is, at best, only weakly related to force ratio, x_0 / y_0.

i. Logarithmic activity ratio, ln (D/A), tends to increase with increasing logarithmic force ratio, $\ln (x_0 / y_0)$.

j. Advantage, $\ln \mu$, tends to decrease with logarithmic force ratio, $\ln (x_0 / y_0)$.

k. Advantage, $\ln \mu$, and logarithmic bitterness, $\ln \epsilon$, are uncorrelated.

l. Advantage, $\ln \mu$, and logarithmic duration, ln t, are uncorrelated.

m. Residual advantage[28] and logarithmic bitterness, $\ln \epsilon$, appear to be uncorrelated.

n. Residual advantage and logarithmic total force, $\ln X$, appear to be uncorrelated.

o. Residual advantage and logarithmic total casualties, $\ln C$, appear to be uncorrelated.

p. With the possible exception of 11 sample battles which occurred between 1757 and 1760, average residual advantage appears to be independent of battle date.

q. Data inaccuracies do not account for all of the observed variability in residual advantage.

r. Comparison of narrative accounts for battles with unusually large and unusually small residual advantage suggests that surprise accompanied by a successful enveloping maneuver may be highly advantageous to an attacker, but that failure to achieve surprise and to execute an enveloping maneuver may be highly disadvantageous to the attacking side.

s. The principal determinants of victory in battle are contained in the residual advantage parameter.

t. Logarithmic force ratio, $\ln (x_0 / y_0)$, exhibits no tendency to increase or decrease with battle date.

u. Logarithmic duration, $\ln t$, appears to be much larger for World War II battles than formerly. In addition, for the other battles

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[28]Residual advantage is the difference between observed advantage, $\ln \mu$, and that represented by the regression line of advantage on logarithmic force ratio, $\ln (x_0 / y_0)$. It represents, in a sense, the amount of advantage remaining after the "contaminating" effects of force ratio have been eliminated

considered in this study, there may be a tendency for duration to gradually increase with battle date.

v. Logarithmic bitterness, ln ϵ, does not appear to show any consistent trend with respect to battle date, although it may have been somewhat lower than average for sample battles in the decade 1840 - 1849 and somewhat higher than average for World War II sample battles.

w. Logarithmic intensity, ln λ, appears to have gradually declined between 1740 and 1880 and to have been markedly lower during World War II than formerly. The activity parameters, A and D, generally follow the same trend as the intensity parameter does.

x. Logarithmic bitterness, ln ϵ, tends to increase with logarithmic duration, ln t.

y. Data inaccuracies are large enough to account for all of the observed variability in residual logarithmic bitterness.[29]

z. Residual logarithmic bitterness does not appear to be normally distributed. In particular, it appears that the sample contains more battles with small values of residual logarithmic bitterness than would be expected on the basis of a normal distribution.

aa. During the decade 1840 - 49, mean residual logarithmic bitterness may have been lower, and the variability in residual logarithmic bitterness greater, than for the other battles considered in this study.

ab. Logarithmic intensity, ln λ, tends to decrease with increasing logarithmic duration, ln t.

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[29] Residual logarithmic bitterness is the difference between observed logarithmic bitterness, ln ϵ, and that represented by the regression line of logarithmic bitterness on logarithmic duration, ln t. It represents, in a sense, the amount of logarithmic bitterness remaining after the "contaminating" effects of duration have been eliminated.

ac. Logarithmic total casualties, ln C, tends to increase with logarithmic total force, ln X.

ad. Logarithmic force ratio, $\ln (x_0 / y_0)$, appears to be uncorrelated with logarithmic total force, ln X.

ae. It appears that no one nation has consistently been tactically superior to its opponents.

DISCUSSION

Range of Applicability of the Findings

In technical terms, what is to be understood here is the problem of identifying the population from which the sample battles were selected. As noted earlier, this is a difficult problem. The author will state his views for the guidance of the reader, although it should be borne in mind that interpretations may differ.

In general terms, the author feels that the population sampled consists of the large (in terms of number of troops involved) battles which occurred under reasonably average conditions between the years 1740 - 1950.

Certain features of the sampling process deserve additional discussion, particularly the possibility of censoring effects.

1. Censoring may act to eliminate from the sample battles characterized by extremely large or extremely small values of force ratio, as such battles would not normally be considered as significant from the standpoint of the whole campaign. Thus, it is possible that more such battles actually occurred than is indicated by the sample data.

2. Battles with extremely large or extremely small values of advantage may be less well documented than those with moderate values of

advantage. This would be the case if extreme advantage values imply disorder, confusion, and the like on the defeated side.

On the other hand, battles with extreme advantage values are frequently of great interest to the military scholar, and such battles may have been more thoroughly studied than those with moderate advantage values. The combination of lack of data and the desire for definitive historical analysis could, unfortunately, result in sharply conflicting battle descriptions.

3. For battles with extremely large bitterness values, the destruction of records may have been considerable. If this actually occurred, then data for battles of great bitterness could be less reliable than for battles of little bitterness. Intensity may also affect records and data reliability in this manner.

Whether or not any or all of the above censoring effects are involved is not known to the author. A more extensive survey might help to solve or eliminate some of the problems associated with censoring and its effects.

Unsolved Problems

1. The unexplained variability in residual advantage is a problem that has been earlier discussed in great detail. It is a problem that continues to elude our grasp.

2. Another unsolved problem is that of battle duration. What is it that terminates a battle? Why doesn't the defeated side fight to the last man? This last did occur on occasion, though principally in circumstances where either no retreat was possible or where retreat was possible only by breaking out of an encirclement. But why is it exceptional rather than common?

The author has no answ[illegible]r these questions, save for the rather ineffectual observation th[illegible], presumably, the defeated side usually has better things to do than merely stand and fight. This observation, in its present form, does not permit us to say why any given battle lasted, e.g., 6 hours rather than 3 hours or 12 hours, and therefore does not represent an effective solution to the problem of predicting battle duration.

3. Similarly, we know very little about what determines battle intensity. Nor do we have any clue to the causes of the decline of battle intensity in recent times. In particular, we are unable to say whether the introduction of tactical nuclear weapons will reverse or continue the downward trend of battle intensity. To the extent that war games are a reliable imitation of combat, they may provide data on which to base an answer to this last problem, but it is important to recognize that this, alone, would not necessarily improve our fundamental understanding of the factors affecting intensity.

4. Lastly, there is the problem of victory. Advantage is closely associated with victory, but advantage does not always follow the victor. It is not clear why this should be, nor is it clear what factors operate to create exceptions to the general trend.

Findings of Value

1. Possibly the most important finding obtained to date is that of a general correspondence between the theoretical Lanchester parameters and real-world phenomena, since in the absence of such a correspondence attempts at prediction would be doomed to failure. The fact of such correspondence encourages the hope that fitting the theoretical parameters to past data will provide an empirical first approximation to some of the essentially important aspects of ground combat.

2. Possibly the second most important finding is that for Lanchester's square-law model the activity ratio and the advantage parameter depend on the initial force ratio. Formerly, investigators involved with the use of Lanchester's square-law model have usually assumed that

activity ratio and advantage were independent of force ratio, and it is now clear that modifications to these views are necessary.

3. Possibly of lesser importance are the findings indicating some of the factors which do not significantly influence the advantage parameter; since these findings are of greatest value to further investigations.

4. Possibly last in importance are the miscellaneous findings of connections, or lack of connection, among various other parameters (e.g., the connection between intensity and duration, and the lack of connection between force ratio and total force.) Although these results are often interesting and sometimes suggestive, we lack a unifying theoretical foundation which would place these elements in satisfying perspective.

APPLICATIONS

We here consider the question whether the findings in their present incomplete and tentative form can be put to use. The author will list some areas in which he believes that immediate application would be of value.

Applications to War Gaming

1. The findings can be used as a rough, general check on the validity of a ground combat war game by computing the parameter values from game results and comparing them with the parameter values estimated from the historical sample battles.

2. Since the Lanchester parameter provide a useful scheme for comparing battles conducted under different conditions, the Lanchester model and the findings of this study could be of considerable value to those involved in the analysis of ground combat war game results

3. The close association between the theoretical parameters and real-world phenomena suggests that the Lanchester square-law model, together with empirical measures of the variability of certain parameters, can provide a simple, approximate, aggregated ground combat model for war games which emphasize command, control, communications, or strategic aspects rather than detailed combat interactions.

Provide Guidance

1. The findings presented in this study can provide useful background information and guidance to various studies of large-unit ground combat, and can serve as general background information for military planners and students of military history.

2. From one point of view, the most important application of this study is as a point of departure and as a source of information and guidance for future studies and theories of ground combat.

APPENDIX A

CALCULATION OF ϵ

APPENDIX A

CALCULATION OF ϵ

FIRST METHOD

We use Equation (6) as a point of departure, writing the hyperbolic functions in their exponential form and writing $g = e^{\epsilon}$ for brevity. Multiplying both equations by g to clear fractions, we obtain after transposition and reduction the simultaneous quadratic equations:

$$(\mu - 1) g^2 + 2ag - (\mu + 1) = 0 \qquad \text{(A-1)}$$

$$(\mu^{-1} - 1) g^2 + 2dg - (\mu^{-1} + 1) = 0 \qquad \text{(A-2)}$$

Dividing Equation (A-1) by $(\mu - 1)$ and Equation (A-2) by $(\mu^{-1} - 1)$ and subtracting the results yields

$$\left(\frac{2a}{\mu - 1} - \frac{2d}{\mu^{-1} - 1}\right) g - \left(\frac{\mu + 1}{\mu - 1} - \frac{\mu^{-1} + 1}{\mu^{-1} - 1}\right) = 0 \qquad \text{(A-3)}$$

which can be solved for g as

$$g = \frac{\dfrac{\mu + 1}{\mu - 1} - \dfrac{\mu^{-1} + 1}{\mu^{-1} - 1}}{\dfrac{2a}{\mu - 1} - \dfrac{2d}{\mu^{-1} - 1}} \qquad \text{(A-4)}$$

Clearing fractions in Equation (A-4), expanding products and regrouping terms yields

$$g = \frac{\mu^{-1} - \mu}{a (\mu^{-1} - 1) - d (\mu - 1)} \qquad \text{(A-5)}$$

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Multiplying the numerator and denominator of the RHS of Equation (A-5) by μ and factoring yields

$$g = \frac{1-\mu^2}{a(1-\mu) - d(\mu^2 - \mu)} = \frac{(1+\mu)(1-\mu)}{a(1-\mu) + d\mu(1-\mu)}$$

$$= \frac{1+\mu}{a + d\mu} \qquad \text{(A-6)}$$

Taking natural logarithms of both sides of Equation (A-6) and recalling the definition of g yields

$$\epsilon = \ln\left(\frac{1+\mu}{a + d\mu}\right) \qquad \text{(A-7)}$$

which is the desired result.

SECOND METHOD

We again start from Equation (6), this time solving equation (6a) for μ and Equation (6b) for μ^{-1}. We obtain

$$\mu = -\frac{a - \cosh\epsilon}{\sinh\epsilon} \qquad \text{(A-8)}$$

$$\mu^{-1} = -\frac{d - \cosh\epsilon}{\sinh\epsilon} \qquad \text{(A-9)}$$

Eliminating μ between (A-8) and (A-9) yields:

$$\frac{a - \cosh\epsilon}{\sinh\epsilon} = \frac{\sinh\epsilon}{d - \cosh\epsilon} \qquad \text{(A-10)}$$

Clearing fractions and regrouping terms yields:

$$ad - (a + d)\cosh\epsilon + \cosh^2\epsilon - \sinh^2\epsilon = 0 \qquad \text{(A-11)}$$

But since we have identically

$$\cosh^2 \epsilon - \sinh^2 \epsilon = 1$$

Equation (A-11) reduces identically to

$$-(a + d) \cosh \epsilon + 1 + ad = 0 \qquad \text{(A-12)}$$

from which we immediately conclude

$$\epsilon = \cosh^{-1} \left(\frac{1 + ad}{a + d} \right) \qquad \text{(A-13)}$$

which is the desired result.

REMARKS

Either method leads to two values of ϵ, one positive and the other negative in sign. Only the positive sign leads to a reasonable result, however, as may be shown by Equation (6). Thus, for $\epsilon > 0$, $a < 1$ and $d < 1$. But for $\epsilon < 0$, $a > \cosh |\epsilon| > 1$ and $d > \cosh |\epsilon| > 1$, which is impossible in terms of the real-world interpretation of a and d. For this reason, only the positive values of ϵ are indicated in the foregoing developments.

APPENDIX B

DISCUSSION OF SPECIFIC SOURCES AND THEIR DATA

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APPENDIX B

DISCUSSION OF SPECIFIC SOURCES AND THEIR DATA

PURPOSE

To investigate the quality of quantitative historical combat data and to estimate the amount of variability in some of the Lanchester square-law parameters which arises from discrepancies between data sources.

METHOD

Individual Sources

Each source consulted in the preparation of this report was examined for statements relevant to the quality of the data contained in that source. Such statements might include information regarding the author's qualifications, the sources which he used, and his own evaluation of the quality of the data presented.

Variability Between Sources

Seven battles which occurred during the American Civil War were selected for study. Four sources were consulted for data on initial strengths and casualties for each battle. For each battle and for each source, the Lanchester parameters, a, d, μ, D/A, ϵ, and x_0 / y_0 were computed as explained in the body of the report (Section II, Method). Total casualty, C, and total force, X, parameters were also obtained.

The sample variance between sources for the parameters $\ln \mu$, $\ln \epsilon$, and $\ln (x_0 / y_0)$ was obtained separately for each battle, and a pooled estimate of the variance between sources was then obtained for the separate sample variances. In addition, logarithmic scatter diagrams were

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prepared for D/A against x_0/y_0, μ against x_0/y_0, ϵ against x_0/y_0, and for C against X.

RESULTS

Individual Sources

Statements from each source relevant to the quality of the data contained in each are quoted below to the extent applicable, and additional notes are appended if necessary.

1. Reference 9

a. When this paper was written, Engel was employed by the Operations Evaluation Group.

b. The chief source of data used by the above reference is The Iwo Jima Operation, prepared by Capt. Clifford P. Morehouse, Historical Division, United States Marine Corps, undated.

c. No evaluation of the data is given in Reference 9. The Iwo Jima Operation describes itself in its Foreword as follows:

"The information has been compiled from official records, from observations and notes made by the writer during the progress of the operation, and from a few supplementary sources noted in the text."

"In its present version, this monograph is tentative and subject to correction."

"It is hoped that a revised version, possibly with illustrations and additional maps, may be published in more permanent form at a later date."

Additional Notes

This writer does not know whether or not the anticipated revised version has been published.

2. Reference 11

a. Various authors have contributed to the separate articles and have not always listed all the source documents used.

b. No general statement can be made regarding self-evaluation by the authors of the data which they present. In several articles there is an implied self-evaluation to the extent that ranges of values are given for strengths or losses rather than a fixed figure, or more than one source is quoted to illustrate the lack of consensus regarding strength and loss figures. In some articles, a phrase such as "it has been variously estimated at" is inserted, presumably to warn the reader that precise data is not available.

3. Reference 15

a. S' .ele states that this work represents a part of his three years' experience as a lecturer in military history at the Army Service Schools at Fort Leavenworth.

b. Eighteen principle sources are listed by Steele in his Preface, and appear to include the major historical works on American campaigns from the French and Indian War to the Spanish-American War, inclusive. Additional sources are indicated by the footnote references contained in the main part of the book.

c. The data given is frequently prefaced by a remark such as "it is difficult to determine how many were engaged (or lost)" in this battle. In some cases, more than one source is quoted to illustrate lack of agreement. In a few cases, Steele gives his own analysis of conflicting reports, together with his conclusions.

4. Reference 16

a. Morison and his series on World War II Naval Operations are well-known to most readers. The series has been described on occasion as "semi-official." In preparing his history, Morison was frequently in the area of operation, has made extensive use of interviews with commanding officers of both sides and of official reports, and has in several instances visited the battlefield involved (though not always at the time the fighting was conducted).

b. Morison includes practically nothing in the way of data evaluation and criticism.

5. Reference 17

a. As of the time of writing, Berndt was a Captain (Hauptmann) in the German Imperial General Staff.

b. Forty-eight sources are listed as consulted, of which five are German Imperial General Staff works.

c. The following is a free translation of the explanatory preface to Berndt's battle data and describes the type of information given. (Only that information pertaining to "battle" was used in this report.)

"The following Tables 27-63 depict the strength of the opponents, as well as their losses, for the most important battles of the Silesian War and of the 19th century, the most significant sea battles, and finally for some of the most important seiges and investments of fortresses.

"In addition to the stated scope of this book, the most remarkable battles of Frederick II will be presented as well as battles of the 19th century, since these battles -- typical of their era -- provide valuable material for comparison with campaigns of a later date.

"The scale of area of graphic representations, the diagrams, is the same within a given section.

"Blue color always designates the victor, red color the vanquished; the former is always placed on the left, the latter on the right.

"In ambiguous cases, the victor is always taken as the one who in battle generally achieved his objectives, or gained possession of the field. Seiges and investments are assigned colors on the same basis.

"The heavy boldface strength numbers, whose magnitude is graphically displayed by the area of the corresponding color rectangle (or, as the case may be, circle), identify the total number of troops which were present on or near the battlefield, whether they were in fact engaged or not.

"For each battle, if a considerable portion of the total strength was not engaged and if this portion can be measured with some claim to correctness, these facts are cited in the diagrams.

"Whenever possible, the strength of the cavalry (in horsemen) and artillery (in guns) is set forth in parentheses below the total strength values; note that the number of horsemen is included in total strength values.

"For the representation of sea battles (Table 54-55), the quantity of effective troops (i. e., the manning of the fleet) will not be used as in the above battles, but the number of ships, which is without doubt appropriate in such cases.

"In representing the losses, the bloody losses (killed and wounded) correspond to darker shading, the unbloody losses (captured, missing, stragglers, deserters, etc.) are represented by lighter shading.

"To count the missing with the unbloody losses may not wholly accord with the facts, since a larger or smaller portion of them should properly be counted with the killed. Since the number of these cannot be estimated in each case, the missing are counted with the unbloody losses throughout.

"The bloody and unbloody together give the totality of losses, which is entered in heavier type beside the appropriate chart.

"The absolute number of losses can provide no scale for comparative evaluation, but the relative number of losses in proportion to the number of personnel is often employed for this purpose. Thus, the losses in percent of total strength have also been ascertained and are printed in boldface on either side of the word ' - Verluste - '. These percentage values indicate how many men of each hundred of total strength were lost on the average.

"In addition, the percent of bloody losses was calculated and appears in smaller type either enclosed in parentheses near the corresponding absolute number or, if space permits, in a separate row beneath the loss-diagram. These percentages indicate how many of each hundred men were killed or wounded.

"It may easily occur that the loss percentage will be calculated from total strength and not, as is proper in certain cases and for many purposes, from the number actually participating in the battle. This occurs, on the one hand, because it is -- especially for battles of early date -- often quite impossible to say which troops actually fought and which not; on the other hand, because it is not at all clear what influence was exerted on the development of the action by the mere arrival or presence of a force not directly engaged, and it would therefore be most objectionable if these forces were simply dropped from the calculations. Moreover, the latter would form a wholly unstable basis, while a solid foundation cannot be dispensed with for the comparisons to be made later on.

"If halfway tolerably reliable sources are available, the losses in officers (already included in total losses) will be separately listed (in parentheses), and also the losses of battle standards and war materiel.

"A central circle and concentric annulus will be used to represent 'sieges and investments,' where the area of the inner circle represents the strength of the defenders (the besieged), that of the annulus the strength of the siegers.

"The last great investment, namely that of Plevna 1877, will not make use of this representation since Plevna was defended by field fortifications rather than by a fortress, and it is therefore a case of a 'battle.'"

d. Additional Notes

The reader will no doubt wish that Berndt had done some things differently. While admitting the controversial nature of some aspects of Berndt's treatment, this author wishes only that the other works which he consulted had been half as conscientious in their description and their display of the data.

Curiously enough, and despite the completeness with which other data are given, Berndt does not include any information which would identify the participants as to attacker or defender, seemingly considering either that this was of little consequence or that the information was general knowledge. As noted in the body of the report, this lack was supplied by appeal to Reference 18.

6. Reference 18

a. The following Preface appears in this work:

"The sad death of Mr. Harbottle, just as this work was going to press, has thrown upon me the onus of correcting the proofs and preparing the index. The necessity for hurrying the work through the press has precluded comparison of the references in every instance with the original sources from which the Author had taken them; if therefore some few printer's errors or varieties of spelling still remain, they may, I hope, be attributed to the imperfections of one, who had to step suddenly into the breach caused by the loss of a valued friend and collaborator, whose patience in research, depth of knowledge and accuracy in compilation, he could never hope to equal.

P. H. Dalbiac"

b. Sources consulted in preparation of the dictionary are not specified.

c. Additional Notes

This dictionary is one of a series (Dutton's Dictionaries of Quotations) with six other, analogous works. Harbottle is credited with authorship of three of these seven dictionaries — Dictionary of Spanish Quotations, and Dictionary of Historical Allusions and Dictionary of Battles (Ref. 18). This writer estimates that Reference 18 above contains between 1500 and 2000 entries, all of which are quite brief.

On the whole, this author has little confidence in the accuracy of the quantitative data contained in this work, and has as far as possible made use of it only when no other source was adequate.

7. Reference 19

a. No author is credited with this article, and this writer presumes that it was prepared by the staff of LIFE magazine.

b. LIFE notes that all casualty figures given for the war are approximate. No indication is given of the sources used by LIFE.

Variability Between Sources

The seven American Civil War Battles and the data collected on each battle from the four separate sources is presented in Table B-I. Note that (1) not all of the sources give complete data, and (2) that we here allow the description "indecisive" for victorious side, provided the reference consulted gives no better indication of the victor.

The parameter values estimated from the data of Table B-I are given in Table B-II, together with the sample variance between sources for each battle separately and the variance estimate obtained by pooling those for the separate battles according to the formula (See Ref. 13.):

$$\text{Pooled estimate} = S_0^2 = \frac{\sum_{i=1}^{k} (n_i - 1) S_i^2}{\sum_{i=1}^{k} (n_i - 1)}$$

Here n_i is the number of parameter values obtained for the ith battle and S_i^2 is the estimated between-sources variance for the ith battle.

Figures B-1 through B-4 present logarithmic scatter diagrams of D/A against x_0/y_0, μ against x_0/y_0, ϵ against x_0/y_0, and C against X. The data are plotted on these figures as numbers in order to identify the battle to which the plotted values refer. For example, battle number 2 (Antietam) is represented on each figure by four small numeral 2's, one for each of the four sources involved. Regression lines, average values, etc., as determined in the body of the report, are indicated on these figures to facilitate comparison with the corresponding figures developed in the body of the report.

TABLE B-I

COMPARISON OF SOURCES, DATA

No.	Battle	Date	Reference[a]	Identification of Attacker	Identification of Defender	Initial Attacker Strength, x_0	Initial Defender Strength, y_0	Attacker's Casualties, C_x	Defender's Casualties, C_y	Victorious side
1	First Bull Run	1861	19	U.S.A.	C.S.A.	30,[illegible]	32,000	2,900[m]	2,000[m]	D
	"	"	15	"	"	18,5[illegible]	18,000	1,584[w]	1,982[m]	D
	"	"	18	"	"	40,[illegible]	30,000	1,492[w]	1,752[l]	D
	"	"	11	"	"	18,[illegible]	18,000	2,896[l]	1,9[illegible][l]	D
2	Antietam	1862	19	U.S.A.	C.S.A.	[illegible]0,000	39,000	12,350[m]	13,700[m]	A
	"	"	15	"	"	7[illegible]	39,200	15,029[w]	11,305[w]	A
	"	"	18	"	"	[illegible]	35,000	12,460[l]	9,000[l]	Indecisive
	"	"	11	"	"	[illegible],000	40,000	12,[illegible]0[l]	9,000[l]	D
3	Fredericksburg	1862	19	U.S.A.	C.S.A.	106,[illegible]	72,500	12,000[m]	5,300[m]	D
	"	"	15	"	"	122,000	78,500	10,208[w]	5,209[l]	D
	"	"	18	"	"	150,000	80,000	13,771[l]	1,800[l]	D
	"	"	11	"	"	122,000	78,00[illegible]	12,650[l]	4,[illegible]00[l]	D
4	Chancellorsville	1863	19	C.S.A.	U.S.A.	[illegible]3,000	7[illegible],000	13,000[m]	17,000[m]	A
	"	"	15	"	"	55,000	132,000	10,9[illegible][w]	11,128[w]	A
	"	"	18	"	"	53,000	120,000	10,000[l]	10,350[m]	A
	"	"	11	"	"	55,000	132,000	12,000[l]	16,000[l]	A
5	Missionary Ridge (Chattanooga)	1863	19	U.S.A.	C.S.A.	56,000	46,[illegible]	5,800[m]	6,600[m]	A
	"	"	15	"	"	56,359	40,929	5,665[w]	2,541[w]	A
	"	"	14	"	"	80,000	40,000	5,616[m]	b	A
	"	"	11		"	60,000	34,000	6,000[l]	6,664[l]	A
6	Franklin	1864	19	C.S.A.	U.S.A.	[illegible]9,000	37,[illegible]	6,250[m]	2,300[m]	A
	"	"	15	"	"	27,000	26,000	4,550[w]	1,222[w]	D
	"	"	18	"	"	40,[illegible]	30,000	4,500[l]	1,5[illegible]0[w]	D
	"	"	11	"	"	27,000	28,000	4,50[illegible][l]	2,236[l]	Indecisive
7	Stones River	1862-3	1[illegible]	C.S.A.	U.S.A.	38,000	[illegible]7,000	12,000[m]	12,000[m]	D
	"	"	15	"	"	38,000	47,000	b	b	D
	Murfreesboro	"	18	"	"	35,000	49,000	8,0[illegible]0[m]	8,000[m]	D
	Stone River	"	11	"	"	37,7[illegible]	44,800	9,239[w]	9,220[w]	D

[a] Numbers given refer to the list of works consulted

[b] Data not given in source consulted

[c] Not applicable

[w] Killed and wounded

[m] Killed, wounded, and missing

[l] Losses

TABLE B-II

COMPARISON OF SOURCES, PARAMETERS

	Battle	Source[a]	Surviving Fraction of Attacker a	Surviving Fraction of Defender d	n	Activity Ratio D/A	Bitterness ε	Force Ratio x_0/y_0	Sample Variance of Advantage ln μ	Sample Variance of Logarithmic Bitterness ln μ	Sample Variance of Logarithmic Force Ratio ln (x_0/y_0)
1	First Bull Run	19	0.903	0.938	1.233	1.325	0.081	0.838			
	"	15	0.914	0.880	0.888	0.632	0.102	1.028			
	"	18	0.963	0.942	0.804	1.148	0.048	1.333			
	"	11	0.843	0.880	1.170	1.465	0.141	1.028	0.044	0.323	0.023
2	Antietam	19	0.824	0.649	0.746	1.790	0.289	1.795			
	"	15	0.786	0.712	0.881	2.476	0.287	1.766			
	"	18	0.869	0.743	0.740	4.029	0.204	2.714			
	"	11	0.829	0.775	0.886	2.403	0.219	1.750	0.620	0.132	0.046
3	Fredericksburg	19	0.887	0.927	1.231	3.241	0.085	1.462			
	"	15	0.916	0.934	1.118	3.018	0.077	1.554			
	"	18	0.906	0.978	1.984	11.642	0.047	1.875			
	"	11	0.886	0.947	1.376	4.631	0.077	1.544	0.063	0.367	0.012
4	Chancellorsville	19	0.824	0.767	1.117	0.433	0.209	0.589			
	"	15	0.301	0.916	1.491	0.388	0.140	0.417			
	"	18	0.871	0.914	1.439	0.40[illegible]	0.137	0.442			
	"	11	0.782	0.879	1.307	0.298	0.178	0.417	0.017	0.574	0.027
[illegible]	Missionary Ridge Chattanooga	[illegible]	0.896	0.867	0.859	1.093	0.130	1.217			
	"	15	0.903	0.938	1.239	2.908	0.081	1.377			
	"	18	b	b	b	b	b	b			
	"	11	0.900	0.745	0.653	1.322	0.178	1.765	0.103	0.458	0.036
6	Franklin	19	0.936	0.928	1.476	3.072	0.116	1.168			
	"	15	0.831	0.956	1.901	3.361	0.091	0.964			
	"	18	0.684	0.950	1.475	3.872	0.078	1.333			
	"	11	0.759	0.920	1.662	2.568	0.151	0.964	0.016	0.335	0.026
7	Stones River	19	0.684	0.745	1.093	0.780	0.334	0.808			
	"	15	b	b	b	b	b	b			
	Murfreesboro	18	0.771	0.800	1.061	0.861	0.241	0.875			
	Stone River	11	0.755	0.794	1.079	0.825	0.254	0.842	0.009	0.092	0.002
					Pooled Estimate of Variance, S_0^2				0.034	0.409	0.025
					Pooled Estimate of Standard Deviation, S_0				0.186	0.639	0.158

[a] Numbers given refer to list of works consulted

[b] Not applicable

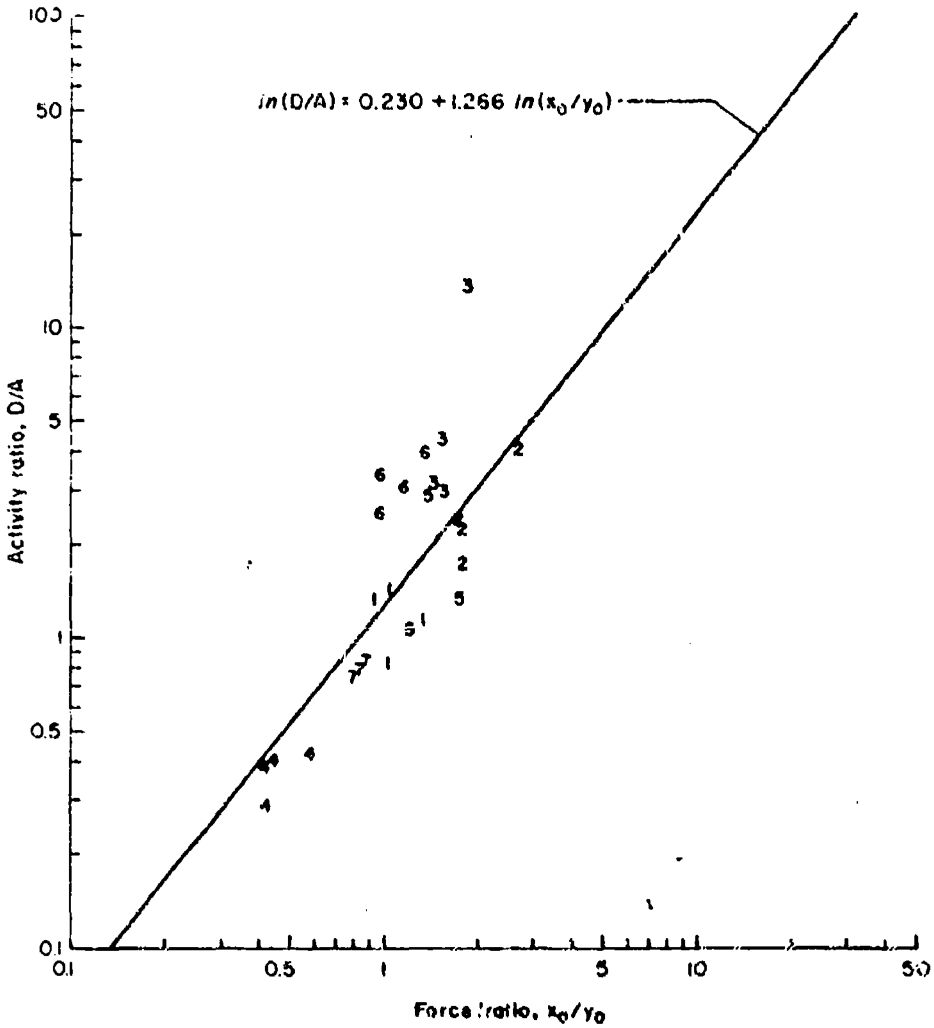

Figure B-1. Comparison of sources: Logarithmic scatter diagram of activity ratio, D/A, against force ratio, x_0/y_0.

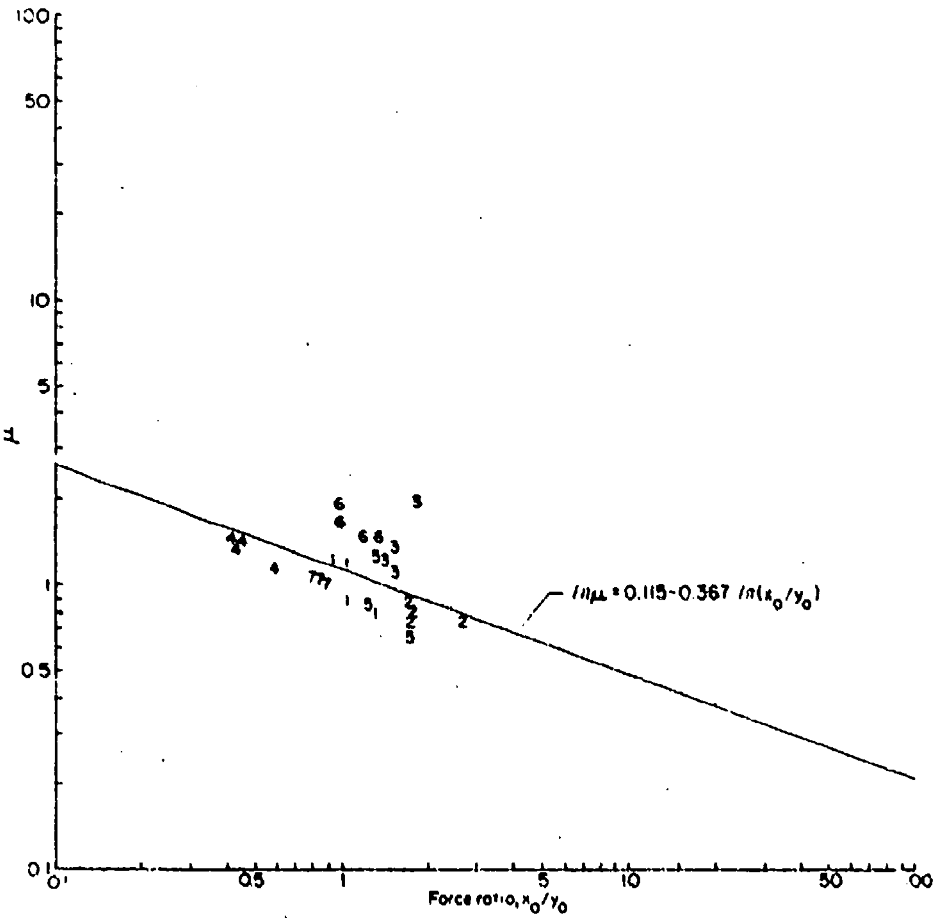

Figure B-2. Comparison of sources: Logarithmic scatter diagram of μ against force ratio, x_0/y_0.

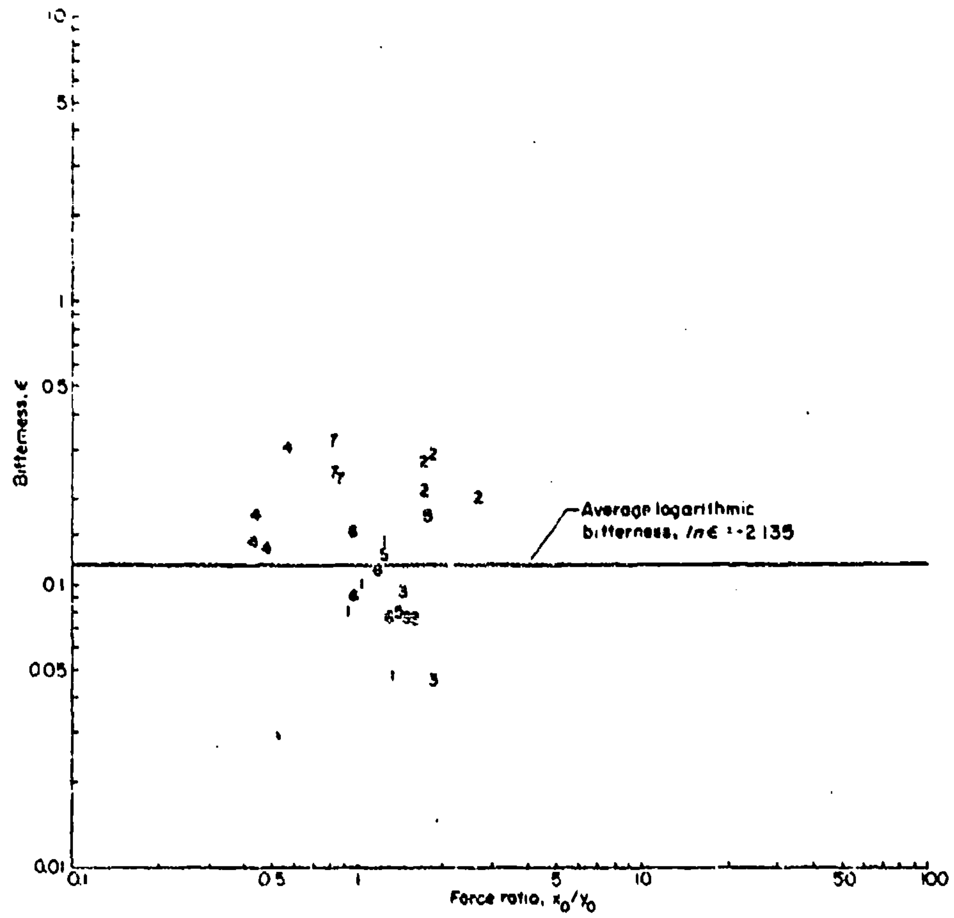

Figure B-3. Comparison of sources: Logarithmic scatter diagram of bitterness, ϵ, against force ratio, x_0/y_0.

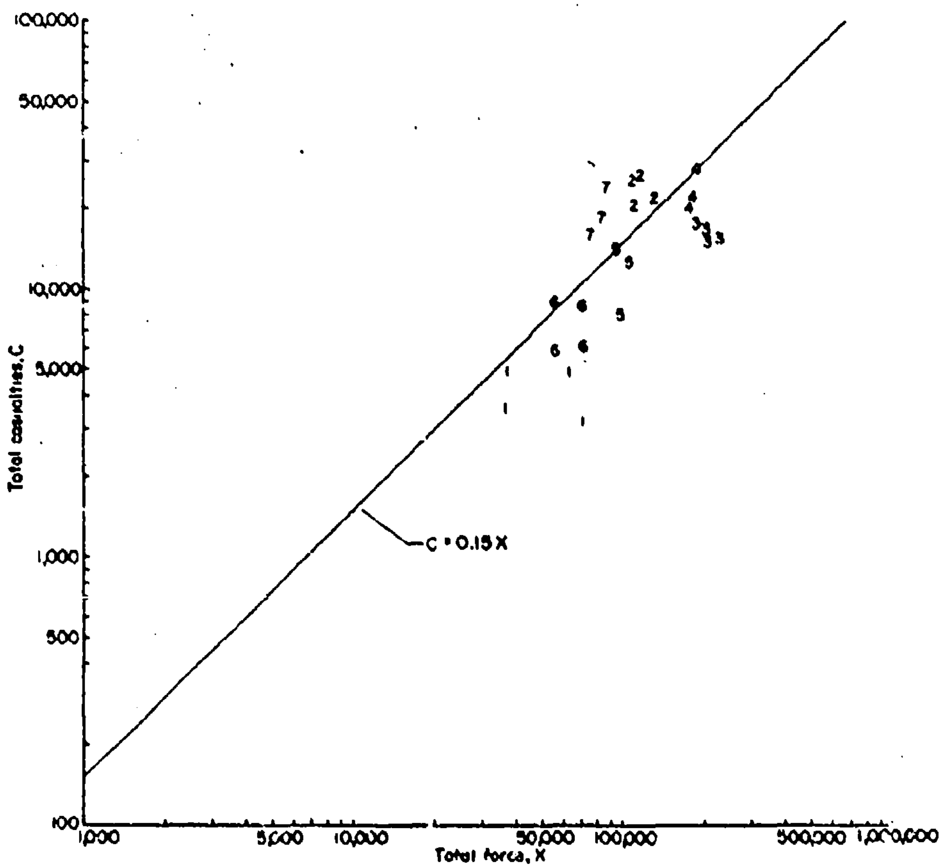

Figure B-4. Comparison of sources: Logarithmic scatter diagram of total casualties, C, against total force, X.

APPENDIX C

SOME NARRATIVE ACCOUNTS FOR BATTLES WITH EXTREME VALUES RESIDUAL ADVANTAGE

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SOME NARRATIVE ACCOUNTS FOR BATTLES WITH EXTREME VALUES RESIDUAL ADVANTAGE

BATTLE OF CONTRERAS, MEXICAN WAR, 1847 A.D., RESIDUAL ADVANTAGE = - 1.064

Narrative from Reference 11, Article on "Contreras"

A hamlet about eight miles S.W. of Mexico City. It was in the vicinity of this small town that Maj. Gen. Winfield Scott, with about 4,200 U.S. troops of his column in the southern campaign of the war between Mexico and the United States (1846-48) encountered first, the difficult barriers of pedregal (lava beds) and lakes surrounding his adversary's captial. In this particular defense, Gen. A. L. deSanta Anna employed about 7,000 Mexicans, although possibly three times that number occupied forts and redoubts at critical points elsewhere about the city. Scott, having circled to the south of his goal, found, upon determined reconnaissance, that San Antonio directly in his front was so heavily defended that an assault might cripple him. He decided to build a road over the pedregal toward the west, thereby pinching out Santa Anna's strong position. Accordingly, on August 19, 1847, Maj. Gen. G. J. Pillow's division was ordered to supply working parties and push forward the road the engineers were building and to brush aside any small resistance without bringing on a general engagement. Pillow, however, after being stopped, decided to attack. In the premature struggle only the natural teamwork of the trained leaders, such as Riley, Smith, and Magruder, saved defeat until Scott, coming upon the field, ordered a concentration of his forces at San Geronimo, the key position. Darkness, aggravated by a storm, cut Scott off from the knowledge of the whereabouts of his troops.

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In the early morning of August 20, Capt. Robert E. Lee, after a voluntary desperate ride across the pedregal, bore Scott the intelligence that Brig. Gen. P. F. Smith had found a way to the enemy's rear and would attack at dawn. Scott prepared Brig. Gen. D. E. Twiggs to cooperate in front, and as a result, San Geronimo was taken in front, flank, and reverse in less than 20 minutes. San Antonio was evacuated and the battle of Churubusco began the same day. The cannon and ammunition captured from the Mexicans aided Scott materially. The Mexican losses were about 1,500 as against fewer than 100 Americans.

Narrative from Reference 15, pp. 112-123, passim

Valencia marched first to San Angel, and then, contrary to Santa Anna's orders, put his command into an isolated position upon an open ridge just north of Contreras and west of the southern end of the Pedregal, a large space covered with volcanic rocks. He placed a part of his troops several hundred yards in front of his main position at the ranch of Padierna. Valencia was planning to fall upon the flank and rear of the Americans as they moved on the capital by way of San Antonio.

The American army reached San Augustin on the 17th of August, and occupied it without serious opposition. This place then became the base of operations. The engineers discovered the position of Valencia's army; they also ascertained by reconnaissance that a road could be made over the Pedregal by which Valencia's position could be turned, his rear attacked, and his line of retreat to the main army and the capital cut off.

On the 18th of August Worth's division moved forward to San Antonio, and on the 19th masked that place. On this day the divisions of Twiggs and Pillow advanced against Valencia's position. While part of this force attacked and captured the advanced post at Padierna driving back the [illegible] brigades, [illegible] made their way across the Pedregal to the woods around San Geronimo, less than 2,000 yards to the left and rear of Valencia's main position.

Contd [illegible]

Here Smith's detachment was wholly separated from the rest of the American Army, and his rear was exposed to attack from the direction of San Angel. Santa Anna, in fact, hurried forward a brigade, which appeared on the Hill of Toro, 1,500 yards north of Geronimo, at 5 P.M.; but before it made any movement of attack, it received orders from Santa Anna to retire to San Angel, where he himself spent the night with other troops. The only assault upon Smith at San Geronimo was made by Valencia's cavalry, which was easily repulsed.

That evening the Americans vacated the ranch of Padierna, which was reoccupied by the Mexicans. During the night General Scott sent Shield's brigade to reinforce Smith. Santa Anna sent an order to Valencia directing him to withdraw to Coyoacan. Valencia refused to obey the order, and held his position.

Leaving Shields to hold Geronimo, protect his rear and cut off the enemy's retreat, Smith moved forward the rest of his detachment before daybreak on the 20th, and fell upon the rear of Valencia's position. Scott had ordered Worth and Quitman to support this attack by assaulting the front of the position. This "secondary attack" was quite unnecessary, for Smith's assault took the enemy by surprise and put him to flight. The victory was complete.

While the Americans were thus routing Valencia's command, Santa Anna was at San Angel, less than three miles distance from the battlefield, with three brigades. He started forward with part of his force, but was almost immediately met by Valencia's flying troops. Thereupon he turned about and hastened toward the city, sending orders for all the Mexican troops to concentrate upon the inner line of defense at the garitas of the capital.

In this engagement, which Americans call the battle of Contreras, and the Mexicans call the battle of Padierna, the Americans engage numbered 4,500; the Mexicans 4,000. The Americans lost fewer than 100 men; the Mexicans lost 700 killed, 813 prisoners.

General Valencia's division was ordered to hold San Angel; but this general moved forward and occupied the advanced position at Contreras. The position was from every point of view untenable. Not only could it be turn easily, and attacked in rear (as was done), but Valencia would have been just as bad off, if Scott had only left a containing force to occupy his attention in front, while he forced a way forward over the San Antonio cause way, and from there crossed over to the San Angel road. This would have cut Valencia off from his communications. Taking position, then, at Contreras was Valencia's strategic mistake. He made an equally bad tactical blunder, in putting a part of his force in front at Padierna — an advanced post beyond the support of his main position.

BATTLE OF ROSSBACH, SEVEN YEARS' WAR, 1757 A.D., RESIDUAL ADVANTAGE = - 0.821

Narrative from Reference 11, Article on "Rossbach"

A village in the Land of Sacony, Germany, in the district of Merseburg, 8 miles S.W. of that place and N.W. of Weissenfels, famous as the scene of Frederick the Great's victory over the allied French and the army of the Empire on November 5, 1757. For the preceding events, see Seven Years' War. The Prussian camp on the morning of the 5th lay between Rossbach (left) and Bedra (right), facing the Allies, who, commanded by the French General, Charles de Rohan, prince de Soubise, and Joseph Frederick William, duke of Saxe-Hildburghausen, General Feldzumeister of the Empire, had maneuvered in the preceding days without giving Frederick an opportunity to bring them to action, and now lay to the westward with their right near Branderoda and their left at Mucheln (see Sketch). The advanced Posts of the Prussians were in the villages immediately west of their camp, those of the Allies on the Schortau hill and the Galgenberg.

The Allies possessed a numerical superiority of two to one in the battle itself, irrespective of detachments,[1] and their advanced post overlooked all parts of Frederick's camp. They had had the best of it in the maneuvers of the previous days, and Hildburghausen determined to take the offensive. He had some difficulty, however, in inducing Soubise to risk a battle, and the Allies did not begin to move off their camping-ground until after eleven o'clock on the 5th, Soubise's intention being probably to engage as late in the day as possible with the idea of gaining what advantages he could in a partial action. The plan was to march the Allied army by Zeuchfeld, round Frederick's left (which was covered by no serious natural obstacle), and to deploy in battle array, facing north, between Reichardtswerben (right) and Pettstadt (left). The duke's proposed battle and the more limited aim of Soubise were equally likely to be attained by taking this position, which threatened to cut off Frederick from the towns on the Saale. This position, equally, could only be gained by marching round the Prussian flank, i.e., by a flank march before the enemy. The obvious risk of interference on the exposed flank was provided against by a considerable flank guard, and in fact it was not in the execution of their original design but in hastily modifying it to suit unfounded assumptions that the Allies met with disaster.

Frederick spent the morning watching them from a house top in Rossbach. The initial stages of their movement convinced him that the Allies were retreating southward towards their magazines, and about noon he went to dinner, leaving Captain Von Gaudi on the watch. This officer formed a different impression of the Allies' intentions, for the columns which from time to time became visible in the undulations of

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[1]V. der Goltz (Rossbach bis Jena, 1906 edition) gives 41,000 Allies and 21,000 Prussians as the combatant strengths. Berndt's statistical work, Zahl im Kriege, gives the respective forces engaged as Allies 43,000, Prussians 21,000. Other accounts give the Allies total strength as 64,000 and the Prussians as 24,000.

the ground were seen to turn eastwards from Zeuchfeld. Gaudi's excited report at first served only to confirm Frederick in his error. But when the king saw for himself that hostile cavalry and infantry were already near Pettstadt, he realized the enemy's intentions. The battle for which he had maneuvered in vain was offered to him, and he took it without hesitation. Leaving a handful of light troops to oppose the French advanced post (or flank guard) on the Schortau hill, the Prussian army broke camp and moved -- half an hour after the king gave the order -- to attack the enemy. The latter were marching in the normal order in two main columns, the first line on the left, the second line on the right; farther to the right was a column consisting of the reserve of foot, and between the first and second lines was the reserve artillery on the road. The right-wing cavalry was of course at the head, the left wing cavalry at the tail of the two main columns. At first regulation distances were preserved, but when wheeling eastward at Zeuchfeld there was much confusion, part of the reserve infantry getting in between the two main columns and hampering the movements of the reserve artillery, and the rest, on the outer flank of the wheel, being unable to keep up with the over-rapid movement of the wheeling pivot. A weak flank guard was thrown out towards Rossbach. When it was seen that the Prussians were moving, as far as could be judged, eastward, it was presumed that they were about to retreat in order to avoid being taken in flank and rear; and the Allied generals thereupon hurried the march, sending the cavalry on ahead.

Frederick had no intention either of forming up parrallel to the enemy or of retreating. As his army could move as a unit twice as fast as the enemy's, he intended to make a detour, screened by the Janus Hugel and the Polzen Hugel, and to fall upon them suddenly from the east. If at the moment of contact the Allies had already formed their line of battle facing north, the attack would strike their right flank. If they were still on the move in column eastwards or north-eastwards, the heads of their columns would be crushed before the rest could deploy in the new direction -- deployment in those days being a lengthy affair. To this end General Von Seydlitz, with every available squadron, hurried eastward from Rossbach, behind the Janus Hugel, to the Polzen Hugel; Colonel Von Moller, with eighteen heavy guns, came into action on the Janus Hugel at 3:15 against the advancing columns of the Allied cavalry; and the infantry followed as fast as possible. When they came under the fire of Moller's guns, the Allied squadrons, which were now north of Reichardtswerben and well ahead of their own infantry, suffered somewhat heavily, but it was usual to employ heavy guns to protect a retreat, and they contented themselves with bringing some fieldguns into action. They were, however, amazed when Seydlitz's thirty-eight squadrons suddenly rode down upon the head and right flank of their columns from the Polzen Hugel "avec une incroyable vitesse." Gallantly as the leading German regiments deployed to meet him, the result was scarcely in doubt for a moment. Seydlitz threw in his last squadron, and then himself fought like a trooper, receiving a severe wound. The melee drifted rapidly southward, past the Allied infantry, and Seydlitz finally rallied his horsemen in a hollow near Tagewerben, ready for fresh service. This first episode was over in half an hour, and by that time the Prussian infantry, in echelon from the left, was descending the Janus Hugel to meet the already confused and disheartened infantry of the Allies. The latter, as their cavalry had done, managed to deploy some regiments on the head of the column, and the French in particular formed one or two columns of attack -- then peculiar to the French army -- and rushed forward with the bayonet. But Moller's guns, which had advanced with the infantry, tore gaps in the close masses, and, when it arrived within effective musketry range, the attack died out before the rapid and methodical volleys of the Prussian line. Meanwhile the Allies were trying in vain to form a line of battle. The two main columns had got too close together in the advance from Pettstadt, part of the reserve which had become entangled between the main columns was extricating itself by degrees and endeavoring to catch up with the rest of the reserve column away to the right, and the reserve artillery was useless in the middle of the infantry. The Prussian infantry was still in echelon from the left, and the left-most battalions that had repulsed the French columns were quickly within musket-shot of this helpless mass. A few volleys directed against the head and left flank of the column sufficed to create disorder, and then from the Tagewerben hollow Seydlitz's rallied squadrons charged, wholly unexpectedly, upon its right flank. The Allied infantry thereupon broke and fled. Soubise and the duke, who was wounded, succeeded in keeping one or two regiments together, but the rest scattered over the countryside. The battle had lasted less than an hour and a half, and the last episode of the infantry fight no more than fifteen minutes. Seven Prussian battalions only were engaged, and these expended five to fifteen rounds per man. Seydlitz and Prince Henry of Prussia, the cavalry and the infantry leaders engaged, were both wounded, but the total loss of the king's army was under 550 officers and men as compared with 7,700 on the part of the Allies.

Narrative from Reference 18, Article on "Rossbach"

Fought November 5, 1757, between 80,000 French and Austrians under Marshal Soubise, and 30,000 Prussians, under Frederick the Great. Frederick, who occupied the heights of Rossbach was attacked by the allies. The Prussian cavalry, however, under Seydlitz, charged down upon the Austrians, and threw them into disorder, and the infantry falling upon the broken columns utterly routed them, with a loss of 4,000 killed and wounded, 7,000 prisoners, including 11 generals and 63 guns. The Prussians lost 3,000 only.

BATTLE OF MORTARA, WAR OF THE ITALIAN RISING, 1849 A. D., RESIDUAL ADVANTAGE = - 0.583

Narrative from Reference 18, Article on "Mortara"

Fought March 21, 1849, between the Piedmontese, under the Duke of Savoy (Victor Emmanuel) and General Darando, and the main Austrian army, under Radetsky. No steps had been taken by the Piedmontese to render Mortara defensible, and little guard was kept, with the result that they were surprised by Radetsky, and driven out of the town in confusion, with a loss of 500 killed and wounded, 2,000 prisoners and 5 guns. The Austrians lost 300 only.

BATTLE OF CERRO GORDO, MEXICAN WAR, 1847 A. D., RESIDUAL ADVANTAGE = - 0.569

Narrative from Reference 15, pp. 108-121, passim

After his defeat at Buena Vista (February 22-23), Santa Anna fell back with his army to San Luis Potose, where he arrived after a march of great hardship with less than 10,000 effectives. After resting here four days, he resumed the march to the capital with two brigades.

A new revolution had meantime, broken out in the City of Mexico, and, when Santa Anna arrived there, he found armed partisans confronting each other in the streets. He managed to compose the dissensions, and on the 2nd of April set out for Cerro Gordo, a strong position on the road to Vera Cruz, where he purposed stopping the progress of Scott's army. A part of the troops he had commanded at Buena Vista, some 5,600, had already turned toward the same point; the rest remained for the present at San Luis Potose. From the capital Santa Anna took the National Guards of that city. Cerro Gordo (Big or Fat Hill), is at the foot of the Sierra Madre -- the last step from the great Plateau of Anahuac to the Tierra Caliente.

Here the road from Vera Cruz crosses a small river and a narrow stretch of level ground, the Plan del Rio; then it zigzags upwards and to the west, following the easiest grade from one bench to the next. To the right of the road, as one goes west, are first rugged c fs and then an impassable ravine. Between the ravine and the road a two prominent wooded knolls, Atalaya and Telegrafo, a few hundred yards beyond which the road passes the Cerro Gordo Ranch. From the Plan del Rio to this ranch, and probably farther westward, the stream flows in an impassable gorge; and about midway of the distance is a commanding table. On this table the right of the Mexican line rested The left was on Telegrafo, and the XX reserve was at the ranch.

The position as occupied was strengthened with parapets, trenches, palisades, and abatis, and the trees were cleared away from the field in front. Artillery was so placed as to command the road and sweep all the approaches to the position.

Owing to the ferocious heat and deep sand, the march of Scott's army across the Tierra Caliente, from Vera Cruz to Plan del Rio, was very hard and trying. Twigg's division, which started on the 8th of April, did not reach Plan del Rio until the evening of the 11th. Twiggs had been informed on the way that Santa Anna was at Jalapa with troops, and he expected to meet him at Cerro Gordo.

As soon as General Scott received report that the divisions of Twiggs and Patterson had found the enemy in force at Cerro Gordo, he hastened forward from Vera Cruz himself, and joined the troops at the front on April 14. Having gotten more transportation, General Worth, also marched his division forward, and had reached the camp of the other divisions at Plan del Rio, by the evening of April 17.

Meantime the American engineer officers had been reconnoitering the Mexican position. They reported that the position, though fortified and very strong in front, could be turned by its left and struck in rear; and that the intrenchments on Telegrafo could be carried by assault. Telegrafo was the key to the position.

On the 17th Twigg's division followed the route picked out by the engineers; and, after an action in which it lost ninety-seven officers and men, it got possession of Atalava. General Scott there upon issued an order for a general attack, for the morning of the 18th. Worth's division of regulars with Shield's brigade of volunteers was to follow up and support the "main attack" against the Mexican left and rear; and Pillow's brigade of volunteers was to make the "secondary attack" against the front. The main attack carried Telegrafo, put the left of the Mexican line to flight, and got possession of the Jalapa road. Seeing escape impossible, the entire right of the Mexican line then surrendered. The American cavalry pursued the routed Mexicans, but was not fleet enough to do them much damage.

General Scott reported his strength at this battle as 8,500, and his killed and wounded, thirty-three officers and 398 enlisted men. He estimated the Mexican strength at 12,000, and the losses 1,000 to 1,200 killed and wounded, and 3,000 captured. The prisoners, like those taken at Vera Cruz, were, for lack of means to care for them, paroled.

Lieutenant - General Robles, the Mexican engineer that fortified Cerro Gordo, gave it as his opinion, that the position was only suitable for "harassing an invading army," -- that is, for fighting a delaying action, a rear-guard action -- and not for fighting a defensive battle with the hope of winning a decisive victory. He reported that the position could be turned precisely as it was afterwards turned by the Americans. He "advised that the main defense be made at Corral Falso, six or eight miles in rear." But Santa Anna insisted upon fortifying Cerro Gordo. It was here that the patriots had made a famous stand against the Spanish in the War of Independence. Nor did Santa Anna make any provision against the turning movement. Even after the Americans had got possession of the hill, Atalaya, on the first day of the Battle, he persisted in believing that the main attack would be made against his right and front, and he made his dispositions accordingly.

BATTLE OF PLEVNA, RUSSO-TURKISH WAR, 1877 A. D., RESIDUAL ADVANTAGE = - 0.515

Narrative from Reference 11, Article on "Plevna," passim

Investment and Fall of Plevna

This was the last open-force attack on Osman's lines. (Third Battle of Plevna) General Todleben, the defender of Sevastopol, was now entrusted with the conduct of the seige, he determined to complete the investment, which was accomplished by October 24, Osman's request to retire from Plevna having been refused by Constantinople. Supplies eventually gave out and a sortie on the night of December 9-10 failed, with the result that he and his army capitulated.

Plevna is a striking example of the futility of the purely passive defense, which is doomed to failure however, tenaciously carried out. Osman Pasha repelled three Russian attacks and practically held the whole Russian army. It remained for the other Turkish forces in the field to take the offensive and by a vigorous counter-stroke to reap the fruits of his successes. Victories which are not followed up are useless.

Narrative from Reference 18, Article on "Plevna," passim

On December 10, Osman Pasha, at the head of 25,000 Turks, [illegible] and wounded in carts, attempted to [illegible] way through the Russian army, now 100,000 strong, under [illegible] Roumania, with Todleben as Chief of Staff. The attempt was made on the east of Plevna, and was directed against the Imperial Grenadiers, under General Ganetzke. Having successfully crossed the Vid, Osman charged down upon the Russians, on a line two miles in length, and carried the first line of entrenchments. Todleben, however, hurried up reinforcements, and the Turks were in turn attacked, and driven back in confusion across the river, Osman being severely wounded. Here they made their last stand, but were overpowered, and driven into Plevna, which before evening capitulated, after a defense lasting 143 days. In this engagement, the Turks lost 5,000, and the Russians 2,000 killed and wounded.

BATTLE OF TSCHERNAJA, CRIMEAN WAR, 1855 A. D., RESIDUAL ADVANTAGE = 0.577

Narrative from Reference 18, Article on "Tschernaja"

Fought August 16, 1855, between three Russian divisions, under General Gortschakoff, and three French and one Sardinian division, under General Marmora. The Russians attacked the allies' position on the Tschernaja, and after severe fighting, were repulsed with a loss of 5,000 killed and wounded. The allies lost 1,200.

BATTLE OF LISAINE, FRANCO-GERMAN WAR, 1871 A. D., RESIDUAL ADVANTAGE = 0.601

Narrative from Reference 11, Article on "Franco-German" War, passim

The last episode is Bourbaki's campaign in the east, with its mournful close at Pontarlier. Before the crisis of the last week of November, the French forces under General Cremer, Cambriel's successor, had been so far successful in minor enterprises that, as mentioned above, the right wing of the Loire army, severed from the left by the battle of Orleans and subsequently held inactive at Bourges and Nevers, was ordered to Franch Comte to take the offensive against the XIV corps and other German troops there, to relieve Belfort and to strike a blow across the invader's line of communications. But there were many delays in execution. The staff work, which was at no time satisfactory in the French armies of 1870, was complicated by the snow, the bad state of the roads, and the mountainous nature of the country, and Bourbaki, a brave general of division in action, but irresolute and pretentious as a commander-in-chief, was not the man to cope with the situation. Only the furious courage and patient endurance of hardships of the rank and file, and the good qualities of some of the generals, such as Clinchart, Cremer, and Billot, and junior staff officers such as Major Brugere (afterwards generalissimo of the French army), secured what success was attained.

Werder, the German commander, warned of the imposing concentration of the French, evacuated Dijon and Dole just in time to avoid the blow and rapidly drew together his forces behind the Ognion above Vesoul. A furious attack on one of his divisions at Villersexel (Jan. 9) cost him 2,000 prisoners as well as his killed and wounded, and Bourbaki, heading for Belfort, was actually nearer to the fortress than the Germans, but at the crisis more time was wasted, Werder (who had almost lost hope of maintaining himself and had received both encouragement and stringent instructions to do so) slipped in front of the French, and took up a long weak line of defense on the river Lisaine, almost within cannon shot of Belfort. The cumbrous French army moved up and attacked him there with 150,000 against 60,000 (Jan. 15-17, 1871). It was last repulsed, thanks chiefly to Bourbaki's inability to handle his forces, and, to the bitter disappointment of officers and men alike, he ordered a retreat, [illegible] to its fate.

BATTLE OF PALO ALTO, MEXICAN WAR, 1846 A. D., RESIDUAL ADVANTAGE = 0.662

Narrative from Reference 11, Article on "Palo Alto"

A battlefield in Cameron County, Texas, between Point Isabel and Matamoras, about nine miles northeast of the latter. There on May 8, 1846, took place the first battle of the war between Mexico and the United States (1846-48). Brigadier General Zachary Taylor's forces of about 2,200 regulars, in an effort to join with beleaguered troops at Fort Brown, came upon a superior number of Mexicans (variously estimated between 4,000 and 6,000) under General Mariano Arista. Between the chaparral and the marshes the two lines were drawn up opposite each other, the Mexicans astride the road to Fort Brown. Taylor's artillery, better handled than the Mexican heavy pieces, cut great swaths in the enemy's lines in process of forming. An attempt to turn the U. S. right by a superior force of Mexican cavalry was met by a hollow square of the 5th Infantry. The grass was set on fire by the shells, so that a dense smoke screen kept the two armies from seeing each other well. In this haze, the trained and disciplined subordinate leaders of the U. S. forces met the Mexican attempt to encircle the left. Samuel Ringgold's and James Duncan's batteries moved and fired handily in spite of the smoke. When darkness closed the issue for the day the Mexican loss was, according to estimate, about seven times that of the U. S. The engagement was the first instance where superiority of training against an organized enemy rested with the Americans.

Narrative from Reference 15, pp. 85-101, passim

At about noon on May 8, Taylor found Arista's army in line of battle near Palo Alto. The right of the Mexican line rested on a sort of low ridge, about 3,000 yards to the east of Taylor's road, which passed along the edge of a swampy chaparral. The right of the line was covered by a squadron of cavalry, and the left by eight squadrons. The guns were in the intervals between the infantry regiments. The line faced nearly north and had a clear field of fire over a stretch of prairie with several swampy places in it. The left of the line rested on a marsh, and there were two large marshes immediately in rear of the line. The left of the Mexican lines was the weaker, and the chaparral on that flank offered some cover for the Americans. So General Taylor made his main attack against that flank. Captain May's squadron covered the left of Taylor's line, while Captain Ker's squadron guarded the exposed right, and the rear where the train was parking. As with the Mexican line, the artillery was posted between the infantry regiments.

Taylor's line advanced in regimental columns until the Mexican artillery opened upon it. Thereupon it deployed. The Mexican cavalry under General Torrejon now moved through the chaparral to attack the right and rear of the American line; but it was repulsed by the 5th Infantry, which was on the right flank, and a section of artillery. The Mexican left was now drawn back. Taylor changed the direction of his line to conform to the new position of the enemy. There was now about an hour's cessation of the battle.

May's squadron was sent round against the Mexican left flank; but finding itself greatly outnumbered it did not charge, but withdrew to the rear.

Just before dark Arista massed his troops in his right wing and moved them against the American left flank and rear, sending Torrejon again with his cavalry against the American right. The main attack was repulsed by Duncan's battery, supported by the 8th Infantry and Ker's squadron of cavalry. The cavalry attack was also repulsed. At this moment there was great confusion in the Mexican line, and, if the Americans had followed up their success by a vigorous counter-attack, the Mexicans would probably have been routed. It was now dark and the Mexicans retired into the chaparral in rear of their position, the Americans bivouacked on the battlefield.

Although General Taylor took the offensive in this battle and advanced to the attack, he found himself outnumbered by Arista's army, especially in cavalry, and was immediately thrown upon the defensive. General Wilcox in his account says: "The action of May 8 on the plains of Palo Alto was, on the part of the United States forces, defensive and mainly of artillery against Mexican artillery and cavalry supported by infantry."

One incident occurred in the combat that might happened, with like effects, even in our day of long-range magazine fire: the prairie grass was set on fire by bursting shells, and the smoke for a time concealed the movements of the hostile armies from each other.

At Palo Alto the Americans numbered 2,288, and the Mexicans were estimated at about 6,000. The Mexican account of the combat gives the number of the Mexicans as only 3,000.

At dawn the next morning the hostile armies were in sight of each other, but by sunrise the Mexicans had begun a retreat for Matamoros. The American army, with its front well covered by scouts, took up the pursuit. About three miles from the battlefield of the day before, Arist[illegible] in his retreat came upon what he conceived to be a very strong position. Here he decided to halt his army. He placed his line in a resaca, or narrow shallow swale, which crossed the road in the midst of a thick chaparr[illegible] and ended in a pool of water at each end. Arista appears not to [illegible] believed that the Americans would seriously attack him in this [illegible]on; he had his tent pitched and was busy writing, when General [illegible] opened fire on his line, and began the battle of Resaca [illegible] Palma. Some Mexican artillery in and near the road on both si[illegible] of the swale could not be driven back; so Captain May was ord[illegible] [illegible]e it with his squadron. On account of the dense gro[illegible] [illegible]d only move in the road, and had to charge in column of fours. He captured seven guns and a Mexican general. He was driven from the battery by Mexican infantry, but the guns were soon retaken by American infantry. The action in the chaparral lasted for some time, but the Mexicans were finally routed. They dispersed and fled, never stopping till they had crossed the Rio Grande.

Ker's dragoons, Duncans's battery, and the 3rd Infantry followed in pursuit. The Mexican account states that if Taylor had pursued with his whole available force, and followed the fugitives across the river, "it is undoubted that he would have completely destroyed them and taken Matamoros without opposition." But Taylor had no means of crossing the river. Mexican batteries at Matamoros opened on the pursuers, and the American guns in Fort Brown (as the fort built opposite Matamoros was afterwards named, in honor of Major Jacob Brown, who was killed there during the siege and attack by the Mexicans) "were directed upon the mass of fleeing Mexicans while crossing at the upper ferry."

The losses of the Americans in the two engagements were 170 killed and wounded, and the estimated loss of the Mexicans, 1,000. Arista's baggage and all sorts of plunder were abandoned to the Americans.

At Palo Alto Taylor's line of battle was formed nearly parallel to the road he was marching on -- his line of communication with his base at Point Isabel. Could the Mexicans have driven him back, or turned his left flank, they might have captured his wagon-train and cut him off from his base. Of course, with so small a force, this was not as serious a consideration as it would be with a great army. Taylor undoubtedly fought the battle in that way, because the chaparral west of the road gave him some cover through which to approach the Mexican left flank. An attack directly from the front, or against the Mexican right, would have had to be made over the open prairie without any sort of cover. Had Taylor been defeated in this combat, he would probably have lost his depot at Point Isabel, and his field-work and garrison at Fort Brown.

At Resaca de la Palma Arista's army fought with the Rio Grande at its back; under like conditions a large army would have been captured. This little force, with no impedimenta, was able to scatter and cross wherever the men could find boats. Many were drowned. We have seen that Taylor could not cross the river and complete the route and dispersion of his enemy for lack of a ponton-train.

In the Combat of Palo Alto, also, there was the appearance of over-caution on the part of the American commander -- it is evident that he believed himself greatly outnumbered. The Mexican account gives Arista only 3,000 men in the engagement; this number is probably nearly right. Taylor had 2,288. If our troops had not been content with repulsing the enemy and holding their ground, but had made a counter-attack at the right moment, they would have routed the enemy. After defeating the Mexicans at Resaca de la Palma all available troops ought to have taken part in the pursuit. But only a regiment of infantry, a squadron and a battery followed the fugitives; the rest of the American army bivouacked on the battlefield. Judged by the sum of their success, however, General Taylor's operations in those few days were well-nigh faultless. They could hardly have achieved more.

Narrative from Reference 13, Article on Palo Alto

Fought May 8, 1846, between the Americans, under General Taylor, and the Mexicans, under Arista. The Mexicans were completely routed, at very small cost to the victors.

APPENDIX D

TABLES

LIST OF TABLES

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LIST OF TABLES (Continued)

TABLE I
BASIC DATA FOR THE SAMPLE BATTLES

No	Battle	Date	Reference	Identification of attacker	Identification of defender	Initial attacker strength, x_0	Initial defender strength, y_0	Casualties to attacker, c_x	Casualties to defender, c_y	Duration of engagement, hours t	Victor[b]
1	Mollwitz	1741	17	Prussia	Austria	22,500	19,000	3,930[w]	2,980[v]	5.0	A
2	Prague	1757	17	Prussia	Austria	64,000	61,000	11,740[w]	9,050[w]	10.0	A
3	Kolin	1757	17	Austria	Prussia	53,500	34,000	6,470[w]	6,710[w]	5.5	A
4	Rossbach	1757	17	Prussia	French	22,000	43,000	540[w]	2,700[w]	2.0	A
5	Breslau	1757	17	Austria	Prussia	80,000	30,000	5,270[w]	8,000[w]	5.0	A
6	Leuthen	1757	17	Prussia	Austria	43,000	72,000	6,200[w]	6,510[w]	6.0	A
7	Zorndorf	1758	17	Prussia	Austria	36,000	42,000	12,000[w]	16,000[w]	12.0	A
8	Hochkirch	1758	17	Austria	Prussia	65,000	37,000	5,300[w]	7,110[w]	4.0	A
9	Kunersdorf	1759	17	Prussia	Russian	43,000	71,000	18,670[i]	15,700[i]	5.0	D
10	Quebec	1759	15	France	England	5,000	3,200	650	644[w]		D
11	Liegnitz	1760	17	Austria	Prussia	30,000	30,000	3,790	3,270[w]	3.0	D
12	Torgau	1760	17	Prussia	Austria	44,000	65,000	9,020	4,042	9.0	A
13	Bunker Hill	1775	15	English	USA	2,500	1,500	1,054[w]	120[w]		A
14	Cowpens	1780	15	English	USA	1,100	900	230[w]	71[w]		D
15	Marengo	1800	17	Austria	French	28,000	28,500	6,500[w]	4,700[w]	12.0	D
16	Hohenlinden	1800	17	Austria	French	57,000	48,000	5,400[w]	2,500[w]	6.0	D
17	Caldiero	1806	17	French	Austria	45,750	49,200	8,300[w]	5,672[i]	7.0	D
18	Austerlitz	1805	11	Austria	French	82,500	65,300	12,200[w]	6,800[w]	12.0	D
19	Eylau	1807	17	Russia	French	65,000	70,000	19,000[w]	18,000[w]	13.0	D
20	Friedland	1807	17	French	Russia	86,000	46,000	11,670[w]	10,000[w]	12.0	A
21	Talavera	1809	17	French	Spain	54,000	54,000	8,210[w]	6,860[w]	10.0	D
22	Aspern	1809	17	French	Austria	90,000	75,000	42,080[w]	22,530[w]	21.0	D
23	Wagram	1809	17	French	Austria	181,700	128,400	33,900[w]	18,110[w]	14.0	A
24	Borodino	1812	17	French	Russia	110,000	121,000	23,500[w]	37,500[w]	15.0	A
25	Salamanca	1812	17	French	English	47,000	44,000	10,000[w]	5,160[w]		D
26	Vittoria	1813	17	French	English	60,000	80,000	6,000[w]	5,000[w]		D
27	Lutzen	1813	17	French	Prussia	110,000	66,000	13,000[w]	10,000[w]	8.0	A
28	Bautzen	1813	17	French	Prussia	163,000	96,500	20,000[w]	13,500[w]	15.0	A
29	Dresden	1813	17	French	Austria	96,000	200,000	10,000[w]	18,000[w]	15.0	A
30	Kulm	1813	17	Russia	French	103,200	37,000	3,310[w]	5,000[w]	15.0	A
31	Katzbach	1813	17	French	Prussia	79,000	96,000	10,000[w]	3,400[w]	6.0	D
32	Gross-Beeren	1813	17	French	Prussia	17,200	31,600	1,500[w]	1,000[w]	3.0	D
33	Dennewitz	1813	17	French	Prussia	58,000	54,000	10,000[w]	7,300[w]	7.0	D
34	Leipzig	1813	17	Russia	French	301,500	171,000	48,000[w]	45,500[w]	72.0	A
35	La Rothiere	1814	17	Austria	French	79,300	41,000	6,660[w]	5,000[w]	7.0	A
36	Laon	1814	17	French	Prussia	52,000	80,000	12,000[w]	8,000[w]	10.0	D
37	Lundy's Lane	1814	15	USA	English	2,000	3,010	743[w]	643[w]	5.0	D
38	Waterloo	1815	17	French	Prussia	72,250	145,200	24,000[w]	22,100[w]	8.0	D
39	Kulevtcha	1829	17	Russia	Turk	28,000	40,300	1,500[i]	3,000[w]	6.0	A
40	Palo Alto	1846	15	Mexican	USA	3,000	2,288	1,300	170[m]	12.0	D
41	Buena Vista	1847	15	Mexican	USA	14,000	4,757	1,700	150	12.0	D
42	Cerro Gordo	1847	15	USA	Mexican	8,500	12,000	431[w]	1,200[w]		A

Footnotes at end of table.

TABLE I

BASIC DATA FOR THE SAMPLE BATTLES (Continued)

[illegible]	Battle	Date	Reference	Identification of attacker	Identification of defender	Initial attacker strength, x_0	Initial defender strength, y_0	Casualties to attacker, c_x	Casualties to defender, c_y	Duration of engagement, hours, T	[illegible]
43	Contreras	1847	15	USA	Mexican	4,500	4,000	100	100		A
44	[illegible]	1848	17	Piedmont	Austria	41,000	18,000	760	350	7.0	D
[illegible]	Custoza	1848	17	Austria	Piedmont	55,000	75,000	960	875	14.0	A
[illegible]	Mortara	1849	17	Austria	Piedmont	15,960	25,240	190	550	2.5	A
[illegible]	Novara	1849	17	Sardinian	Austria	40,000	41,600	2,940	2,260	9.0	D
[illegible]	[illegible]	1849	17	Hungarian	Austria	30,000	34,000	1,200	600	6.0	D
[illegible]	Temesvar	1849	17	Austria	Hungarian	28,000	54,000	270	400	7.0	A
[illegible]	Alma	1854	17	French	Russia	57,000	33,600	4,110	5,665	3.5	A
[illegible]	Inkermann	1854	17	Russia	English	36,400	18,500	8,770	4,[illegible]	9.0	D
[illegible]	Tchernaja	1855	17	Russia	French	63,000	30,340	8,[illegible]	1,7[illegible]	4.0	D
[illegible]	Magenta	1859	17	French	Austria	48,000	49,630	3,[illegible]	5,730	9.0	A
[illegible]	Solferino	1859	17	French	Austria	151,200[c]	135,250[c]	14,137	13,100	11.[illegible]	A
[illegible]	First Bull Run	1861	15	USA	CSA	18,500[c]	18,000	1,564	1,932	10.6	D
[illegible]	Gaines's Mill	1862	15	CSA	USA	65,000	30,000	6,000	6,837	7.0	A
[illegible]	Seven Day's Battle	1862	15	USA	CSA	90,000[c]	105,060[c]	19,13[illegible]	9,796	70.0	A
[illegible]	Second Bull Run	1862	15	CSA	USA	55,000[c]	70,000	8,474	10,1[illegible]	30.0	A
[illegible]	Antietam	1862	16	USA	CSA	70,300	39,200	15,029	11,305	10.0	A
[illegible]	Fredericksburg	1862	15	USA	CSA	122,000	78,500	10,208	5,309	10.0	D
[illegible]	Stone River	1862-3	15	CSA	USA	37,712[c]	44,800[c]	9,[illegible]	9,[illegible]	19.0	D
[illegible]	Chancellorsville	1863	15	CSA	USA	55,000	132,000	10,960	11,12[illegible]	24.0	A
[illegible]	Chickamauga	1863	15	CSA	USA	71,551	58,965	15,060	11,700	16.0	A
[illegible]	Gettysburg	1863	16	USA	CSA	58,356	40,920	5,4[illegible]	2,541	17.0	A
[illegible]	Duppeler Schanzen	1864	17	Prussia	Dane	25,000	13,000	1,200	1,210	3.5	A
[illegible]	Wilderness	1864	15	USA	CSA	101,895[c]	41,025[c]	14,283	7,750	16.0	A
[illegible]	Franklin	1864	15	CSA	USA	27,000	28,000	4,550	1,222	7.0	D
[illegible]	Langensalza	1866	17	Hannoverian	Prussia	19,100	8,690	1,100	810		A
[illegible]	Nachod	1866	17	Austria	Prussia	3[illegible],000	24,000	3,646	1,122	6.0	D
[illegible]	Trautenau	1866	17	Austria	Prussia	26,800	32,000	3,611	1,282	13.0	A
[illegible]	Skalitz	1866	17	Austria	Prussia	22,300	28,000	3,290	1,365	4.0	D
[illegible]	Jicin	1866	17	Austria	Prussia	44,000	28,000	2,910	1,555	4.0	D
[illegible]	Koniggratz	1866	17	Austria	Prussia	215,134	220,982	23,598	8,894	8.0	D
[illegible]	Worth	1870	17	German	French	82,100	48,500	9,270	6,000	8.5	A
[illegible]	Spicheren	1870	17	German	French	[illegible]	2[illegible],800	4,[illegible]	1,99[illegible]	8.5	A
[illegible]	Mars-la-Tour	1870	17	French	German	113,[illegible]00	63,000	[illegible]	14,830	10.0	D
[illegible]	Gravelotte	1870	17	German	French	187,600	112,800	19,640	7,[illegible]	9.0	A
[illegible]	Beaumont	1870	17	French	German	58,700	67,700	3,500	3,[illegible]	7.[illegible]	D
[illegible]	Sedan	1870	17	French	German	90,000	154,000	17,000	8,[illegible]	12.5	D
[illegible]	Beaune-la-Rolande	1870	17	French	German	61,[illegible]00	41,000	1,300	900	8.0	D
[illegible]	[illegible]	[illegible]	17	German	French	83,700	60,70[illegible]	[illegible]	2,000		A
[illegible]	[illegible]-Barey	[illegible]	11	French	German	[illegible]	30,500	3,670	4,000	4.0	A
[illegible]	Servigny	[illegible]	21	French	German	150,000	60,000	3,[illegible]	3,000	12.0	D
[illegible]	[illegible]	[illegible]	17	French	German	140,000	45,000	7,500	[illegible]		A
[illegible]	Plevna I	1877	17	Russian	Turk	16,000	14,000	2,[illegible]	2,500	4.0	D
[illegible]	Plevna II	1877	17	Russia	Turk	32,[illegible]	20,000	[illegible]	2,000	10.0	D
[illegible]	Plevna III	1877	17	Russia	Turk	95,000	35,000	16,000	4,000	144.0	D

Footnotes at end of table

TABLE I

BASIC DATA FOR THE SAMPLE BATTLES (Concluded)

No.	Battle	Date	Reference[a]	Identification of attacker	Identification of defender	Initial attacker strength, x_0	Initial defender strength, y_0	Casualties to attacker, c_x	Casualties to defender, c_y	Duration engagement, hours, T	Victor[b]
88	Plevna	1877	17	Turk	Russian	30,000	120,000	5,000[w]	1,900[w]	2.0	D
89	Driniumor River	1944	14	Japan	USA	20,000	40,000	10,000[l]	3,000[w]	720.0	D
90	Guam	1944	16	USA	Japan	54,891	20,000	7,083[m]	8,000[w]	480.0	A
91	Saipan	1944	16	USA	Japan	67,500	26,000	16,500[m]	24,000[w]	600.0	A
92	Iwo Jima	1945	8	USA	Japan	73,000	21,500	20,860[m]	21,500[m]	864.0	A

b - "A" designates victory by attacking side; "D" designates victory by defending side
a - numbers given refer to the list of references
c - data from reference
l - identified as losses
m - killed, wounded, missing
w - killed, wounded
k - killed

TABLE II

ESTIMATED LANCHESTER SQUARE-LAW PARAMETERS FOR THE SAMPLE BATTLE

No.	Battle	Surviving Fraction of Attacker a	Surviving Fraction of Defender c	[illegible]	Activity Ratio D/A	Bitterness ϵ	Intensity λ	Defender's Initial Fractional Loss Rate η	Attacker's Initial Fractional Loss Rate δ	Attacker's Activity A	Defender's Activity D	Force Ratio x_0/y_0
[illegible]	[illegible]	0.126	0.943	1.350	1.547	0.181	0.900	0.827	0.912	0.606	1.090	1.196
[illegible]	[illegible]	0.617	0.452	1.181	1.335	0.180	0.633	0.393	0.477	0.355	0.500	1.345
[illegible]	[illegible]	0.879	0.914	0.620	1.485	0.163	0.717	0.464	0.563	0.164	0.481	1.146
[illegible]	[illegible]	[illegible]	0.937	0.631	0.186	0.040	0.481	0.782	0.84	1.481	0.156	[illegible]
[illegible]	[illegible]	[illegible]	0.700	0.500	1.777	0.155	0.748	1.492	1.175	0.559	0.996	[illegible]
[illegible]	[illegible]	[illegible]	0.910	1.24	0.553	0.121	0.466	0.730	0.645	0.663	0.361	[illegible]
[illegible]	[illegible]	0.647	[illegible]	0.968	0.406	0.475	0.952	1.048	0.884	1.222	0.791	0.927
[illegible]	[illegible]	[illegible]	0.706	0.671	1.190	0.136	0.908	1.274	0.542	0.685	2.952	1.757
[illegible]	[illegible]	2.44	0.779	1.315	0.334	0.376	1.503	1.183	1.918	1.86	1.196	2.966
[illegible]	[illegible]	[illegible]	0.789	0.419	1.640	0.177	-	-	-	-	-	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.377	1.145	0.125	0.999	0.542	1.078	0.932	1.076	1.966
[illegible]	[illegible]	[illegible]	[illegible]	1.740	1.400	0.171	0.384	0.308	0.635	0.307	0.430	0.877
[illegible]	[illegible]	[illegible]	[illegible]	1.175	3.134	0.423	-	-	-	-	-	1.643
[illegible]	[illegible]	0.791	0.319	1.551	3.594	0.141	-	-	-	-	-	1.222
[illegible]	[illegible]	0.768	0.935	1.145	1.409	0.218	0.536	0.375	0.508	0.381	0.497	0.992
[illegible]	[illegible]	0.906	0.949	1.354	2.482	0.073	0.290	0.214	0.203	0.184	0.458	1.163
[illegible]	[illegible]	0.462	0.195	1.096	1.825	0.135	0.662	0.426	0.302	0.657	0.467	0.930
[illegible]	[illegible]	[illegible]	0.956	1.175	2.325	0.133	0.266	0.226	0.313	0.176	0.397	1.49
[illegible]	[illegible]	[illegible]	[illegible]	1.117	1.875	0.280	0.518	0.482	0.576	0.498	0.535	0.929
[illegible]	[illegible]	0.464	[illegible]	0.868	1.262	0.149	0.378	0.488	0.305	0.250	0.571	1.874
[illegible]	[illegible]	0.453	[illegible]	1.099	1.248	0.147	0.352	0.348	0.378	0.318	0.390	1.031
[illegible]	[illegible]	0.532	[illegible]	1.106	2.022	0.473	0.541	0.656	0.540	0.380	0.768	1.204
[illegible]	[illegible]	0.815	[illegible]	0.828	1.721	0.148	0.253	0.674	0.235	0.193	0.312	1.413
[illegible]	[illegible]	[illegible]	0.685	0.860	0.635	0.306	0.480	0.562	0.425	0.418	0.246	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.31	1.965	0.173	-	-	-	-	-	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.253	0.883	0.082	-	-	-	-	-	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	2.998	2.224	0.136	0.409	0.441	0.343	0.274	0.610	[illegible]
[illegible]	[illegible]	[illegible]	0.965	0.947	2.328	0.148	0.225	0.230	0.211	0.141	0.357	[illegible]
[illegible]	[illegible]	[illegible]	0.915	1.172	0.136	0.052	0.148	0.127	0.173	0.254	0.993	[illegible]
[illegible]	[illegible]	0.545	0.456	[illegible]	1.244	0.069	0.069	0.230	0.815	0.078	0.154	[illegible]
[illegible]	[illegible]	0.473	[illegible]	1.437	2.383	0.078	0.201	0.153	0.516	0.104	0.625	1.432
[illegible]	[illegible]	0.913	0.344	1.634	0.790	0.054	0.434	0.255	0.709	0.686	0.387	2.146
[illegible]	[illegible]	[illegible]	0.465	1.119	1.442	0.168	0.549	0.504	0.495	0.473	0.492	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	0.961	1.893	0.230	0.677	0.066	0.541	0.054	0.108	1.753
[illegible]	[illegible]	0.541	[illegible]	[illegible]	1.863	0.049	0.306	0.431	0.217	0.223	0.615	1.834
[illegible]	[illegible]	0.769	0.967	1.341	1.272	0.186	0.465	0.363	0.595	0.418	0.516	2.467
[illegible]	[illegible]	0.620	0.790	1.336	0.691	0.331	1.580	1.256	2.012	1.912	1.321	0.667
[illegible]	[illegible]	[illegible]	0.649	1.406	0.468	0.280	0.773	0.551	1.050	1.106	0.540	0.438
[illegible]	[illegible]	[illegible]	0.925	1.307	0.578	0.065	0.342	0.314	[illegible]	0.449	0.260	[illegible]
[illegible]	[illegible]	0.667	0.906	1.670	4.675	0.176	0.153	0.179	0.994	0.136	0.910	[illegible]
[illegible]	[illegible]	[illegible]	0.94	0.844	4.753	0.150	0.290	0.379	0.364	0.185	0.778	4.963
[illegible]	[illegible]	0.842	[illegible]	0.72	1.261	0.074	-	-	-	-	-	0.79
[illegible]	[illegible]	0.879	[illegible]	0.371	[illegible]	0.068	-	-	-	-	-	1.125
[illegible]	[illegible]	0.391	0.997	[illegible]	4.695	0.018	0.364	0.064	0.364	0.064	0.136	[illegible]
[illegible]	[illegible]	[illegible]	0.954	1.398	7.57	0.014	0.033	0.020	0.039	0.938	0.729	[illegible]
[illegible]	[illegible]	[illegible]	0.914	[illegible]	[illegible]	0.116	0.156	0.216	0.116	0.341	0.313	0.632
[illegible]	[illegible]	[illegible]	0.96	0.944	1.912	[illegible]	0.102	0.151	[illegible]	0.133	0.197	1.461

TABLE II

ESTIMATED LANCHESTER SQUARE-LAW PARAMETERS FOR THE SAMPLE BATTLE (Concluded)

No.	Battle	Surviving Fraction of Attacker a	Surviving Fraction of Defender d	e	Activity Ratio D/A	Bitterness ϵ	Intensity λ	Defender's Initial Fractional Loss Rate α	Attacker's Initial Fractional Loss Rate δ	Attacker's Activity A	Defender's Activity D	Force Ratio x_0/y_0
49	[illegible]	0.976	0.976	1.010	[illegible]	0.024	0.096	0.096	0.097	0.046	0.143	1.471
19	[illegible]	0.983	0.993	[illegible]	0.350	0.007	0.025	0.025	0.025	0.049	0.013	0.519
50	[illegible]	0.924	0.845	[illegible]	[illegible]	0.113	0.777	1.075	0.801	0.634	0.303	1.696
51	[illegible]	[illegible]	0.744	1.043	4.118	0.274	0.730	[illegible]	0.741	0.360	[illegible]	2.000
[illegible]	[illegible]	0.849	0.954	1.400	[illegible]	0.090	0.109	0.286	0.806	0.170	1.290	1.401
[illegible]	[illegible]	0.919	0.997	0.934	[illegible]	0.093	0.242	0.259	0.228	0.331	0.174	0.790
54	[illegible]	0.946	0.972	0.986	1.251	0.192	0.213	0.216	0.209	0.190	0.238	1.135
[illegible]	[illegible]	0.814	[illegible]	0.986	0.832	0.102	0.348	0.276	0.218	0.269	0.224	1.329
[illegible]	[illegible]	[illegible]	[illegible]	0.640	2.947	0.150	0.562	0.821	0.264	0.779	0.774	2.147
[illegible]	[illegible]	0.787	0.867	1.464	[illegible]	0.153	0.053	0.036	0.077	0.042	0.066	[illegible]
[illegible]	[illegible]	0.928	0.884	[illegible]	0.713	0.173	0.138	[illegible]	0.149	0.167	0.117	0.790
[illegible]	[illegible]	0.765	[illegible]	0.481	2.476	0.287	0.690	0.741	0.606	0.437	1.083	1.796
[illegible]	[illegible]	0.930	0.934	[illegible]	[illegible]	0.077	0.196	0.166	0.290	0.107	0.323	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.679	0.075	0.254	0.305	[illegible]	0.129	0.336	0.277	0.442
[illegible]	[illegible]	0.981	0.916	1.491	0.308	0.149	0.140	0.094	0.206	0.225	0.047	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.005	[illegible]	0.233	0.369	0.346	0.352	0.275	0.442	[illegible]
64	[illegible]	0.903	0.932	[illegible]	2.908	0.041	0.114	[illegible]	0.141	0.067	0.195	1.117
[illegible]	[illegible]	0.952	0.933	[illegible]	1.391	[illegible]	[illegible]	[illegible]	0.341	0.340	1.471	[illegible]
66	[illegible]	0.980	0.973	[illegible]	[illegible]	0.143	0.215	0.205	0.226	0.133	0.276	[illegible]
67	[illegible]	0.931	0.956	[illegible]	[illegible]	0.091	0.111	0.164	[illegible]	0.170	0.570	0.964
[illegible]	[illegible]	[illegible]	0.905	[illegible]	3.318	0.395	-	-	-	-	-	1.873
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4.063	0.877	0.318	0.109	0.412	0.154	0.423	[illegible]
[illegible]	[illegible]	0.866	0.941	1.807	[illegible]	0.076	0.146	0.077	[illegible]	0.992	0.213	0.641
[illegible]	[illegible]	[illegible]	0.956	[illegible]	1.403	0.086	0.518	0.236	0.906	0.330	0.074	0.743
[illegible]	[illegible]	[illegible]	0.944	1.090	2.981	0.643	0.190	[illegible]	[illegible]	0.119	0.213	[illegible]
[illegible]	[illegible]	[illegible]	0.000	1.023	2.473	0.040	0.267	1.128	0.226	0.131	0.327	0.974
[illegible]	[illegible]	[illegible]	[illegible]	0.870	2.017	0.147	0.415	[illegible]	0.348	0.282	0.580	1.603
[illegible]	[illegible]	0.670	0.926	1.386	2.772	0.102	0.287	0.217	0.250	0.172	0.470	1.257
[illegible]	Mars-la-Tour	0.996	[illegible]	[illegible]	1.400	0.169	0.405	0.606	0.275	0.311	0.494	[illegible]
[illegible]	[illegible]	0.886	[illegible]	1.216	[illegible]	[illegible]	0.130	0.194	0.280	0.110	0.443	[illegible]
[illegible]	[illegible]	0.940	0.951	1.395	0.901	0.456	0.179	0.163	0.156	0.188	0.170	[illegible]
[illegible]	[illegible]	0.911	[illegible]	1.615	1.225	0.107	0.206	0.113	0.373	0.134	0.210	[illegible]
[illegible]	[illegible]	0.919	0.974	2.962	2.164	0.022	0.065	0.067	0.064	0.044	0.096	[illegible]
[illegible]	[illegible]	0.931	0.987	0.766	1.300	0.025	-	-	-	-	-	[illegible]
[illegible]	[illegible]	0.929	[illegible]	0.487	1.306	0.112	0.474	0.980	0.463	0.500	0.740	[illegible]
[illegible]	[illegible]	0.971	[illegible]	0.768	[illegible]	0.039	0.078	0.101	0.040	0.061	0.130	[illegible]
[illegible]	[illegible]	0.966	0.971	1.368	[illegible]	0.040	-	-	-	-	-	[illegible]
[illegible]	[illegible]	[illegible]	0.421	1.726	0.767	0.257	[illegible]	1.256	1.058	1.759	1.349	[illegible]
[illegible]	[illegible]	[illegible]	0.416	[illegible]	[illegible]	0.232	[illegible]	0.516	[illegible]	0.180	0.976	[illegible]
[illegible]	[illegible]	[illegible]	0.406	1.196	[illegible]	0.150	2.025	[illegible]	[illegible]	0.006	0.091	[illegible]
[illegible]	[illegible]	0.943	0.944	[illegible]	0.608	0.004	0.047	0.207	1.018	0.029	0.564	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.003	0.017	0.007	0.009	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.006	0.009	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.771	[illegible]	0.044	0.019	0.017	0.049	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5.647	[illegible]	[illegible]	0.034	0.017	0.007	[illegible]	[illegible]

TABLE IIIa

NUMBER OF BATTLES WON BY SIDE AND BY SIGN OF ADVANTAGE, $\ln \mu$

	$\ln \mu > 0$; Positive defender advantage	$\ln \mu < 0$; Negative defender advantage	Total
Victor = D; Battle won by defending side	38	7	45
Victor = A; Battle won by attacking side	17	30	47
Total	55	37	92

Chi-square = 20.32 at 1 degree of freedom

TABLE IIIb

PERCENT OF VICTORIOUS BATTLES BY SIGN OF ADVANTAGE, $\ln \mu$, FOR VICTORIES BY EACH SIDE

	$\ln \mu > 0$; Positive defender advantage	$\ln \mu < 0$; Negative defender advantage	Total
Victor = D; Battle won by defending side	84.4	15.6	100
Victor = A; Battle won by attacking side	36.2	63.8	100

TABLE IV

ADVANTAGE PARAMETER, $\ln \mu$, AND VICTORIOUS SIDE[a]

$\ln \mu$	Victor	$\ln \mu$	Victor	$\ln \mu$	Victor
-0.992	A	-0.014	A	0.281	A
-0.693	A	0.003	D	0.299	D
-0.693	A	0.009	A	0.303	D
-0.531	A	0.010	D	0.339	D
-0.460	A	0.015	D	0.381	A
-0.420	A	0.046	A	0.399	A
-0.416	A	0.049	A	0.439	D
-0.399	A	0.067	D	0.443	D
-0.387	D	0.070	D	0.483	D
-0.375	A	0.076	D	0.491	D
-0.357	A	0.076	D	0.519	D
-0.348	A	0.077	A	0.558	A
-0.327	A	0.083	D	0.560	D
-0.326	A	0.083	A	0.592	A
-0.300	A	0.091	D	0.596	D
-0.280	A	0.094	D	0.608	D
-0.264	D	0.096	A	0.642	D
-0.222	A	0.111	D	0.678	D
-0.213	A	0.112	D	0.824	D
-0.200	D	0.112	D	1.138	D
-0.198	A	0.153	D		
-0.176	A	0.157	A		
-0.164	A	0.161	A		
-0.140	A	0.161	D		
-0.127	A	0.170	D		
-0.124	D	0.171	A		
-0.120	A	0.179	D		
-0.119	D	0.196	A		
-0.097	A	0.204	D		
-0.075	A	0.213	A		
-0.068	A	0.219	A		
-0.061	A	0.226	D		
-0.058	D	0.236	D		
-0.028	A	0.248	D		
-0.019	D	0.272	D		
-0.018	A	0.274	D		

[a] A = Attacking side victorious

D Defending side victorious

TABLE Va

NUMBER OF BATTLES WON BY SIDE AND BY MAGNITUDE OF ADVANTAGE, $\ln \mu$

	$\ln \mu > 0.3$	$0.3 \geq \mu > -0.3$	$-0.3 \geq \ln \mu$	Total
Victor = D	14	30	1	45
Victor = A	4	29	14	47
Total	18	59	15	92

Chi-square = 16.80 at 2 degrees of freedom

TABLE Vb

PERCENT OF VICTORIOUS BATTLES BY MAGNITUDE OF ADVANTAGE, $\ln \mu$, FOR VICTORIES BY EACH SIDE

	$\ln \mu > 0.3$	$0.3 \geq \ln \mu > -0.3$	$-0.3 \geq \ln \mu$	Total
Victor = D	31.2	66.6	2.2	100
Victor = A	8.5	61.7	29.8	100

TABLE VI

THEORETICAL (NORMAL DISTRIBUTION) AND OBSERVED FREQUENCY OF BATTLES WITH VARIOUS LOGARITHMIC FORCE RATES, $\ln x_0/y_0$

$\ln(x_0/y_0)$ From	$\ln(x_0/y_0)$ To	Theoretical Frequency (Normal Distribution)	Observed Frequency	Grouped Theoretical Frequency	Grouped Observed Frequency
$-\infty$	-0.8	2.85	2		
-0.8	-0.7	1.47		9.19	11
-0.7	-0.6	2.02	5		
-0.6	-0.5	2.85	3		
-0.5	-0.4	3.50	2	8.01	7
-0.4	-0.3	4.51	5		
-0.3	-0.2	5.06	4	5.06	4
-0.2	-0.1	6.07	7	6.07	7
-0.1	0.0	6.81	6	6.81	6
0.0	0.1	6.81	6	6.81	6
0.1	0.2	7.27	5	7.27	5
0.2	0.3	7.18	6	7.18	6
0.3	0.4	6.81	6	6.81	6
0.4	0.5	5.98	7	5.98	7
0.5	0.6	5.43	12	5.43	12
0.6	0.7	4.32	5	8.09	7
0.7	0.8	3.77	2		
0.8	0.9	2.76	0		
0.9	1.0	2.30	3		
1.0	1.1	1.47	3	9.56	8
1.1	1.2	1.10	1		
1.2	$+\infty$	1.93	1		

Chi-square = 10.56 at 10 degrees of freedom

TABLE VII

THEORETICAL (NORMAL DISTRIBUTION) AND OBSERVED FREQUENCY OF BATTLES FOR VARIOUS RANGES OF DEFENDER ADVANTAGE, ln μ

ln μ		Theoretical Frequency (Normal Distribution)	Observed Frequency	Grouped Theoretical Frequency	Grouped Observed Frequency
From	To				
$-\infty$	-0.6	2.76	3		
-0.6	-0.5	2.76	1	8.74	7
-0.5	-0.4	3.22	3		
-0.4	-0.3	5.06	8	5.06	8
-0.3	-0.2	7.36	5	7.36	5
-0.2	-0.1	9.20	8	9.20	8
-0.1	0.0	10.12	9	10.12	9
0.0	0.1	10.12	16	10.12	16
0.1	0.2	9.20	11	9.20	11
0.2	0.3	8.74	10	8.74	10
0.3	0.4	7.82	4	7.82	4
0.4	0.5	5.52	4	5.52	4
0.5	0.6	3.68	5	9.20	10
0.6	$+\infty$	5.52	5		
Total		91.08	92		

Chi-square = 9.39 at 8 degrees of freedom

TABLE VIII

THEORETICAL (NORMAL DISTRIBUTION) AND OBSERVED FREQUENCY OF BATTLES FOR VARIOUS RANGES OF LOGARITHMIC BITTERNESS, $\ln \epsilon$

$\ln \epsilon$ From	$\ln \epsilon$ To	Theoretical Frequency (Normal Distribution)	Observed Frequency	Grouped Theoretical Frequency	Grouped Observed Frequency
$-\infty$	-3.70	2.76	6	6.44	7
-3.70	-3.40	3.68	1		
-3.40	-3.10	5.52	3	5.52	3
-3.10	-2.80	8.28	5	8.28	5
-2.80	-2.50	11.04	13	11.04	13
-2.50	-2.20	12.88	12	12.88	12
-2.20	-1.90	12.88	16	12.88	16
-1.90	-1.60	11.96	16	11.96	16
-1.60	-1.30	9.20	8	9.20	8
-1.30	-1.00	6.44	6	6.44	6
-1.00	$+\infty$	7.36	6	7.36	6
Total		92.00	92		

Chi-square = 5.46 at 7 degrees of freedom

TABLE IX

SOME STATISTICS OF SURVIVING FRACTIONS

Victorious Side	No. of Battles	Average surviving fraction of attacker, a	Standard deviation of a	Average surviving fraction of defender, d	Standard deviation of d
[illegible]	47	0.873	0.086	0.[illegible]19	0.[illegible]
Defender	45	0.827	0.117	0.893	0.073

TABLE X

OBSERVED DISTRIBUTION OF SURVIVING FRACTIONS

Range of surviving fraction values	Number of Battles in Stated Range			
	Attacking Side Victorious (47 Battles)		Defending Side Victorious (45 Battles)	
	Surviving fraction of attacker, a	Surviving fraction of defender, d	Surviving fraction of attacker, a	Surviving fraction of defender, d
0.95 - 1.00	8	5	5	14
0.90 - 0.95	12	15	8	9
0.85 - 0.90	13	8	12	9
0.80 - 0.85	5	7	5	5
0.75 - 0.80	6	3	7	7
0.70 - 0.75	1	4	2	1
0.65 - 0.70	1	1	2	0
0.60 - 0.65	0	0	1	0
0.55 - 0.60	1	2	1	0
0.50 - 0.55	0	0	2	0
0.00 - 0.50	0	2	0	0

TABLE XIa

NUMBER OF BATTLES WON BY SIDE AND BY VALUE OF FORCE RATIO

	Defender Numerically Superior ($x_0/y_0 < 1$)	Attacker Numerically Superior ($x_0/y_0 > 1$)	Total
Victor = D	19	26	45
Victor = A	16	31	47
Total	35	57	92

Chi-square = 0.35 at 1 degree of freedom

TABLE XIb

PERCENT OF BATTLES WON BY SIDE FOR NUMERICAL SUPERIORITY

	Defender Numerically Superior ($x_0/y_0 < 1$)	Attacker Numerically Superior ($x_0/y_0 > 1$)
Victor = D	54.3	45.6
Victor = A	45.7	54.5
Total	100.0	100.0

TABLE XII

NUMBER OF BATTLES WON BY SIDE AND BY THREE LEVELS OF FORCE RATIO

	Force Ratio			
	Level 1 ($0.000 < x_0/y_0 < 0.900$)	Level 2 ($0.900 < x_0/y_0 < 1.500$)	Level 3 ($1.500 < x_0/y_0 < \infty$)	Total
Victor = D	14	18	13	45
Victor = A	15	11	21	47
Total	29	29	34	92

Chi-square = 3.56 at 2 degrees of freedom

TABLE XIII

RESULTS OF REGRESSION OF ADVANTAGE, $\ln \mu$ ON LOGARITHMIC FORCE RATIO, $\ln x_0/y_0$

Regression Line: $\ln \mu = b + c \ln (x_0/y_0)$

Number of Data Points: 92

Estimated Value of Regression Coefficients ± 95% Confidence Limits:

$$b = 0.115 \pm 0.064$$

$$c = 0.367 \pm 0.122$$

Standard Error of Estimate: $s_{\ln \mu \mid \ln (x_0/y_0)} = 0.297$

Correlation Coefficient: $r = -0.537$

Sample Mean:

of $\ln \mu = 0.058$

of $\ln (x_0/y_0) = 0.156$

Sample Variance:

of $\ln \mu = 0.122$

of $\ln (x_0/y_0) = 0.262$

TABLE XIV

THEORETICAL (NORMAL DISTRIBUTION) AND OBSERVED FREQUENCY OF BATTLES WITH VARIOUS RESIDUAL ADVANTAGES

Residual Advantage		Theoretical Frequency (Normal Distribution)	Observed Frequency	Grouped Theoretical Frequency	Grouped Observed Frequency
From	To				
- ∞	-0.5	4.140	4	14.260	9
-0.5	-0.3	10.120	5		
-0.3	-0.2	8.648	11	8.648	11
-0.2	-0.1	10.856	9	10.856	9
-0.1	0.0	12.236	21	12.236	21
0.0	0.1	12.236	10	12.236	10
0.1	0.2	10.856	11	10.856	11
0.2	0.3	8.648	7	8.648	7
0.3	0.5	10.120	9	14.260	14
0.5	+ ∞	4.140	5		

Chi-square = 9.90 at 5 degrees of freedom

TABLE XV

RESULTS OF REGRESSION OF LOGARITHMIC BITTERNESS, $\ln \epsilon$, on ADVANTAGE, $\ln \mu$

Regression Line: $\ln \epsilon = b + c \ln \mu$

Number of Data Points: 92

Estimated Value of Regression Coefficients ± 95% Confidence Limits:

$$b = -2.152 \pm 0.177$$

$$c = -0.076 \pm 0.502$$

Standard Error of Estimate: $s_{\ln \epsilon \mid \ln \mu} = 0.838$

Correlation Coefficient: $r = -0.032$

Sample Mean:

of $\ln \epsilon$ = -2.157

of $\ln \mu$ = 0.058

Sample Variance:

of $\ln \epsilon$ = 0.696

of $\ln \mu$ = 0.122

TABLE XVI

RESULTS OF REGRESSION OF ADVANTAGE, $\ln \mu$ ON LOGARITHMIC BATTLE DURATION, $\ln t$

Regression Line: $\ln \mu = b + c \ln T$

Number of Data Points: 82

Estimated Value of Regression Coefficients ± 95% Confidence Limits:

$$b = 0.135 \pm 0.173$$

$$c = -0.027 \pm 0.222$$

Standard Error of Estimate: $s_{\ln \mu | \ln t} = 0.341$

Correlation Coefficient: $r = -0.092$

Sample Mean:

of $\ln \mu = 0.070$

of $\ln t = 2.391$

Sample Variance:

of $\ln \mu = 0.116$

of $\ln t = 1.345$

TABLE XVIIa

NUMBER OF BATTLES WON BY SIDE AND BY SIGN OF RESIDUAL ADVANTAGE

	Residual Advantage Positive	Residual Advantage Negative	Total
Victor = D	30	15	45
Victor = A	12	35	47
Total	42	50	92

Chi-square = 14.06 at 1 degrees of freedom

TABLE XVIIb

PERCENT OF VICTORIOUS BATTLES BY SIGN OF RESIDUAL ADVANTAGE FOR VICTORIES BY EACH SIDE

	Residual Advantage Positive	Residual Advantage Negative	Total
Victor = D	66.7	33.3	100
Victor = A	25.6	74.4	100

TABLE XVIII

RESIDUAL ADVANTAGE AND VICTORIOUS SIDE[a]

Residual Advantage	Victor	Residual Advantage	Victor	Residual Advantage	Victor
-1.064	A	-0.058	A	0.215	A
-0.821	A	-0.045	A	0.234	A
-0.583	A	-0.045	D	0.243	D
-0.569	A	-0.037	A	0.250	A
-0.446	A	-0.035	D	0.267	A
-0.431	A	-0.035	D	0.284	D
-0.374	A	-0.034	D	0.300	A
-0.307	A	-0.033	D	0.336	D
-0.304	A	-0.032	D	0.358	D
-0.290	A	-0.031	D	0.397	D
-0.286	D	-0.029	A	0.414	A
-0.275	A	-0.025	D	0.422	D
-0.274	A	-0.024	A	0.425	D
-0.269	A	-0.022	A	0.430	D
-0.247	A	0.001	A	0.485	D
-0.247	A	0.004	A	0.513	D
-0.236	A	0.005	D	0.514	D
-0.227	A	0.016	D	0.577	D
-0.224	D	0.016	A	0.601	D
-0.221	A	0.023	D	0.662	D
-0.185	A	0.031	D		
-0.166	A	0.037	D		
-0.163	A	0.061	D		
-0.158	A	0.087	A		
-0.151	D	0.119	A		
-0.129	A	0.122	D		
-0.126	A	0.133	D		
-0.125	D	0.139	D		
-0.102	D	0.145	D		
-0.098	A	0.154	D		
-0.098	A	0.154	D		
-0.085	A	0.157	D		
[illegible]	A	0.159	D		
-0.076	D	0.170	D		
-0.063	A	0.181	D		
-0.059	D	0.209	A		

[a] A Attacking Side Victorious

D Defending Side Victorious

TABLE XIXa

NUMBER OF BATTLES WON BY SIDE AND BY MAGNITUDE OF RESIDUAL ADVANTAGE

	Residual Advantage > 0.2	0.2 ≥ Residual Advantage > −0.2	0.2 ≥ Residual Advantage	Total
Victor = D	14	29	2	45
Victor = A	7	22	18	47
Total	21	51	20	92

Chi-square = 16.06 at 2 degrees of freedom

TABLE XIXb

PERCENT OF VICTORIOUS BATTLES BY MAGNITUDE OF RESIDUAL ADVANTAGE FOR VICTORIES BY EACH SIDE

	Residual Advantage > 0.2	0.2 ≥ Residual Advantage > −0.2	−0.2 ≥ Residual Advantage	Total
Victor = D	31.1	64.5	4.4	100
Victor = A	14.9	46.8	38.3	100

TABLE XX

RESULTS OF REGRESSION OF LOGARITHMIC BITTERNESS, $\ln \epsilon$, ON LOGARITHMIC DURATION, $\ln t$, for "t" IN HOURS

Regression Line: $\ln \epsilon = b + c \ln t$

Number of Data Points: 82

Estimated Value of Regression Coefficients ± 95% Confidence Limits:

$b = -2.944 \pm 0.380$

$c = 0.338 \pm 0.144$

Standard Error of Estimate: $s_{\ln \epsilon \mid \ln t} = 0.747$

Correlation Coefficient: $r = 0.468$

Sample Mean:

of $\ln \epsilon = -2.135$

of $\ln t = 2.391$

Sample Variance:

of $\ln \epsilon = 0.704$

of $\ln t = 1.345$

TABLE XXI

THEORETICAL (NORMAL DISTRIBUTION) AND OBSERVED FREQUENCY OF BATTLES FOR VARIOUS RANGES OF RESIDUAL LOGARITHMIC BITTERNESS

Residual Logarithmic Bitterness		Theoretical Frequency (Normal Distribution)	Observed Frequency	Grouped Theoretical Frequency	Grouped Observed Frequency
From	To				
- ∞	-2.000	0.23	2		
-2.000	-1.500	1.48	3	7.38	7
-1.500	-1.200	2.62	1		
-1.200	-1.000	2.95	1		
-1.00	-0.800	4.26	0	10.00	5
-0.800	-0.600	5.74	5		
-0.600	-0.400	6.80	5	6.80	5
-0.400	-0.200	8.11	11	8.11	11
-0.200	0.000	8.70	10	8.70	10
0.000	0.200	8.70	8	8.70	8
0.200	0.400	8.11	10	8.11	10
0.400	0.600	6.80	14	6.80	14
0.600	0.800	5.74	2	10.00	7
0.800	1.000	4.26	5		
1.000	1.200	2.95	2		
1.200	1.500	2.62	3	7.38	5
1.500	2.000	1.48	0		
2.000	+ ∞	0.33	0		

Chi-square = 14.00 at 7 degrees of freedom

TABLE XXII

NUMBERS OF VICTORIES AND DEFEATS BY PARTICIPATING NATION AND BY SIDE

Participating Nation	Side on Which Nation Participated						
	Attacking Side			Defending Side			
	Number of victories	Number of defeats	Total partici-pation on attacking side	Number of victories	Number of defeats	Total partici-pation on defending side	Total partici-pation
France	10	17	27	5	8	13	40
Prussia (or Germany)	11	1	12	16	8	24	36
Austria	8	8	16	5	9	14	30
United States of America	8	3	11	6	6	12	23
Russia	3	6	9	2	3	5	14
Confederate States of America	5	2	7	2	3	5	12
Britain (or England)	1	1	2	5	0	5	7
Turkey	0	1	1	3	1	4	5
Mexico	0	2	2	0	2	2	4
Japan	0	1	1	0	3	3	4
Piedmont	0	1	1	0	2	2	3
Hungary	0	1	1	0	1	1	2
Spain	0	0	0	1	0	1	1
Sardinia	0	1	1	0	0	0	1
Denmark	0	0	0	0	1	1	1
[illegible]	1	0	1	0	0	0	1
Total	47	45	92	45	47	92	184

TABLE XXIII

NUMBER AND FRACTION OF VICTORIES BY PARTICIPATING NATION

Participating Nation	Number of victories	Total participation	Victories as a fraction of total participation	Approximate 95% confidence interval for probability of victory[a]
France	15	40	0.38	0.2 - 0.5
Prussia (or Germany)	27	36	0.75	0.6 - 0.9
Austria	13	30	0.43	0.2 - 0.6
United States of America	14	23	0.61	0.4 - 0.8
Russia	5	14	0.36	0.2 - 0.6
Confederate States of America	7	12	0.58	0.3 - 0.8
Britain (or England)	6	7	0.86	0.4 - 1.0
Turkey	3	5	0.60	0.2 - 0.9
Mexico	0	4	0.00	0.0 - 0.5
Japan	0	4	0.00	0.0 - 0.5
Piedmont	0	3	0.00	0.0 - 0.6
Hungary	0	2	0.00	0.0 - 0.8
Spain	1	1	1.00	0.0 - 1.0
Sardinia	0	1	0.00	0.0 - 1.0
Denmark	0	1	0.00	0.0 - 1.0
Hanover	1	1	1.00	0.0 - 1.0

[a]Estimates given are based on 95% confidence limits for binomial probability parameter, equating observed fraction of victories to observed proportion of success and total participation to sample size. Confidence intervals for each nation were independently determined and the interval limits were rounded to the nearest tenth prior to incorporation in the table.

APPENDIX E

FIGURES

LIST OF FIGURES

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LIST OF FIGURES (Continued)

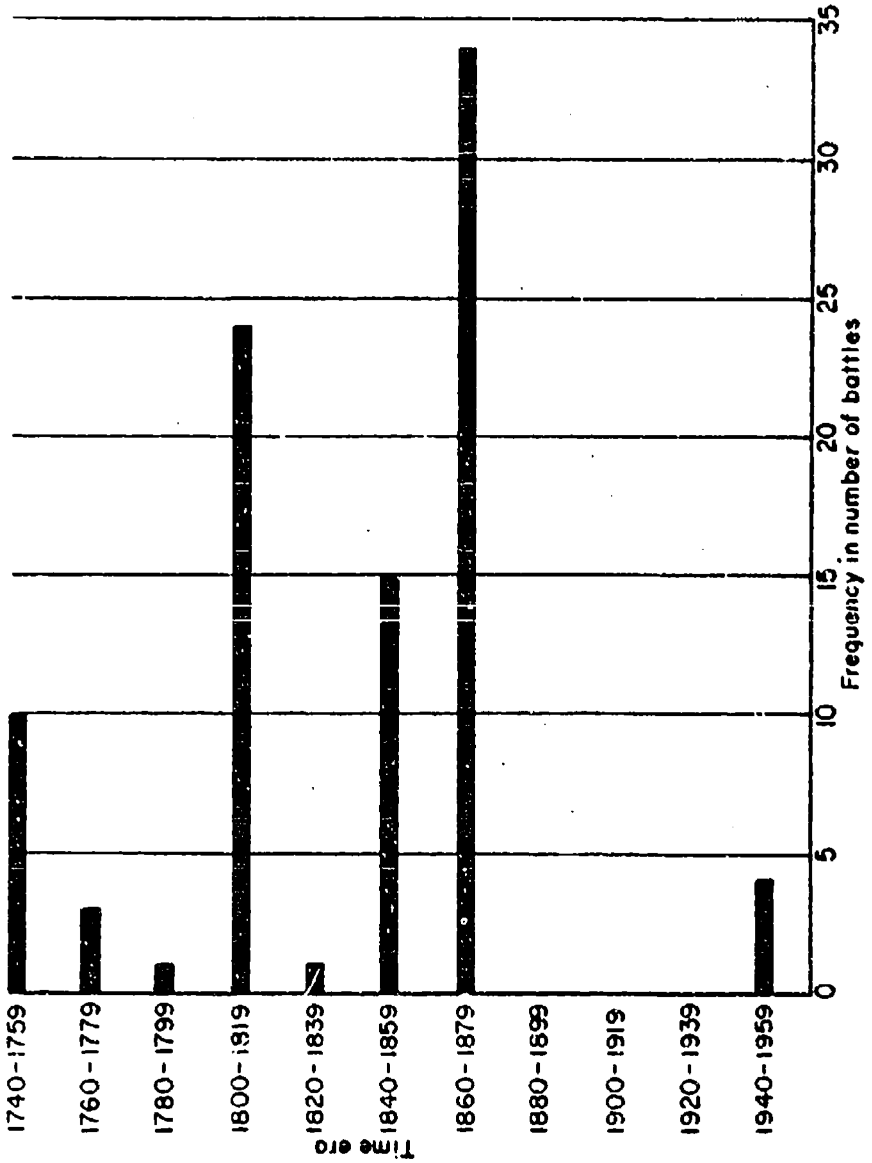

Figure 1. Distribution of battles in time.

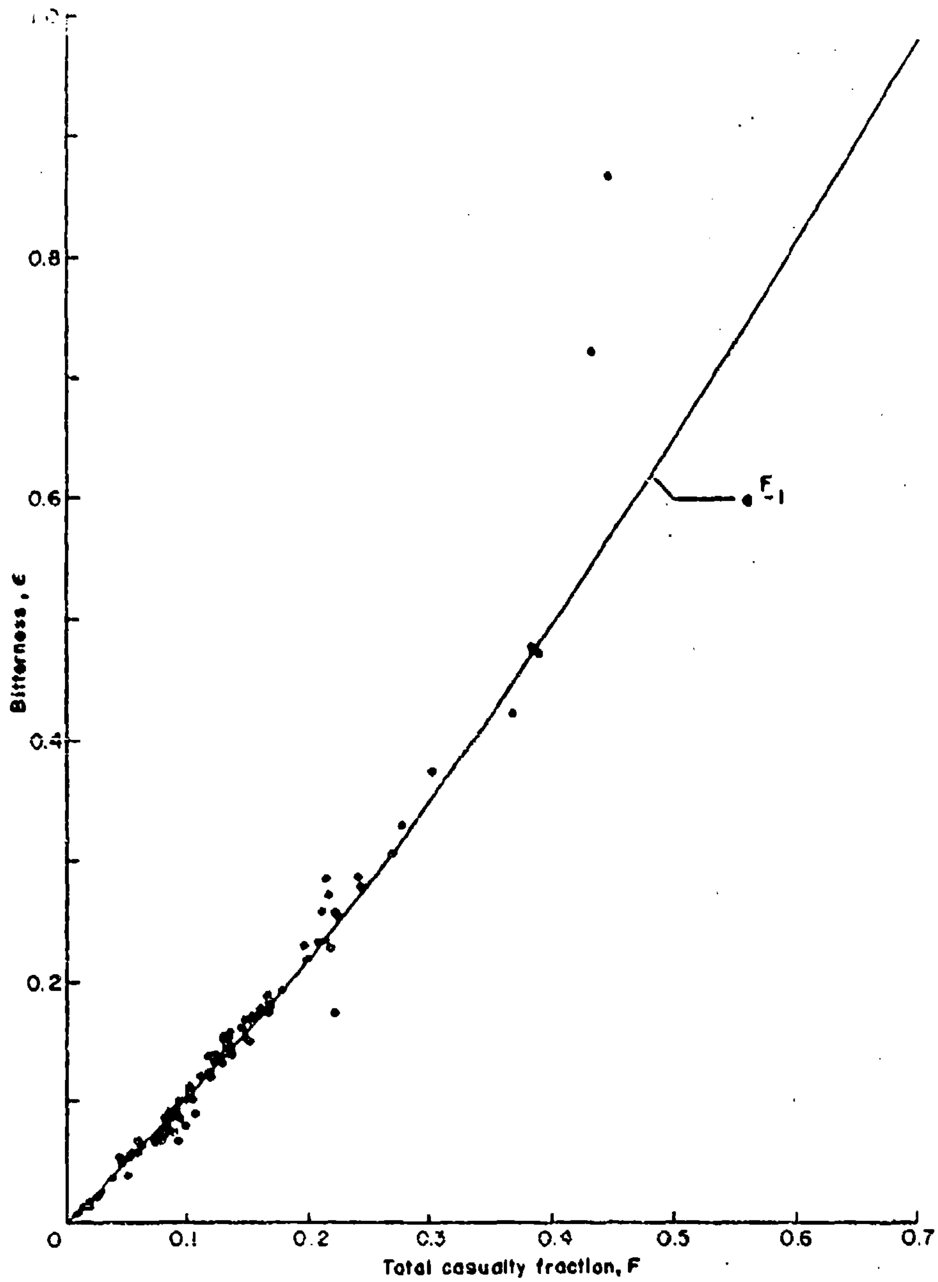

Figure 2. Scatter diagram of bitterness, ϵ, against total casualty fraction, F.

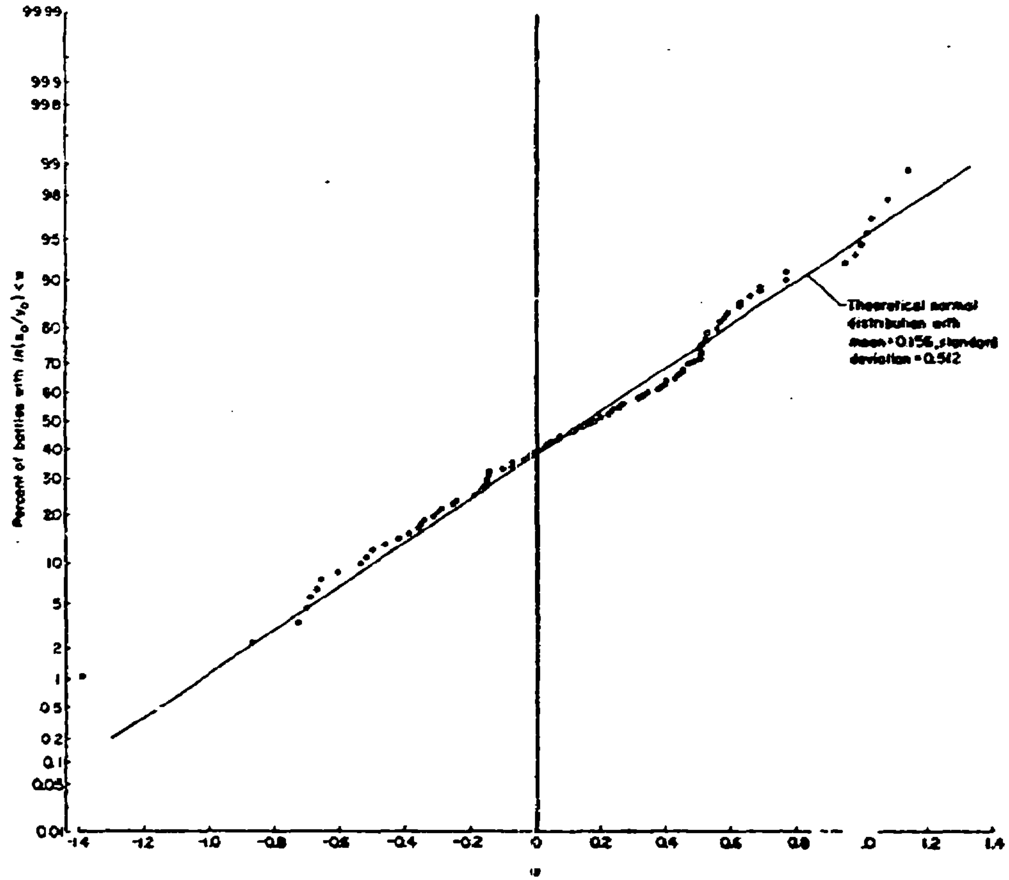

Figure 3. Theoretical and observed cumulative distribution of logarithmic force ratio, $\ln(x_o/y_o)$.

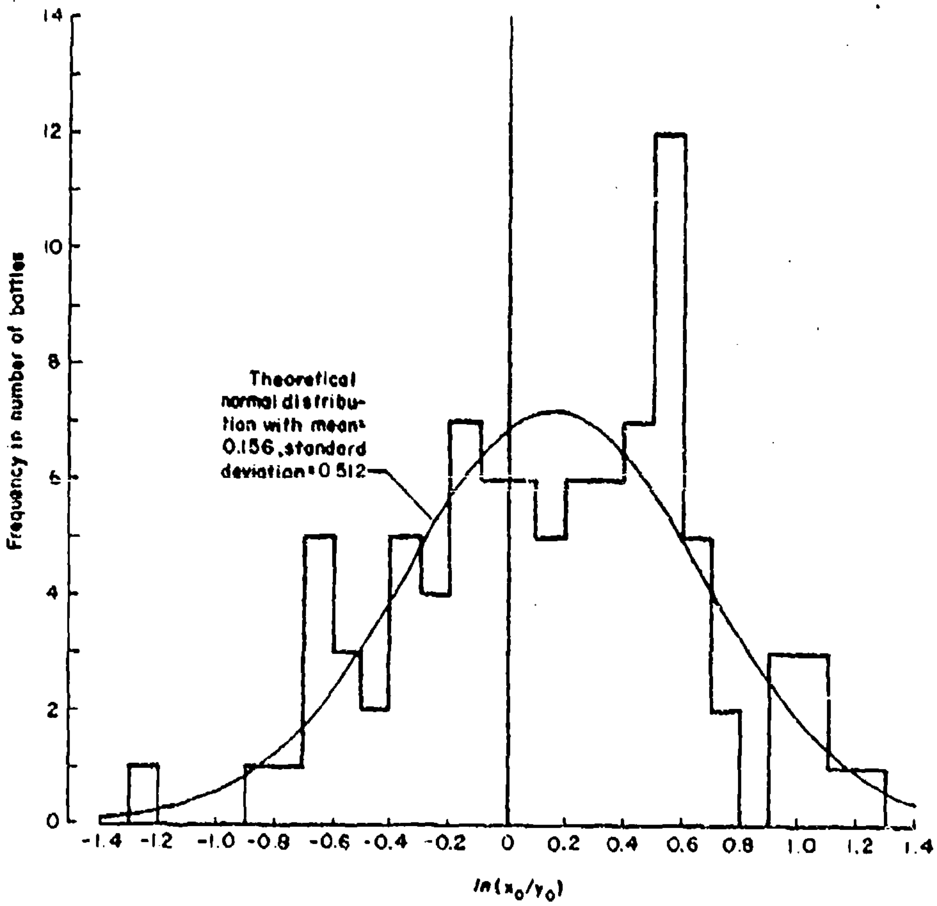

Figure 4. Theoretical and observed frequency distribution of logarithmic force ratio, $\ln(x_0/y_0)$.

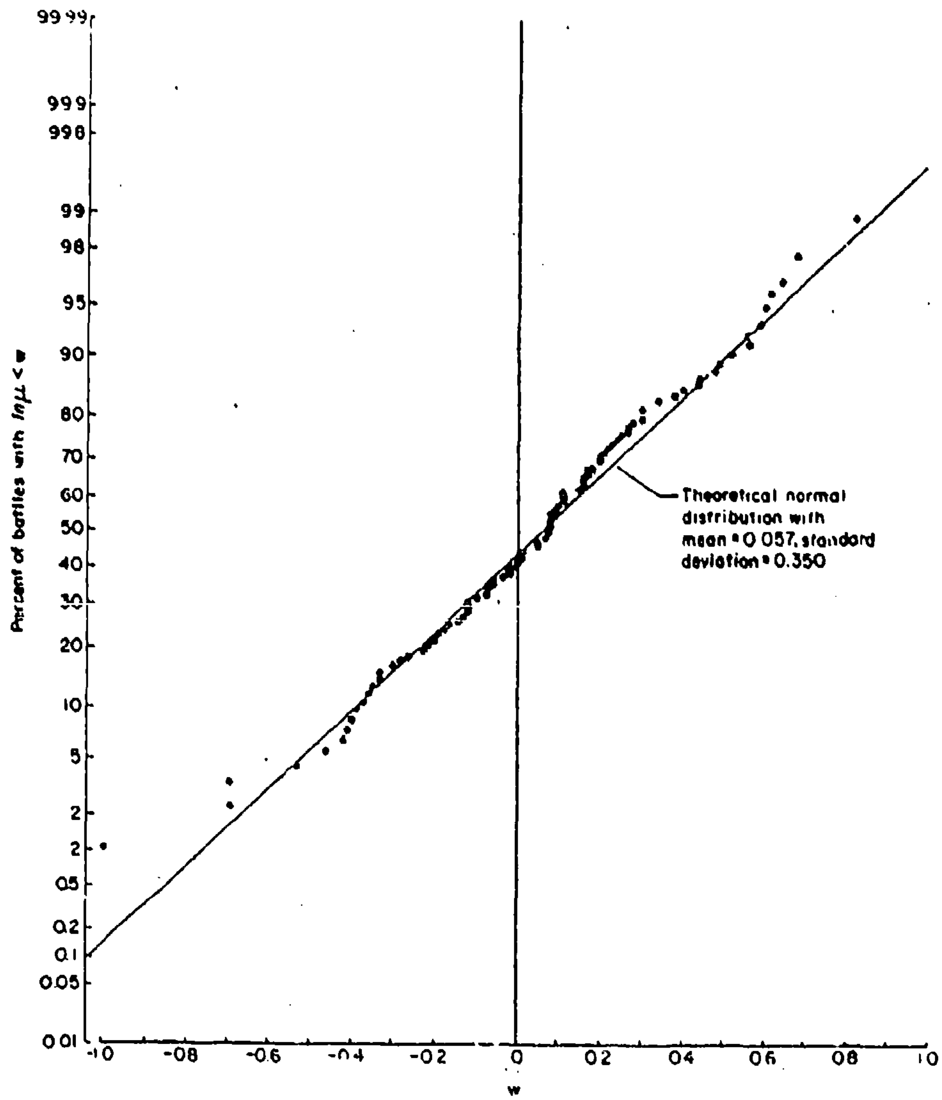

Figure 5. Theoretical and observed cumulative distribution of defender advantage, $\ln \mu$.

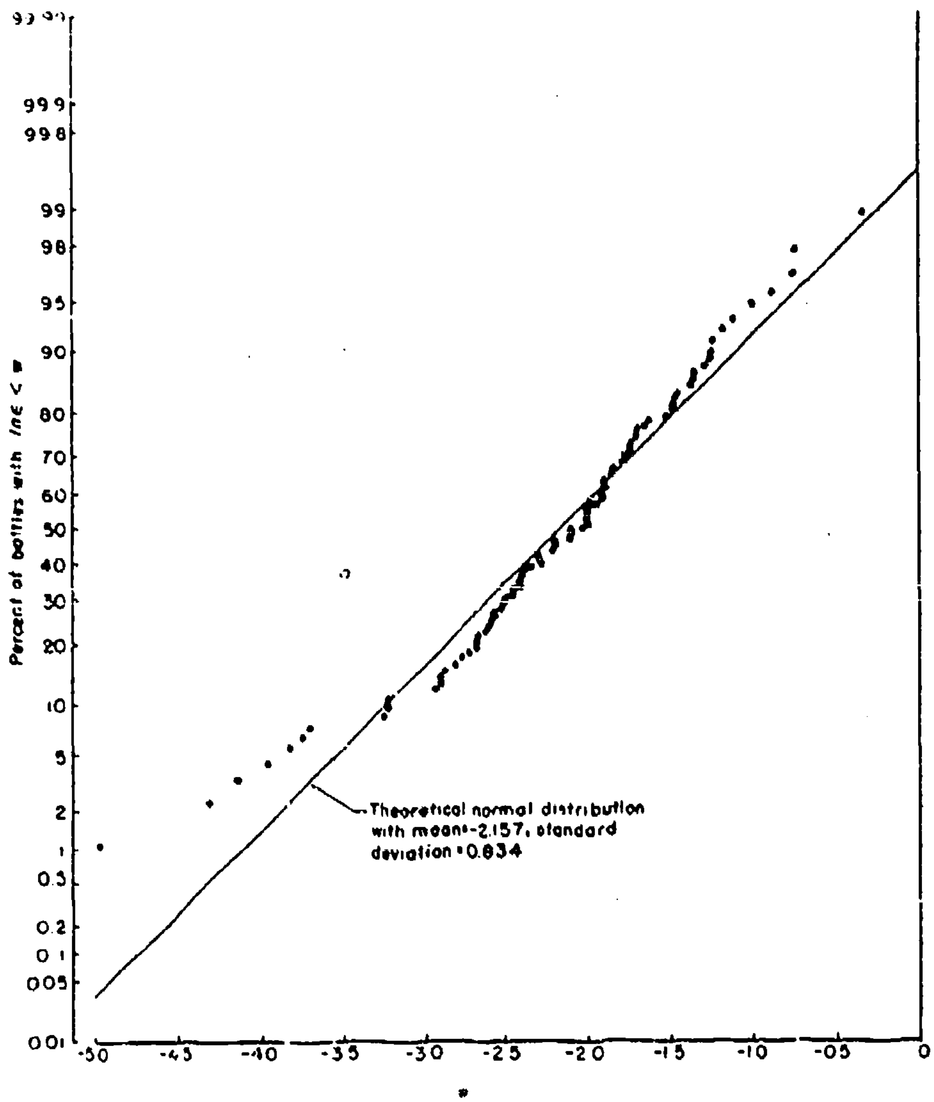

Figure 6. Theoretical and observed cumulative distribution of logarithmic bitterness, lnε.

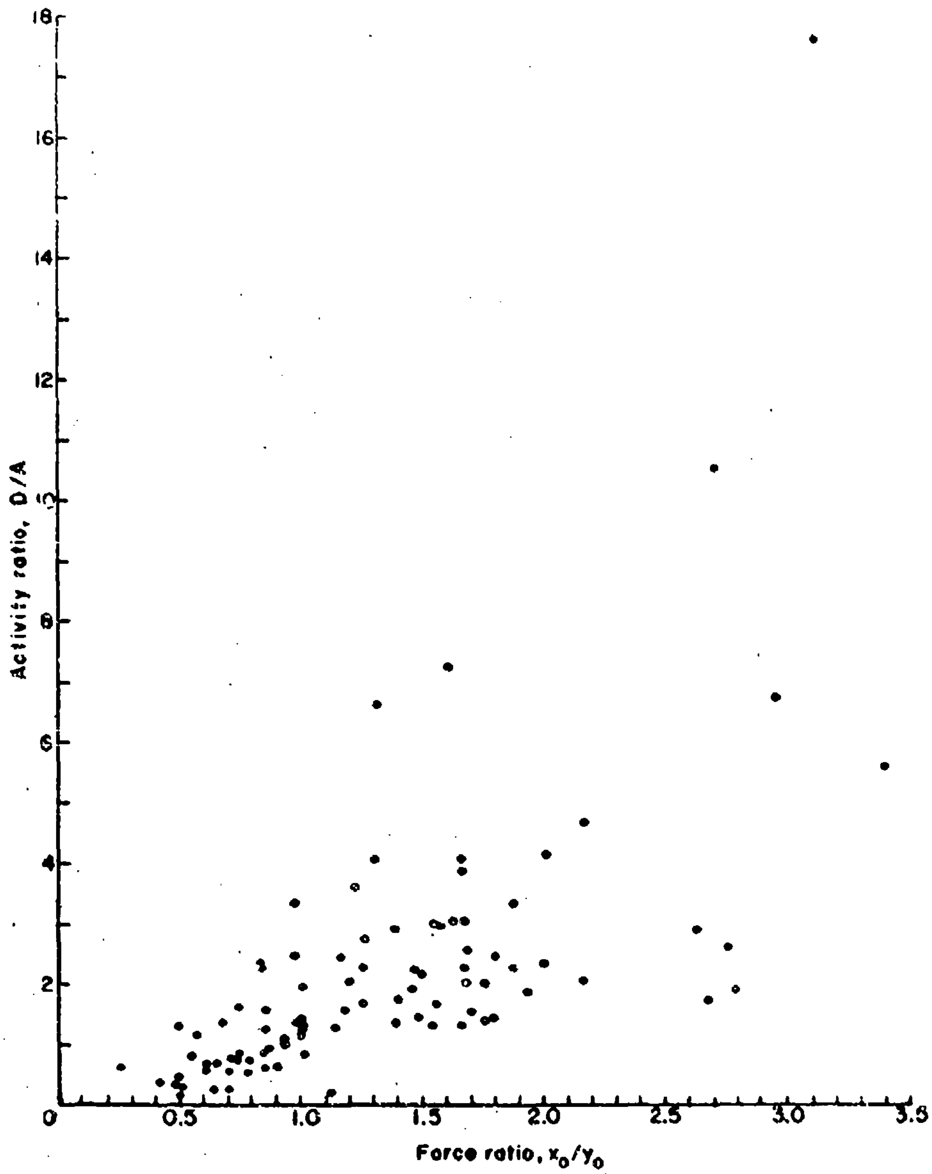

Figure 7. Linear scatter diagram of activity ratio, D/A, against force ratio, x_0/y_0.

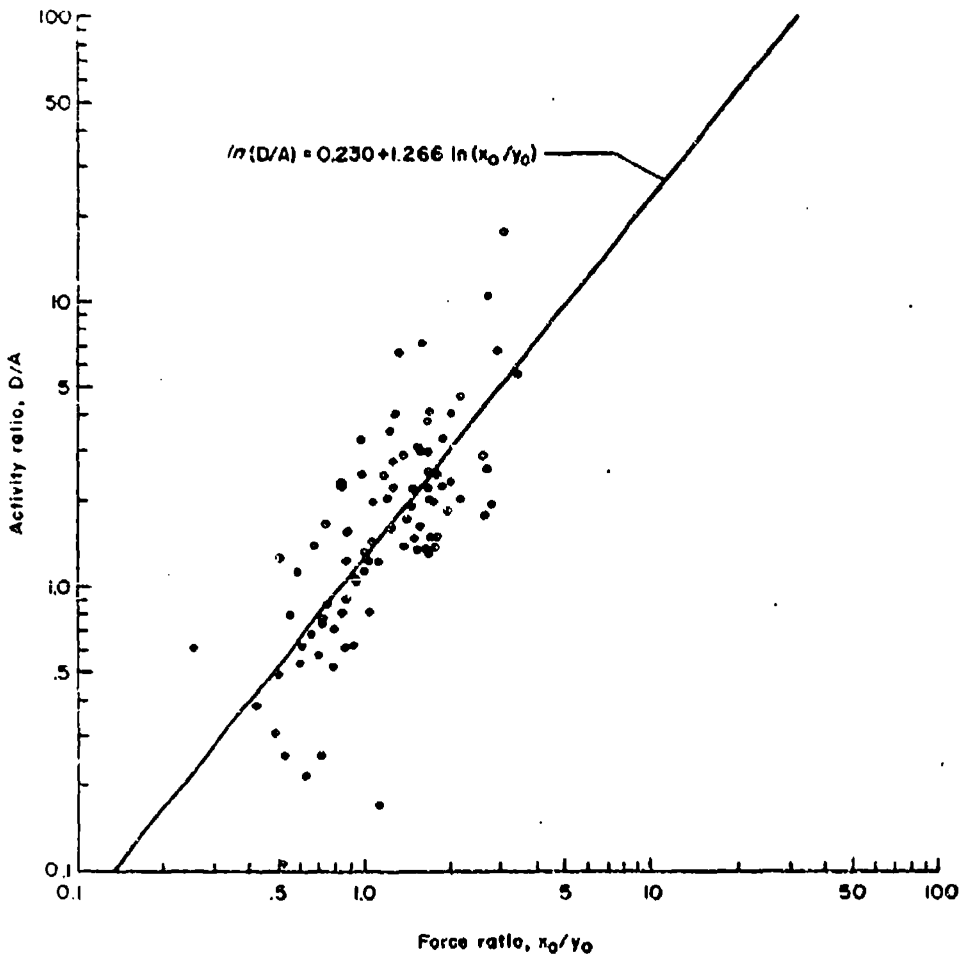

Figure 8. Logarithmic scatter diagram of activity ratio, D/A, against force ratio, x_o/y_o.

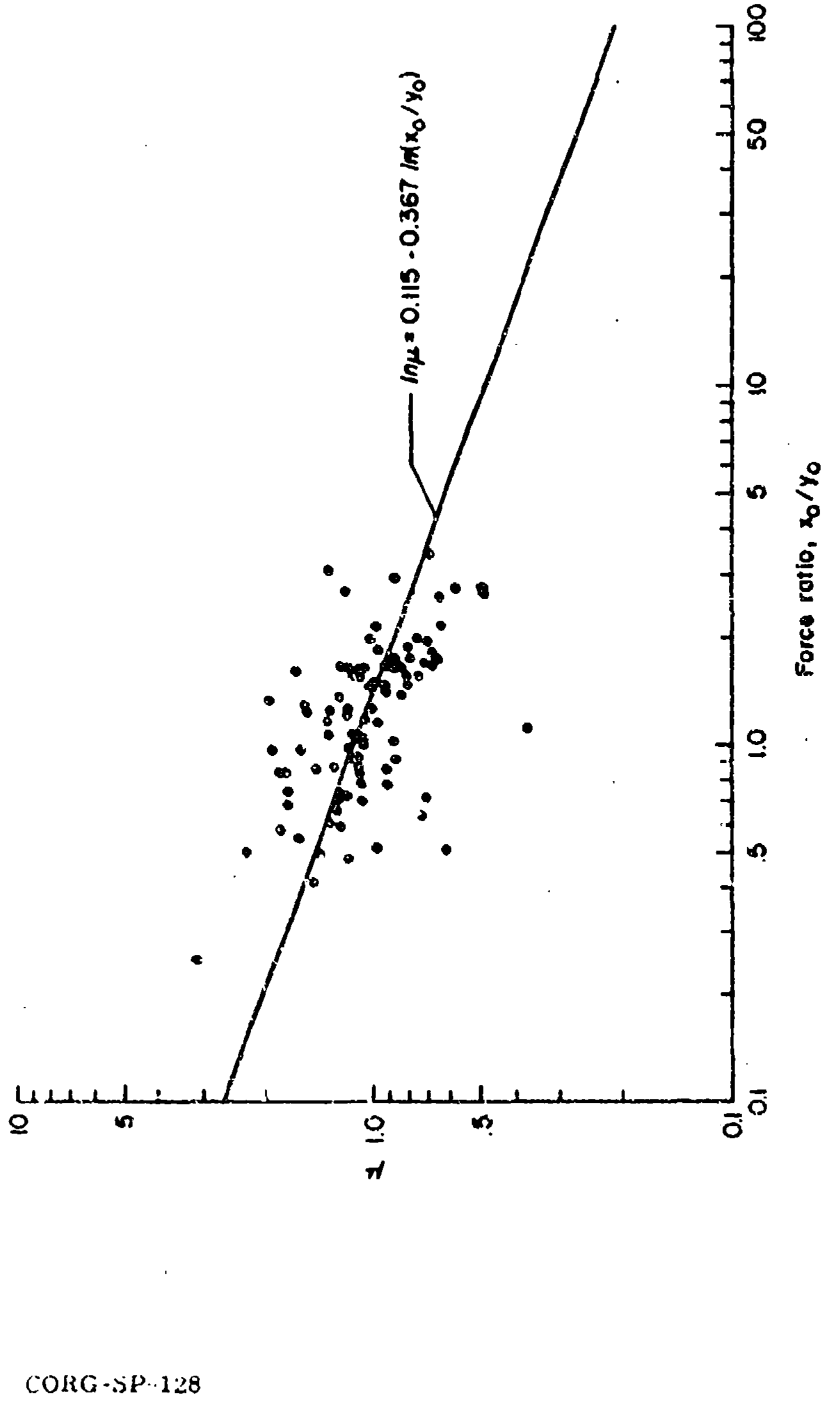

Figure 9 Logarithmic scatter diagram of μ against force ratio, x_o/y_o.

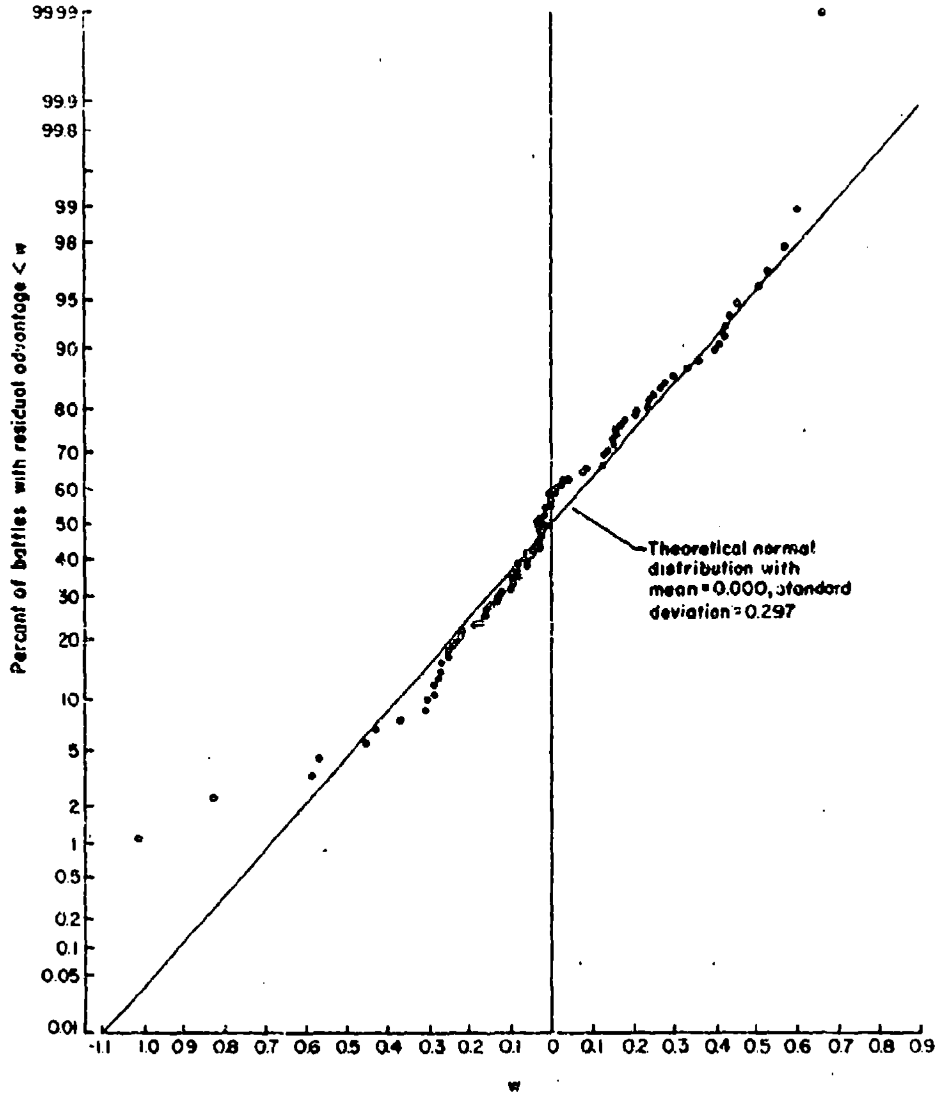

Figure 10. Theoretical and observed cumulative distribution of residual advantage.

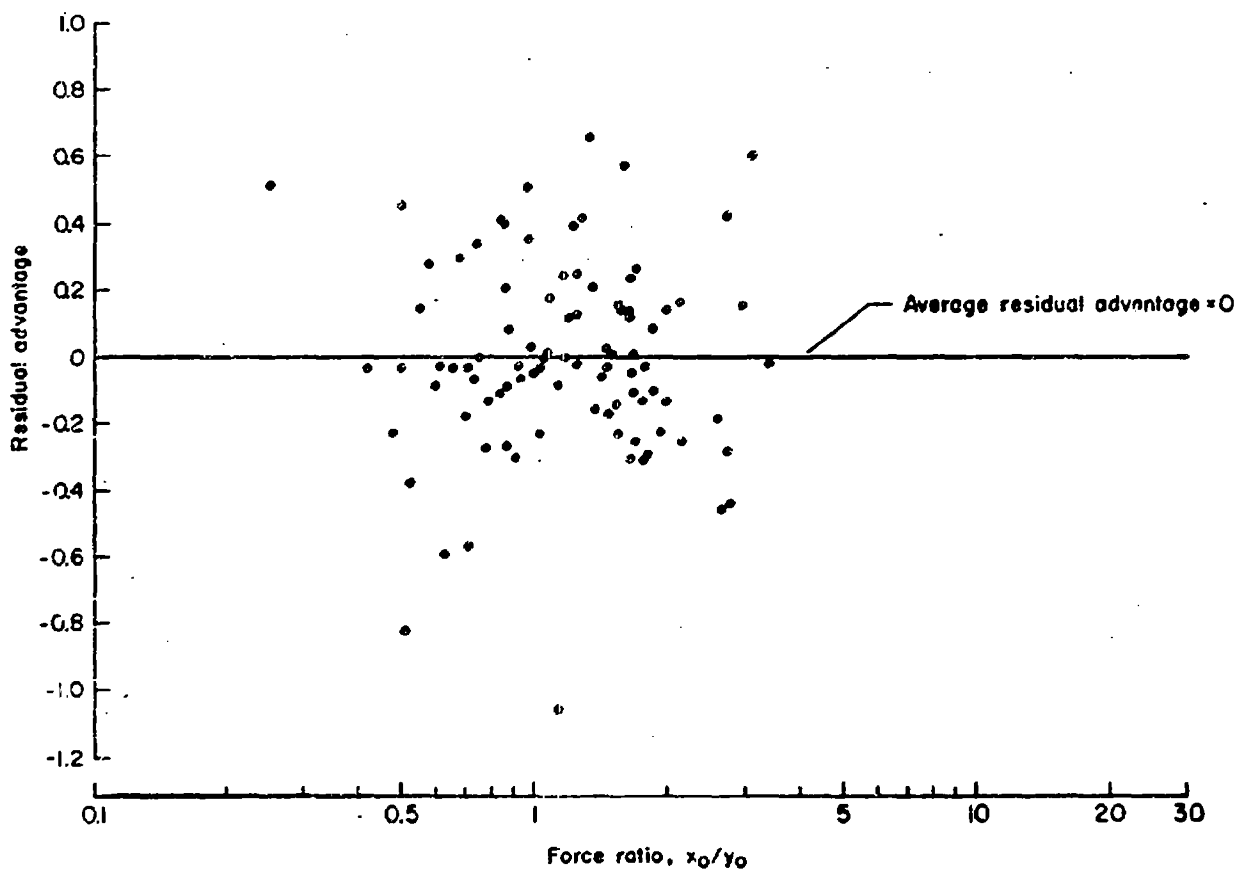

Figure 11. Semilogarithmic scatter diagram of residual advantage against force ratio, x_0/y_0.

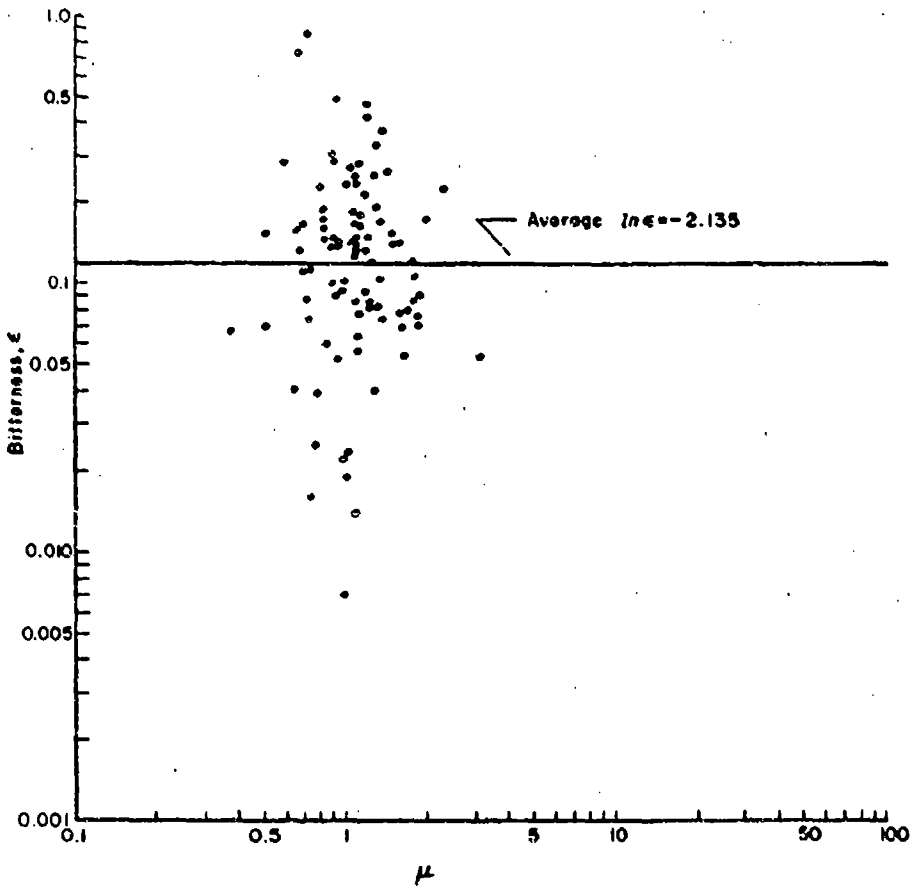

Figure 12. Logarithmic scatter diagram of bitterness, ϵ, against μ.

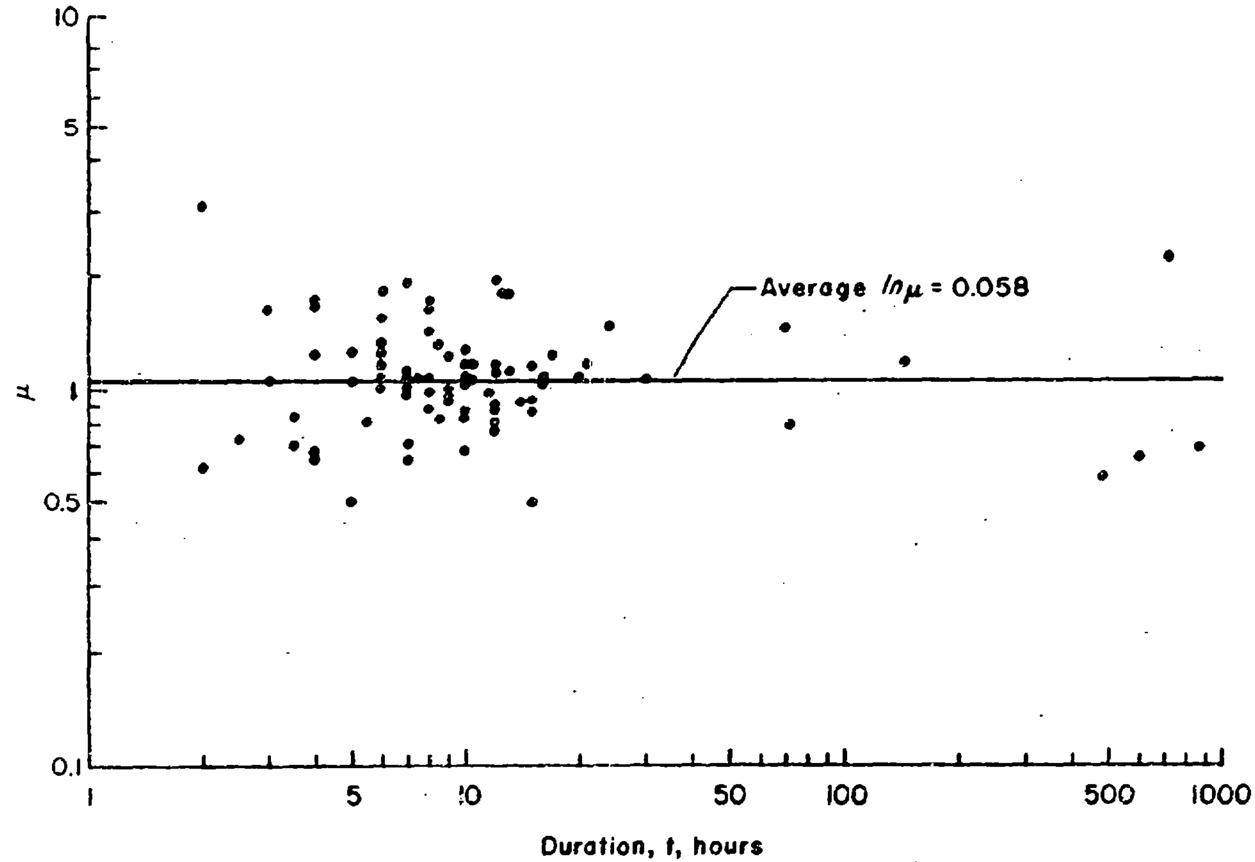

Figure 13. Logarithmic scatter diagram of μ against battle duration, t.

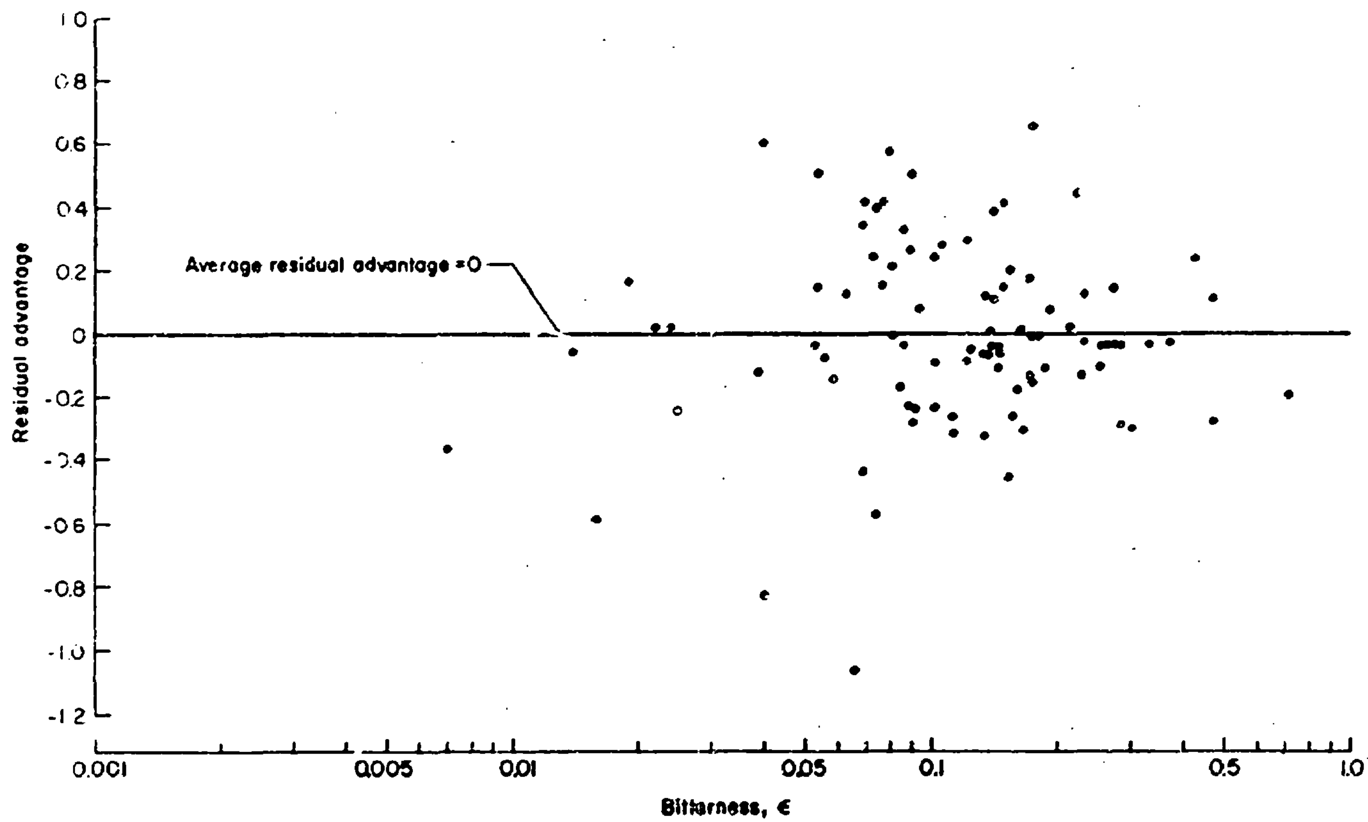

Figure 14. Semilogarithmic scatter diagram of residual advantage against bitterness, ε.

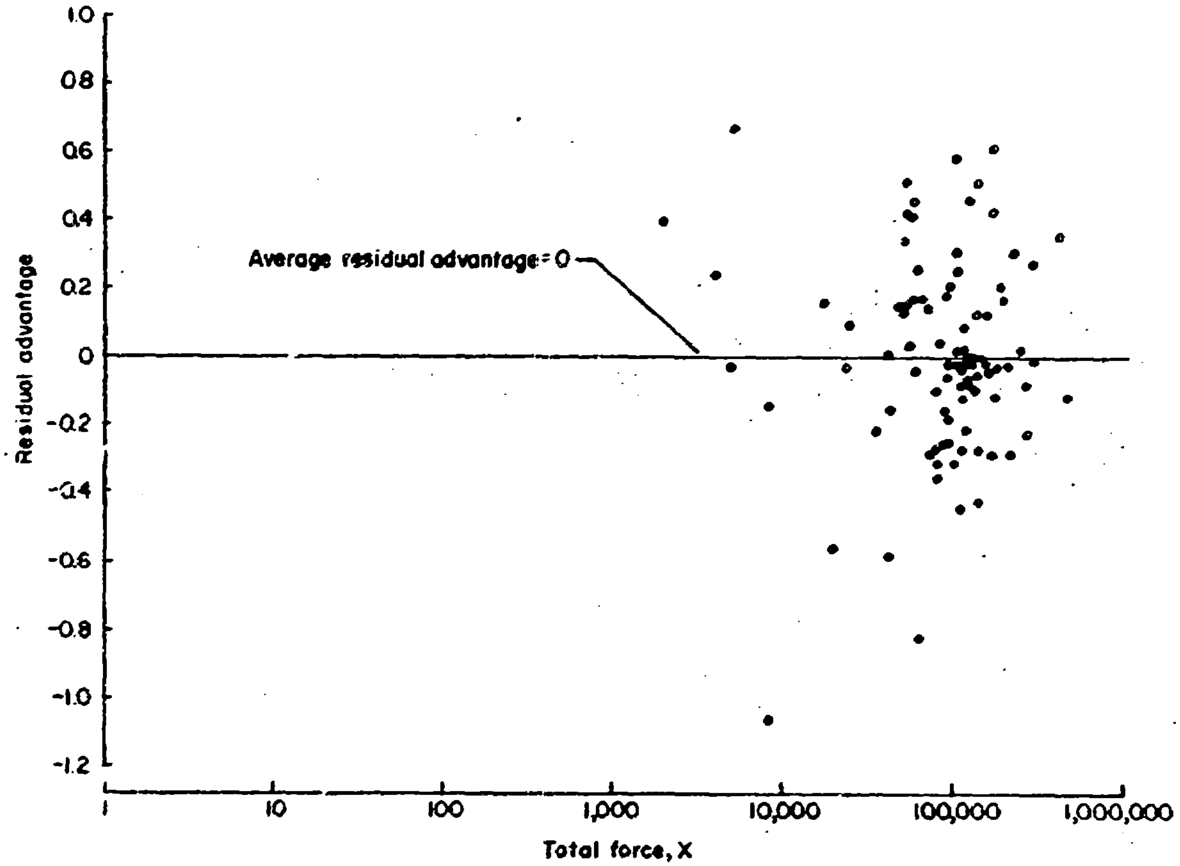

Figure 15. Semilogarithmic scatter diagram of residual advantage against total force, X.

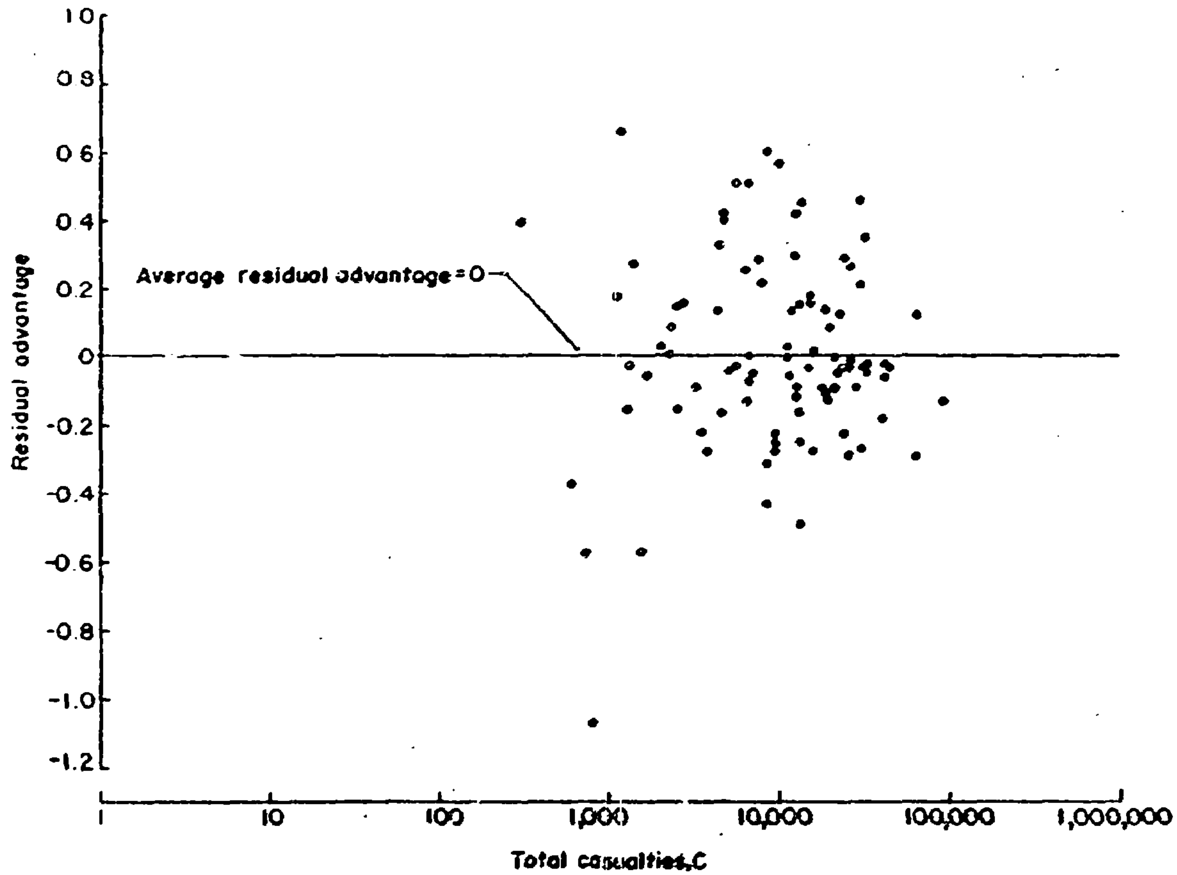

Figure 16. Semilogarithmic scatter diagram of residual advantage against total casualties, C.

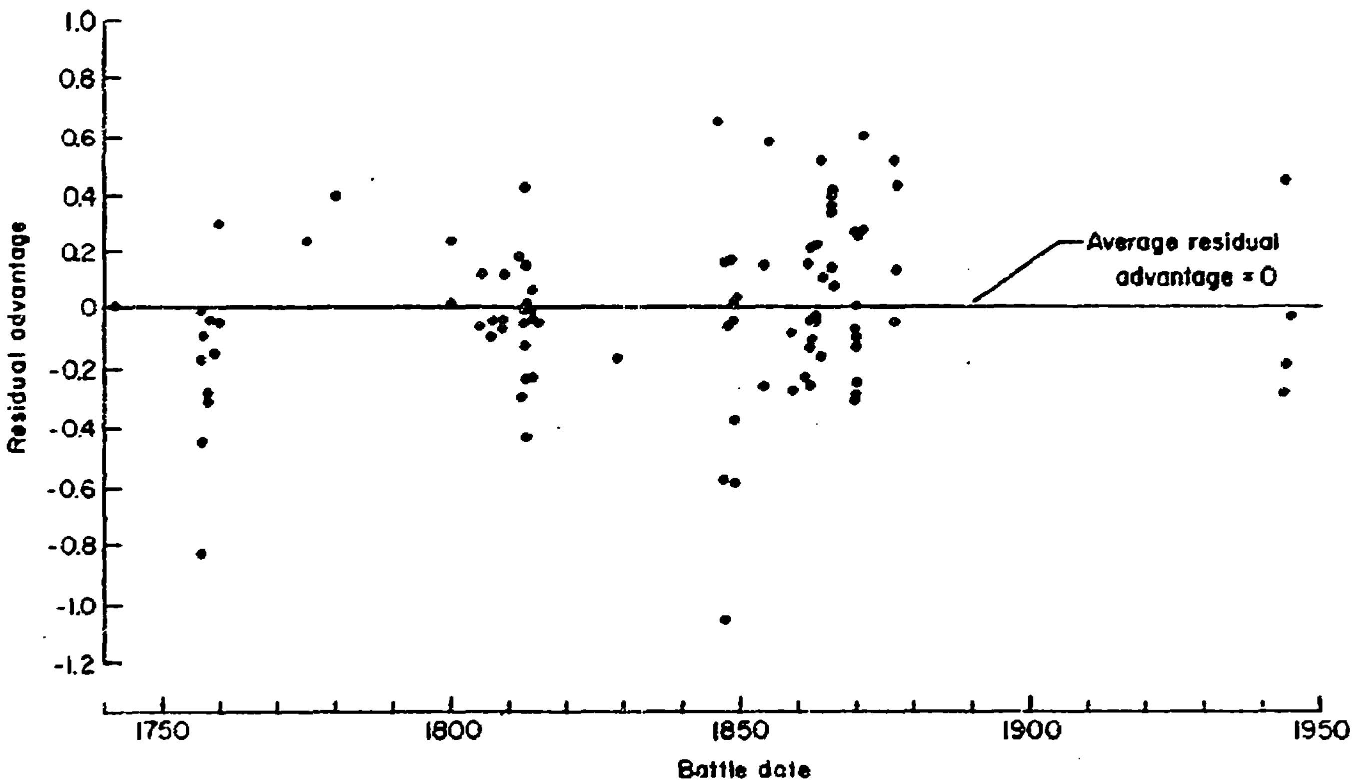

Figure 17. Linear scatter diagram of residual advantage against battle date.

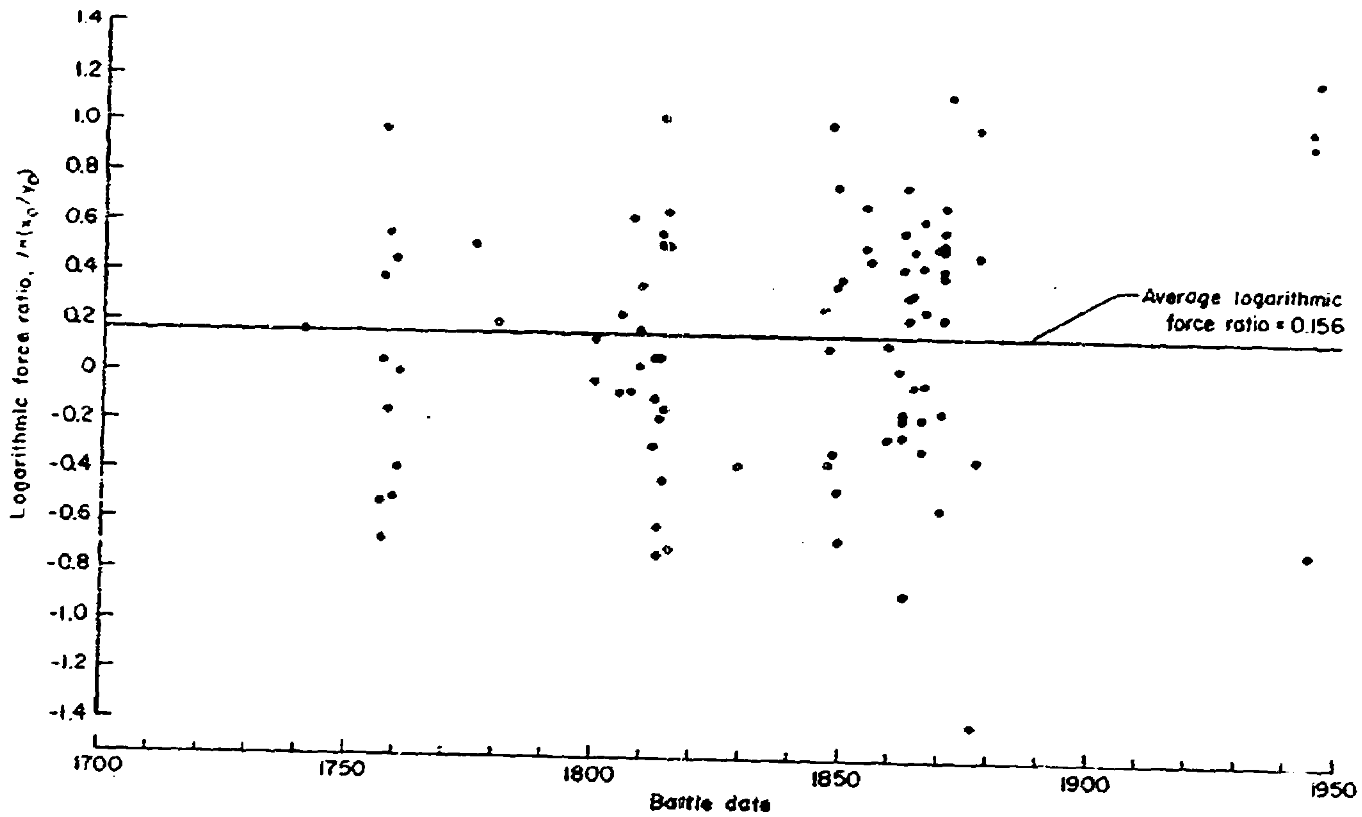

Figure 18. Linear scatter diagram of logarithmic force ratio, $\ln(x_o/y_o)$, against battle date.

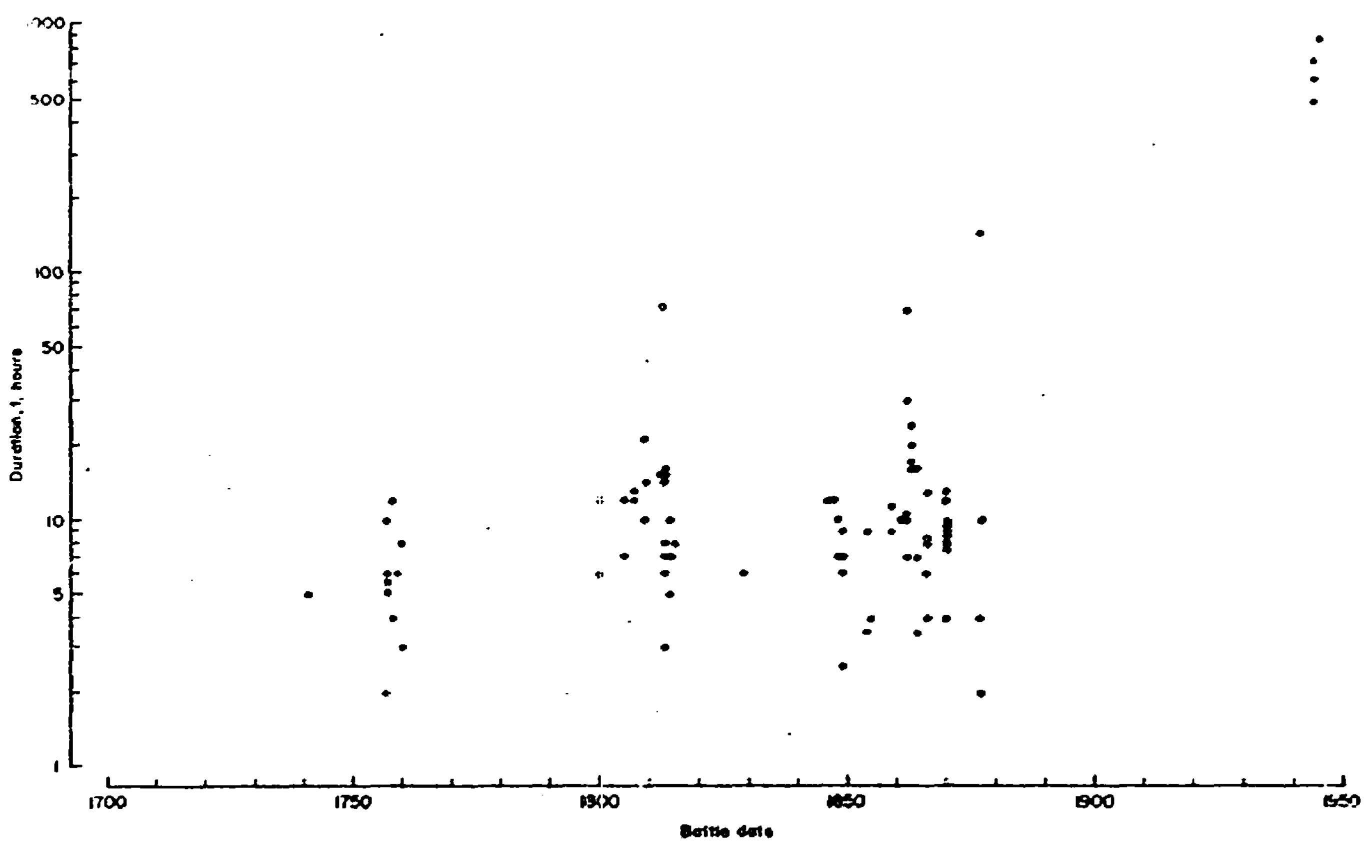

Figure 19. Semilogarithmic scatter diagram of battle duration, t, against battle date.

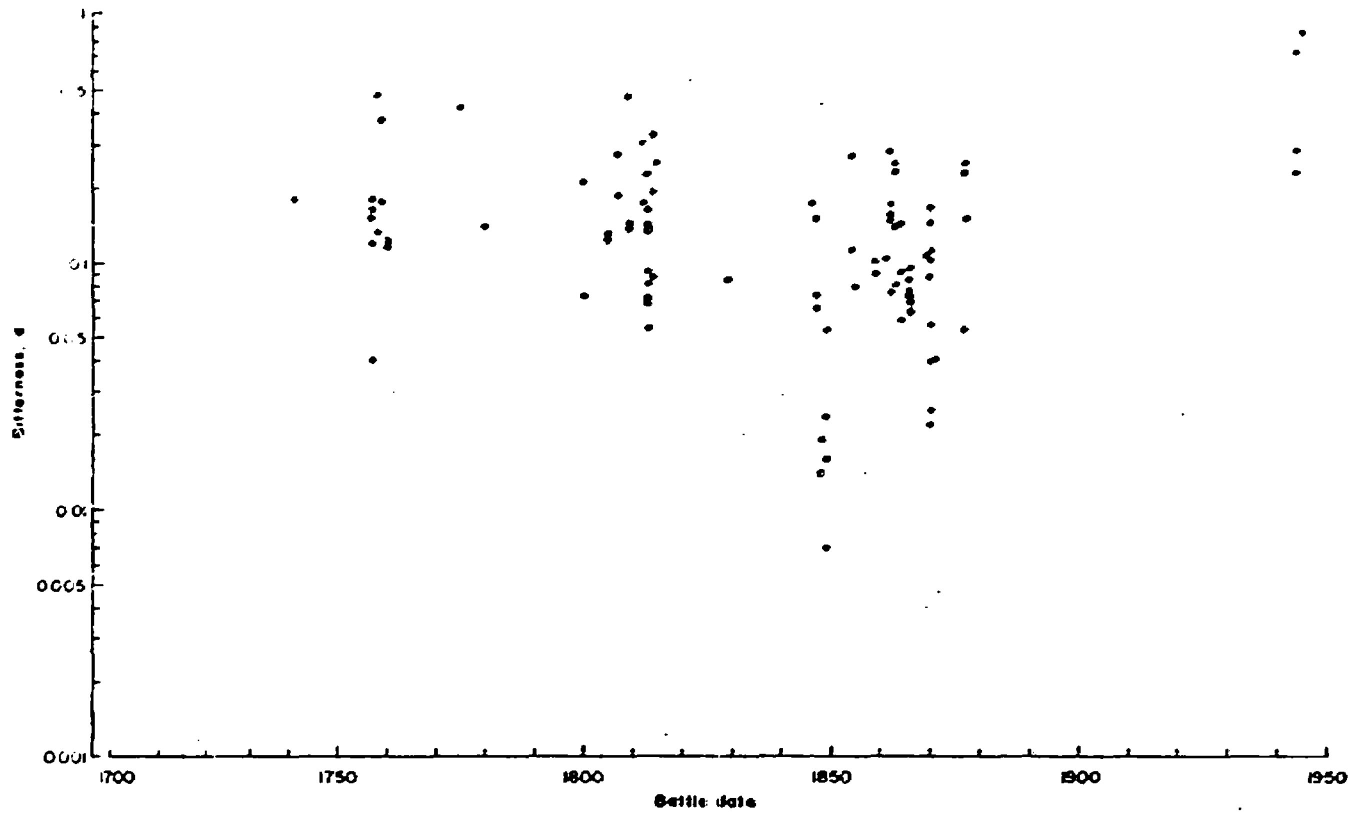

Figure 20. Semilogarithmic scatter diagram of bitterness, ϵ, against battle date.

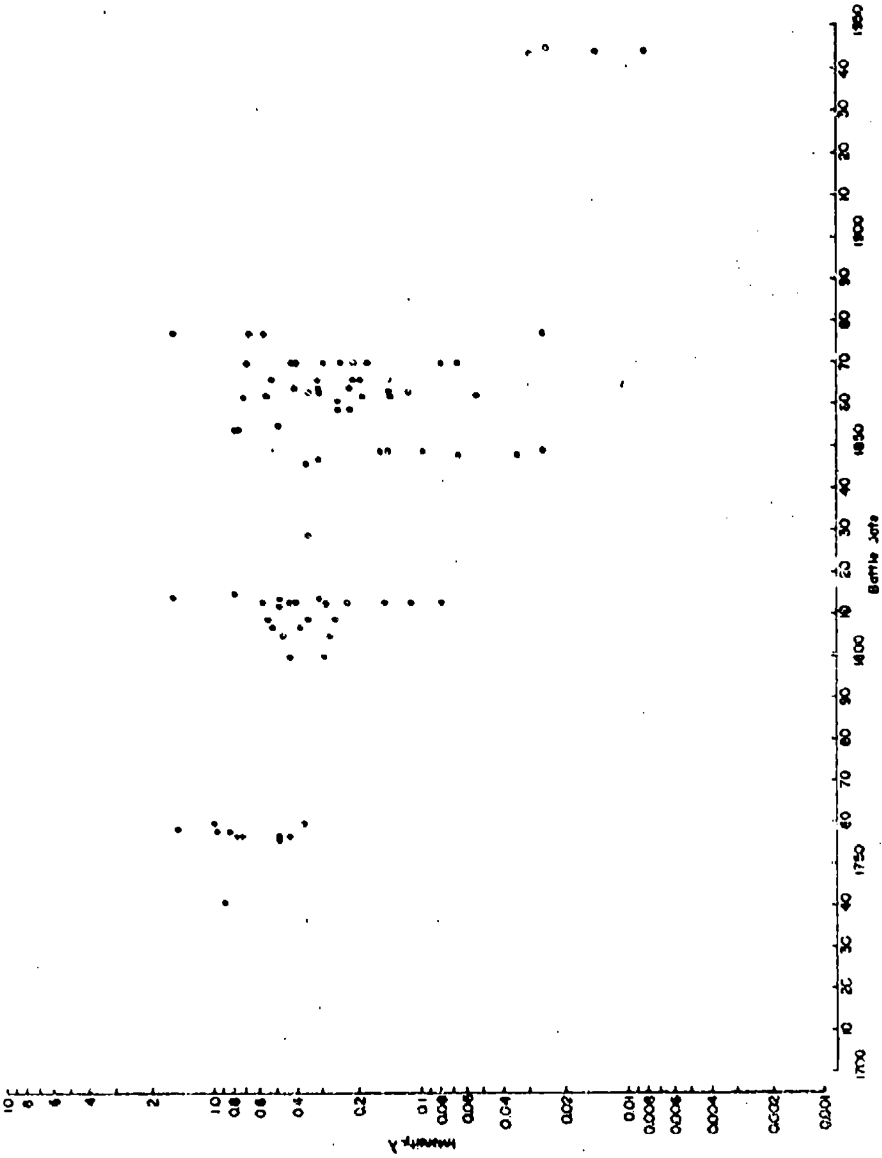

Figure 21. Semilog...thmic scatter diagram of intensity, λ, against battle date.

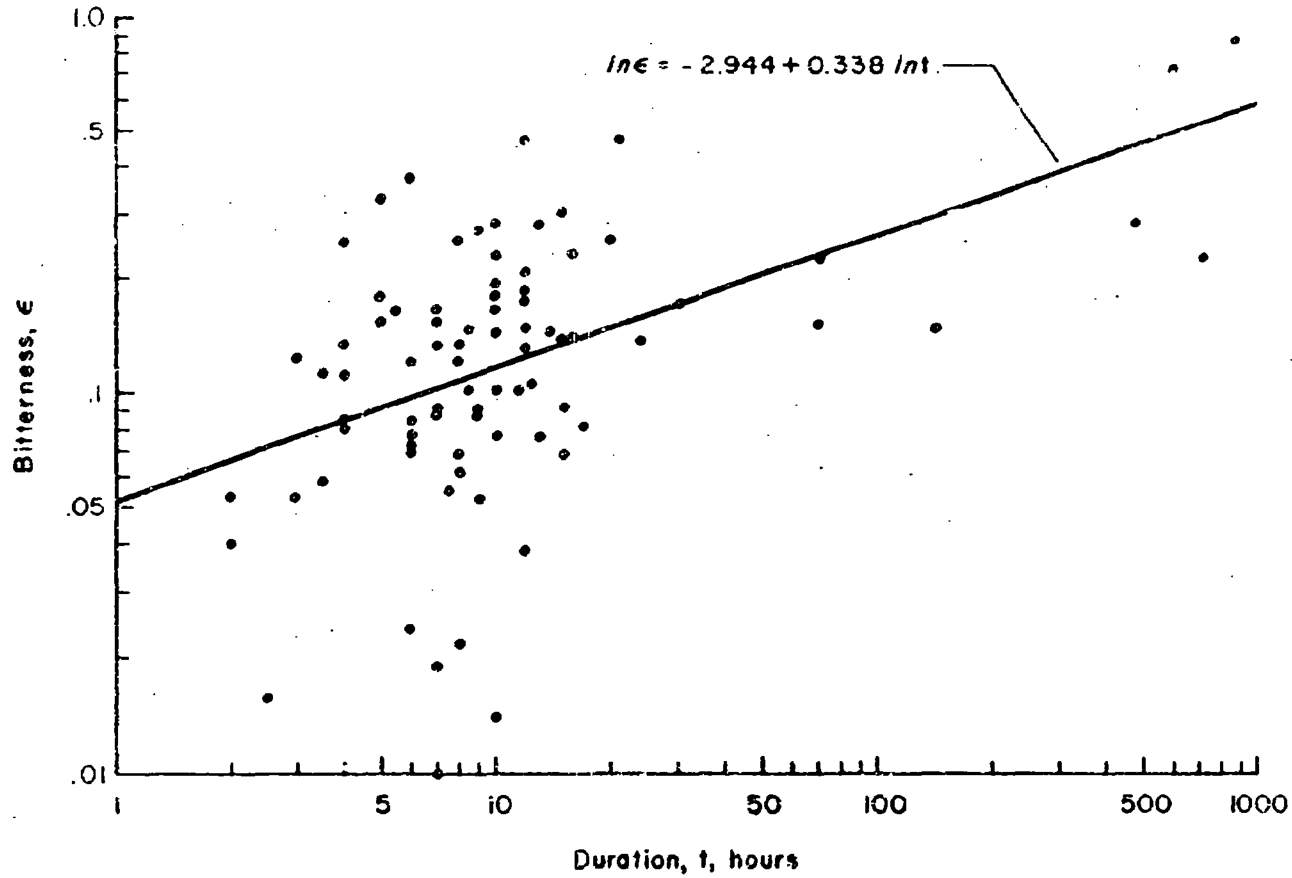

Figure 22. Logarithmic scatter diagram of bitterness, ϵ, against duration, t.

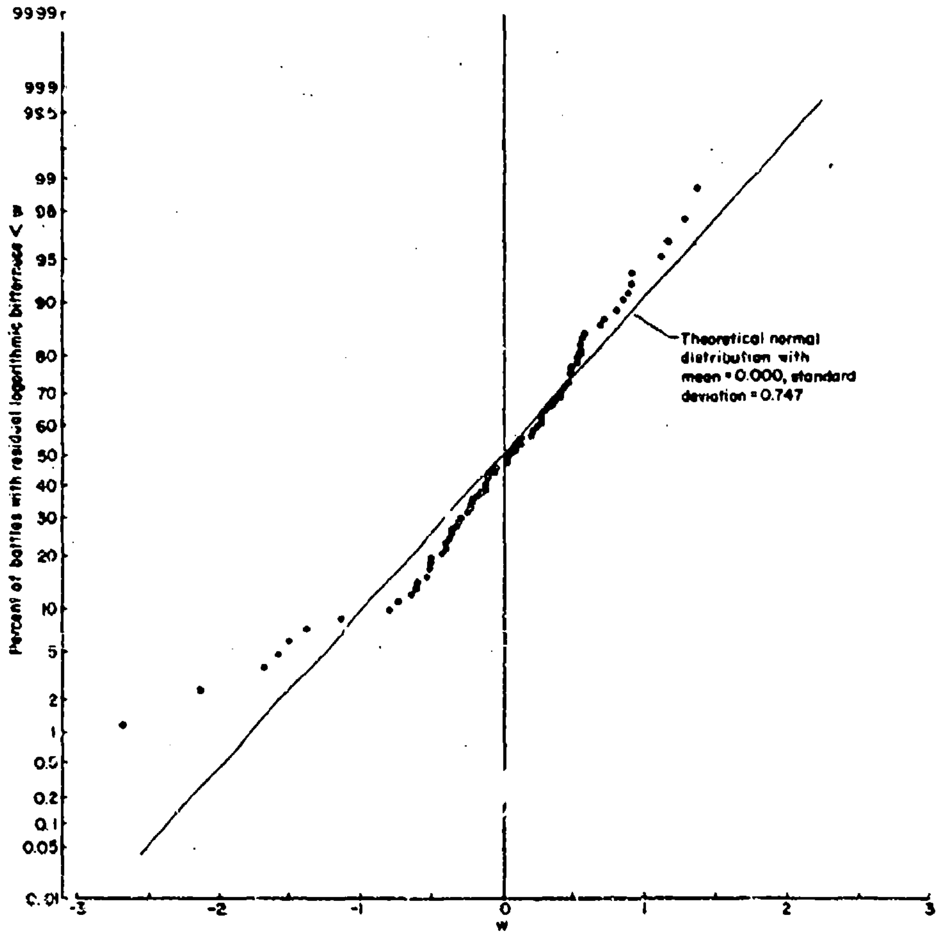

Figure 23. Theoretical and observed cumulative distribution of residual logarithmic bitterness.

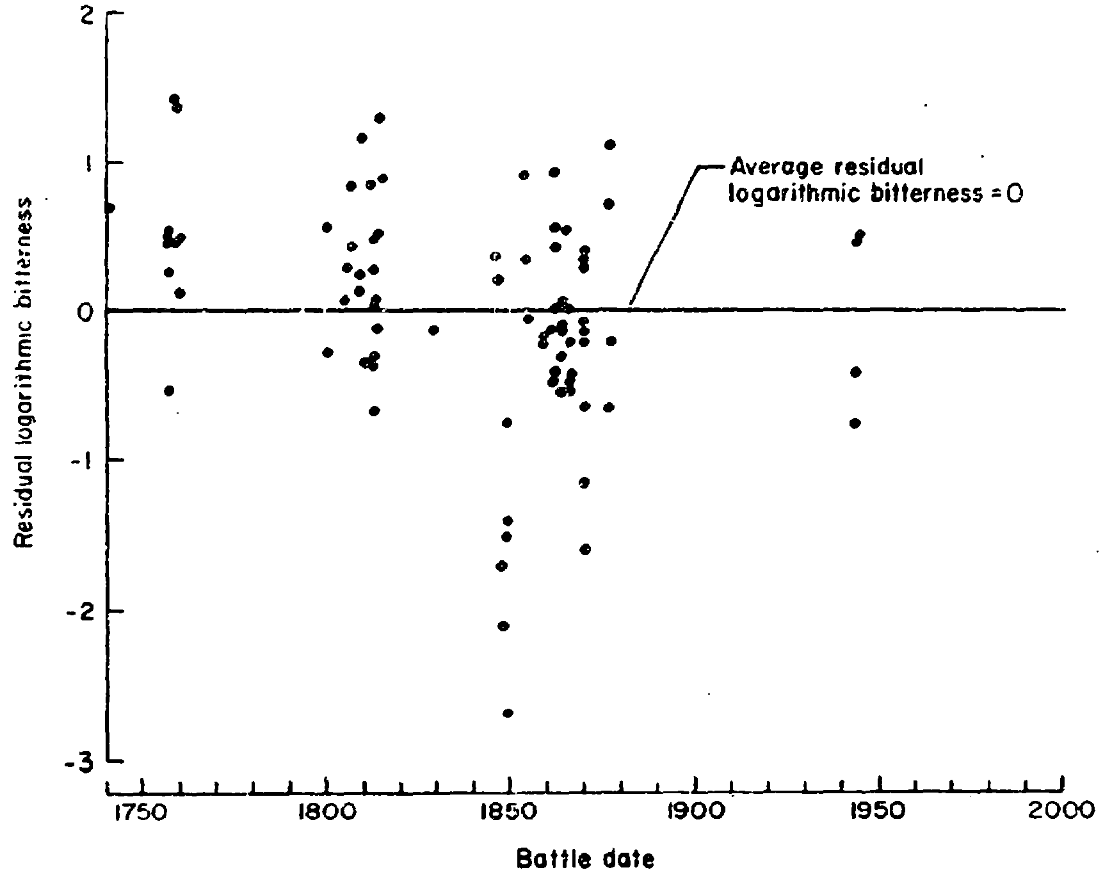

Figure 24. Linear scatter diagram of residual logarithmic bitterness against battle date.

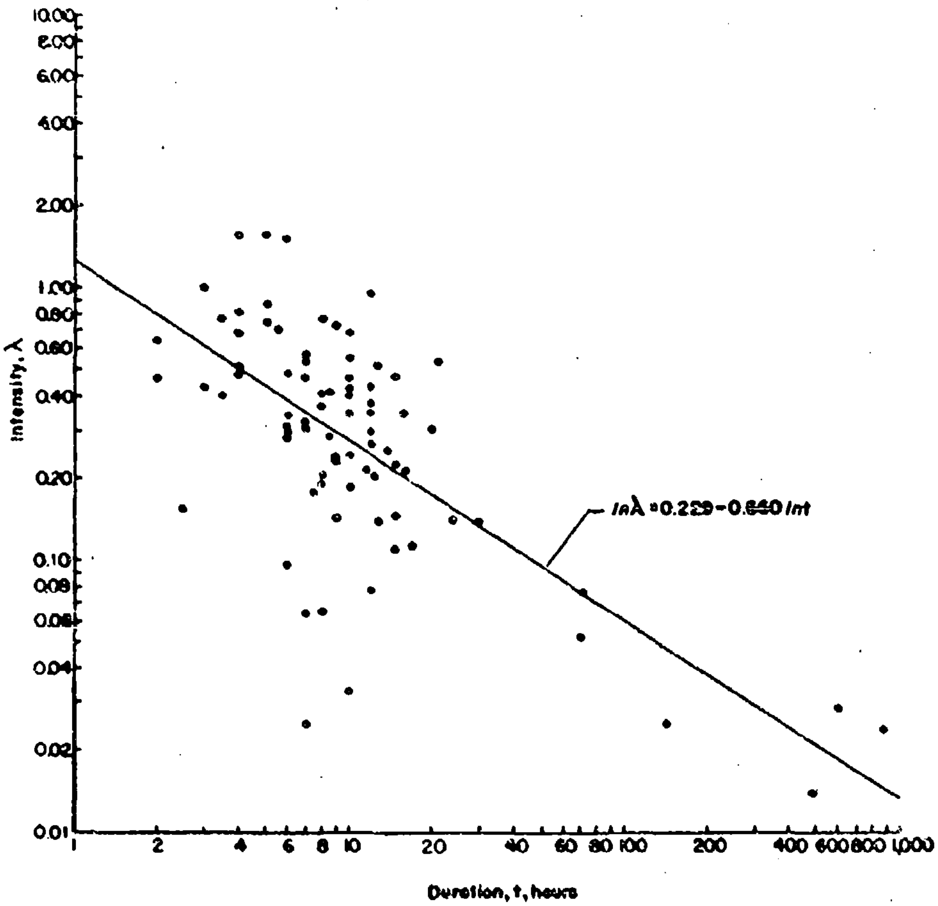

Figure 25. Logarithmic scatter diagram of intensity, λ, against duration, t.

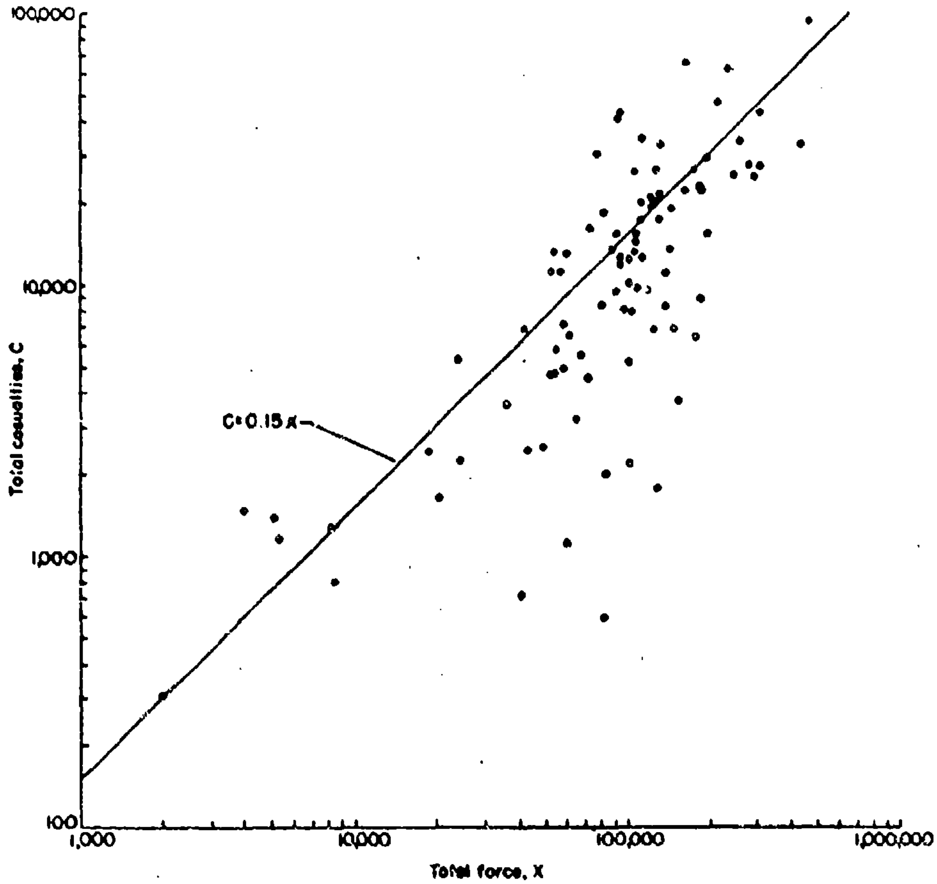

Figure 26. Logarithmic scatter diagram of total casualties, C, against total force, X.

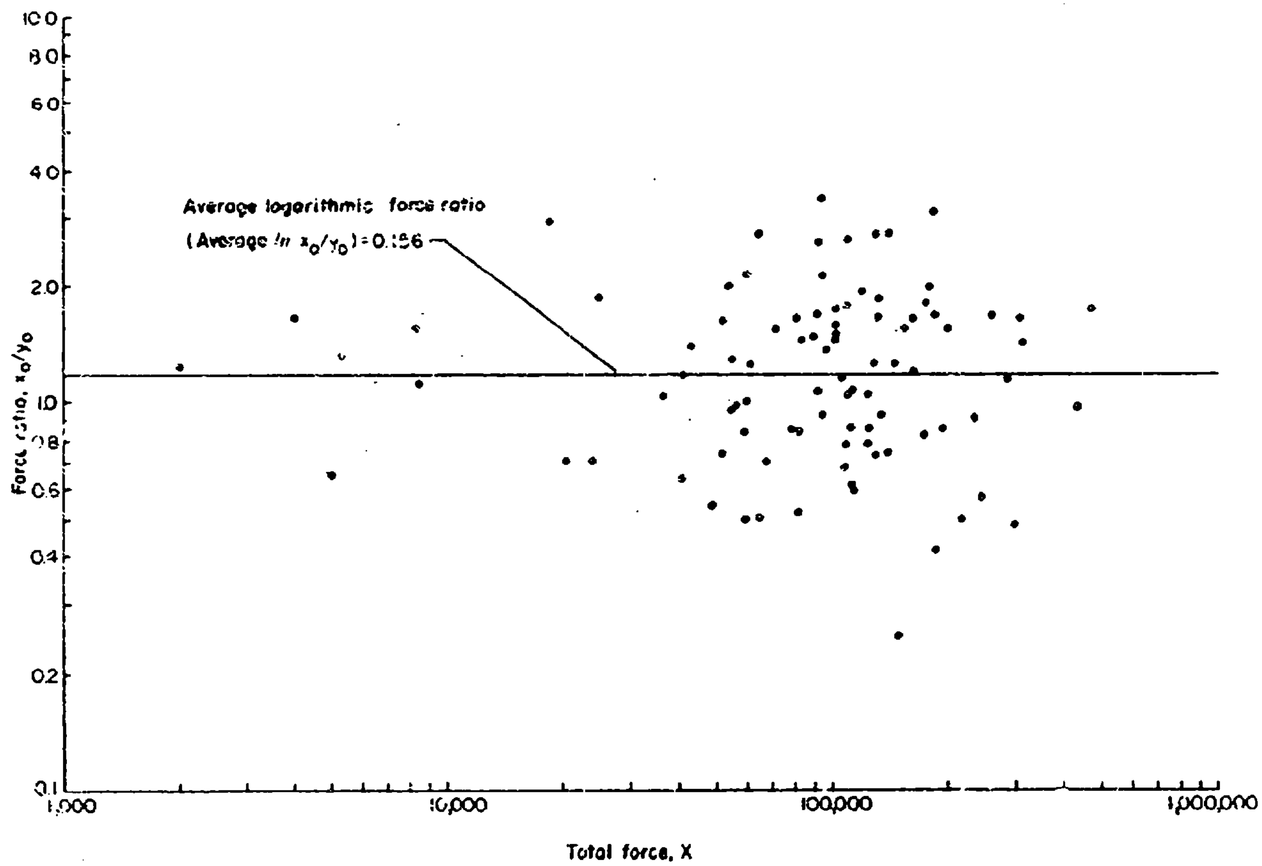

Figure 27. Logarithmic scatter diagram of force ratio, x_o/y_o, against total force, $X = x_o + y_o$.