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A COMPARISON OF GEOSTROPHIC RELATIVE VORTICITY AT 500 MB USING HARMONIC ANALYSIS OF THE HEIGHT FIELD WITH ITS SQUARE GRID COMPUTATION

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ABSTRACT

Geostrophic relative vorticity is computed along three latitude circles generally encompassing the United States, using a Fourier analysis of the 500-mb height field for the period 3-6 November 1961.

A statistical comparison of vorticity obtained in this manner with it*s square grid computation is made.

The writers are deeply indebted to Professor F. L. Martin of the United States Naval Postgraduate School for his suggestion of the topic and his continued help throughout the investigation and during the preparation of this paper.

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LIST OF SYMBOLS USED

Symbol	Definition
ζ_g	vertical component of geostrophic relative vorticity
g	acceleration due to gravity
f	coriolis parameter
z	contour height
a	radius of earth
Ω	angular velocity of earth
ϕ	latitude
$\Delta\phi$	difference in latitude
λ	longitude
β	$2\Omega \cos\phi / a$
$\tilde{z}$	zonally averaged contour height
z^*	perturbation of zonal contour height
$\bar{z}$	space-mean contour height
d	grid mesh = 381 km
n	wave number
N	latitude circles numbered 1 through 27 grid lengths from the north pole
C_n	amplitude of nth wave
θ_n	phase angle of nth wave
m	map factor
r	radius of earth at latitude ϕ, $r = a \cos\phi$
ζ_{FNWF}	vertical component of geostrophic relative vorticity as computed by Fleet Numerical Weather Facility
F	F test for equality of variances
p	number of predictors (independent variables)
u_g	geostrophically-computed zonal component of the wind
U_g	mean zonal wind computed geostrophically

Symbol	Definition
RMS	root mean square
R	multiple correlation coefficient
η	number of cases in sample

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1. Introduction.

The advent of the electronic computer coupled with increased data coverage has removed the primary obstacles to the use of harmonic analysis as a useful instrument in atmospheric research.

Use is made of a Fourier analysis of the 500-mb height field. A computer program devised by G. Arnason and M. Reese and made available by Fleet Numerical Weather Facility[1] was used to obtain the harmonic analysis around latitude circles at distances of 1 through 27 grid lengths d = 381 km from the pole. An equation for relative vorticity using the results of the spherical harmonic analysis is derived and tested statistically against the more usual numerical computation of geostrophic relative vorticity made using a square grid.

The problem undertaken was to compute relative vorticity at 500 mb using the geostrophic vorticity equation in its spherical coordinate form [1, p. 358].

$$\zeta_g = \frac{g}{f}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \qquad \frac{u_g}{a}\tan\phi + \qquad (1)$$

In equation (1), z was replaced by its expression in terms of a Fourier analysis

$$z = \tilde{z} + \sum_{n=1}^{M}\left[c_n \sin(n\lambda - e_n)\right] = \tilde{z} + z^* \qquad (2)$$

Here $\tilde{z}$ is the zonally averaged contour height, whereas z^* is the perturbation quantity which remains when the $\tilde{z}$ field is subtracted from the actual height field. The computer printout included maps of each of these three fields at 500 mb for the period under investigation, 3 through 6 November 1961. Geostrophic relative vorticity was computed by the above mentioned technique for 17 longitudes along each of three latitude circles generally

[1] Henceforth referred to as FNWF

encompassing the north-south extent of the United States. This gave a total sample size of 204 computed relative vorticities which were then correlated with square-grid computations of relative vorticities obtained from FNWF charts of absolute vorticity for exactly the same set of points and times.

2. The geostrophic relative vorticity in terms of a harmonically analyzed contour field.

The adequacy of the Fourier representation of meteorological parameters will not be discussed here; the interested reader is referred to [2].

It was previously mentioned that use was made of the FNWF computer program for a Fourier analysis of the 500 mb contour field. A brief description of the theory and numerical evaluation of the Fourier coefficients is given in [3] and only the applicable portions of the program will be mentioned here.

The zonal mean height $\tilde{z}$ is defined as

$$\tilde{z} = \frac{1}{2\pi}\int_0^{2\pi} z \, d\lambda \tag{3}$$

and its zonal perturbation, in accordance with equation (2), is

$$z^* = z - \tilde{z} \tag{4}$$

Expanding z in a Fourier series one obtains at a given latitude

$$z = \tilde{z} + \sum_{n=1}^{M}\left[A_n \sin n\lambda + B_n \cos n\lambda\right] \tag{5}$$

where A_n and B_n are the coefficients of the nth harmonic functions in the Fourier expansion of z, and M is the total number of harmonic waves necessary to resolve the contour field. Replacing A_n and B_n in (5) by an amplitude factor C_n and a phase angle θ_n, (5) may be rewritten as

$$z = \tilde{z} + \sum_{n=1}^{M} C_n \sin(n\lambda - \theta_n) \tag{6}$$

where

$$C_n = \sqrt{A_n^2 + B_n^2} \qquad \text{and} \qquad \tan\theta_n = -B_n/A_n$$

Equation (6) has been programmed for the CDC-1604 digital computer by FNWF, and the program includes: a contour grid map of the zonal perturbation field z^*, and a tabular list of the Fourier parameters C_n, θ_n and $\tilde{z}$ for each of 27 latitude circles. The last three parameters are actually computed in relation to a zonal band of width equal to the grid mesh d of the FNWF map grid, but may be considered applicable at the center of the band.

Using spherical coordinates λ and ϕ (longitude and latitude, respectively) the derivative operators

$$\frac{\partial}{\partial x} = \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{1}{a}\cdot\frac{\partial}{\partial\phi}$$

apply, and we have

$$\zeta_g = \frac{g}{a^2 f}\left(\frac{1}{\cos^2\phi}\frac{\partial^2 z}{\partial\lambda^2} + \frac{\partial^2 z}{\partial\phi^2}\right) + \frac{2}{a} u_g \csc 2\phi \tag{7}$$

since

$$\beta/f = \frac{1}{a}\cot\phi$$

If now z is replaced in (7) by its expansion in a Fourier series as given in equation (2) we obtain

$$\zeta_g = \frac{g}{a^2 f}\left\{\frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2}\left[\sum_{n=1}^{M} C_n \sin(n\lambda - \theta_n)\right]\right.$$

$$\left. + \frac{\partial^2}{\partial\phi^2}\left[\sum_{n=1}^{M} C_n \sin(n\lambda - \theta_n)\right]\right\} + \frac{2}{a} u_g \csc 2\phi \tag{8}$$

Remembering that C_n and θ_n are independent of longitude λ but not of latitude ϕ, we may perform the indicated differentiations and combine terms, with the result that ζ_g in spherical coordinates becomes

$$\zeta_g = \frac{g}{fa^2}\Big\{-\frac{1}{\cos^2\phi}\sum_{n=1}^{M}\Big[n^2C_n\sin(n\lambda-\Theta_n)\Big] + \sum_{n=1}^{M}\Big[\sin(n\lambda-\Theta_n)\Big(\frac{\partial^2 C_n}{\partial\phi^2} - C_n\Big(\frac{\partial\Theta_n}{\partial\phi}\Big)^2\Big)\Big]$$

$$-\sum_{n=1}^{M}\Big[\cos(n\lambda-\Theta_n)\Big(2\frac{\partial C_n}{\partial\phi}\frac{\partial\Theta_n}{\partial\phi} + C_n\frac{\partial^2\Theta_n}{\partial\phi^2}\Big)\Big]\Big\} + \frac{2}{a}u_g\csc 2\phi \qquad (9)$$

Equation (9) may be shown to have the form

$$\zeta_g = \frac{g}{f}\sum_{n=1}^{M}\Big[-\frac{n^2}{a^2\cos^2\phi} + \frac{1}{a^2C_n}\Big(\frac{\partial^2 C_n}{\partial\phi^2} - C_n\Big(\frac{\partial\Theta_n}{\partial\phi}\Big)^2\Big)\Big]C_n\sin(n\lambda-\Theta_n)$$

$$-\frac{g}{fa^2}\sum_{n=1}^{M}\frac{1}{nC_n}\Big[2\frac{\partial C_n}{\partial\phi}\frac{\partial\Theta_n}{\partial\phi} + C_n\frac{\partial^2\Theta_n}{\partial\phi^2}\Big]nC_n\cos(n\lambda-\Theta_n) \qquad (10)$$

$$+\frac{2}{a}u_g\csc 2\phi\,.$$

Since

$$z = \sum_{n=1}^{M}C_n\sin(n\lambda-\Theta_n) \qquad \text{and} \qquad \frac{\partial z}{\partial\lambda} = \sum_{n=1}^{M}nC_n\cos(n\lambda-\Theta_n),$$

equation (10) may be approximated by

$$\zeta_g - \frac{2u_g}{a}\csc 2\phi \cong \frac{g}{f}\Big\{\overline{-\frac{n^2}{a^2\cos^2\phi} + \frac{1}{a^2C_n}\Big[\frac{\partial^2 C_n}{\partial\phi^2} - C_n\Big(\frac{\partial\Theta_n}{\partial\phi}\Big)^2\Big]}\Big\}z$$

$$-\frac{g}{fa^2}\Big\{\overline{\frac{1}{nC_n}\Big(2\frac{\partial C_n}{\partial\phi}\frac{\partial\Theta_n}{\partial\phi} + C_n\frac{\partial^2\Theta_n}{\partial\phi^2}\Big)}\Big\}\frac{\partial z}{\partial\lambda} \qquad (11)$$

Here the superscript bar denotes an average with respect to n over all significant waves n = 1, ..., M.

The FNWF maps are stereographic maps true at latitude 60N, and elsewhere the grid mesh, d = 381 km, is related to the <u>true distance</u> d_T by the relationship $d = md_T$, where $m = (1 + \sin 60)/(1 + \sin\phi)$ and is the so-called map factor.

The partial derivatives in (11) may be evaluated by centered finite differences. The numerical values of C_n and Θ_n were simply taken from the tabular list of computer output of the harmonic analysis at the various latitude circles. For example, $\partial^2 c_n / a^2 \partial\phi^2$ may be written using finite differences between adjacent latitude circles (for which $a\Delta\phi = d/m$) in the form

$$\frac{1}{a^2}\frac{\partial^2 c_n}{\partial\phi^2} = \frac{c_{N-1,n} - 2c_{N,n} + c_{N+1,n}}{d^2/m^2} .$$

The latitude circles of the Fourier analysis employed in the computation of ζ_g corresponded to N = 12, 15, and 18 grid mesh distances from the pole, which in turn corresponded to the latitude circles: 48.9N, 38.6N, and 28.3N (see fig. 1). However, in the computation of first and second derivatives of the nth Fourier wave parameters, these parameters at latitude circles N-1 and N+1 are also needed.

The term $\partial z^*/\partial\lambda$ is more tractible in the present case if the finite-difference substitution $r\Delta\lambda = 2d/m$ is made; thus, the centered derivative $\partial z_i^*/\partial\lambda$ in finite differences, assumes the form

$$\frac{\partial}{\partial\lambda} z_i^* = \frac{m r}{2d}\left(z^*_{i+1} - z^*_{i-1}\right) \qquad (12)$$

where $r = a\cos\phi$ is the radius of the latitude circle, and the subscript i+1 is one grid mesh eastward of the ith point along the latitude circle.

Including the form of $\partial z^*/\partial\lambda$ given in (12), equation (11) may now be written in the more compact form

$$\zeta_g - \frac{2}{g}\,\omega\,\csc 2\phi = W_1 z_i^* + W_2\left(z^*_{i+1} - z^*_{i-1}\right) \qquad (13)$$

where

$$W_1 = \frac{g}{f}\left\{-\frac{n^2}{a^2\cos^2\phi} + \frac{1}{a^2 c_n}\left[\frac{\partial^2 c_n}{\partial\phi^2} - c_n\left(\frac{\partial e_n}{\partial\phi}\right)^2\right]\right\}$$

$$W_2 = -\frac{g\,m\,r}{2fa^2 d}\left[\frac{1}{n c_n}\left(2\frac{\partial c_n}{\partial\phi}\frac{\partial e_n}{\partial\phi} + c_n\frac{\partial^2 e_n}{\partial\phi^2}\right)\right]$$

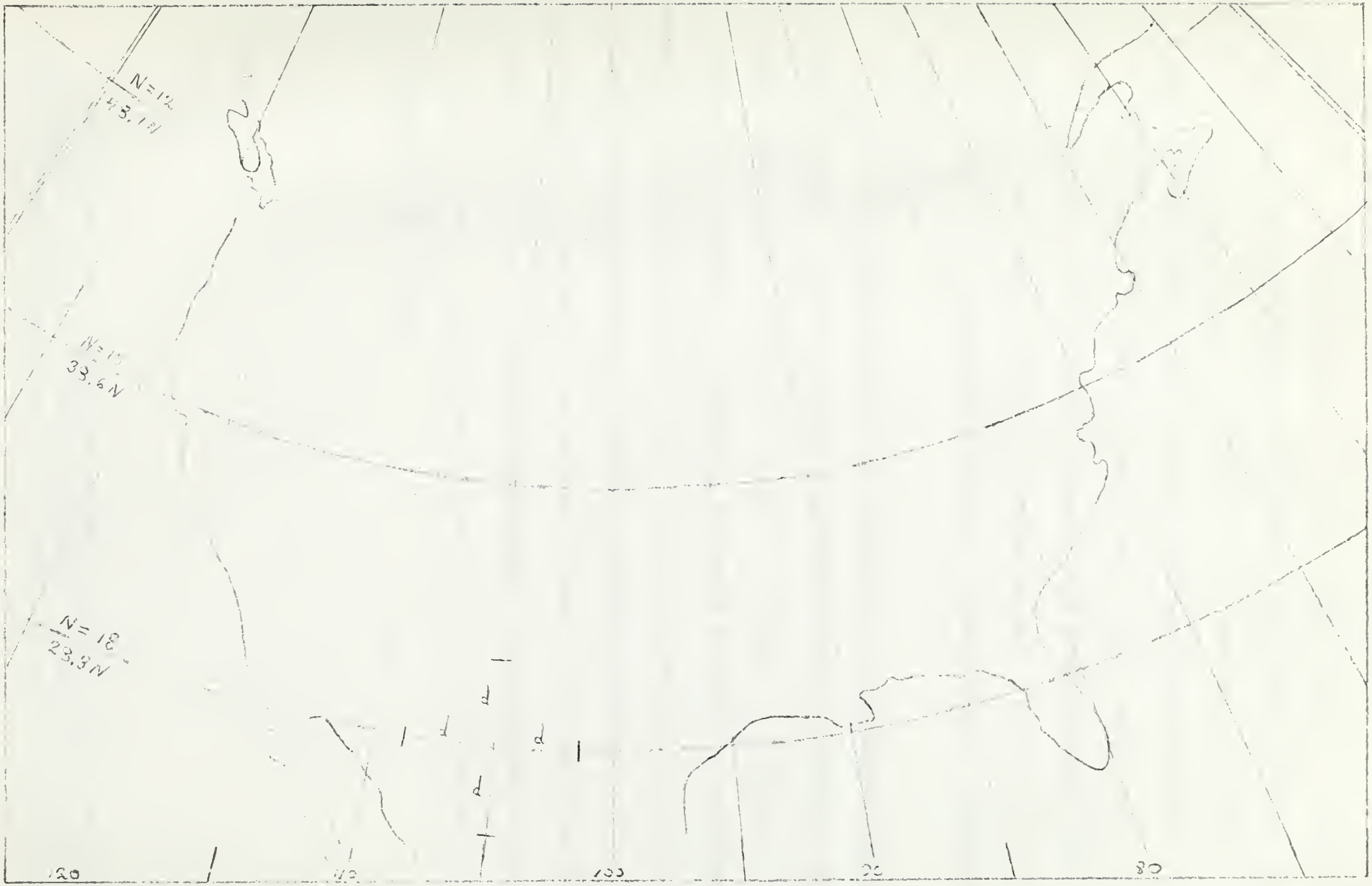

Fig. 1. Grid overlay depicting latitude circles N=12, 15, 18 and the grid mesh length d. Geostrophic relative vorticity is computed at the intersections of the latitude circles with all 5° longitude lines lying within the area bounded by the double-line meridians.

The averaging with respect to n in the expressions for W_1 and W_2 is assumed to be valid, according to the mean value theorem. Actually however the average was approximated by a simple arithmetic mean of the contributions for n = 1, ..., M. Thus it was necessary to compute contributions for each n and divide the result by the number of waves.

Van Mieghem [2] points out that waves of space number higher than 12 to 17 contribute little to the variance of the contour-height field. Consequently, the Fourier analysis was truncated at M = 12 at the two higher latitude circles,[1] although wave numbers 1 through 15 were employed at the lowest latitude.

The coefficients W_1 and W_2 are constants for a particular time and latitude circle, and can be computed directly from the tabulated harmonic data. These values are listed in table 1. However, in applying equation (13) it was necessary to interpolate values of z^* from the perturbation map at the points indicated in fig. 1 along the specified latitude circles. These points were arbitrarily chosen at integral multiples of five degrees of longitude, encompassing the west-east extent of the United States. Therefore, for ease in selecting data points, a grid overlay was constructed presenting all latitude circles N=11,....,19, drawn so as to incorporate the map factor.

[1]At the two higher latitudes, W_1 was negative ensuring that lows corresponded to positive relative vorticity. At the lowest latitude (N=18), it was necessary to use M=15 in order to ensure that this condition was met.

TABLE 1: Values of W_1 and W_2 to be applied in equation ( ). All values are in units of 10^{-6} sec^{-1} ft^{-1}.

DATE	LATITUDE CIRCLE		
	N = 12	N = 15	N = 18
3 Nov. 1961	W = -.1467 W = -.0079	W = -.1635 W = -.0370	W = -.2696 W = -.0303
4 Nov. 1961	W = -.1490 W = -.0311	W = -.0617 W = .0316	W = -.0420 W = .0138
5 Nov. 1961	W = -.1355 W = -.0398	W = -.1013 W = -.0230	W = -.1819 W = .1682
6 Nov. 1961	W = -.0551 W = .0757	W = -.1460 W = .0349	W = -.3510 W = -.7610

Note that the term $(2 u_g \sin 2\phi)/a$ in both (11) and (13) also varies from point to point around any latitude circle. However, the variation of this term was so small that the mean zonal wind for the latitude was used. The mean zonal wind U_g was computed using the latitudinal gradient of zonal mean height $\bar{z}$, that is from

$$U_g = \frac{g}{f} m \frac{\tilde{z}_{N-1} - \tilde{z}_{N+1}}{2d} ,$$

again using finite differences.

Along each of the three latitude circles, 17 values of $W_1 z^*$ and $W (z^*_{i+1} - z^*_{i-1})$ were obtained for each of the four days, giving a total of 204 separate values of these quantities.

3. Statistical results.

For precisely the same set of points and days as described above, corresponding values of ζ_g were obtained from grid maps of geostrophic relative vorticity made available by FNWF. Their maps essentially presented the field of ζ_g computed using their square-grid mesh according to the usual formula

$$\zeta_{FNWF} = \frac{4gm^2}{fd^2}\left(\bar{z} - z\right). \tag{14}$$

where $\bar{z}$ is the four-point space-mean contour height and d = 381 km at latitude 60N.

Since the averaging process in equation (11) is somewhat approximate, it was felt that better verification would result if the relationship between ζ_{FNWF} and $W_1 z_i^*$ and $W_2(z_{i+1}^* - z_{i-1}^*)$ was written

$$\zeta_{FNWF} - \frac{2}{a} U_g \tan\phi = A_0 + A_1(W_1 z_i^*) + A_2[W_2(z_{i+1}^* - z_{i-1}^*)] \tag{15}$$

Here A_0, A_1, and A_2 are regression coefficients[1] to be determined by a least squares technique applied to the 204 simultaneous values of $W_1 z_i^*$ and $W_2(z_{i+1}^* - z_{i-1}^*)$. Because of occasional large gradients of ζ_{FNWF} over 5° longitude intervals, the values of ζ_{FNWF} were smoothed by the three-point binomial running mean formula

$$\bar{X}_i = (X_{i-1} + 2X_i + X_{i+1})/4$$

where X_i is any value of ζ_{FNWF} interpolated from the FNWF vorticity-grid map at each point. The regression equation actually involves the correlation of the smoothed ζ_{FNWF} values with the independent variables already noted.

Using the BIMD 06 statistical program as adapted to the CDC-1604 computer, it was necessary to type 204 IBM cards containing the smoothed ζ_{FNWF} values and the simultaneous values of the independent variables of the right side of equation (15). Best-fitting values of the regression coefficients were

[1]These regression coefficients are non-dimensional since the units of $W_1 z_i^*$ and $W_2(z_{i+1}^* - z_{i-1}^*)$ are in sec^{-1}.

$A_0 = -8.1719$, $A_1 = 0.4585$, and $A_2 = 0.0624$

The standard error of estimate[1] and the multiple linear correlation coefficient for the regression equation were 20.3523 and 0.7632 respectively.

Panofsky and Brier [4, p. 113] present an F test for the analysis of the variance explained by the regression equation. The F value may be defined by the ratio of the explained variance to the residual variance relative to the regression equation. This formula is

$$F = \frac{R^2(n-p-1)}{(1-R^2)p} \quad . \tag{16}$$

With $p = 2$, the number of independent variables, and $n = 204$, the number of random samples, the value of F from equation (16) turns out to be 139.33. However, assuming normally distributed independent and dependent variables, the critical F at the 0.99 significance level with 2 and 201 degrees of freedom is 4.71. Thus the regression equation (15) is significant at a very high confidence level.

The partial correlation coefficients of $W_1 z_i^*$ and $W_2(z_{i+1}^* - z_{i-1}^*)$ were 0.7620 and 0.0382, respectively, indicating very little contribution to the regression equation for the latter variable. An intuitive explanation for the insignificant contribution of $W_2(z_{i+1}^* - z_{i-1}^*)$ may be obtained from fig. 2. The differences, $z_{i+1}^* - z_{i-1}^*$, taken over the pressure centers would be zero, whereas in actuality the largest absolute values of ζ_g would be found at these points. Likewise for a sinusoidal contour wave pattern maximum magnitudes of $z_{i+1}^* - z_{i-1}^*$ would exist near contour inflection points, where $\zeta_g \approx 0$.

The possibility of "automatic" correlation, viz, correlation between the two independent variables, was examined and none was found to exist.

[1] All values of ζ_g are in units x 10^{-6} sec^{-1}.

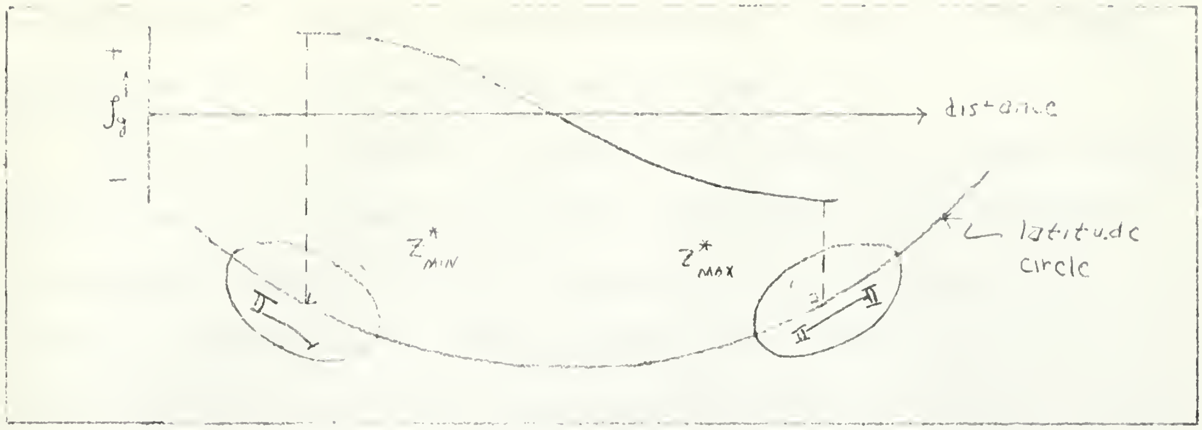

Fig. 2. Distribution of $\dot{J}_g$ along a latitude circle between contour maximum and minimum, in the case of a sinusoidal contour wave.

The individual latitude circles were statistically examined in a like manner. The two northernmost latitude circles gave a multiple linear correlation coefficient of 0.84, whereas there is no significant correlation between $\dot{J}_{FNWF}$ and the two independent variables $W_1 z^*_i$ and $W_2 (z^*_{i+1} - z^*_{i-1})$ at latitude 28.3N. This inconsistency at low latitudes is attributable to "noise" in the Fourier representation of the contour field. The "noise" is due to the small variance of the height field at this latitude, coupled with the relatively large RMS error possible in the 500 mb analysis in the subtropics during the period of study. The use of second derivatives of the Fourier parameters also increases the amount of "noise" generated in the determination of $\dot{J}_g$.

4. Conclusions.

Theoretically there should have been a large positive multiple linear correlation between $\dot{J}_{FNWF}$ and $W_1 z^*_i$ and $W_2 (z^*_{i+1} - z_{i-1})$. This proved to be the case at latitudes 48.9N and 38.6N where the contour field was well resolved by twelve harmonic waves. However, at the lowest latitude circle, 28.3N, the several sources of errors listed at the end of section 3 could, and probably did, influence the low correlations that were obtained. These were basically due to the lack of resolution of the contour field by the

Fourier analysis. However, the same reasons would cause the square-grid computation of ζ_g to be subject to large errors.

Based upon the results at the two higher latitudes, it seems apparent that the use of a Fourier analysis of a meteorological field presents a very valuable tool for atmospheric research. With the advent of the electronic computer and machine-analyzed contour charts, the use of harmonic analysis is an effective and economically feasible approach to the computation of synoptic aids for the practising meteorologist.

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