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RELATIVISTIC CONTINUUM MECHANICS

by

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to

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Richard A. Grot* and A. Cemal Eringen**

ABSTRACT

The basic laws of a special relativistic theory of continuous media suitable for the treatment of electromagnetic interactions with materials are formulated. The kinematics, dynamics and thermodynamics of a continuum are discussed from a relativistic viewpoint. Constitutive equations are deduced for thermoelastic solids, thermoviscous fluids and electromagnetic materials.

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INTRODUCTION

In this article there is presented a nonlinear relativistically (special) invariant theory of continuous media. The object of such a study is a consistent treatment of the interaction of electromagnetic fields with the deformation of matter. In recent years several theoretical works have appeared concerning the simultaneous action of large deformations and electromagnetic fields on material bodies. In general these researches have either considered only static deformations (cf. Toupin [1], Eringen [2], Jordan and Eringen [3], [4]) or developed a dynamical theory along nonrelativistic lines (cf. Toupin [5], and Dixon and Eringen [6]).

It is well known that the invariance group of the basic equations of electromagnetism is the Lorentz group. The modern theories of continuum mechanics, however, make use of the Galilean group for the basic laws of motion and the invariance under the group of rigid motions for the constitutive theory. At the turn of the century, the invariance of the laws of mechanics under the Galilean group was discarded at least for the physical phenomena which fall within the scope of relativistic considerations. Thus we believe that a satisfactory and consistent theory of electromagnetic interactions with deforming materials cannot be obtained until the ground rule (the invariance principles) for mechanics and electromagnetism is taken to be the same. Within the scope of the special theory of relativity the most natural

and the simplest invariance principle is the Lorentz group of trans-
formations.

Following the scheme of classical theories of continuum mechanics,
Eringen [7], Truesdell and Toupin [8], we begin our study with the
geometrical and kinematical description of deformation and motion.
Afterwards the basic laws concerning the motion and physical phenomena
are introduced. The formulation is then completed with the consti-
tutive theories in accordance with certain methodological principles
set forth in [9].

In the literature the kinematics of continuous media is seldom
discussed from a relativistic viewpoint. Exceptions are the works of
Bressan [10] and Toupin [11]. Bressan has formulated a relativisti-
cally (general) invariant kinematics by noting that the motion of a
material particle with undeformed coordinates $X^K(K = 1, \ldots, 3)$ can
be described by $x^\mu = x^\mu (X^K, \bar{t})$ $(\mu = 1, \ldots, 4)$ where $\bar{t}$ is some
temporal parameter. Observing that such a temporal parameterization
is highly arbitrary, he stipulates that the description of the de-
formation is independent of the choice of $\bar{t}$.

Our approach to kinematics is based on an extension of the work
of Toupin [11]. Our strain measures are invariant under the Lorentz
group and they coincide with those of Bressan under appropriate modifi-
cations. The world velocity that is selected is a kinematical one.
In several works the world velocity of a continuous body is defined as
either the time-like eigenvector of the energy-momentum tensor (cf.

Synge [12] or Lichnerowicz [13]) or as the unit vector parallel to the momentum density (cf. Møller [14] or Landau and Lifehitz [15]. Møller actually employs also the kinematical velocity.) This velocity of energy propagation will coincide with our kinematical velocity only in special cases, for example, when the heat conduction and electromagnetic phenomena are neglected.

For basic mechanical and thermomechanical balance conditions we eelect, in Chapter II, the conservation of particle-number, the balance of energy-momentum, the balance of moment of energy-momentum and the eecond law cf thermodynamics. We consider the conservation of particle number as the appropriate generalization of the conservation of mass in nonrelativistic continuum mechanics. Thie point of view ie accepted by several authors (for example, Landau and Lifehitz [15], Van Lantzig [16] and Eckart [17]). An alternate viewpoint ie that the claseical conservation of mase is obtained through the equation of energy in the limit as the epeed of light approaches infinity (for example, Bergmann [18], Thomas [19] or Edelen [20]). From thie point of view it ie difficult to consider thermodynamics without an additional axiom. In nonrelativietic theories the thermodynamical balance equations are deduced from the energy equation. It would seem appropriate to do the eame in relativietic theories. Thus, for example, the energy equation should be ueed to derive the equation of heat conduction.

The form of the balance of energy-momentum ie well known and hardly needs any explanation. It should be noted that in a relativistic

theory part of the momentum is due to nonmechanical sources such as the heat flux and stress tensor. In most of the works on relativistic theories of continua the energy-momentum tensor is assumed to be symmetric. This is usually deduced from the balance of angular momentum. In the classical theory of continuum mechanics the stress tensor is symmetric only if there are no torques acting on the body and the body does not possess an intrinsic spin. We generalize this idea to the relativistic case by introducing a spin tensor and a body torque tensor in four dimensions. The idea of a spin tensor can be found in such works as Bogoliubov and Shirkov [21] and Papapetrou [22]. If the spin and torque tensors vanish, it follows from the balance of moment of energy-momentum that the energy-momentum tensor is symmetric.

The second law of thermodynamics is formulated in a manner analogous to the law given by Eckart [17] with the provision that it is considered as a restriction on the form of the constitutive equations rather than on the electromechanical process.

In Chapter III the methodological principles for formulating constitutive equations are set down. They are obvious generalizations of those employed in nonrelativistic continuum mechanics. The major difference between these principles and those of modern continuum mechanics is the requirement that the constitutive equations are to be form-invariant under the Lorentz group. This is adequate for the theories treated in this paper. A different approach using the idea of non-sentient response is to be found elsewhere, Bragg [23].

As examples of the above formulation of constitutive equations, relativistic theories of thermosolids and thermoviscous fluids are deduced. The second law of thermodynamics forces the acceptance of the heat conduction law of Eckart [17] as opposed to those of Bressan [24] or Pham Mau Quan [25] who attempt to generalize Fourier's law.

In Chapter IV the Minkowski form of the equations of electromagnetism are formulated in the usual four dimensional form. In Chapter V the interaction between electromagnetic fields and matter is treated. An interaction term is written down using the analysis of Dixon and Eringen [6]. If one wishes to treat the interaction of electromagnetic fields and matter one needs a physical model (cf. Dixon and Eringen [6], Jordan and Eringen [3], or the appendix of Fano, Chu and Adler [26]). The seemingly fond wish to use the Minkowski stress tensor or its symmetric part is erroneous unless the material has no polarization or magnetization, in which case this tensor is symmetric.

In Chapter VI the constitutive theory of electromagnetic materials is set down for an elastic solid and a viscous fluid. Such effects as heat conduction, electrical conduction, polarization, and magnetization are included in the theory. The consequences of the second law of thermodynamics are fully investigated.

Finally an appendix on the invariants of tensors and vectors for the Lorentz group is included. In special cases it is shown that the use of results of the three dimensional orthogonal group is permissible. This facilitates greatly the construction of constitutive equations since a reworking of the theory of invariants of tensors and vectors for the Lorentz group would be lengthy and tedious.

Notation

In this paper we use the standard tensor notation and summation convention of repeated indices. The signature of the Lorentz metric $(\gamma^{\alpha\beta})$ is $(+ + + -)$, i.e. $\gamma^{11} = \gamma^{22} = \gamma^{33} = 1$, $\gamma^{44} = -1$, and all other $\gamma^{\alpha\beta} = 0$. It is convenient to use the following convention:

$$\epsilon_{\alpha\beta\gamma\delta} \equiv e_{\alpha\beta\gamma\delta}$$

where $e_{\alpha\beta\gamma\delta}$ is the permutation symbol and define

$$\epsilon^{\alpha\beta\gamma\delta} \equiv - e^{\alpha\beta\gamma\delta}$$

Since $\det(\gamma^{\mu\nu}) = -1$

$$\epsilon^{\alpha\beta\gamma\delta} = \gamma^{\alpha\alpha_1} \gamma^{\beta\beta_1} \gamma^{\gamma\gamma_1} \gamma^{\delta\delta_1} \epsilon_{\alpha_1\beta_1\gamma_1\delta_1}$$

The short hand notation

$$\overset{*}{\epsilon}{}^{\alpha\beta\gamma} \equiv \epsilon^{\alpha\beta\gamma\delta} u_\delta$$

is often employed to define the "cross product" between two four vectors a^α, b^α, e.g., $\underset{\sim}{a} \times \underset{\sim}{b}$ is the vector

$$(\underset{\sim}{a} \times \underset{\sim}{b})^\gamma \equiv \overset{*}{\epsilon}{}^{\gamma\alpha\beta} a_\alpha b_\beta$$

By the outer product notation $\underset{\sim}{a} \otimes \underset{\sim}{b}$ we will mean

$$(\underset{\sim}{a} \otimes \underset{\sim}{b})_{\alpha\beta} \equiv a_\alpha b_\beta$$

For a second order tensor $\Delta_{\alpha\beta}$ and a four vector we will mean by the notation $\underline{a} \times \Delta$ the second order tensor

$$(\underline{a} \times \Delta)_{\alpha\beta} \equiv \overset{*}{\epsilon}_{\alpha\delta\gamma} a^{\delta} \Delta^{\gamma}_{\beta}$$

and by $\Delta \times \underline{a}$ the second order tensor

$$(\Delta \times \underline{a})_{\alpha\beta} \equiv \Delta_{\alpha}^{\gamma} \overset{*}{\epsilon}_{\beta\gamma\delta} a^{\delta}$$

where we raise and lower Greek indices with the metric $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})$. If $\Delta_{\alpha\beta}$ is a second order tensor and a^{α} a vector by $\Delta \underline{a}$ we mean the vector

$$(\Delta \underline{a})_{\alpha} = \Delta_{\alpha\beta} a^{\beta}$$

For two vectors a^{α}, b^{α} the inner product $\underline{a} \cdot \underline{b}$ is defined by

$$\underline{a} \cdot \underline{b} = a^{\alpha} b_{\alpha}$$

Units

For simplicity the speed of light c is set equal to unity, $c = 1$. In the electromagnetic section of this article rationalized natural units are employed. Accordingly we take the dielectric constant of free space $\epsilon_{o} = 1$ and since $c = 1$, the permeability of free space $\mu_{o} = 1$.

CHAPTER I

RELATIVISTIC KINEMATICS OF A CONTINUOUS MEDIUM

Motion

Although some accounts on the relativistic treatment of the dynamics of a continuous medium may be found in various texts on relativity (cf. Møller [14], Tolman [27], Bergmann [18], Landau and Lifshitz [15, 28], Synge [12]), very little discussion has been directed towards a relativistically invariant kinematics of a continuous body. The present chapter is, therefore, devoted to a discussion of deformation and motion (kinematics) from the point of view of the special theory of relativity.

In classical continuum mechanics the deformation of a body is described by specifying the mapping of one configuration of a body, B , onto another configuration, B_t .
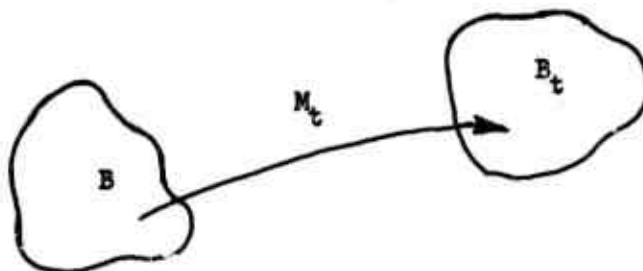

Fig. 1. Deformation

The configuration B , called the reference state, (usually taken as the undeformed body) can be made explicit by specifying the rectangular coordinates X^K , $(K = 1, 2, 3)$ of a set of material points, (B) , in

throughout body B except possibly some singular surfaces, lines and points. In fact, unless otherwise stated we shall not only assume the validity of (1.3) throughout B but also the existence of partial derivatives of (1.1) and (1.2) as many times as we need.

In classical mechanics the parameter t is identified as time. Here it is considered as the fourth coordinate in space-time $x^\mu = (x^k, ct)$, (set c = 1) which has the metric

$$(1.4) \quad \gamma^{\mu\nu} = \gamma_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (c = 1)$$

The following convention is used: The small Latin subscripts or superscripts will always assume the values 1, 2, 3 . They will signify the spatial coordinates of the space-time of events. The small Greek subscripts or superscripts will always assume the values 1, 2, 3, 4 . They will designate the coordinates of space-time. They will be raised and lowered by the metric $\gamma^{\mu\nu}$. The large Latin subscripts or superscripts assume the values 1, 2, 3 and will be raised and lowered by the Kronecker delta δ_{KL} . They will signify the coordinates of the reference state.

Similar to (1.2) the inverse motion of a continuous medium may be described by three functions

$$(1.5) \quad X^K = X^K(x^\mu) \quad (K = 1, 2, 3 \ ; \ \mu = 1, 2, 3, 4)$$

The domain of $X^K()$ is the material tube swept out by the body: $D = (x^\mu : \ x^k \in B_t \ , \ 0 < t < \infty)$ in space - time. The range of $X^K()$ is the set (B). In the following it will be implicitly assumed that $\underline{x} \in D$ and $\underline{X} \in B$ and no further reference will be made to this fact.

The three functions $X^K()$ are assumed to be invariant functions of x^μ under the group of homogeneous Lorentz transformations, Λ ,

$$(1.6) \quad \begin{array}{l} \hat{x}^\mu = \Lambda^\mu{}_\nu x^\nu \\[1em] \Lambda^\alpha{}_\nu \Lambda_\beta{}^\nu = \Lambda_\nu{}^\alpha \Lambda^\nu{}_\beta = \delta^\alpha{}_\beta \end{array}$$

that is:

$$(1.7) \quad X^K(x^\mu) = X^K(\hat{x}^\mu)$$

Mathematically this is all that is needed to develop a relativistically invariant kinematics. However, for a physical theory, some meaning must be attached to the functions $X^K()$. Physically, consider an undeformed body B which before deformation is at rest in some frame. In that frame the body B is described by a set (B) in the three dimensional space with coordinates X^K . The three

functions $X^K()$ describe the mapping of the set D onto B. It will be assumed that such a reference state exists. Thus X^K can be used to describe lengths and angles in that frame.

It is also assumed that (1.5) is invertible in the first three variables x^k, that is

$$x^k = x^k(X^K, x^4)$$

(1.8) $\quad \det (x^k_{,K}) \neq 0$

$$x^k_{,K} \equiv \frac{\partial x^k(X^K, x^4)}{\partial X^K}$$

World Lines, Velocity, Deformation Gradients

It is now meaningful to define the world line of a particle A^K by the curve in space - time defined by

(1.9) $\quad A^K = X^K(x^\mu)$

The particle velocity is defined as

(1.10) $\quad v^k = \frac{\partial x^k(X^K, x^4)}{\partial x^4} \Bigg|_{X^K}$

The world velocity vector is

$$u^\alpha(x^K, x^4) \equiv \left(\frac{v^k}{\sqrt{1-v^2}} \; , \; \frac{1}{\sqrt{1-v^2}} \right)$$

(1.11)
$$u^\alpha(x^\beta) \equiv u^\alpha(x^K(x^\beta), x^4)$$

$$u^\alpha u_\alpha = -1 \quad , \quad v^2 \equiv \delta_{k\ell} v^k v^\ell$$

Since $u^\alpha(x^\beta)$ is the unit tangent to the world line passing through x^β, u^α transforms as a four vector under Lorentz transformation.

The following notation is convenient. Consider a tensor function $\phi = \phi(x^\mu)$ we define

(1.12)
$$\phi_{,\beta} \equiv \frac{\partial \phi(x^\mu)}{\partial x^\beta} \bigg|_{x^\mu \neq x^\beta}$$

From (1.8) the function ϕ can also be considered as a function of x^K, x^4. The same symbol ϕ will be used for both $\phi(x^\mu)$ and $\phi(x^k(x^K, x^4), x^4) = \phi(x^K, x^4)$. We define $\phi_{,K}$ and $\frac{\partial \phi}{\partial x^4}$ by

(1.13)
$$\phi_{,K} = \frac{\partial \phi(x^L, x^4)}{\partial x^K} \bigg|_{x^L \neq x^K \, ; \, x^4}$$

$$\frac{\partial \phi}{\partial x^4} = \frac{\partial \phi(x^K, x^4)}{\partial x^4} \bigg|_{x^K}$$

Obviously the four numbers $(x^K_{,\beta})$ (K fixed) are components of a four vector. Using (1.7), (1.8) and (1.10) one sees that

$$0 = x^k_{,4} = x^k_{,K} x^K_{,4} + v^k$$

$$\delta^k_\ell = x^k_{,\ell} = x^k_{,K} x^K_{,\ell}$$

$$\delta^K_L = x^K_{,L} = x^K_{,\ell} x^\ell_{,L}$$

Thus the following useful identities are derived:

$$x^k{}_{,K} \; X^K{}_{,\ell} \; = \; \delta^k{}_{\ell}$$

$$(1.14) \quad X^K{}_{,\ell} \; x^{\ell}{}_{,L} \; = \; \delta^K{}_{L}$$

$$X^K{}_{,4} \; = \; - \, v^k \, X^K{}_{,k}$$

The last equation yields the identity

$$(1.15) \quad u^{\alpha} \, X^K{}_{,\alpha} \; = \; 0$$

Material Surfaces and Volumes

It is convenient to introduce the invariant derivative

$$(1.16) \quad D \, \phi \; \equiv \; u^{\alpha} \, \phi{}_{,\alpha}$$

If ϕ is a tensor under Lorentz transformations, $D\phi$ is a tensor under Lorentz transformations. This is the relativistic generalization of the material derivative $\frac{D}{Dt}$, in fact

$$D \; = \; \frac{1}{\sqrt{1-v^2}} \qquad \frac{D}{Dt}$$

The concept of a material surface is important for the formulation of the dynamical laws of a continuous medium.

<u>Def.</u>   A surface $f(x^\alpha) = 0$ is a material surface if and only if

(1.17) $f(x^k(x^K, x^4), x^4) = F(x^K)$

A necessary and sufficient condition for the surface $f(x^\alpha) = 0$
to be material is that

(1.18) $Df = 0$

The proof is obvious.

From (1.17) or (1.18) it follows that if the surface determined
by $f(x^\alpha) = 0$ is a material surface, the surface $f(\hat{x}^\alpha) = 0$, where
$\hat{x}^\alpha$ is related to x^α by (1.6), is also a material surface. That is,
the concept of a material surface is invariant under Lorentz trans-
formations.

A material volume is a three dimensional volume containing
material points.

Decomposition of Vectors

A four vector f^α is called space-like if $f^\alpha f_\alpha > 0$, null if
$f^\alpha f_\alpha = 0$, and time-like if $f_\alpha f^\alpha < 0$. From (1.11) the four vector
u^α is a unit time-like vector. In the sequel it will be found convenient

to decompose all vectors and tensors into space-like and time-like components.

An arbitrary four vector F^α can be decomposed into a time-like component parallel to u^α and a space-like component perpendicular to u^α. To this end we introduce the projectors $s^\alpha{}_\beta$

$$(1.19) \quad \begin{aligned} s^\alpha{}_\beta &= \delta^\alpha{}_\beta + u^\alpha u_\beta \\ s_\beta{}^\alpha &\equiv \gamma_{\beta\sigma} \gamma^{\alpha\tau} s^\sigma{}_\tau = \delta^\alpha{}_\beta + u^\alpha u_\beta = s^\alpha{}_\beta \end{aligned}$$

The projectors satisfy the identities

$$(1.20) \quad \begin{aligned} s^\alpha{}_\beta s^\beta{}_\gamma &= s^\alpha{}_\gamma \\ u_\beta s^\beta{}_\alpha &= u^\beta s^\alpha{}_\beta = 0 \end{aligned}$$

In general the four vector F^α can be uniquely written in the form

$$(1.21) \quad \begin{aligned} F^\alpha &= f^\alpha + u^\alpha f \\ f^\alpha u_\alpha &= 0 \end{aligned}$$

From (1.20) and (1.21) it follows that

$$(1.22) \quad f = -F^\alpha u_\alpha$$

$$(1.23) \quad f^\alpha = s^\alpha{}_\beta F^\beta$$

It can be easily seen that

$$f^\alpha f_\alpha \geq 0$$

That is f^α is space-like or a null vector (light signal).

From (1.15) and (1.19) it follows that

$$(1.24) \qquad \delta^\mu_{\ \alpha} \, x^K_{\ ,\mu} = x^K_{\ ,\alpha}$$

Therefore the vectors $x^K_{\ ,\mu}$ are space-like.

Deformation Tensors

The deformation of bodies can be described by the deformation gradients $x^K_{\ ,\mu}$. The invariant strain measure (Toupin [11])

$$(1.25) \qquad \overset{-1}{c}{}^{KL} = \gamma^{\alpha\beta} \, x^K_{\ ,\alpha} \, x^L_{\ ,\beta}$$

$$\overset{-1}{c}{}^{KL} = \overset{-1}{c}{}^{LK}$$

is the relativistic generalization of the inverse Green deformation tensor. These are six Lorentz invariant scalars. From $(1.14)_3$

$$(1.26) \qquad \overset{-1}{c}{}^{KL} = \delta^{ij} x^K_{\ ,i} \, x^L_{\ ,j} - x^K_{\ ,4} x^L_{\ ,4} = (\delta^{ij} - v^i v^j) \, x^K_{\ ,j} \, x^L_{\ ,i}$$

From (1.26) it is clear that $\overset{-1}{c}{}^{KL}$ is the inverse Green deformation tensor of a local instantaneous rest frame.

From (1.26) we calculate

$$\det (\overset{-1}{C}{}^{KL}) = \det (\delta^{ij} - v^i v^j) [\det (x^K{}_{,i})]^2$$

(1.27)
$$= (1 - v^2) [\det (x^K{}_{,i})]^2 > 0$$

The inequality follows from $(1.8)_2$ and the physical assumption that no body can move faster than the speed of light $(v^2 < 1)$. The matrix $\overset{-1}{C}{}^{KL}$ is invertible and it is easily verified that

(1.28) $C_{KL} = (\delta_{ij} + u_i u_j) x^i{}_{,K} x^j{}_{,L}$

is the inverse of $\overset{-1}{C}{}^{KL}$, i.e.,

(1.29) $C_{KM} \overset{-1}{C}{}^{ML} = \overset{-1}{C}{}^{LM} C_{MK} = \delta^L{}_K$

The six quantities C_{KL} are scalar invariants under the Lorentz group since they are invariant functions of the invariants $\overset{-1}{C}{}^{KL}$.

The matrix C_{KL} has a very simple physical meaning: it gives the changes in length and angle due to the deformation as viewed by an observer in a local instantaneous rest frame. To an observer in a local instantaneous rest frame $v^k = 0$, $u^k = 0$ and (1.26) and (1.28) revert to the forms known to us from non-relativistic continuum mechanics for which the connection of C_{KL} to length and angle changes is well known (cf. Eringen [7 , Art. 7]).

The invariant strain tensor E_{KL} is defined as

$$(1.30) \quad 2E_{KL} = C_{KL} - G_{KL}$$

The strain measures which have been introduced so far are the relativistic generalization of the Lagrangian strain measures of classical continuum mechanics. To introduce the Eulerian strain measures, note that

$$(1.31) \quad C_{KL} = 8_{k\ell} \, x^k_{,K} \, x^\ell_{,L} = 8_k^\alpha \, 8_{\alpha\ell} \, x^k_{,K} \, x^\ell_{,L}$$

This may also be expressed as

$$(1.32) \quad C_{KL} = \gamma_{\alpha\beta} \, x^\alpha_{\ K} \, x^\beta_{\ L}$$

where

$$(1.33) \quad x^\alpha_{\ K} \equiv 8^\alpha_k \, x^k_{\ ,K}$$

With the help of (1.14) and (1.20) it is simple to show that

$$(1.34) \quad
\begin{aligned}
8^\alpha_{\ \beta} \, x^\beta_{\ K} &= x^\alpha_{\ K} \\
x^K_{\ ,\beta} \, x^\beta_{\ L} &= \delta^K_{\ L} \\
x^\alpha_{\ K} \, x^K_{\ ,\beta} &= 8^\alpha_{\ \beta} \\
u_\alpha \, x^\alpha_{\ K} &= 0
\end{aligned}$$

<u>Theorem</u>: The quantities $x^\alpha_{\;K}$ (K fixed) are four vectors.

    <u>Proof</u>:

$$\hat{x}^\alpha \;=\; \Lambda^\alpha_{\;\beta}\, x^\beta$$

From (1.34) we have

$$x^\alpha_{\;K}\, X^K_{\;,\beta} \;=\; S^\alpha_{\;\beta}$$

Since $S^\alpha_{\;\beta}$ is a tensor

$$\hat{S}^\alpha_{\;\beta} \;=\; \Lambda^\alpha_{\;\gamma}\, \Lambda_\beta^{\;\delta}\, S^\gamma_{\;\delta}$$

Thus

$$\Lambda^\alpha_{\;\gamma}\, \Lambda_\beta^{\;\delta}\, x^\gamma_{\;K}\, X^K_{\;,\delta} \;=\; \hat{S}^\alpha_{\;\beta} \qquad , \qquad \Lambda^\alpha_{\;\gamma}\, x^\gamma_{\;K}\, \hat{X}^K_{\;,\beta} \;=\; \hat{S}^\alpha_{\;\beta}$$

since

$$\hat{X}^K_{\;,\beta} \;=\; \Lambda_\beta^{\;\gamma}\, X^K_{\;,\gamma}$$

But from $(1.34)_2$

$$\hat{X}^K_{\;,\beta}\, \hat{x}^\beta_{\;L} \;=\; \delta^K_{\;L}$$

Consequently

$$\hat{S}^\alpha_{\;\beta}\, \hat{x}^\beta_{\;L} \;=\; \Lambda^\alpha_{\;\gamma}\, x^\gamma_{\;K}\, \hat{X}^K_{\;,\beta}\, \hat{x}^\beta_{\;L} \;=\; \Lambda^\alpha_{\;\gamma}\, x^\gamma_{\;L}$$

or in view of $(1.34)_1$

$$\hat{x}{}^{\alpha}{}_{K} = \Lambda^{\alpha}{}_{\beta} \, x^{\beta}{}_{K}$$

which is the proof. Thus $x^{\beta}{}_{K}$ is a two point tensor. (The fact that $x^{\beta}{}_{K}$ is a vector under change of X^{K}, with β fixed, is more or less obvious).

The relativistic generalizations of the Cauchy deformation tensors are:

$$(1.35) \qquad c_{\alpha\beta} \equiv \delta_{KL} \, X^{K}{}_{,\alpha} \, X^{L}{}_{,\beta}$$

$$(1.36) \qquad \overset{-1}{c}{}^{\alpha\beta} \equiv \delta^{KL} \, x^{\alpha}{}_{K} \, x^{\beta}{}_{L} = g^{\alpha}{}_{k} \, g^{\beta}{}_{\ell} \, \overset{-1}{c}{}^{k\ell}$$

where

$$\overset{-1}{c}{}^{k\ell} = \delta^{KL} \, x^{k}{}_{,K} \, x^{\ell}{}_{,L}$$

From these and (1.34) we can establish the identity

$$(1.37) \qquad c^{\alpha\gamma} \, \overset{-1}{c}{}_{\gamma\beta} = g^{\alpha}{}_{\beta}$$

The following interpretation can be given to $c_{\alpha\beta}$. An infinitesimal spatial measurement by an observer in a Lorentz frame is the separation of two simultaneous events, (x^{k}, x^{4}) and $(x^{k} + dx^{k}, x^{4})$.

Since two events simultaneous in one Lorentz frame are not necessarily simultaneous in another Lorentz frame, the act of spatial measurement is not an invariant operation. The difference between two events x^β and $x^\beta + dx^\beta$ is a spatial measurement in the local instantaneous rest frame at x^β if and only if

$$u_\beta \, dx^\beta = 0 \qquad \text{at } \underset{\sim}{x}$$

If the two pairs of events $[x^\beta, x^\beta + \overset{1}{dx}{}^\beta]$ and $[x^\beta, x^\beta + \overset{2}{dx}{}^\beta]$ are spatial measurements of two material lines in the local instantaneous rest frame at $\underset{\sim}{x}$, then

$$\frac{c_{\alpha\beta} \, \overset{1}{dx}{}^\alpha \, \overset{2}{dx}{}^\beta}{\sqrt{c_{\alpha\beta} \, \overset{1}{dx}{}^\alpha \, \overset{1}{dx}{}^\beta} \, \sqrt{c_{\alpha\beta} \, \overset{2}{dx}{}^\alpha \, \overset{2}{dx}{}^\beta}}$$

is the cosine between the material lines before deformation.

Invariants of Strain, Volume Change₁

By use of the defining equations (1.35), (1.36) of the Cauchy deformation tensors and (1.25) and (1.32) of Green deformation measures the relations of invariants of $\underset{\sim}{c}^{-1}$ and $\underset{\sim}{c}$ to those of material strain measures can be deduced. Thus,

$$(1.38) \quad \operatorname{tr} \underset{\sim}{c} = \operatorname{tr} \overset{-1}{\underset{\sim}{c}} \quad , \quad \operatorname{tr} \underset{\sim}{c}^2 = \operatorname{tr} (\overset{1}{c})^2 \quad , \quad \operatorname{tr} \underset{\sim}{c}^3 = \operatorname{tr} (\overset{1}{\underset{\sim}{c}})^3$$

$$(1.39) \quad \operatorname{tr} \underset{\sim}{c} = \operatorname{tr} \overset{-1}{\underset{\sim}{c}} \quad , \quad \operatorname{tr} \underset{\sim}{c}^2 = \operatorname{tr} (\overset{-1}{c})^2 \quad , \quad \operatorname{tr} \underset{\sim}{c}^3 = \operatorname{tr} (\overset{-1}{c})^3$$

where we used the notation

$$\operatorname{tr} \overset{-1}{\underset{\sim}{c}} = \overset{-1}{c}{}^{\alpha}{}_{\alpha} \quad , \quad \operatorname{tr} \underset{\sim}{c} = c^{\alpha}{}_{\alpha} \quad , \quad \operatorname{tr} \underset{\sim}{c}^2 = c^{\alpha}{}_{\beta} \, c^{\beta}{}_{\alpha} \quad , \quad \ldots$$

$$\operatorname{tr} \overset{-1}{\underset{\sim}{c}} = \overset{-1}{c}{}^{K}{}_{K} \quad , \quad \operatorname{tr} \underset{\sim}{c} = c^{K}{}_{K} \quad , \quad \ldots$$

Suppose that the material volume dV is deformed into dv in the Lorentz frame x^{α}. In an instantaneous rest frame coinciding with $\tilde{x}^{\alpha}$ the deformed volume is dv_0 given by

$$(1.40) \quad \frac{dv_0}{dV} = J$$

where J is the Jacobian defined by

$$(1.41) \quad J = \frac{1}{6} \epsilon^{\alpha\beta\gamma\delta} \, \epsilon_{KLM} \, x^{K}{}_{,\alpha} \, x^{L}{}_{,\beta} \, x^{M}{}_{,\gamma} \, u_{\delta}$$

Alternatively

$$(1.42) \quad \frac{dV}{dv_0} = j$$

where

$$(1.43) \qquad j = \frac{1}{6} \, \epsilon^{KLM} \, \epsilon_{\alpha\beta\gamma\delta} \, x^{\alpha}{}_{K} \, x^{\beta}{}_{L} \, x^{\gamma}{}_{M} \, u^{\delta} = \frac{1}{J}$$

From (1.41) and (1.43) we can also deduce the identities

$$(1.44) \qquad \epsilon_{\alpha\beta\gamma\delta} \, u^{\delta} = j \, \epsilon_{KLM} \, X^{K}{}_{,\alpha} \, X^{L}{}_{,\beta} \, X^{M}{}_{,\gamma}$$

$$(1.45) \qquad \epsilon_{KLM} = J \, \epsilon_{\alpha\beta\gamma\delta} \, x^{\alpha}{}_{K} \, x^{\beta}{}_{L} \, x^{\gamma}{}_{M} \, u^{\delta}$$

The above identities are useful in the reduction of constitutive equations for isotropic materials.

Compatibility Conditions

In classical continuum mechanics the compatibility conditions for the deformations tensors are deduced from the requirement that the space remain Euclidean. This is not the case in the relativistic theory presented here. Suppose the deformation tensor $c_{\alpha\beta}$ is given at each event $(\underline{x})$, then the system of equations for $X^{K}(\underline{x})$

$$(1.46) \qquad c_{\alpha\beta} = \delta_{KL} \, X^{K}{}_{,\alpha} \, X^{L}{}_{,\beta}$$

is over determined. Differentiate (1.46) with respect to x^{γ}.

(1.47) $c_{\alpha\beta,\gamma} = \delta_{KL} x^K_{,\alpha\gamma} x^L_{,\beta} + \delta_{KL} x^K_{,\alpha} x^L_{,\beta\gamma}$

Define a Christoffel symbol of the second kind by

(1.48) $[\alpha\beta,\gamma]_{\mathfrak{L}} = \frac{1}{2}[c_{\alpha\gamma,\beta} + c_{\beta\gamma,\alpha} - c_{\alpha\beta,\gamma}]$

Then using (1.47), (1.48) we see that

(1.49) $[\alpha\beta,\gamma]_{\mathfrak{L}} = \delta_{KL} x^K_{,\alpha\beta} x^L_{,\gamma}$

But (1.49) is equivalent to

(1.50) $x^K_{,\alpha\beta} = x^K_{,\gamma} \{^{\gamma}_{\alpha\ \beta}\}$

where

(1.51) $\{^{\gamma}_{\alpha\ \beta}\} = \overset{-1}{c}^{\gamma\delta} [\alpha\beta,\delta]_{\mathfrak{L}}$

is called Christoffel's symbol of the second kind.

The system (1.50) is completely integrable if and only if

(1.52)

$$x^K_{,\gamma} \{^{\gamma}_{\alpha\ \beta}\}_{,\delta} + x^K_{,\gamma} \{^{\gamma}_{\epsilon\ \delta}\} \{^{\epsilon}_{\alpha\ \beta}\}$$

$$= x^K_{,\gamma} \{^{\gamma}_{\alpha\ \delta}\}_{,\beta} + x^K_{,\gamma} \{^{\gamma}_{\epsilon\ \beta}\} \{^{\epsilon}_{\alpha\ \delta}\}$$

(See Eisenhart [29], p. 1 and pp. 186-188.)

Thus a necessary and sufficient condition that (1.50) is completely integrable is that the "curvature" tensor vanish, i.e.

$$(1.53) \qquad \overset{*}{R}{}^{\alpha}_{\beta\gamma\delta} = 0$$

where

$$(1.54) \qquad \overset{*}{R}{}^{\alpha}_{\beta\gamma\delta} = S^{\alpha}_{\tau} \left[\left\{ {}^{\tau}_{\beta\ \delta} \right\}_{,\gamma} - \left\{ {}^{\tau}_{\beta\ \gamma} \right\}_{,\delta} + \left\{ {}^{\tau}_{\sigma\ \gamma} \right\} \left\{ {}^{\sigma}_{\beta\ \delta} \right\} - \left\{ {}^{\tau}_{\sigma\ \delta} \right\} \left\{ {}^{\sigma}_{\beta\ \gamma} \right\} \right]$$

This is the relativistic generalization of the compatibility conditions for $c_{\alpha\beta}$.

Rates of Deformation Gradients and Strain

The following lemma is useful in the kinematics of continuous media

$$(1.55) \qquad DX^{K}_{,\beta} = - u^{\alpha}_{,\beta} X^{K}_{,\alpha}$$

To prove this we note from the definition (1.16) of the operator D that

$$DX^{K}_{,\beta} = u^{\alpha} X^{K}_{,\beta\alpha} = (u^{\alpha} X^{K}_{,\alpha})_{,\beta} - u^{\alpha}_{,\beta} X^{K}_{,\alpha}$$

According to (1.15) the first term on the right hand side is zero and

we get (1.55).

A dual to (1.55) is

$$(1.56) \qquad S^{\alpha}_{\ \beta} \, D \, x^{\beta}_{\ K} \ = \ x^{\beta}_{\ K} \, u^{\alpha}_{\ ,\beta}$$

For the proof of this we use $(1.34)_2$.

$$D(x^{K}_{\ ,\beta} \, x^{\beta}_{\ L}) \ = \ D \, x^{K}_{\ ,\beta} \, x^{\beta}_{\ L} + x^{K}_{\ ,\beta} \, D \, x^{\beta}_{\ L} \ = \ 0$$

Multiplying this by $x^{\alpha}_{\ K}$ and using (1.55) and (1.34) we obtain (1.56). It is useful to introduce

$$(1.57) \qquad \overset{*}{u}{}^{\alpha}_{\ \beta} \ = \ S^{\mu}_{\ \beta} \, u^{\alpha}_{\ ,\mu}$$

in terms of which (1.56) may be expressed as

$$(1.58) \qquad S^{\alpha}_{\ \beta} \, D \, x^{\beta}_{\ K} \ = \ \overset{*}{u}{}^{\alpha}_{\ \beta} \, x^{\beta}_{\ K}$$

Using $(1.34)_4$ we have

$$u_{\alpha} \, D \, x^{\alpha}_{\ K} \ = \ - \, x^{\alpha}_{\ K} \, D \, u_{\alpha}$$

When this and the expression $(1.19)_1$ are employed in (1.58) we find

$$(1.59) \qquad D \, x^{\alpha}_{\ K} \ = \ \overset{*}{u}{}^{\alpha}_{\ \beta} \, x^{\beta}_{\ K} + u^{\alpha} \, x^{\beta}_{\ K} \, D \, u_{\beta}$$

The invariant derivative of C_{KL} may now be calculated by

$$(1.60) \qquad D\, C_{KL} = 2\, \overset{*}{d}_{\alpha\beta}\, x^{\alpha}{}_{K}\, x^{\beta}{}_{L}$$

where

$$(1.61) \qquad \overset{*}{d}_{\alpha\beta} = \frac{1}{2}(\overset{*}{u}_{\alpha\beta} + \overset{*}{u}_{\beta\alpha}) = \overset{*}{u}_{(\alpha\beta)}$$

The proof of (1.60) is immediate from taking the invariant derivative of (1.32) and using (1.59).

The tensor $\overset{*}{d}$ is the relativistic generalization of the deformation rate tensor. It is clear that for locally rigid motions $D\, C_{KL} = 0$; consequently, we have

Theorem: A necessary and sufficient condition for locally rigid motion is

$$(1.62) \qquad \overset{*}{d}_{\alpha\beta} = 0$$

This is the relativistic generalization of the Killing's theorem of differential geometry.

The relativistic generalization of spin is defined by

$$(1.63) \qquad \overset{*}{\omega}_{\alpha\beta} = \frac{1}{2}(\overset{*}{u}_{\alpha\beta} - \overset{*}{u}_{\beta\alpha}) = \overset{*}{u}_{[\alpha\beta]}$$

Upon adding this to (1.61) we have

$$(1.64) \qquad \overset{*}{u}_{\alpha\beta} = \overset{*}{d}_{\alpha\beta} + \overset{*}{\omega}_{\alpha\beta}$$

The invariant derivatives of various strain measures and other tensors can be calculated by use of the apparatus set above. Here we give

$$(1.65) \qquad D c_{\alpha\beta} + u^{\gamma}{}_{,\beta} e_{\alpha\gamma} + u^{\gamma}{}_{,\alpha} e_{\gamma\beta} = 0$$

which is obtained by taking the invariant derivative of (1.35) and using (1.55). This expression will be found useful in the treatment of isotropic materials.

Another useful result is the identity

$$(1.66) \qquad Dj = - j \overset{*}{u}{}^{\alpha}{}_{\alpha}$$

which can be proven by (1.43) and (1.59).

CHAPTER II

RELATIVISTIC BALANCE LAWS FOR A CONTINUOUS MEDIUM

General Balance Laws

In nonrelativistic continuum mechanics the balance laws are written in the form

$$(2.1) \qquad \frac{d}{dt} \int_{V(t)} \phi \, dv = \int_{\mathcal{S}(t)} \underline{i} \cdot d\underline{a} + \int_{V(t)} s \, dv$$

where ϕ is some quantity whose influx is $\underline{i}$ and whose supply is s. Equation (2.1) holds for an arbitrary material volume $V(t)$ whose enclosing surface is $\mathcal{S}(t)$. This equation can be integrated from t_1 to t_2 for arbitrary $t_1 < t_2$ to obtain

$$(2.2) \qquad \int_{V(t_2)} \phi \, dv - \int_{V(t_1)} \phi \, dv - \int_{t_1}^{t_2} dt \int_{\mathcal{S}(t)} \underline{i} \cdot d\underline{a} = \int_{t_1}^{t_2} dt \int_{V(t)} s \, dv$$

Under the proper smoothness conditions the left hand side of (2.2) can be replaced by an integral over the three dimensional circuit in space-time enclosing the four dimensional volume (called material tube) swept out by the material volume $V(t)$ in the interval (t_1, t_2). Hence

$$\oint \phi^\alpha \, ds_{3\alpha} = \int s \, dv_4$$

$$(2.3) \qquad dv_4 = dv \, dx^4$$

where

$$(2.4) \quad \phi^k = -i^k + \phi \, v^k \quad (k = 1, 2, 3)$$

$$\phi^4 = \phi$$

and $ds_{3\alpha}$ and dv_4 respectively denote the elements of three dimensional oriented surface and the volume.

To obtain (2.3) we recall that a material volume in space-time is a three dimensional surface with parametization

$$(2.5) \quad x^k = x^k (X^K, \tau) \qquad K = 1, 2, 3$$

$$x^4 = \tau \qquad (\tau = \text{constant})$$

Also a three dimensional surface in space-time swept out by a material surface can be parametrized by

$$(2.6) \quad x^k = x^k (X^K (u_1, u_2), u_3) \quad , \quad x^4 = u_3$$

so that the oriented three-dimensional surface element $ds_{3\mu}$ can be expressed by

$$(2.7) \quad ds_{3\mu} = \epsilon_{\mu\alpha\beta\gamma} \frac{\partial x}{\partial u_1}^{[\alpha} \frac{\partial x}{\partial u_2}^{\beta} \frac{\partial x}{\partial u_3}^{\gamma]} du^1 \, du^2 \, du^3$$

where a bracket enclosing indices indicates skew-symmetry as defined by

$$A^{[\alpha\beta\gamma]} \equiv \frac{1}{6} (A^{\alpha\beta\gamma} + A^{\gamma\alpha\beta} + A^{\beta\gamma\alpha} - A^{\beta\alpha\gamma} - A^{\gamma\beta\alpha} - A^{\alpha\gamma\beta})$$

The quantity $\bullet$ will be said to be conserved if $i^k = 0$, $s = 0$. From (2.3) and (2.4) the quantity $\bullet$ is conserved if and only if

$$\oint \bullet^\alpha \, d s_{3\alpha} = 0$$

(2.8)
$$\bullet^\alpha = \bullet_0 \, u^\alpha$$

$$\bullet_0 = -\bullet^\alpha u_\alpha = \bullet \sqrt{1 - v^2}$$

So far all this holds for either classical continuum mechanics or relativistic continuum mechanics. The only distinction between them is the group of transformations in space-time for which the integral

$$(2.9) \qquad \phi\,[\bullet,\, s_3] = \int_{\varepsilon_3} \bullet^\alpha \, ds_{3\alpha}$$

has a specified transformation law -- t..e Galilean group for classical mechanics or the Lorentz group for relativistic mechanics. In the following subsections the laws of conservation of mass, balance of energy-momentum, balance of angular momentum, and the second law of thermodynamics are formulated in a relativistically invariant manner. In general, these laws will have the form (2.3) with a transformation law for a set of quantities of the form (2.9) specified for the Lorentz group.

Conservation of Particle Number

In nonrelativistic mechanics the postulate of conservation
of mass and the conservation of the number of particles are equivalent
for nonreacting systems. It is well known that this is not the case
in relativistic mechanics. In relativity the mass is closely related
to the energy of a particle. In general the mass, even the rest
mass, of a particle varies. The concept of the non-creation or
indestructability of particles still remains valid in classical
relativistic theories of nonreacting substances (the word classical is
used in opposition to relativistic quantum theories where the idea
of creating or destroying particles is essential). It is assumed that
the number of particles contained in a material volume $V(t)$ is
a constant of the motion.

To formulate* the law of conservation of particle number we
assign, to every three dimensional surface s_3 , a positive scalar
$N[s_3]$ of the form

---

*
Each law formulated in these subsections is deduced along the
lines leading from (2.1) to (2.3) from the corresponding classical
law. Though the correspondence (2.4) is usually listed, it should be
noted that this is not needed for a relativistic formulation of the
laws of mechanics. Only the form of the equations is necessary. For
a physical feeling and interpretation of the various components of
space-time tensors and vectors the correspondence (2.4) is indispensable
and will usually be given.

$$(2.10) \qquad N[s_3] = \int_{s_3} n^\alpha \, ds_{3\alpha}$$

which is postulated to be invariant under the group of Lorentz transformations. The law of conservation of particle-number states that for every material volume, $V(t)$, $N[s_3]$ is conserved. From (2.8) this is equivalent to

$$\oint n^\alpha \, ds_{3\alpha} = 0$$

$$(2.11) \qquad n^\alpha = n_0 \, u^\alpha$$

$$n_0 = - n^\alpha \, u_\alpha > 0$$

where the integral is over an arbitrary material tube. The scalar n_0 is called the rest frame particle number and is related to the particle number , n , by

$$(2.12) \qquad n = \frac{n_0}{\sqrt{1-v^2}}$$

The use of Green-Gauss theorem in the integral equation $(2.11)_1$ leads, in the usual way, to the following differential equation and jump condition:

$$(2.13) \qquad n^\alpha{}_{,\alpha} = 0 \quad , \quad [n^\alpha] \, \Sigma_{,\alpha} \quad \text{across} \quad \Sigma \, (x^\mu) = 0$$

where $[f] = f^+ - f^-$ denotes the jump of f at any discontinuity surface $\Sigma(x^\mu) = 0$.

It is convenient to write (2.13) as

$$(2.14) \qquad Dn_o + n_o \overset{*\beta}{u}{}_\beta = 0$$

where the following identity has been employed

$$(2.15) \qquad \overset{*\beta}{u}{}_\beta = u^\beta{}_{,\beta}$$

By using (1.66) one can show that (2.14) has a solution of the form

$$(2.16) \qquad n_o = \overset{o}{n}{}_o \, j$$

where $\overset{o}{n}{}_o$ is a function of X^K.

In the following, various scalars ϕ will appear in the balance equations. It is convenient to define another scalar ϕ_o related to ϕ by

$$(2.17) \qquad \phi = n_o \phi_o$$

From (2.14)

$$(2.18) \qquad (\phi u^\beta)_{,\beta} = n_o \, D\phi_o$$

This identity is used frequently in the formulation of the remaining balance laws.

Balance of Energy-Momentum

In relativity theory the laws of balance of momentum and energy
are closely connected. They are the components of a four vector.
For physical reasons it is expected that the time rate of change of
the momentum is equal to the forces applied to the body and that
the time rate of change of the energy is equal to the work done on
the body. In a classical nonrelativistic theory of a continuous
medium these laws take the form: the time rate of change of the mo-
mentum contained in a material volume is equal to the sum of the
forces applied to the volume. These forces can be decomposed into
two parts, one arising from surface tractions and the other from the
body forces. The time rate of change of the energy contained in a
material volume is equal to the heat flow through the surface plus
a heat supply inside the volume plus the work done by the surface
tractions and body forces. In nonrelativistic mechanics the momentum
is usually due to the motion (kinetic momentum) and the energy is
the sum of the kinetic energy and the internal energy. In relativistic
theories it is possible to have momentum of nonkinetic origin. There
is no advantage in decomposing, a priori, the energy momentum tensor
into kinetic, thermodynamic, and other parts.

The energy-momentum of a material body is determined by assigning
to every three-dimensional subspace, s_3 , of a four-dimensional
material tube four functions $P^{\mu}[s_3]$ which are components of a four
vector under Lorentz transformations. The functions $P^{\mu}[s_3]$ have

the form:

$$(2.19) \quad P^{\mu}[s_3] = \int_{s_3} T^{\mu\nu} \, ds_{3\nu}$$

where $T^{\mu\nu}$ is a second order tensor. The balance of energy-momentum states that for every material tube we have

$$(2.20) \quad \oint T^{\mu\nu} \, ds_{3\nu} = \int f^{\mu} \, dv_4$$

where the four vector f^{μ} represents the body forces and energy supply per unit volume.

In nonrelativistic continuum mechanics we have the identifications:

$$(2.21) \quad T^{i4} = p^i \ , \quad T^{ij} = -t^{ij} + p^i v^j \ , \quad T^{44} = e \ ,$$

$$T^{4i} = q^i - t^{ji} v_j + e v^i$$

and

$$(2.22) \quad f^i = f^i \ , \quad f^4 = h + \underline{f} \cdot \underline{v}$$

where p^i is the momentum density, t^{ij} is the stress tensor, e is the energy density (internal plus kinetic) and q^i is the heat flux, f^i is the body force and h is the body heat supply. This identifi-

cation is useful for establishing a familiarity with the physical meaning of the components of the energy-momentum tensor; however, it is not essential for a treatment of relativistic mechanics. The only assumption employed in decomposing the energy-momentum tensor is the existence of a world velocity vector in the four dimensional material tube in space-time.

In the usual manner (2.20) leads to

$$(2.23) \qquad T^{\mu\nu}{}_{,\nu} = f^{\mu} \quad , \quad [T^{\mu\nu}] \Sigma_{,\nu} = 0 \quad \text{across} \quad \Sigma(x^{\mu}) = 0$$

The tensor $T^{\alpha\beta}$ can be decomposed into a scalar, two spatial vectors and a spatial tensor by applying the projection operators $s^{\alpha}{}_{\beta}$. To this end define

$$(2.24) \qquad \begin{aligned} \omega &= T^{\alpha\beta} u_{\alpha} u_{\beta} \\ q^{\alpha} &= -s^{\alpha}{}_{\beta} T^{\gamma\beta} u_{\gamma} \\ p^{\alpha} &= -s^{\alpha}{}_{\beta} T^{\beta\gamma} u_{\gamma} \\ t^{\alpha\beta} &= -s^{\alpha}{}_{\gamma} s^{\beta}{}_{\delta} T^{\gamma\delta} \end{aligned}$$

It is easily shown that

$$(2.25) \qquad T^{\alpha\beta} = \omega u^{\alpha} u^{\beta} + u^{\alpha} q^{\beta} + p^{\alpha} u^{\beta} - t^{\alpha\beta}$$

It should be noted that this decomposition is perfectly general and depends only on the character of the world velocity field. We have the following physical interpretations: The tensor $\omega\, u^\alpha u^\beta$ is the kinetic energy-momentum tensor with the mass density given by the famous formula of Einstein $E = mc^2$. Thus there is a contribution to the mass density due to the internal energy of the body. The four vector q^β in the local instantaneous rest frame reduces to $[q^1, 0]$, where q^1 in the heat flow vector. Thus q^β is called the heat flow four vector. The four vector p^α becomes $[p^1, 0]$ in the local instantaneous rest frame, where p^1 is the nonmechanical momentum. Thus p^α is called the nonmechanical momentum four vector. The four tensor $t^{\alpha\beta}$ is the relativistic stress tensor since it reduces to

$$
\begin{bmatrix} t^{1j} & 0 \\ 0 & 0 \end{bmatrix}
$$

in the local instantaneous rest frame.

From (2.24) and (1.20) the following identities follow:

$$
q^\alpha u_\alpha = 0
$$

$$
(2.26) \quad p^\alpha u_\alpha = 0
$$

$$
t^{\alpha\beta} u_\beta = t^{\beta\alpha} u_\beta = 0
$$

The fourth component of $(2.23)_1$ is the balance of energy. Consequently the first law of thermodynamics is contained in these equations;

however, it is not the fourth component of (2.23). To obtain the
first law of thermodynamics one must take the projection of (2.23)
onto u_α . This operation in the relativistic generalization is
the counterpart of taking the scalar product of Cauchy's equations
and the velocity and subtracting the result from the energy
equation. We recall that this operation eliminates the kinetic
energy. Hence

$$u_\alpha T^{\alpha\beta}_{,\beta} = f^\alpha u_\alpha$$

or

(2.27) $\quad (u_\alpha T^{\alpha\beta})_{,\beta} - T^{\alpha\beta} u_{\alpha,\beta} = f^\alpha u_\alpha$

But from (2.25), using (2.26), we have

(2.28)
$$u_\alpha T^{\alpha\beta} = -\omega u^\beta - q^\beta$$
$$T^{\alpha\beta} u_{\alpha,\beta} = p^\alpha Du_\alpha - t^{\alpha\beta} u_{\alpha,\beta}$$

Thus equation (2.27) reduces to

(2.29) $\quad -(\omega u^\beta)_{,\beta} - q^\beta_{,\beta} - p^\alpha Du_\alpha + t^{\alpha\beta} u_{\alpha,\beta} = f^\alpha u_\alpha$

Define ϵ by

(2.30) $\quad \omega = n_0 \epsilon$

From (2.14), (2.29) becomes

$$(2.31) \quad n_o \ D\epsilon + q^\beta{}_{,\beta} + p^\beta \ Du_\beta - t^{\alpha\beta} \ u_{\alpha,\beta} \ = \ - f^\alpha u_\alpha$$

This is the relativistic generalization of the first law of thermo-dynamics for a continuous medium (see Eckart [17]).

The generalization of Cauchy's laws can be obtained by applying $s^\gamma{}_\alpha$ to (2.23). The result is:

$$(2.32) \quad n_o \ \epsilon \ Du^\alpha + n_o \ s^\alpha{}_\gamma \ D(\frac{p^\gamma}{n_o}) + q^\beta \ u^\alpha{}_{,\beta} - t^{\alpha\beta}{}_{,\beta} + t^{\beta\nu} \ u_{\beta,\nu} \ u^\alpha = s^\alpha{}_\beta \ f^\beta$$

Only three of the four equations (2.32) are independent.

Principle of Moment of Energy-Momentum

In the classical theory of continuum mechanics, for nonpolar materials, there is a balance law of angular momentum which by the arguments presented at the beginning of this chapter can be written in the form:

$$(2.33) \quad \oint x^{[j} \ T^{i]\alpha} \ ds_{3\alpha} \ = \ \int x^{[j} \ f^{i]} \ dv_4 \ , \quad (i,j = 1, \ 2, \ 3)$$

for an arbitrary material tube. (2.33) is the spatial part of the four tensor

$$(2.34) \qquad \oint x^{[\alpha}T^{\beta]\mu} \, ds_{3\mu} = \int x^{[\alpha} f^{\beta]} \, dv_4$$

If it is required that the angular momentum is balanced in every Lorentz frame then (2.33) implies (2.34). The remaining three components of (2.34), other than (2.33), i.e.,

$$(2.35) \qquad \oint x^{[4}T^{1]\mu} \, ds_{3\mu} = \int x^{[4} f^{1]} \, dv_4$$

are the expressions of equivalence of energy flux and momentum flow. If (2.35) is assumed to hold in every Lorentz frame, (2.33) and (2.34) must hold. This would indicate that (2.34) must be a basic law for nonpolar materials.

That there should be a skew-symmetric four tensor balance law to replace (2.33) is indicated also by the following argument: Consider a closed conservative system. In modern physics the conservation of momentum is interpreted as the invariance under spatial translation, while the conservation of energy is implied by the invariance under time displacements. In classical mechanics this leads to one vector law and one scalar law because the Galilean group is used. In relativistic mechanics this leads to a four vector equation since the Lorentz group is used. In classical mechanics the Galilean group allows spatial rotations and invariance under these leads to the conservation of angular momentum. The Lorentz group, however, allows rotations in the four dimensional space of events and

thus implies three extra conservation laws. (See Bogoliubov and Shirkov [21] for an excellent discussion of these points.) Continuum mechanics treats open systems. The conservation laws of a closed system are replaced by corresponding balance laws. The balance of energy-momentum for a continuous medium has already been formulated. The following assumption formulates the principle of moment of energy-momentum which serves as the relativistic generalization of the balance of angular momentum.

To every three dimensional subspace, s_3 , of the four dimensional material tube assign a skew symmetric tensor function $M^{\alpha\beta} [s_3]$

(2.36)
$$M^{\alpha\beta}[s_3] = \int_{s_3} M^{\alpha\beta\mu} ds_{3\mu}$$

$$M^{(\alpha\beta)\mu} = 0$$

The tensor $M^{\alpha\beta\mu}$ is usually written as

(2.37)
$$M^{\alpha\beta\mu} = x^{[\alpha} T^{\beta]\mu} + s^{\alpha\beta\mu}$$

$$s^{(\alpha\beta)\mu} = 0$$

$s^{\alpha\beta\mu}$ is called the spin tensor[1] (cf. Papapetrou [22]). For nonpolar materials $s^{\alpha\beta\mu} = 0$.

---

[1]This tensor includes such effects as the couple stress and the intrinsic spin of continuum mechanics.

The law of balance of moment of energy-momentum is that for every material tube

$$\oint M^{\alpha\beta\mu} \, ds_{3\mu} = \int \{x^{[\alpha} f^{\beta]} + L^{\alpha\beta}\} \, dv_4$$

(2.38)

$$L^{(\alpha\beta)} = 0$$

where $L^{\alpha\beta}$ is the four dimensional analogue of the body torque. Equation (2.38) leads to

$$M^{\alpha\beta\mu}_{,\mu} = x^{[\alpha} f^{\beta]} + L^{\alpha\beta}$$

(2.39)

$$[M^{\alpha\beta\mu}_{\sim}] \Sigma_{\sim,\mu} = 0 \qquad \text{across} \quad \Sigma(\underset{\sim}{x}) = 0$$

Using (2.23) and (2.37), (2.39) leads to

(2.40) $\qquad s^{\alpha\beta\mu}_{,\mu} - T^{[\alpha\beta]} = L^{\alpha\beta}$

For nonpolar materials $L^{\alpha\beta} = 0$ and (2.40) reduces to

(2.41) $\qquad T^{[\alpha\beta]} = 0$

From (2.25), (2.41) is equivalent to

(2.42) $\qquad t^{[\alpha\beta]} = 0 \qquad , \qquad p^{\alpha} = q^{\alpha}$

Throughout this work $s^{\alpha\beta\mu} = 0$. In the case of electromagnetic interaction with a material body, a body torque $L^{\alpha\beta}$ of electromagnetic origin will be introduced. The inclusion of $s^{\alpha\beta\mu}$ is necessary if one wishes to formulate a relativistic generalization of mechanical theories of couple stresses. A fuller investigation of the properties of this tensor is left for further research.

Second Law of Thermodynamics

In modern continuum mechanics the second law of thermodynamics is considered to be a restriction on the form of the constitutive equations. However, formulation of this law is independent of the character of the media under consideration. This law is expressed in the form of an inequality called the Clausius-Duhem inequality which has the form (2.1) with the equality replaced by an inequality.

To formulate the relativistic extension of this inequality, assign to every three dimensional subspace, s_3 , of a material tube a scalar invariant $H[s_3]$ of the form:

$$(2.43) \qquad H[s_3] = \int_{s_3} \eta^\alpha \, ds_{3\alpha}$$

The second law of thermodynamics is the statement

$$(2.44) \qquad \oint \eta^\alpha \, ds_{3\alpha} + \int r \, dv_4 \geq 0$$

for every material tube. Here the scalar invariant r is the supply of entropy from extraneous sources. Inequality (2.44) leads to the local law

$$\eta^{\alpha}{}_{,\alpha} + r \geq 0$$

(2.45)

$$[\eta^{\alpha}] \Sigma_{,\alpha} \geq 0 \qquad \text{across } \Sigma(x^{\mu}) = 0$$

It is convenient to decompose η^{α} into its space-like and time-like components.

$$(2.46) \qquad s^{\alpha} = g^{\alpha}{}_{\beta} \eta^{\beta} \qquad , \qquad \eta_{o} = -\eta^{\alpha} u_{\alpha}$$

so that

$$(2.47) \qquad \eta^{\mu} = \eta_{o} u^{\mu} + s^{\mu}$$

A simple thermodynamics process is one for which

$$s^{\mu} = \frac{q^{\mu}}{\theta} \qquad , \qquad r \equiv \frac{h_{o}}{\theta}$$

(2.48)

$$h_{o} \equiv f^{\mu} u_{\mu} \quad , \qquad \theta > 0$$

The quantity θ is a scalar invariant called the temperature, and h_{o} is the heat supply term. Define η_{oo} by

$$(2.49) \qquad \eta_{o} = n_{o} \eta_{oo}$$

The second law (2.45) is then

$$(2.50) \qquad n_o \, D\eta_{oo} + s^\beta_{,\beta} + h \geq 0$$

For a simple thermodynamic process (2.50) becomes:

$$(2.51) \qquad n_o \, D\eta_{oo} + \left(\frac{q^\beta}{\theta}\right)_{,\beta} + \frac{h_o}{\theta} \geq 0$$

This is the form of the second law introduced by Eckhart [17] for fluids. Eliminating h_o from (2.31) and (2.51) we get

$$(2.52) \qquad n_o \left(D\eta_{oo} - \frac{1}{\theta} D\varepsilon\right) - \frac{q^\beta}{\theta^2} \theta_{,\beta} - \frac{p^\alpha Du_\alpha}{\theta} + \frac{t^{\alpha\beta} u_{\alpha,\beta}}{\theta} \geq 0$$

The inequality (2.52) is useful for the reduction of constitutive equations.

CHAPTER III

CONSTITUTIVE THEORY OF MECHANICAL MATERIALS

General Principles

In general the system of balance laws proposed in Chapter II is inadequate for the treatment of problems of relativistic continua except in some special cases. The properties of the medium are brought into consideration through a set of constitutive equations. In general a constitutive theory should satisfy certain principles[1]:

i) <u>Principle of Causality</u>: The behavior of the material at the event $\underline{x}$ is determined only by events lying in the past light cone at $\underline{x}$. That is only by those events $\hat{\underline{x}}$ which satisfy the following inequalities.

$$(\underline{x} - \hat{\underline{x}}) \cdot (\underline{x} - \hat{\underline{x}}) \leq 0$$

$$(\hat{\underline{x}} - \underline{x}) \cdot \underline{u} \leq 0$$

where $\underline{u}$ is the world velocity at $\underline{x}$.

ii) <u>Principle of Locality</u>: The behavior of the material at an event depends strongly on the properties of the material in the neighborhood of the event.

iii) <u>Lorentz Invariance</u>: The constitutive equations are covariant under the orthochronous proper inhomogeneous Lorentz group, i.e., the

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[1]For a discussion on non-relativistic constitutive theory see Eringen [7, Ch. V] and [9].

group for which $\det \underline{\Lambda} = +1$, $\Lambda^4_4 > 0$.

iv) <u>Material Invariance</u>: The constitutive equations are invariant under the symmetry group which characterizes the material in the Lagrangian frame X^K .

v) <u>Consistency</u>: The constitutive equations must be consistent with the balance laws of particle-number, energy-momentum, moment of energy-momentum, the laws of electromagnetism (to be formulated in the next chapter), and the second law of thermodynamics.

vi) <u>Equipresence</u>: An independent variable that appears in one constitutive equation should appear in all constitutive equations unless excluded by one of the above principles i. - v.

A word is in order about the requirement of invariance under Lorentz transformations. In the modern theories of continuum mechanics the constitutive equations are covariant under rigid body motions. Although it appears that classically the invariant group of mechanics is the Galilean group [30], [31], the formulation of modern continuum theories decomposes the forces acting on the body into external forces and internal forces and assumes that the internal forces are objective under rigid motions while the external forces are not objective [32]. In this article we make no such distinction between forces. The objectivity under Lorentz transformations is adequate for the theories presented in the following sections.

In the next two subsections, relativistically invariant constitutive theories for thermoelastic solids and thermoviscous fluids

are presented. These are simple relativistic generalizations of the corresponding classical theories of nonpolar materials for which $s^{\alpha\beta\mu} = 0$, $L^{\alpha\beta} = 0$. Thus (2.42) is valid.

Thermoelastic Solid

For the construction of constitutive equations of a thermoelastic solid, an appropriate set of independent variables is:

$$(3.1) \qquad \theta \; ; \; X^{K}_{\ ,\beta} \; ; \; \overset{*}{\theta}_{\beta} \; , \; x_{\beta}$$

where

$$(3.2) \qquad \overset{*}{\theta}_{\beta} = S_{\beta}^{\ \alpha} (\theta_{,\alpha} + \theta \, Du_{\alpha})$$

The choice of $\overset{*}{\theta}_{\beta}$ as an independent variable appears to be a natural one through the examination of the second law of thermodynamics (2.52) for nonpolar materials:

$$(3.3) \qquad n_{o} (D\eta_{oo} - \frac{1}{\theta} D\epsilon) - \frac{q^{\beta}}{\theta^{2}} \overset{*}{\theta}_{\beta} + \frac{t^{\alpha\beta}}{\theta} \overset{*}{d}_{\alpha\beta} \geq 0$$

where (2.42) has been used.

We now write constitutive equations for the dependent variables ϵ , η_{oo} , q^{α} , $t^{\alpha\beta}$ in the general forms

$$\epsilon = \epsilon(\theta, x^K{}_{,\beta}, \overset{*}{\theta}_\beta, x_\beta)$$

$$\eta_{oo} = \eta_{oo}(\theta, x^K{}_{,\beta}, \overset{*}{\theta}_\beta, x_\beta)$$

(3.4)

$$q^\alpha = q^\alpha(\theta, x^K{}_{,\beta}, \overset{*}{\theta}_\beta, x_\beta)$$

$$t^{\alpha\beta} = t^{\alpha\beta}(\theta, x^K{}_{,\beta}, \overset{*}{\theta}_\beta, x_\beta)$$

The invariance of (3.4) under space-time displacements eliminates the dependence on x_β. Thus (3.4) becomes:

$$\epsilon = \epsilon(\theta, x^K{}_{,\beta}, \theta^K)$$

$$\eta_{oo} = \eta_{oo}(\theta, x^K{}_{,\beta}, \theta^K)$$

(3.5)

$$q^\alpha = q^\alpha(\theta, x^K{}_{,\beta}, \theta^K)$$

$$t^{\alpha\beta} = t^{\alpha\beta}(\theta, x^K{}_{,\beta}, \theta^K)$$

where we have made the variable change

(3.6) $$\theta^K = x^K{}_{,\alpha} \overset{*}{\theta}{}^\alpha$$

By chain rule differentiation and the use of (1.55), the second law of thermodynamics (3.3) reduces to:

(3.7)
$$-\frac{n_o}{\theta}\left(\frac{\partial \psi_o}{\partial \theta} + \eta_{oo}\right) D\theta - n_o \frac{\partial \psi_o}{\partial \theta^K} D\theta^K - \frac{q_K \theta^K}{\theta^2}$$

$$-\frac{1}{\theta}\left(t^\beta{}_\alpha + n_o \frac{\partial \psi_o}{\partial x^K{}_{,\beta}} x^K{}_{,\alpha}\right) x^\alpha{}_L DX^L{}_{,\beta} \geq 0$$

where

(3.8) $\psi_0 \equiv \epsilon - \theta\, \eta_{\infty}$

(3.9) $q_K \equiv x^\alpha_{K}\, q_\alpha$

The quantity ψ_0 is called the free energy.

At any event $\underline{x}$ the following quantities

(3.10) θ ; $D\theta$; $x^K_{,\alpha}$; θ^K ; $D\theta^K$; $Dx^K_{,\alpha}$ $(K = 1,2,3 ; \alpha = 1,2,3,4)$

are independent in the sense that there always exists a motion such that for an arbitrary set of numbers

(3.11) T ; B ; F^K_{α}; T^K ; A^K ; D^K_{α} $(\alpha = 1,2,3,4 ; K = 1,2,3)$

subject to $T > 0$, $\det(\gamma^{\alpha\beta}\, F^K_{\alpha}\, F^L_{\beta}) > 0$, there exists an allowable motion such that

(3.12) $\theta = T$; $D\theta = B$; $x^K_{,\alpha} = F^K_{\alpha}$; $\theta^K = T^K$; $D\theta^K = A^K$; $Dx^K_{,\alpha} = D^K_{\alpha}$

at an arbitrary event $\underline{x}$. The inequality (3.7) contains terms of the form:

(3.13) $g(\ )\, y + f(\ ) \geq 0$

where y is from the set (3.10) and $g(\ )$ and $f(\ )$ do not depend on y . Since y is independent in the above sense, a necessary and sufficient condition that (3.13) holds, for arbitrary values of y , is that

(3.14) $g(\ ) = 0$, $f(\ ) \geq 0$

A repeated application of this argument to (3.7) leads to the following result: A necessary and sufficient condition that (3.7) holds is that:

$$\eta_{oo} = -\frac{\partial \psi_o}{\partial \theta} \quad , \quad \frac{\partial \psi_o}{\partial \theta^K} = 0$$

(3.15)

$$t^{\beta}{}_{\alpha} = -n_o \frac{\partial \psi_o}{\partial x^K{}_{,\beta}} x^K{}_{,\alpha} \quad , \quad q_K \theta^K \leq 0$$

The equation $(3.15)_2$ implies ψ_o is independent of θ^K . By using $(2.26)_3$ we have

(3.16) $\dfrac{\partial \psi_o}{\partial x^K{}_{,\beta}} u^{\beta} = 0$

We also note that $t_{\alpha\beta} = t_{\beta\alpha}$. Consequently we have the following results

$$(3.17) \quad t_{\alpha\beta} = n_0 \frac{\partial \psi_0}{\partial x^K_{,\gamma}} s_{\gamma(\alpha} x^K_{,\beta)}$$

$$(3.18) \quad \frac{\partial \psi_0}{\partial x^K_{,\gamma}} s_{\gamma[\alpha} x^K_{,\beta]} = 0$$

The most general form of the constitutive equations for an anisotropic thermoelastic solid satisfying the requirements of Lorentz invariance and nonnegative entropy production is the following:

i. The free energy ψ_0 has the functional form:

$$(3.19) \quad \psi_0 = \psi_0(\theta, \overset{-1}{c}{}^{KL}) \quad , \quad \overset{-1}{c}{}^{KL} = \gamma^{\alpha\beta} x^K_{,\alpha} x^L_{,\beta}$$

ii. The entropy η_{00} and the stress tensor $t^{\alpha\beta}$ are determined from the free energy ψ_0 by

$$(3.20) \quad \eta_{00} = -\frac{\partial \psi_0}{\partial \theta}$$

$$(3.21) \quad t_{\alpha\beta} = -2n_0 \, x^K_{,\alpha} \, x^L_{,\beta} \, \frac{\partial \psi_0}{\partial \overset{-1}{c}{}^{KL}}$$

iii. The heat flow vector q^{α} has the form:

$$(3.22) \quad q_{\alpha} = Q_K \, x^K_{,\alpha}$$

where

$$(3.23) \quad Q_K = Q_K(\theta, \overset{-1}{c}{}^{KL}, \theta^K)$$

and satisfies the inequality

(3.24) $\quad Q_K \, \theta^K \leq 0$

The inequality (3.24) implies that

(3.25) $\quad Q_K(\theta \, , \, \overset{-1}{C}{}^{KL} \, , \, 0) \; = \; 0$

The proof of statement (i) follows from $(3.15)_2$ and Lorentz invariance. According to $(3.15)_2$ we have

$$\psi_0 \; = \; \psi_0(\theta \, , \, x^K{}_{,\alpha})$$

Now ψ_0 is an invariant function so that

$$\psi_0(\theta \, , \, \Lambda_\alpha{}^\beta \, x^K{}_{,\beta}) \; = \; \psi_0(\theta \, , \, x^K{}_{,\alpha})$$

for an arbitrary Lorentz transformation $\Lambda^\alpha{}_\beta$. There exists at each event $\underline{x}$ a Lorentz transformation, $\overset{\bullet}{\Lambda}$, such that

$$\overset{\bullet}{\Lambda}_1{}^\beta \, x^K{}_{,\beta} \; = \; \hat{x}{}^K{}_{,1}$$

$$\overset{\bullet}{\Lambda}_4{}^\beta \, x^K{}_{,\beta} \; = \; 0$$

From (1.14) the matrix $\hat{x}{}^K{}_{,1}$ is invertible, therefore it has a unique polar decomposition

$$\hat{x}{}^K{}_{,1} \; = \; V^{KL} \, R_{Li}$$

where

$$V^{KL} V^M_{\ L} = \overset{-1}{C}{}^{KM}$$

and R_{Li}

$$R_{Li} R^{K\,i} = \delta^K_{\ L} \quad , \quad R_{Mi} R^{Mj} = \delta^j_{\ i}$$

Thus

$$R^{Li} \overset{.}{\Lambda}{}^{\ \beta}_1 X^K_{\ ,\beta} = V^{KL}$$

Therefore at each event x , there exists a Lorentz transformation $\Lambda^\alpha_{\ \beta}$ such that

$$\hat{\Lambda}^{K\beta} = R^{Ki} \overset{.}{\Lambda}{}^{\ \beta}_i \quad , \quad \overset{\wedge 4\beta}{\Lambda} = \overset{.4\beta}{\Lambda}$$

and

$$\hat{\Lambda}^{L\beta} X^K_{\ ,\beta} = V^{KL} \quad , \quad \overset{\wedge 4\beta}{\Lambda} X^K_{\ ,\beta} = 0$$

Thus Ψ_o can be considered as a function

$$\Psi_o = \Psi_o(\theta\,,\,V^{KL})$$

Since V^{KL} is a function of $\overset{-1}{C}{}^{KL}$ we have the proof of (i). The proof of the first part of (ii), (eq. 3.20), was already given by (3.15). The second part (eq. 3.21) follows using

$$\frac{\partial \Psi_o}{\partial X^M_{\ ,\beta}} = \frac{\partial \Psi_o}{\partial \overset{-1}{C}{}^{KL}} \left(X^K_{\ ,\beta}\, \delta^L_{\ M} + X^L_{\ ,\beta}\, \delta^K_{\ M} \right)$$

in equation (3.17). Equations (3.16) and (3.18) are now satisfied identically. The proof of statement (iii) follows from the fact that $Q_K(\theta, x^K_{,\beta}, \theta^K)$ is an invariant function under Lorentz transformations:

$$Q_K(\theta, \Lambda_\alpha^\beta x^K_{,\beta}, \theta^K) = Q_K(\theta, x^K_{,\alpha}, \theta^K)$$

Using the same argument that was employed above for ψ_0, Q_K must have the form:

$$Q_K = Q_K(\theta, \overset{-1}{C}{}^{KL}, \theta^K)$$

Therefore (3.22) and (3.23) follow.

The inequality (3.24) implies that if $\theta_2 = \theta_3 = 0$, $Q^1 > 0$ if $\theta_1 < 0$ and $Q^1 < 0$ if $\theta_1 > 0$. If we assume that Q^K is a continuous function then $Q^1(\theta, \overset{-1}{C}{}^{KL}, 0) = 0$. A similar argument being valid for Q^2 and Q^3 we have the proof of (3.25).

The constitutive functions may also be expressed in terms of C_{KL}. For this we note that $\overset{-1}{C}{}^{KL}$ is invertible. Hence we may write

(3.26) $\quad \psi_0 = \psi_0(\theta, C_{KL})$

(3.27) $\quad Q_K = Q_K(\theta, C_{KL}, \theta_K)$

(3.28) $\quad C_{KL} = \gamma_{\alpha\beta} x^\alpha{}_K x^\alpha{}_L$

Equation (3.21) for the stress tensor becomes in this case:

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Continuum Mechanics						
Electromagnetic Theory						
Electro-Mechanical Interactions						
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$$(3.29) \quad t^{\alpha\beta} = 2n_o \, x^{\alpha}_{K} \, x^{\beta}_{L} \, \frac{\partial \psi_o}{\partial c_{KL}}$$

The argument for material symmetry restrictions follows the classical lines for various crystal classes. In particular, for an isotropic material with a center of symmetry ψ_o is a scalar invariant under the full orthogonal group of transformations, $\{Q\}$,

$$Q_{MK} \, Q^{ML} = Q_{KM} \, Q^{LM} = \delta^{L}_{K}$$

The free energy is thus a function of the three invariants

$$I_1 = \operatorname{tr} \overset{-1}{c} = \operatorname{tr} c$$

$$(3.30) \quad I_2 = \operatorname{tr}(\overset{-1}{c})^2 = \operatorname{tr} c^2$$

$$I_3 = \operatorname{tr}(\overset{-1}{c})^3 = \operatorname{tr} c^3$$

For an isotropic material ψ_o can be considered a function of θ and $c_{\alpha\beta}$.

$$\psi_o = \psi_o(\theta, c_{\alpha\beta})$$

One can also show that (3.21) is equivalent to

$$(3.31) \quad t^{\alpha}_{\beta} = -2n_o \, \frac{\partial \psi_o}{\partial c_{\alpha\gamma}} \, c_{\gamma\beta}$$

From the reduced form of the Clayley-Hamilton theorem (A.41)

we have

$$\underset{\sim}{c}^3 = I_1 \underset{\sim}{c}^2 - \frac{(I_1^2 - I_2)}{2} \underset{\sim}{c} + \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{6} \underset{\sim}{S}$$

The stress tensor $t_{\alpha\beta}$ is finally

$$(3.32) \quad t_{\alpha\beta} = -n_o(I_1^3 + 2I_3 - 3I_2 I_1) \frac{\partial \psi_o}{\partial I_3} S_{\alpha\beta} - n_o[2\frac{\partial \psi_o}{\partial I_1} - 3(I_1^2 - I_2)\frac{\partial \psi_o}{\partial I_3}]c_{\alpha\beta}$$

$$-n_o[4 \frac{\partial \psi_o}{\partial I_2} + 6 I_1 \frac{\partial \psi_o}{\partial I_3}] c_{\alpha\gamma} c^\gamma{}_\beta$$

For isotropic materials, following a similar argument to that of nonrelativistic continuum mechanics [33], it can be shown that

$$(3.33) \quad q_\beta = (K_1 s^\alpha{}_\beta + K_2 c^\alpha{}_\beta + K_3 c^\alpha{}_\gamma c^\gamma{}_\beta) (\theta_{,\alpha} + \theta Du_\alpha)$$

where K_1 , K_2 and K_3 are scalar functions of the invariants

$$I_1 , I_2 , I_3$$

$$(3.34)$$

$$\theta_1 = \overset{*}{\theta}_\beta \overset{*\beta}{\theta} ; \quad \theta_2 = \overset{*}{\theta}_\alpha c^{\alpha\beta} \overset{*}{\theta}_\beta ; \quad \theta_3 = \overset{*}{\theta}_\alpha c^\alpha{}_\beta c^{\beta\gamma} \overset{*}{\theta}_\gamma$$

The inequality (3.24) becomes in the isotropic case

$$(3.35) \quad K_1 \theta_1 + K_2 \theta_3 + K_3 \theta_3 \leq 0$$

The only other theory of this order of generality is that of Bressan [24]. The stress-strain law which was deduced in this section

from the second law of thermodynamics was arrived at by Bressan by
assuming that a stress function exists. The heat conduction law (3.33)
is a generalization of the one proposed by Eckart [17]. Bressan,
as does Pham Man Quan [25], attempts to generalize Fourier's law of
heat conduction without any regard for the second law. It seems that
it is impossible to satisfy the second law (as stated by (2.52))
without eliminating the heat flux if one assumes their forms of the
heat conduction law.

Thermoviscous Fluids

For a viscous fluid with heat conduction an appropriate set
of independent variables is[1]:

$$(3.36) \quad n_o \ , \ \theta \ , \ \overset{*}{\theta}_\beta \ , \ \overset{*}{d}_{\alpha\beta} \ , \ u_\beta$$

In the classical theory one usually starts with the velocity
gradients $v_{i,j}$ and employs the principle of objectivity to deduce
that since $\omega_{ij} = v_{[i,j]}$ is not invariant under rigid motions of
the spatial frame, it cannot appear in a constitutive equation. By
starting with $\overset{*}{u}_{\alpha\beta}$ it is impossible to show that the dependence
of the constitutive functions on $\overset{*}{\omega}_{\alpha\beta}$ can be eliminated by the
requirement of Lorentz invariance. Since only $\overset{*}{d}_{\alpha\beta}$ occurs in the
entropy production one would expect that this tensor would be sufficient
to describe a wide class of simple materials. Thus in the present

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[1]Note that these are independent variables as tensors; however, not
all components of these tensors are independent, cf. footnote, p. 61.

framework the dependence of the constitutive equations on $\overset{*}{d}_{\alpha\beta}$ is a constitutive assumption defining a class of materials.

The procedure followed here is identical to that used for thermoelastic solids. The second law of thermodynamics for non-polar materials reduces to:

$$- \frac{n_o}{\theta} \left(\frac{\partial \psi_o}{\partial \theta} + \eta_{oo} \right) D\theta - \frac{n_o}{\theta} \frac{\partial \psi_o}{\partial u_\beta} Du_\beta - \frac{n_o}{\theta} \frac{\partial \psi_o}{\partial \overset{*}{\theta}_\beta} D\overset{*}{\theta}_\beta - \frac{n_o}{\theta} \frac{\partial \psi_o}{\partial \overset{*}{d}_{\alpha\beta}} D\overset{*}{d}_{\alpha\beta}$$

(3.37)

$$- \frac{q^\beta}{\theta^2} \overset{*}{\theta}_\beta + \frac{1}{\theta} \left(t^{\alpha\beta} + n_o^2 \frac{\partial \psi_o}{\partial n_o} s^{\alpha\beta} \right) \overset{*}{d}_{\alpha\beta} \geq 0$$

where we used (2.14) and

$$\psi_o \equiv \epsilon - \theta \eta_{oo}$$

At any event x the following quantities

(3.38) ϵ , $D\epsilon$, $\overset{*}{\epsilon}_i$, $D\overset{*}{\epsilon}_i$, $\overset{*}{d}_{ij}$, $D\overset{*}{d}_{ij}$, u_i , Du_i , $(i,j = 1,2,3)$

can be varied independently in the sense described by (3.10). By repeated use of the argument employed to deduce (3.14) from (3.13) one finds that a necessary and sufficient condition that the second law (3.37) is satisfied is that[1]

(3.39) $\eta_{oo} = \dfrac{\partial \psi_o}{\partial \theta}$

---

[1] Since all the components of the tensors in the set (3.36) are not independent. in deducing (3.40) to (3.42) one should consider ψ_o as a function of θ , n_o , $\overset{*}{d}_{ij}$, $\overset{*}{\theta}_i$, v_i .

$$(3.40) \qquad \frac{\partial \psi_o}{\partial u_\beta} = 0$$

$$(3.41) \qquad \frac{\partial \psi_o}{\partial \overset{*}{\theta}_\beta} = 0$$

$$(3.42) \qquad \frac{\partial \psi_o}{\partial \overset{*}{d}_{\alpha\beta}} = 0$$

$$(3.43) \qquad \frac{q^\beta \overset{*}{\theta}_\beta}{\theta} - {}_D t^{\alpha\beta} \overset{*}{d}_{\alpha\beta} \leq 0$$

where

$$(3.44) \qquad {}_D t^{\alpha\beta} = t^{\alpha\beta} + n_o^2 \frac{\partial \psi_o}{\partial n_o} s^{\alpha\beta}$$

is the dissipative stress tensor.

By using the results given in the appendix on matrix functions of Lorentz invariant functions and the conclusions (3.39 to (3.44) for the satisfaction of the second law, one deduces that:

The constitutive equations of a thermoviscous fluid satisfy the second law of thermodynamics and the requirement of Lorentz invariance if and only if

1. The free energy ψ_o depends only on the temperature and particle number.

$$(3.45) \qquad \psi_o = \psi_o(\theta, n_o)$$

ii. The entropy and stress tensor are determined from:

$$(3.46) \qquad \eta_{oo} = -\frac{\partial \psi_o}{\partial \theta} \ , \quad t = -n_o^2 \frac{\partial \psi_o}{\partial n_o}\, \mathbf{g} + {}_D t$$

iii. The energy flux $\underset{\sim}{q}$ and the dissipative part of the stress tensor ${}_D t$ have the form:[1]

$$\begin{aligned}
{}_D t =\ & \lambda_1 \underset{\sim}{S} + \lambda_2 \overset{*}{\underset{\sim}{d}} + \lambda_3 \overset{*}{\underset{\sim}{d}}{}^2 + \lambda_4 \overset{*}{\underset{\sim}{e}} \otimes \overset{*}{\underset{\sim}{e}} + \lambda_5 [\overset{*}{\underset{\sim}{e}} \otimes \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}} + \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}} \otimes \overset{*}{\underset{\sim}{e}}] \\[4pt]
& + \lambda_6 [\overset{*}{\underset{\sim}{e}} \otimes \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}} + \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}} \otimes \overset{*}{\underset{\sim}{e}}] + \lambda_7 [\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}} - \overset{*}{\underset{\sim}{d}} \times \overset{*}{\underset{\sim}{e}}] \\[4pt]
& + \lambda_8 [\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}{}^2 - \overset{*}{\underset{\sim}{d}}{}^2 \times \overset{*}{\underset{\sim}{e}}] + \lambda_9 [\overset{*}{\underset{\sim}{d}}(\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}{}^2) - \overset{*}{\underset{\sim}{d}}{}^2(\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}})] \\[4pt]
& + \lambda_{10} [\overset{*}{\underset{\sim}{e}} \otimes (\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}}) + (\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}}) \otimes \overset{*}{\underset{\sim}{e}}] \\[4pt]
& + \lambda_{11} [\overset{*}{\underset{\sim}{e}} \otimes (\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}}) + (\overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}}) \otimes \overset{*}{\underset{\sim}{e}}]
\end{aligned} \qquad (3.47)$$

and

$$\begin{aligned}
\underset{\sim}{q} =\ & \mu_1 \overset{*}{\underset{\sim}{e}} + \mu_2 \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}} + \mu_3 \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}} + \mu_4 \overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}} + \mu_5 \overset{*}{\underset{\sim}{e}} \times \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}} \\[4pt]
& + \mu_6 (\overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}}) \times (\overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}})
\end{aligned} \qquad (3.48)$$

where $\lambda_1, \ldots, \lambda_{11}$ and $\mu_1, \ldots, \mu_6$ are functions of the invariants

$$\begin{aligned}
& I_d = \mathrm{tr}\,\overset{*}{\underset{\sim}{d}} \ , \quad II_d = \mathrm{tr}\,\overset{*}{\underset{\sim}{d}}{}^2 \ , \quad III_d = \mathrm{tr}\,\overset{*}{\underset{\sim}{d}}{}^3 \ , \quad \Theta_1 = \overset{*}{\underset{\sim}{e}} \cdot \overset{*}{\underset{\sim}{e}} \ , \\[4pt]
& \Theta_2{}^d = \overset{*}{\underset{\sim}{e}}\, \overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}} \ , \quad \Theta_3{}^d = \overset{*}{\underset{\sim}{e}}\, \overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}} \ , \quad \Theta_4{}^d = \overset{*}{\underset{\sim}{e}} \cdot (\overset{*}{\underset{\sim}{d}}\, \overset{*}{\underset{\sim}{e}}) \times (\overset{*}{\underset{\sim}{d}}{}^2\, \overset{*}{\underset{\sim}{e}})
\end{aligned} \qquad (3.49)$$

where ${}_D t$ and $\underset{\sim}{q}$ must satisfy the inequality (3.43).

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[1] We have not assumed that the fluid possesses a center of symmetry. In the latter case further reductions are possible. For this use conclusions made in the apperdix.

CHAPTER IV

ELECTROMAGNETIC THEORY

The basic laws of electromagnetic theory are the conservation of charge, the conservation of magnetic flux, and Ampere's and Guass' Laws. The relativistic formulation of these laws is well known (cf. Post [34], Truesdell and Toupin [8] and Møller [14]).

Conservation of Charge

To formulate the law of conservation of charge assign to every three dimensional subspace in space-time a scalar function $Q[s_3]$ of the form:

$$(4.1) \qquad Q[s_3] = \int_{s_3} \sigma^\alpha \, ds_{3\alpha}$$

where σ^α is called charge-current vector. The law of conservation of charge states that $Q[s_3]$ vanishes for every three dimensional circuit.

$$(4.2) \qquad \oint \sigma^\alpha \, ds_{3\alpha} = 0$$

By familiar arguments of the Green-Gauss theorem this leads to the following differential equation and jump conditions.

$$(4.3) \qquad \sigma^{\alpha}{}_{,\alpha} = 0 \qquad , \qquad [\underset{\sim}{\sigma^{\alpha}}] \, \Sigma_{,\alpha} = 0 \quad \text{across} \; \Sigma(x^{\mu}) = 0$$

It is useful to resolve the vector σ^{α} into a space-like vector and a time-like vector, i.e.,

$$\sigma \equiv - \sigma^{\alpha} u_{\alpha}$$

$$(4.4) \qquad j^{\beta} \equiv s^{\beta}_{\alpha} \sigma^{\alpha} \qquad ; \qquad j^{\alpha} u_{\alpha} = 0$$

$$\sigma \equiv n_{o} \sigma_{o}$$

so that

$$(4.5) \qquad \sigma^{\alpha} = n_{o} \sigma_{o} u^{\alpha} + j^{\alpha}$$

Equation $(4.3)_{1}$ with the use of (2.14) becomes

$$(4.6) \qquad n_{o} D\sigma_{o} + j^{\alpha}{}_{,\alpha} = 0$$

The physical interpretations of σ and j^{α} are:

σ is the charge density in the instantaneous local rest frame.

j^{α} reduces to $[j^{1}, 0]$ in the instantaneous local rest frame where j^{1} is the conduction current.

Conservation of Magnetic Flux

The conservation of magnetic flux is obtained by assigning to every two dimensional subspace in space-time a scalar quantity $\phi[s_2]$, called the magnetic flux.[1]

$$(4.7) \qquad \phi[s_2] = \frac{1}{2} \int_{s_2} \phi_{\alpha\beta} \, ds_2^{\alpha\beta} \qquad ; \qquad \phi_{\alpha\beta} = -\phi_{\alpha\beta}$$

The law of conservation of magnetic flux is the statement that the scalar $\phi[s_2]$ vanishes for every two dimensional circuit.

$$(4.8) \qquad \oint \phi_{\alpha\beta} \, ds_2^{\alpha\beta} = 0$$

By use of Stokes' theorem, this leads to the well known equations:

$$(4.9) \qquad \epsilon^{\alpha\beta\gamma\delta} \phi_{\gamma\delta,\beta} = 0$$

$$[\epsilon^{\alpha\beta\gamma\delta} \phi_{\gamma\delta}] \Sigma_{,\beta} = 0 \qquad \text{across} \quad \Sigma(x^\mu) = 0$$

The tensor $\phi_{\alpha\beta}$ is physically

$$(4.10) \qquad \phi_{\alpha\beta} = [\text{dual } \underset{\sim}{B}, \underset{\sim}{E}]$$

where $\underset{\sim}{B}$ is the density of magnetic flux and $\underset{\sim}{E}$ is the electric field.

---

[1] The surface element of the surface s_2 with parameterization u_1, u_2 is defined by $ds_2^{\alpha\beta} = 2 \dfrac{\partial x^{[\alpha}}{\partial u_1} \dfrac{\partial x^{\beta]}}{\partial u_2} \, du_1 \, du_2$.

That is:

$$(4.11) \quad \phi_{ij} = \epsilon_{ijk} B^k \quad , \quad \phi_{i4} = -\phi_{4i} = E_i \quad , \quad \phi_{44} = 0$$

Maxwell's equations (4.9) are often encountered in the literature in the form

$$(4.12) \quad \phi_{\alpha\beta,\gamma} + \phi_{\beta\gamma,\alpha} + \phi_{\gamma\alpha,\beta} = 0$$

This is obtained by multiplying $(4.9)_1$ by $\epsilon_{\alpha\sigma\tau\mu}$ and using equation (A.33). Sometimes it is convenient to use the dual of ϕ , $\hat{\phi}$

$$(4.13) \quad \hat{\phi}^{\alpha\beta} = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \phi_{\gamma\delta}$$

In terms of $\underset{\sim}{E}$ and $\underset{\sim}{B}$, $\hat{\phi}$ is:

$$\hat{\phi}^{\alpha\beta} = [\text{dual } \underset{\sim}{E} , \underset{\sim}{B}]$$

The conservation of magnetic flux (4.9) in terms of $\hat{\phi}$ reduces to:

$$(4.14) \quad \hat{\phi}^{\alpha\beta}{}_{,\beta} = 0 \quad , \quad [\hat{\phi}^{\alpha\beta}] \Sigma_{,\beta} = 0$$

It is possible to decompose $\phi_{\alpha\beta}$ (cf. Møller [14]) into

(4.15) $\quad \phi_{\alpha\beta} = \mathcal{E}_\beta u_\alpha - \mathcal{E}_\alpha u_\beta + \overset{*}{\phi}_{\alpha\beta}$

where

(4.16)
$$\mathcal{E}_\alpha = \phi_{\alpha\beta} u^\beta \; ; \; \mathcal{E}_\alpha u^\alpha = 0 \; ;$$
$$\overset{*}{\phi}_{\alpha\beta} = s_\alpha^{\;\gamma} s_\beta^{\;\delta} \phi_{\gamma\delta} \; ; \; \overset{*}{\phi}_{\alpha\beta} u^\beta = 0$$

The spatial tensor $\overset{*}{\phi}_{\alpha\beta}$ is conveniently written as

(4.17)
$$\overset{*}{\phi}_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta} \mathcal{B}^\gamma u^\delta \; , \; \mathcal{B}^\alpha u_\alpha \equiv 0$$
$$\mathcal{B}^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \phi_{\beta\gamma} u_\delta = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \overset{*}{\phi}_{\beta\gamma} u_\delta$$

The magnetic flux tensor $\phi_{\alpha\beta}$ becomes:

(4.18) $\quad \phi_{\alpha\beta} = \mathcal{E}_\beta u_\alpha - \mathcal{E}_\alpha u_\beta + \epsilon_{\alpha\beta\gamma\delta} \mathcal{B}^\gamma u^\delta$

and the dual of ϕ , $\overset{\wedge}{\phi}{}^{\alpha\beta}$, can be decomposed into:

(4.19) $\quad \overset{\wedge}{\phi}{}^{\alpha\beta} = \mathcal{B}^\alpha u^\beta - \mathcal{B}^\beta u^\alpha + \epsilon^{\alpha\beta\gamma\delta} \mathcal{E}_\gamma u_\delta$

The two decomposition (4.18) and (4.19) are useful for the formulation of constitutive equations for the electromagnetic quantities and for expressing the interaction of electromagnetic fields with matter. In terms of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ the four vector $\mathcal{E}^\alpha$ and $\mathcal{B}^\alpha$ are

$$(4.20) \qquad \mathcal{E}^\alpha = [\frac{\underline{E} + \underline{v} \times \underline{B}}{\sqrt{1 - v^2}} , \frac{\underline{v} \cdot \underline{E}}{\sqrt{1-v^2}}] , \quad \mathcal{B}^\alpha = [\frac{\underline{B} - \underline{v} \times \underline{E}}{\sqrt{1 - v^2}} , \frac{\underline{v} \cdot \underline{B}}{\sqrt{1-v^2}}]$$

Upon substituting (4.19) into $(4.14)_1$ and multiplying the results by u_α and $s^\gamma_{\ \alpha}$ and using (A.26) we find the following equations which are equivalent to $(4.14)_1$.

$$(4.21) \qquad s^\gamma_{\ \beta} \, \mathcal{B}^\beta_{\ ,\gamma} - \overset{*\alpha\beta\gamma}{\epsilon} \, \mathcal{E}_\alpha \, u_{\beta,\gamma} = 0$$

$$(4.22) \qquad \overset{*\alpha\beta\gamma}{\epsilon} \, \mathcal{E}_{\gamma,\beta} - \overset{*\alpha\beta\gamma}{\epsilon} \, \mathcal{E}_\beta \, D_{\gamma} + s^\alpha_{\ \beta} \, D\mathcal{B}^\beta + \mathcal{B}^\alpha \, u^\beta_{\ ,\beta} - \mathcal{B}^\beta \, u^\alpha_{\ ,\beta} = 0$$

It should be observed that only three of the four equations of (4.22) are independent.

Ampere's and Gauss' Laws

Ampere's and Gauss' laws are combined into one invariant law by assigning to every two dimensional subspace s_2 a scalar invariant $\Gamma[s_2]$

$$(4.23) \qquad \Gamma[s_2] = \int_{s_2} G^{\alpha\beta} \, d\hat{s}_{2\alpha\beta} \quad ; \quad G^{\alpha\beta} = G^{\beta\alpha} \quad ; \quad d\hat{s}_{2\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \, ds_2^{\ \gamma\delta}$$

Ampere's and Gauss' laws state that for every closed two dimensional circuit enclosing a three dimensional subspace s_3

$$(4.24) \qquad \oint G^{\alpha\beta} \, ds_{2\alpha\beta} = \int \sigma^{\alpha} \, ds_{3\alpha}$$

The differential form of (4.24) and the jump conditions are:

$$(4.25) \qquad G^{\alpha\beta}{}_{,\beta} = \sigma^{\alpha} \quad , \quad [\underset{\sim}{G^{\alpha\beta}}] \, \Sigma_{,\beta} = 0 \quad \text{across} \quad \Sigma(x^{\mu}) = 0$$

The tensor $G^{\alpha\beta}$ is the electric displacement-magnetic field intensity tensor:

$$(4.26) \qquad G^{\alpha\beta} = [\text{dual } \underset{\sim}{H} \, , \, -\underset{\sim}{D}]$$

where $\underset{\sim}{D}$ is the electric displacement and $\underset{\sim}{H}$ is the magnetic field intensity. Explicitly (4.26) gives

$$(4.27) \qquad G^{ij} = e^{ijk} H_{k} \quad , \quad G^{4i} = -G^{i4} = D^{i} \quad , \quad G^{44} = 0$$

Similar to (4.15) we decompose $G^{\alpha\beta}$ into

$$(4.28) \qquad G^{\alpha\beta} = \mathcal{D}^{\beta} u_{\alpha} - \mathcal{D}^{\alpha} u_{\beta} + \epsilon^{\alpha\beta\gamma\delta} \mathcal{H}_{\gamma} \dot{u}_{\delta}$$

where

$$\mathcal{D}^{\beta} = G^{\beta\alpha} u_{\alpha} \quad , \quad \mathcal{D}^{\beta} u_{\beta} = 0 \quad , \quad \mathcal{H}_{\beta} u^{\beta} = 0$$

$$(4.29) \qquad \overset{*}{G}{}^{\alpha\beta} = S^{\alpha}{}_{\gamma} S^{\beta}{}_{\delta} G^{\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta} \mathcal{H}_{\gamma} u_{\delta}$$

$$\mathcal{H}_{\alpha} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} G^{\beta\gamma} u^{\delta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \overset{*}{G}{}^{\beta\gamma} u^{\delta}$$

In terms of the electromagnetic fields $\underline{D}$ and $\underline{H}$, $\mathcal{D}^{\beta}$ and $\mathcal{H}^{\beta}$ are:

$$(4.30) \qquad \mathcal{D}^{\beta} = [\frac{\underline{D} + \underline{v} \times \underline{H}}{\sqrt{1 - v^2}} \ , \ \frac{\underline{v} \cdot \underline{D}}{\sqrt{1-v^2}}]$$

$$\mathcal{H}^{\beta} = [\frac{\underline{H} - \underline{v} \times \underline{D}}{\sqrt{1 - v^2}} \ , \ \frac{\underline{v} \cdot \underline{H}}{\sqrt{1-v^2}}]$$

By substituting (4.28) into $(4.25)_1$ in the same way as done in obtaining (4.21) and (4.22) we get

$$(4.31) \qquad s^{\gamma}_{\ \beta} \, \mathcal{D}^{\beta}_{\ ,\gamma} + \overset{*}{\epsilon}{}^{\alpha\beta\gamma} \mathcal{H}_{\alpha} u_{\beta,\gamma} = n_o \sigma_o$$

$$(4.32) \qquad \overset{*}{\epsilon}{}^{\alpha\beta\gamma} \mathcal{H}_{\gamma,\beta} - \overset{*}{\epsilon}{}^{\alpha\beta\gamma} \mathcal{H}_{\beta} \, Du_{\gamma} - s^{\alpha}_{\ \beta} \, D\mathcal{D}^{\beta} + \mathcal{D}^{\beta} u^{\alpha}_{\ ,\beta} - \mathcal{D}^{\alpha} u^{\beta}_{\ ,\beta} = j^{\alpha}$$

Only three of the four equations (4.32) are independent.

CHAPTER V

ELECTROMAGNETIC INTERACTIONS WITH PONDERABLE MATTER

The presence of matter in an electromagnetic field has been the
object of researches since the beginning of electromagnetic theory.
There are several approaches to this problem. The one that is adopted
in this chapter was originated by Lorentz [35] in order to derive the
electromagnetic field equations for ponderable matter. According to
this approach, the interaction between matter and the electric fields
can be deduced from a microscopic model to within the order of approxi-
mation desired. The form of the interactions so deduced can then be
taken as the starting point for the development of a continuum theory.
The force on a material body can also be determined by assuming an
effective current distribution and postulating that the body force is
the Lorentz force on this distribution of charge-current. By redefining
the energy-momentum tensor one can show that these two approaches are
equivalent. A third approach to this problem is to attempt directly
to write down an energy-momentum for the material body and the electric
field. Such attempts are usually guided by one of the above approaches.
There is still no widely accepted form of the energy-momentum tensor.

The point of view adopted in this chapter is that matter is acted
upon by forces of various types, one of which is due to the electromagnetic
interaction with the molecular and atomic structure of matter. By

applying the arguments of Dixon and Eringen [6], it is physically reasonable to assume that electromagnetic interaction with matter (neglecting electric quadrupoles and higher terms) is due to a body force[1]

(5.1) $f_e{}^\mu = \pi^{\beta\alpha} \phi^\mu{}_{\alpha,\beta} + \sigma^\alpha \phi^\mu{}_\alpha$

and a body couple

(5.2) $L^{\mu\nu} = -\pi_\alpha{}^{[\mu} \phi^{\nu]\alpha}$

where $\pi^{\alpha\beta}$ is defined by

(5.3) $G^{\alpha\beta} \equiv \phi^{\alpha\beta} - \pi^{\alpha\beta}$

The polarization tensor $\pi^{\alpha\beta}$ is a skew-symmetric tensor of the form

(5.4) $\pi^{\alpha\beta} = [\text{dual } \underline{M} + \text{dual } (\underline{v} \times \underline{P}) , \underline{P}]$

where $\underline{P}$ is the polarization vector and $\underline{M}$ is the magnetization vector. This definition of $\pi^{\alpha\beta}$ is that of Lorentz. It is well known that it lacks a symmetry in the transformation of the magnetization vector and

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[1]In body force F_i of [6, eq. 3.21] we drop the quadrupole moment tensor q^{ij} and take $\underline{f} = \overline{\underline{f}} - \partial(\underline{p} \times \underline{b})/\partial t$ where $\overline{\underline{f}}$ is that used by Dixon and Eringen. This rate term will appear later in the momentum, cf. eq. (5.18) below.

The magnetic term in the electromagnetic body force (5.1) is based on the Amperian current model. Some researchers prefer a magnetic dipole model (cf. Fano, Chu and Adler [26] and Penfield and Haus [36]). If one uses this model then instead of (5.1) one would assume

$$f_e^{\mu} = (\wp^{\beta} u^{\alpha} - \wp^{\alpha} u^{\beta}) \phi^{\mu}{}_{,\beta} + (\mathcal{M}^{\alpha} u^{\beta} - \mathcal{M}^{\beta} u^{\alpha})\hat{\phi}^{\mu}{}_{,\beta} + \sigma^{\alpha} \phi^{\mu}{}_{\alpha}$$

where $\wp^{\alpha}$ and $\mathcal{M}^{\alpha}$ are defined by (5.8) and (5.9) respectively. Since the existence of magnetic poles is doubted we prefer the Amperian model. It should be noted that both (5.1) and the form listed in this footnote for the magnetic dipole model lead to the same form of the first law of thermodynamics (5.22). Thus the conclusions of Chapter VI remain valid for the magnetic dipole model.

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the polarization vector. This has no effect on electromagnetic phenomena.

The balance equations (2.23) and (2.40) are now

$$(5.5) \qquad T^{\mu\beta}{}_{,\beta} = \pi^{\beta\alpha} \phi^{\mu}{}_{\alpha,\beta} + \sigma^{\alpha} \phi^{\mu}{}_{\alpha} + f^{\mu}$$

$$(5.6) \qquad T^{[\mu\nu]} = \pi_{\alpha}{}^{[\mu} \phi^{\nu]\alpha}$$

It is useful to decompose $\pi^{\alpha\beta}$ into

$$(5.7) \qquad \pi^{\alpha\beta} = \wp^{\alpha} u^{\beta} - \wp^{\beta} u^{\alpha} + \overset{*}{\pi}{}^{\alpha\beta}$$

where

$$(5.8) \qquad \wp^{\beta} = \pi^{\alpha\beta} u_{\alpha} \quad , \quad \overset{*}{\pi}{}^{\alpha\beta} = s^{\alpha}{}_{\gamma} s^{\beta}{}_{\delta} \pi^{\gamma\delta}$$

One can also define

(5.9) $\quad \mathcal{M}_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \pi^{\beta\gamma} u^\delta = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \overset{*}{\pi}{}^{\beta\gamma} u^\delta$

Thus

(5.10) $\quad \overset{*}{\pi}{}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} \mathcal{M}_\gamma u_\delta$

The polarization tensor $\pi^{\alpha\beta}$ reduces to

(5.11) $\quad \pi^{\alpha\beta} = \mathcal{P}^\alpha u^\beta - \mathcal{P}^\beta u^\alpha + \epsilon^{\alpha\beta\gamma\delta} \mathcal{M}_\gamma u_\delta$

It should be noted that

(5.12) $\quad \mathcal{P}^\beta u_\beta = 0 \;, \quad \mathcal{M}^\beta u_\beta = 0 \;, \quad \overset{*}{\pi}{}^{\alpha\beta} u_\beta = 0$

Using (4.18) and (4.28) we see that (5.3) is equivalent to

(5.13) $\quad \mathcal{D}^\alpha = \mathcal{E}^\alpha + \mathcal{P}^\alpha \;, \quad \mathcal{H}^\alpha = \mathcal{B}^\alpha - \mathcal{M}^\alpha$

The spatial part of the four vector $\mathcal{P}^\beta$ reduces to the polarization vector and the spatial part of $\mathcal{M}^\beta$ to the magnetization vector in the local instantaneous rest frame.

This interaction model is a physically reasonable description of a charged, conducting, polarizable, magnetizable material. The following observations allow the construction of various subclasses of materials.

a) If $\sigma_\alpha u^\alpha \equiv 0$, the material is charge-free in the local instantaneous rest frame.

b) If $s^\alpha{}_\beta \sigma^\beta = j^\alpha \equiv 0$, the material is a non-conductor in the local instantaneous rest frame.

c) If $\overset{*}{\pi}{}^{\alpha\beta} \equiv 0$ $(\mathcal{M}_\alpha = 0)$, the material is unmagnetized in the local instantaneous rest frame.

d) If $\wp^\beta \equiv 0$, the material is unpolarized in tho local instantaneous rest frame.

From (5.7) and (4.18) one can show that

$$(5.14) \quad \pi^{\alpha\beta} \phi_{\mu\beta} = -\wp^\beta \mathcal{E}_\beta u^\alpha u_\mu + \overset{*}{\pi}{}^{\alpha\beta} \mathcal{E}_\beta u_\mu + \wp^\alpha \mathcal{E}_\mu$$

$$- \overset{*}{\phi}_{\mu\beta} \wp^\beta u^\alpha + s^\alpha{}_\mu \mathcal{B}^\beta \mathcal{M}_\beta - \mathcal{B}^\alpha \mathcal{M}_\mu$$

where (4.16), (5.12) and (A.36)$_1$ of the appendix were used. Therefore the right hand side of (5.2) becomes:

$$(5.15) \quad - \pi_\beta{}^{[\alpha} \phi^{\mu]\beta} = \wp^{[\alpha} \mathcal{E}^{\mu]} - \mathcal{E}_\beta \overset{*}{\pi}{}^{\beta[\alpha} u^{\mu]} + \mathcal{P}_\beta \overset{*}{\phi}{}^{\beta[\mu} u^{\alpha]} + \mathcal{M}^{[\alpha} \mathcal{B}^{\mu]}$$

From (2.25) the left hand side of (3.2) is

$$(5.16) \qquad T^{[\alpha\beta]} = u^{[\alpha} q^{\beta]} + p^{[\alpha} u^{\beta]} - t^{[\alpha\beta]}$$

Equation (5.6), using (5.15) and (5.16), leads to

$$(5.17) \qquad p^{[\alpha} u^{\beta]} - q^{[\alpha} u^{\beta]} - t^{[\alpha\beta]} = \mathcal{P}_\gamma \overset{*}{\phi}{}^{\gamma[\alpha} u^{\beta]} + \mathcal{E}_\gamma \overset{*}{\pi}{}^{\gamma[\alpha} u^{\beta]}$$
$$- \mathcal{P}^{[\alpha} \mathcal{E}^{\beta]} - \mathcal{M}^{[\alpha} \mathcal{B}^{\beta]}$$

By taking the projection of (5.17) along u_α, one obtains

$$(5.18) \qquad p^\alpha = q^\alpha + \mathcal{P}_\gamma \overset{*}{\phi}{}^{\gamma\alpha} + \mathcal{E}_\gamma \overset{*}{\pi}{}^{\gamma\alpha}$$

Applying the projector $\underline{\underline{g}}$ to (5.17), one deduces that

$$(5.19) \qquad t^{[\alpha\beta]} = \mathcal{P}^{[\alpha} \mathcal{E}^{\beta]} + \mathcal{M}^{[\alpha} \mathcal{B}^{\beta]}$$

Therefore a necessary condition that the balance of angular momentum (5.2) is satisfied is that (5.18) and (5.19) are valid. It is easily shown that (5.18) and (5.19) are also sufficient for (5.2).

To obtain the first law of thermodynamics, we substitute $f_e^{\ \mu} + f^\mu$ (where $f_e^{\ \mu}$ is given by (5.1)) into (2.31). Hence

$$n_o \, D\epsilon + q^\beta_{\ ,\beta} + p^\alpha \, Du_\alpha - t^{\alpha\beta} \overset{*}{u}_{\alpha\beta} = \pi^{\alpha\beta} \phi^\mu_{\ \alpha,\beta} u_\mu - \sigma^\alpha \phi^\mu_{\ \alpha} u_\mu - f^\mu u_\mu$$

$$(5.20) \qquad = - \mathcal{P}^\alpha \, D\mathcal{E}_\alpha - \mathcal{P}^\alpha \overset{*}{\phi}_{\beta\alpha} \, Du^\beta - \mathcal{M}^\alpha \, D\mathcal{B}_\alpha - \mathcal{E}_\alpha \overset{*}{\pi}{}^{\beta\alpha} \, Du_\beta$$
$$+ j^\alpha \mathcal{E}_\alpha - f^\mu u_\mu$$

Where (4.4), (4.16), and the identity

$$(5.21) \qquad u_\mu \, \pi^{\beta\alpha} \bullet^\mu{}_{\alpha,\beta} \;=\; \mathcal{P}^\alpha \, D\mathcal{E}_\alpha + \mathcal{P}^\alpha \, \overset{*}{\phi}_{\beta\alpha} \, Du^\beta + \mathcal{M}^\alpha \, D\mathcal{B}_\alpha + \mathcal{E}_\alpha \, \overset{*}{\pi}{}^{\beta\alpha} \, Du_\beta$$

have been employed. To prove (5.21), one uses (4.15) and (5.7). With
the help of (4.16) and (5.12),the first two terms on the right hand
side fall out almost immediately; the remaining two terms are arrived
at by using (A.36) of the appendix and (4.22).

Substituting (5.18) and (5.19) into (5.20) we obtain

$$(5.22) \qquad
\begin{aligned}
& n_0 \, D\epsilon \;+\; q^\beta{}_{,\beta} + q^\beta \, Du_\beta - t^{(\alpha\beta)} \, \overset{*}{d}_{\alpha\beta} - \mathcal{P}^{[\alpha} \mathcal{E}^{\beta]} u_{\alpha,\beta} \\
& \quad - \mathcal{M}^{[\alpha} \mathcal{B}^{\beta]} u_{\alpha,\beta} + \mathcal{P}^\alpha \, D\mathcal{E}_\alpha + \mathcal{M}^\alpha \, D\mathcal{B}_\alpha - J^\alpha \mathcal{E}_\alpha = -f^\mu u_\mu
\end{aligned}$$

Equation (5.22) is the first law of thermodynamics for an electro-
mechanical material. It is often convenient to rewrite (5.22) as:

$$(5.23) \qquad
\begin{aligned}
& n_0 \, D\epsilon \;+\; q^\beta{}_{,\beta} + q^\beta \, Du_\beta - [\, t^{(\alpha\beta)} - \mathcal{P}^{(\alpha} \mathcal{E}^{\beta)} - \mathcal{M}^{(\alpha} \mathcal{B}^{\beta)}] \\
& \quad + \mathcal{P}^\alpha \, [D\mathcal{E}_\alpha - \mathcal{E}^\beta u_{\alpha,\beta}] + \mathcal{M}^\alpha \, [D\mathcal{B}_\alpha - \mathcal{B}^\alpha u_{\alpha,\beta}] \\
& \quad - J^\alpha \mathcal{E}_\alpha = -f^\mu u_\mu
\end{aligned}$$

An equivalent third form of (5.22) is

$$n_o D[\epsilon + \frac{\mathcal{P}^\alpha \mathcal{E}_\alpha}{n_o} + \frac{\mathcal{M}^\alpha \mathcal{B}_\alpha}{n_o}] + q^\beta_{,\beta} + q^\beta Du_\beta$$

$$(5.24) \quad - [t^{(\alpha\beta)} + \mathcal{P}^{(\alpha} \mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha} \mathcal{B}^{\beta)} + \mathcal{P}^\gamma \mathcal{E}_\gamma s^{\alpha\beta} + \mathcal{P}^\gamma \mathcal{E}_\gamma s^{\alpha\beta}]\overset{*}{d}_{\alpha\beta}$$

$$- \mathcal{E}^\alpha [D\mathcal{P}_\alpha - \mathcal{P}^\beta u_{\alpha,\beta}] - \mathcal{B}^\alpha [D\mathcal{M}_\alpha - \mathcal{M}^\beta u_{\alpha,\beta}] - j^\alpha \mathcal{E}_\alpha = -r^\mu u_\mu$$

Each of these forms may be found useful in various situations. For example (5.22) is convenient if one wishes to describe a fluid. The form (5.23) will be used for a solid when the independent variables are $\mathcal{E}^\alpha$ and $\mathcal{B}^\alpha$. If the independent variables are $\mathcal{P}^\alpha$ and $\mathcal{M}^\alpha$, then (5.24) is the appropriate form of the first law.

According to the entropy inequality (2.51)

$$(5.25) \quad n_o D\eta_{oo} + (\frac{q^\beta}{\theta})_{,\beta} + \frac{r^\mu \cdot u_\mu}{\theta} \geq 0$$

Employing (5.22), (5.25) becomes

$$n_o(D\eta_{oo} - \frac{1}{\theta} D\epsilon) - \frac{q^\beta}{\theta^2} \overset{*}{\theta}_\beta + \frac{t^{(\alpha\beta)}}{\theta} \overset{*}{d}_{\alpha\beta} + \frac{\mathcal{P}^{[\alpha} \mathcal{E}^{\beta]}}{\theta} \overset{*}{\omega}_{\alpha\beta}$$

$$(5.26)$$

$$+ \frac{\mathcal{M}^{[\alpha} \mathcal{B}^{\beta]}}{\theta} \overset{*}{\omega}_{\alpha\beta} - \frac{\mathcal{P}^\alpha D\mathcal{E}_\alpha}{\theta} - \frac{\mathcal{M}^\alpha D\mathcal{B}_\alpha}{\theta} - + \frac{j^\alpha \mathcal{E}_\alpha}{\theta} \geq 0$$

where $\overset{*}{\theta}_\beta$ is defined by (3.2).

Using (5.23) in (5.25), an equivalent expression to (5.26) useful fo. solids is

$$n_o(D\eta_{oo} - \frac{1}{\theta}D\epsilon) - \frac{q^\beta}{\theta^2}\overset{*}{\theta}_\beta + [t^{(\alpha\beta)} - \mathscr{P}^{(\alpha}\mathscr{E}^{\beta)} - \mathscr{M}^{(\alpha}\mathscr{B}^{\beta)}]\frac{\overset{*}{d}_{\alpha\beta}}{\theta}$$

$$(5.27) \qquad - \frac{\mathscr{P}^\alpha}{\theta}[D\mathscr{E}_\alpha - \mathscr{E}^\beta u_{\alpha,\beta}] - \frac{\mathscr{M}^\alpha}{\theta}[D\mathscr{B}_\alpha - \mathscr{B}^\beta u_{\alpha,\beta}] + \frac{\jmath^\alpha\mathscr{E}_\alpha}{\theta} \geq 0$$

Another form of the second law is possible if one uses (5.24).

Using (4.25) and (5.3) to eliminate σ^α from (5.1) and employing (4.12), it is possible to express the body force as the divergence of a tensor

$$f_e^{\ \mu} = -T_e^{\ \mu\beta}{}_{,\beta}$$

$$(5.28) \qquad T_e^{\ \mu\beta} = \phi^\mu_{\ \gamma}\pi^{\gamma\beta} - \phi^\mu_{\ \gamma}\phi^{\gamma\beta} - \frac{1}{4}\phi^{\alpha\gamma}\phi_{\gamma\alpha}\gamma^{\mu\beta} = -\phi^\mu_{\ \gamma}G^{\gamma\beta} - \frac{1}{4}\phi^{\alpha\gamma}\phi_{\gamma\alpha}\gamma^{\mu\beta}$$

In the derivation of the jump conditions (2.23) it was implicitly assumed that body force is continuous across the singular surface Σ. This is not necessarily the case for (5.1) since the electromagnetic fields may suffer a discontinuity. The forms of the balance of energy momentum and its jump conditions, taking into account this possibility, are

$$(5.29) \qquad (T^{\mu\beta} + T_e^{\ \mu\beta})_{,\beta} = f^\mu \ , \quad [T^{\mu\beta} + T_e^{\ \mu\beta}]\Sigma_{,\beta} = 0 \quad \text{across} \ \Sigma(x^\mu) = 0$$

The balance of moment of energy-momentum (5.6) becomes:

$$(5.30) \qquad T^{[\mu\beta]} + T_e^{\ [\mu\beta]} = 0$$

Thus it is possible to write a "total" energy-momentum tensor for the interaction model presented in this chapter. This tensor is symmetric as a consequence of the principle of balance of moment of energy-momentum. Whether it is possible to obtain such a tensor for any interaction model is an open question. The important point to remember is the jump condition $(5.29)_2$. Whether the Maxwell stresses are "real" stresses is a much discussed point. It seems unlikely that they are (see Dixon and Eringen [6] for a discussion of this point). The introduction of the interaction energy-momentum tensor is a mathematical convenience. The system consisting of the body forces (5.1) and the body couple (5.2) is equipollent to a system of "surface tractions" given by (5.28).

CHAPTER VI

CONSTITUTIVE THEORY OF DEFORMABLE ELECTROMAGNETIC MATERIALS

A constitutive theory of deformable electromagnetic materials can be formulated following the principles enunciated in Chapter III. In this chapter relativistic theories of polarizable, magnetizable, conducting solids and fluids are presented. The requirement that the constitutive theory be thermodynamically admissible leads to a considerable reduction in the form of the constitutive equations. The following theories are sufficiently general to include thermal, electrical, and mechanical effects with restricted spatial and temporal variations. Thus such effects as gyrotropic phenomena, optical activity and heredity are excluded.

Electromagnetic Solid

For an elastic, magnetized dielectric which is also a conductor, an appropriable set of independent variables is:

$$(6.1) \qquad \theta \;, \; x^{\beta}{}_{K} \;, \; \theta^{K} \;, \; \mathcal{E}^{K} \;, \; \mathcal{B}^{K}$$

where θ^{K}, $\mathcal{E}^{K}$ and $\mathcal{B}^{K}$ are defined as:

$$\mathcal{E}^K = x^K_{,\beta} \, \mathcal{E}^\beta$$

$$(6.2) \qquad \mathcal{B}^K = x^K_{,\beta} \, \mathcal{B}^\beta \, \text{sgn} \, (^{x^i}/_X K)$$

$$\mathcal{B}^K = x^K_{,\beta} \, \overset{*}{\mathcal{B}}{}^\beta$$

The sgn $(^{x^i}/_X K)$, which signifies the sign of the jacobian, is introduced so that $\mathcal{B}^\beta$ remains as an axial vector. The dependent variables are

$$(6.3) \qquad \epsilon \, , \, \eta_{oo} \, , \, q^\beta \, , \, t^{(\alpha\beta)} \, , \, \mathcal{P}^\alpha \, , \, \mathcal{M}^\alpha \, , \, j^\alpha$$

with p^α and $t^{[\alpha\beta]}$ determined from (5.18) and (5.19).

One can now proceed in a manner identical to that used in Chapter III for mechanical materials: write the entropy production for this set of constitutive equations and find the necessary and sufficient conditions for it to be non-negative. To this end we use the chain rule of differentiation in calculating $D\epsilon$ and $D\eta_{oo}$ and

$$D\mathcal{E}^K = x^K_{,\beta}(D\mathcal{E}^\beta - \mathcal{E}^\alpha u^\beta_{,\alpha})$$

$$D\mathcal{B}^K = x^K_{,\beta}(D\mathcal{B}^\beta - \mathcal{B}^\alpha u^\beta_{,\alpha})$$

Recalling the identity (1.55), the second law (5.27) becomes

$$- \frac{n_o}{\theta} \left(\frac{\partial \psi_o}{\partial \theta} + \eta_{oo} \right) D\theta - \frac{1}{\theta} \left\{ n_o \frac{\partial \psi_o}{\partial x^\mu_{\ K}} \right.$$

$$- [t^{(\alpha\beta)} - \mathscr{P}^{(\alpha} \mathscr{E}^{\beta)} - \mathscr{M}^{(\alpha} \mathscr{B}^{\beta)}] \gamma_{\alpha\mu} x^K_{\ ,\beta} \} Dx^\mu_{\ K}$$

$$(6.4) \qquad - \frac{n_o}{\theta} \frac{\partial \psi_o}{\partial \theta^K} D\theta^K - \frac{1}{\theta} (\mathscr{P}_K + n_o \frac{\partial \psi_o}{\partial \mathscr{E}^K}) D\mathscr{E}^K$$

$$- \frac{1}{\theta} (\mathscr{M}_K + n_o \frac{\partial \psi_o}{\partial \mathscr{B}^K}) D\mathscr{B}^K + \frac{J_K \mathscr{E}^K}{\theta} + \frac{Q_K \theta^K}{\theta^2} \geq 0$$

where $\mathscr{P}_K$, $\mathscr{M}_K$, J_K and Q_K are defined as

$$\mathscr{P}_K = \mathscr{P}_\alpha x^\alpha_{\ K}$$

$$\mathscr{M}_K = \mathscr{M}_\alpha x^\alpha_{\ K} \ \text{sgn}(x^1/_x K)$$

$$(6.5) \qquad J_K = J_\alpha x^\alpha_{\ K}$$

$$Q_K = q_\alpha x^\alpha_{\ K}$$

and ψ_o is the free energy $\psi_o = \epsilon - \theta \eta_{oo}$. At any point, $\underline{x}$, in space-time the following quantities can be assigned arbitrarily:

$$(6.6) \qquad \theta \ , \ x^\beta_{\ K} \ , \ \theta^K \ , \ \mathscr{E}^K \ , \ \mathscr{B}^K \ , \ D\theta \ , \ Dx^\beta_{\ K} \ ,$$
$$D\theta^K \ , \ D\mathscr{E}^K \ , \ D\mathscr{B}^K$$

Thus by the argument leading from (3.13) to (3.14), a necessary and sufficient condition that the entropy production (6.4) is non-negative

is that the following identities hold:

$$(6.7) \qquad \eta_{00} = - \frac{\partial \psi_0}{\partial \theta}$$

$$(6.8) \qquad t^{(\alpha\beta)} = n_0 \, s^{\gamma(\alpha} x^{\beta)}_{\ \ K} \frac{\partial \psi_0}{\partial x^\gamma_{\ K}} + \mathcal{P}^{(\alpha} \mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha} \mathcal{B}^{\beta)}$$

$$(6.9) \qquad \mathcal{P}_K = - n_0 \, \frac{\partial \psi_0}{\partial \mathcal{E}^K}$$

$$(6.10) \qquad \mathcal{M}_K = -n_0 \frac{\partial \psi_0}{\partial \mathcal{B}^K}$$

$$(6.11) \qquad \frac{\partial \psi_0}{\partial \theta^K} = 0$$

$$(6.12) \qquad \frac{\partial \psi_0}{\partial x^\beta_{\ K}} u^\beta = 0$$

$$(6.13) \qquad s^{\gamma[\alpha} x^{\beta]}_{\ \ K} \frac{\partial \psi_0}{\partial x^\gamma_{\ K}} = 0$$

$$(6.14) \qquad J_k \mathcal{E}^K - \frac{Q_K \theta^K}{\theta} \geq 0$$

As in Chapter III, in order that ψ_0 be invariant under the proper Lorentz group, ψ_0 must be a function of the form:

$$(6.15) \qquad \psi_0 = \psi_0(\theta, C_{KL}, \mathcal{E}^K, \mathcal{B}^K)$$

Since

$$\frac{\partial \Psi_0}{\partial x^\alpha_K} = \frac{\partial \Psi_0}{\partial c_{MN}} \frac{\partial c_{MN}}{\partial x^\alpha_K} = \frac{\partial \Psi_0}{\partial c_{KN}} \gamma_{\alpha\beta} x^\beta_N + \frac{\partial \Psi_0}{\partial c_{NK}} \gamma_{\alpha\beta} x^\beta_N$$

equation (6.12) and (6.13) are satisfied identically and (6.8) becomes:

$$(6.16) \quad t^{(\alpha\beta)} = 2n_0 x^\alpha_K x^\beta_L \frac{\partial \Psi_0}{\partial c_{KL}} + \mathcal{P}^{(\alpha} \mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha} \mathcal{B}^{\beta)}$$

In a similar way J_K and Q_K must have the forms:

$$(6.17) \quad
\begin{aligned}
J_K &= J_K(\theta, \theta^K, \mathcal{E}^K, \mathcal{B}^K, c_{KL}) \\
Q_K &= Q_K(\theta, \theta^K, \mathcal{E}^K, \mathcal{B}^K, c_{KL})
\end{aligned}$$

We have thus arrived at the following important conclusion:

The most general form of the constitutive equations for an aniso-
tropic elastic dielectric with heat conduction satisfying the require-
ment of Lorentz invariance, non-negative entropy production, and the
balance of moment of energy-momentum is the following:

i. The free energy Ψ_0 assumes the form

$$(6.18) \quad \Psi_0 = \Psi_0(\theta, c_{KL}, \mathcal{E}^K, \mathcal{B}^K)$$

ii. The entropy η_{00}, the stress tensor $t^{\alpha\beta}$, the
polarization vector $\mathcal{P}_\alpha$, and the magnetization
vector $\mathcal{M}_\alpha$ are determined from the free energy by

$$(6.19) \quad \eta_{oo} = -\frac{\partial \Psi_o}{\partial \theta}$$

$$(6.20) \quad t^{\alpha\beta} = 2n_o \, x^{\alpha}_{,K} \, x^{\beta}_{,L} \frac{\partial \Psi_o}{\partial c_{KL}} + \mathscr{P}^{\alpha} \mathscr{E}^{\beta} + \mathscr{M}^{\alpha} \mathscr{B}^{\beta}$$

$$(6.21) \quad \mathscr{P}_{\alpha} = -n_o \, x^{K}_{,\alpha} \frac{\partial \Psi_o}{\partial \mathscr{E}^{K}}$$

$$(6.22) \quad \mathscr{M}_{\alpha} = -n_o \, x^{K}_{,\alpha} \frac{\partial \Psi_o}{\partial \mathscr{B}^{K}} \quad \text{sgn} \, (x^{1}/_{X^{K}})$$

where (5.19) is employed.

iii. The nonmechanical momentum p is determined through (5.18)

$$(6.23) \quad \underset{\sim}{p} = \underset{\sim}{q} - \mathscr{P} \times \underset{\sim}{\mathscr{B}} - \underset{\sim}{\mathscr{E}} \times \mathscr{M}$$

iv. The conduction current j_{α} and the heat flow vector q_{α} are determined from the equations

$$(6.24) \quad j_{\alpha} = x^{K}_{,\alpha} \, j_{K}$$

$$(6.25) \quad q_{\alpha} = x^{K}_{,\alpha} \, q_{K}$$

where

$$(6.26) \quad j_{K} = j_{K}(\theta, \theta^{K}, \mathscr{E}^{K}, \mathscr{B}^{K}, c_{KL})$$

$$(6.27) \quad q_{K} = q_{K}(\theta, \theta^{K}, \mathscr{E}^{K}, \mathscr{B}^{K}, c_{KL})$$

v. The conduction current j_K and the heat flow vector Q_K must satisfy the inequality

(6.28) $\qquad j_K \mathcal{E}^K - \dfrac{Q_K \theta^K}{\theta} \geq 0$

which in view of the continuity of j_K and Q_K implies

(6.29) $\qquad j_K(\theta, 0, 0, \mathcal{B}^K, c^{KL}) = 0$

(6.30) $\qquad Q_K(\theta, 0, 0, \mathcal{B}^K, c^{KL}) = 0$

Further reduction in the form of the free energy ψ_0, the heat flow vector Q_K, and the conduction current j_K can be obtained if one knows the crystal class of the material. In particular, for an isotropic material with a center of symmetry, a minimal integrity basis for the symmetric tensor $\overset{-1}{c}^{KL}$,[1] the vector $\mathcal{E}^K$, and the axial vector $\mathcal{B}^K$ (see Smith [33]) is:

$$I_1 = \overset{-1}{c}{}^K_{\ K}, \quad I_2 = \overset{-1}{c}{}^{KM}\overset{-1}{c}_{MK}, \quad I_3 = \overset{-1}{c}{}^{KM}\overset{-1}{c}_{MN}\overset{-1}{c}{}^N_{\ K}$$

(6.31)

$$E_0 = \mathcal{E}^K\mathcal{E}_K, \quad E_1 = \mathcal{E}^K\overset{-1}{c}_{KL}\mathcal{E}^L, \quad E_2 = \mathcal{E}^K\overset{-1}{c}_{KM}\overset{-1}{c}{}^M_{\ L}\mathcal{E}^L$$

_____

[1] It is convenient to consider ψ_0 as a function of $\overset{-1}{c}_{KL}$ instead of c_{KL}. The form of (6.20) corresponding to (3.21) is easy to deduce.

(Eq. 6.31) continued)

$$B_0 = \mathcal{B}^K \mathcal{B}_K \,, \quad B_1 = \mathcal{B}^K \overset{-1}{c}_{KL} \mathcal{B}^L \,, \quad B_2 = \mathcal{B}^K \overset{-1}{c}_{KM} \overset{-1}{c}{}^M_L \mathcal{B}^L$$

$$B_3 = \epsilon_{KLM} \overset{-1}{c}{}^K_N \mathcal{B}^N \mathcal{B}^L \overset{-1}{c}{}^{MR} \overset{-1}{c}_{RS} \mathcal{B}^S$$

$$B_4 = (\mathcal{E}_K \mathcal{B}^K)^2 \,, \quad B_5 = (\mathcal{E}^R \mathcal{B}_R)(\mathcal{B}^K \overset{-1}{c}_{KL} \mathcal{E}^L)$$

$$B_6 = (\mathcal{E}_R \mathcal{B}^R)(\mathcal{B}^K \overset{-1}{c}{}^M_K \overset{-1}{c}_{ML} \mathcal{E}^L)$$

$$B_7 = \epsilon_{KLM} \mathcal{E}^K \overset{-1}{c}{}^L_N \mathcal{E}^N \mathcal{B}^M \,, \quad B_8 = \epsilon_{KLM} \mathcal{E}^K \overset{-1}{c}{}^L_N \overset{-1}{c}{}^N_R \mathcal{E}^R \mathcal{B}^M$$

$$B_9 = \epsilon_{KLM} \overset{-1}{c}{}^K_R \mathcal{E}^R \overset{-1}{c}{}^L_N \overset{-1}{c}{}^N_S \mathcal{E}^S \mathcal{B}^M$$

$$B_{10} = (\mathcal{E}^R \mathcal{B}_R) \, \epsilon_{KLM} \mathcal{E}^K \overset{-1}{c}{}^L_N \mathcal{B}^N \mathcal{B}^M$$

$$B_{11} = (\mathcal{E}^S \mathcal{B}_S) \, \epsilon_{KLM} \mathcal{E}^K \overset{-1}{c}{}^L_N \overset{-1}{c}{}^N_R \mathcal{B}^R \mathcal{B}^M$$

It can be shown that the set (6.31) can also be expressed in the from:

$$I_1 = \text{tr}\,\mathfrak{c} \,, \quad I_2 = \text{tr}\,\mathfrak{c}^2 \,, \quad I_3 = \text{tr}\,\mathfrak{c}^3$$

(6.32) $\quad E_0 = \mathcal{E} \, \mathfrak{c} \, \mathcal{E} \,, \quad E_1 = \mathcal{E} \, \mathfrak{c}^2 \, \mathcal{E} \,,$

$$E_2 = \mathcal{E} \cdot \mathcal{E} \, \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{6} - \frac{(I_1^2 - I_2)}{2} \, \mathcal{E} \, \mathfrak{c} \, \mathcal{E} + I_1 \, \mathcal{E} \, \mathfrak{c}^2 \, \mathcal{E}$$

(Equation continued next page)

(Equation 6.32 continued)

$$B_0 = \mathscr{B} \mathscr{L} \mathscr{B} \, , \quad B_1 = \mathscr{B} \mathscr{L}^2 \mathscr{B}$$

$$B_2 = \mathscr{B} \cdot \mathscr{B} \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{6} - \frac{(I_1^2 - I_2)}{2} \mathscr{B} \mathscr{L} \mathscr{B} + I_1 \mathscr{B} \mathscr{L}^2 \mathscr{B}$$

$$B_3 = |J| \, \mathscr{B} \mathscr{L} (\mathscr{B} \times \mathscr{L}^2 \mathscr{B}) \, , \quad B_4 = (\mathscr{E} \mathscr{L} \mathscr{B})^2 \, , \quad B_5 = (\mathscr{E} \mathscr{L} \mathscr{B})(\mathscr{B} \mathscr{L}^2 \mathscr{E})$$

$$B_6 = (\mathscr{E} \mathscr{L} \mathscr{B}) \, [\mathscr{B} \cdot \mathscr{E} \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{6} - \frac{(I_1^2 - I_2)}{2} \mathscr{B} \mathscr{L} \mathscr{E} + I_1 \mathscr{B} \mathscr{L}^2 \mathscr{E}]$$

$$B_7 = |J| \, (\mathscr{L} \mathscr{E}) \times \mathscr{B} \cdot \mathscr{E} \, , \quad B_8 = |J| \, (\mathscr{L}^2 \mathscr{E}) \times \mathscr{B} \cdot \mathscr{E}$$

$$B_9 = |J| \, (\mathscr{L} \mathscr{E}) \times \mathscr{L}^2 \mathscr{E} \cdot \mathscr{E} \, , \quad B_{10} = |J| \, (\mathscr{E} c \mathscr{B})(\mathscr{L} \mathscr{B}) \times \mathscr{B} \cdot \mathscr{E}$$

$$B_{11} = (\mathscr{E} \mathscr{L} \mathscr{B}) \, (\mathscr{L}^2 \mathscr{B}) \times \mathscr{B} \cdot \mathscr{E} \, |J|$$

where the following notation has been used

$$\mathscr{E} \mathscr{L} \mathscr{E} = \mathscr{E}^\alpha \, c_{\alpha\beta} \mathscr{E}^\beta \, , \quad (\mathscr{L} \mathscr{E})_\alpha = c_{\alpha\beta} \mathscr{E}^\beta$$

$$(\mathscr{B} \times \mathscr{E})_\alpha = \epsilon_{\alpha\beta\gamma\delta} \mathscr{B}^\beta \mathscr{E}^\gamma u^\delta$$

From the work of Wineman and Pipkin [37], for an isotropic material
with a center of symmetry, ψ_0 must be a function of the invariants (6.32).

$$(6.33) \quad \psi_0 = \psi_0(\theta \, , \, I_a \, , \, E_b \, , \, B_c) \qquad \begin{pmatrix} a = 1,2,3 \\ b = 0,1,2 \\ c = 0,1,\ldots,11 \end{pmatrix}$$

From (6.32) after some algebraic manipulations we can show that (6.20),
(6.21), (6.22) assume the following forms:

$$
\begin{aligned}
\underline{t} =\ & \tau_1 \underline{\varepsilon} + \tau_2 \underline{c} + \tau_3 \underline{c}^2 + \tau_4 \underline{c}\, \underline{\varepsilon} \otimes \underline{c}\,\underline{\varepsilon} + \tau_5[\underline{c}\,\underline{\varepsilon} \otimes \underline{c}^2\underline{\varepsilon} + \underline{c}^2\underline{\varepsilon} \otimes \underline{c}\,\underline{\varepsilon}] \\
& + \tau_6 \underline{c}\,\underline{B} \otimes \underline{c}\,\underline{B} + \tau_7[\underline{c}\,\underline{B} \otimes \underline{c}^2\underline{B} + \underline{c}^2\underline{B} \otimes \underline{c}\,\underline{B}] + \tau_8[\underline{c}\,\underline{B} \otimes (\underline{B} \times \underline{c}^2\underline{B}) \\
& + (\underline{B} \times \underline{c}^2\underline{B}) \otimes \underline{c}\,\underline{B} - (\underline{B} \times \underline{c}\,\underline{B}) \otimes \underline{c}^2\underline{B} - \underline{c}^2\underline{B} \otimes (\underline{B} \times \underline{c}\,\underline{B}) \\
& - \underline{c}\,\underline{B} \otimes \underline{c}(\underline{B} \times \underline{c}\,\underline{B}) - \underline{c}(\underline{B} \times \underline{c}\,\underline{B}) \otimes \underline{c}\,\underline{B}] + \tau_9[\underline{c}\,\underline{B} \otimes \underline{c}\,\underline{\varepsilon} + \underline{c}\,\underline{\varepsilon} \otimes \underline{c}\,\underline{B}] \\
& + \tau_{10}[\underline{c}^2\underline{B} \otimes \underline{c}\,\underline{\varepsilon} + \underline{c}^2\underline{\varepsilon} \otimes \underline{c}\,\underline{B} + \underline{c}\,\underline{\varepsilon} \otimes \underline{c}^2\underline{B} + \underline{c}\,\underline{B} \otimes \underline{c}^2\underline{\varepsilon}] \\
& - \chi_7[\underline{c}\,\underline{\varepsilon} \otimes (\underline{B} \times \underline{\varepsilon}) + (\underline{B} \times \underline{\varepsilon}) \otimes \underline{c}\,\underline{\varepsilon}] - \chi_8[(\underline{B} \times \underline{\varepsilon}) \otimes \underline{c}^2\underline{\varepsilon} \\
& + \underline{c}\,\underline{\varepsilon} \otimes \underline{c}(\underline{B} \times \underline{\varepsilon}) + \underline{c}^2\underline{\varepsilon} \otimes (\underline{B} \times \underline{\varepsilon}) + \underline{c}(\underline{B} \times \underline{\varepsilon}) \otimes \underline{c}\,\underline{\varepsilon}] - \chi_9[(\underline{B} \times \underline{c}\,\underline{\varepsilon}) \otimes \underline{c}^2\underline{\varepsilon} \\
& + \underline{c}^2\underline{\varepsilon} \otimes (\underline{B} \times \underline{c}\,\underline{\varepsilon}) + \underline{c}\,\underline{\varepsilon} \otimes \underline{c}(\underline{B} \times \underline{c}\,\underline{\varepsilon}) + \underline{c}(\underline{B} \times \underline{c}\,\underline{\varepsilon}) \otimes \underline{c}\,\underline{\varepsilon} \\
& - \underline{c}\,\underline{\varepsilon} \otimes (\underline{B} \times \underline{c}^2\underline{\varepsilon}) - (\underline{B} \times \underline{c}^2\underline{\varepsilon}) \otimes \underline{c}\,\underline{\varepsilon}] - \chi_{10}[\underline{c}\,\underline{B} \otimes (\underline{\varepsilon} \times \underline{B}) \\
& + (\underline{\varepsilon} \times \underline{B}) \otimes \underline{c}\,\underline{B}] - \chi_{11}[\underline{c}\,\underline{B} \otimes \underline{c}(\underline{\varepsilon} \times \underline{B}) + \underline{c}(\underline{\varepsilon} \times \underline{B}) \otimes \underline{c}\,\underline{B} \\
& + (\underline{\varepsilon} \times \underline{B}) \otimes \underline{c}^2\underline{B} + \underline{c}^2\underline{B} \otimes (\underline{\varepsilon} \times \underline{B})] + \underline{P} \otimes \underline{\varepsilon} + \underline{M} \otimes \underline{B}
\end{aligned}
$$

(6.34)

$$
\begin{aligned}
\underline{P} =\ & -\chi_1 \underline{\varepsilon} - \chi_2 \underline{c}\,\underline{\varepsilon} - \chi_3 \underline{c}^2\underline{\varepsilon} - \chi_4 \underline{B} - \chi_5 \underline{c}\,\underline{B} - \chi_6 \underline{c}^2\underline{B} - \chi_7[\underline{c}(\underline{B} \times \underline{\varepsilon}) \\
& - \underline{B} \times \underline{c}\,\underline{\varepsilon}] - \chi_8[\underline{c}^2(\underline{B} \times \underline{\varepsilon}) - \underline{B} \times \underline{c}^2\underline{\varepsilon}] - \chi_9[\underline{c}^2(\underline{B} \times \underline{c}\,\underline{\varepsilon}) \\
& - \underline{c}(\underline{B} \times \underline{c}^2\underline{\varepsilon})] - \chi_{10}\underline{B} \times \underline{c}\,\underline{B} - \chi_{11}\underline{B} \times \underline{c}^2\underline{B}
\end{aligned}
$$

(6.35)

$$
\begin{aligned}
\underline{M} =\ & -\mu_1 \underline{B} - \mu_2 \underline{c}\,\underline{B} - \mu_3 \underline{c}^2\underline{B} - \mu_4[\underline{c}(\underline{B} \times \underline{c}^2\underline{B}) + (\underline{c}^2\underline{B}) \times (\underline{c}\,\underline{B}) \\
& - \underline{c}^2(\underline{B} \times \underline{c}\,\underline{B}] - \chi_4 \underline{\varepsilon} - \chi_5 \underline{c}\,\underline{\varepsilon} - \chi_6 \underline{c}^2\underline{\varepsilon} - \chi_7 \underline{\varepsilon} \times \underline{c}\,\underline{\varepsilon} - \chi_8 \underline{\varepsilon} \times \underline{c}^2\underline{\varepsilon} \\
& - \chi_9 (\underline{c}\,\underline{\varepsilon}) \times (\underline{c}^2\underline{\varepsilon}) - \chi_{10}[\underline{c}(\underline{\varepsilon} \times \underline{B}) - \underline{\varepsilon} \times \underline{c}\,\underline{B}] \\
& - \chi_{11}[\underline{c}^2(\underline{\varepsilon} \times \underline{B}) - \underline{\varepsilon} \times \underline{c}^2\underline{B}]
\end{aligned}
$$

(6.36)

where

$$\tau_1 = -n_0 \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{3} \frac{\partial \Psi_0}{\partial I_3} \ , \quad \tau_2 = -2n_0 \frac{\partial \Psi_0}{\partial I_1} + n_0 (I_1^2 - I_2) \frac{\partial \Psi_0}{\partial I_3}$$

$$\tau_3 = -2n_0 \frac{\partial \Psi_0}{\partial I_2} - 2n_0 I_1 \frac{\partial \Psi_0}{\partial I_3} \ , \quad \tau_4 = -2n_0 \frac{\partial \Psi_0}{\partial E_1} \ , \quad \tau_5 = -2n_0 \frac{\partial \Psi_0}{\partial E_2}$$

$$\tau_6 = -2n_0 \frac{\partial \Psi_0}{\partial B_1} \ , \quad \tau_7 = -2n_0 \frac{\partial \Psi_0}{\partial B_2} \ , \quad \tau_8 = -|J| n_0 \frac{\partial \Psi_0}{\partial B_3}$$

$$\tau_9 = -n_0 \mathcal{E} \, \varepsilon \, \mathcal{B} \frac{\partial \Psi_0}{\partial B_5} \ , \quad \tau_{10} = -n_0 (\mathcal{E} \, \varepsilon \, \mathcal{B}) \frac{\partial \Psi_0}{\partial B_6}$$

$$\chi_1 = n_0 \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{3} \frac{\partial \Psi_0}{\partial E_2} \ , \quad \chi_2 = 2n_0 \frac{\partial \Psi_0}{\partial E_0} - n_0 (I_1^2 - I_2) \frac{\partial \Psi_0}{\partial E_2}$$

(6.37)

$$\chi_3 = 2n_0 \frac{\partial \Psi_0}{\partial E_1} + 2n_0 I_1 \frac{\partial \Psi_0}{\partial E_2} \ , \quad \chi_4 = n_0 \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{6} \mathcal{B} \, \varepsilon \, \mathcal{E} \frac{\partial \Psi_0}{\partial B_6}$$

$$\chi_5 = 2n_0 \mathcal{E} \, \varepsilon \, \mathcal{B} \frac{\partial \Psi_0}{\partial B_4} + n_0 (\mathcal{B} \varepsilon^2 \mathcal{E}) \frac{\partial \Psi_0}{\partial B_5} + n_0 \frac{\partial \Psi_0}{\partial B_6} [I_1 (\mathcal{B} \varepsilon^2 \mathcal{E}) - (I_1^2 - I_2)(\mathcal{B} \varepsilon \mathcal{E})$$

$$\qquad + \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{6} \mathcal{B} \cdot \mathcal{E}] + n_0 |J| \, \mathcal{E} \cdot (\varepsilon \mathcal{B}) \times \mathcal{B} \frac{\partial \Psi_0}{\partial B_{10}}$$

$$\qquad + n_0 |J| \, \mathcal{E} \ (\varepsilon^2 \mathcal{B}) \times \mathcal{B} \frac{\partial \Psi_0}{\partial B_{11}}$$

$$\chi_6 = n_0 \mathcal{B} \, \varepsilon \, \mathcal{B} \frac{\partial \Psi_0}{\partial B_5} + n_0 I_1 \mathcal{B} \, \varepsilon \, \mathcal{B} \frac{\partial \Psi_0}{\partial B_6} \ , \quad \chi_7 = n_0 |J| \frac{\partial \Psi_0}{\partial B_7}$$

$$\chi_8 = n_0 |J| \frac{\partial \Psi_0}{\partial B_8} \ , \quad \chi_9 = n_0 |J| \frac{\partial \Psi_0}{\partial B_9} \ , \quad \chi_{10} = -n_0 |J| \frac{\partial \Psi_0}{\partial B_{10}} \mathcal{B} \, \varepsilon \, \mathcal{E}$$

$$\chi_{11} = -n_0 |J| \mathcal{B} \, \varepsilon \, \mathcal{E} \frac{\partial \Psi_0}{\partial B_{11}} \ , \quad \mu_1 = n_0 \frac{(I_1^3 + 2I_3 - 3I_2 I_1)}{3} \frac{\partial \Psi_0}{\partial B_2}$$

$$\mu_2 = 2n_0 \frac{\partial \Psi_0}{\partial B_0} - n_0 (I_1^2 - I_2) \frac{\partial \Psi_0}{\partial B_2} \ , \quad \mu_3 = 2n_0 \frac{\partial \Psi_0}{\partial B_1} + 2n_0 I_1 \frac{\partial \Psi_0}{\partial B_2}$$

$$\mu_4 = n_0 \frac{\partial \Psi_0}{\partial B_3} |J|$$

Also using the results of Smith [33], we can show that the conduction current and heat flow have the following forms:

$$
\begin{aligned}
\underline{j} = {}& \sigma_1 \mathcal{E} + \sigma_2 \varepsilon \mathcal{E} + \sigma_3 \varepsilon^2 \mathcal{E} + \sigma_4 \overset{*}{\mathcal{E}} + \sigma_5 \varepsilon \overset{*}{\mathcal{E}} + \sigma_6 \varepsilon^2 \overset{*}{\mathcal{E}} + \sigma_7 \mathcal{B} \times \mathcal{E} + \\
& \sigma_8 \varepsilon(\mathcal{B} \times \mathcal{E}) + \sigma_9 \varepsilon^2(\mathcal{B} \times \mathcal{E}) + \sigma_{10} \mathcal{B} \times \varepsilon \mathcal{E} + \sigma_{11} \mathcal{B} \times \varepsilon^2 \mathcal{E} + \\
& \sigma_{12} \mathcal{B} \times \overset{*}{\mathcal{E}} + \sigma_{13} \varepsilon(\mathcal{B} \times \overset{*}{\mathcal{E}}) + \sigma_{14} \varepsilon^2(\mathcal{B} \times \overset{*}{\mathcal{E}}) + \sigma_{15} \mathcal{B} \times \varepsilon \overset{*}{\mathcal{E}} + \\
& \sigma_{16} \mathcal{B} \times \varepsilon^2 \overset{*}{\mathcal{E}} + \sigma_{17}[\varepsilon(\mathcal{B} \times \varepsilon^2 \mathcal{E}) - \varepsilon^2(\mathcal{B} \times \varepsilon \mathcal{E})] + \sigma_{18}[\varepsilon(\mathcal{B} \times \varepsilon^2 \overset{*}{\mathcal{E}}) \\
& - \varepsilon^2(\mathcal{B} \times \varepsilon \overset{*}{\mathcal{E}})] + \sigma_{19} \mathcal{B} + \sigma_{20} \varepsilon \mathcal{B} + \sigma_{21} \varepsilon^2 \mathcal{B} + \sigma_{22} \mathcal{B} \times \varepsilon \mathcal{B} \\
& + \sigma_{23} \mathcal{B} \times \varepsilon^2 \mathcal{B}
\end{aligned}
\tag{6.38}
$$

$$
\begin{aligned}
\underline{q} = {}& \kappa_1 \overset{*}{\mathcal{E}} + \kappa_2 \varepsilon \overset{*}{\mathcal{E}} + \kappa_3 \varepsilon^2 \overset{*}{\mathcal{E}} + \kappa_4 \mathcal{E} + \kappa_5 \varepsilon \mathcal{E} + \kappa_6 \varepsilon^2 \mathcal{E} + \kappa_7 \mathcal{B} \times \overset{*}{\mathcal{E}} \\
& + \kappa_8 \varepsilon(\mathcal{B} \times \overset{*}{\mathcal{E}}) + \kappa_9 \varepsilon^2(\mathcal{B} \times \overset{*}{\mathcal{E}}) + \kappa_{10} \mathcal{B} \times \varepsilon \overset{*}{\mathcal{E}} + \kappa_{11} \mathcal{B} \times \varepsilon^2 \overset{*}{\mathcal{E}} + \kappa_{12} \mathcal{B} \times \mathcal{E} \\
& + \kappa_{14} \varepsilon^2(\mathcal{B} \times \mathcal{E}) + \kappa_{15} \mathcal{B} \times \varepsilon \mathcal{E} + \kappa_{16} \mathcal{B} \times \varepsilon^2 \mathcal{E} + \kappa_{17}[\varepsilon(\mathcal{B} \times \varepsilon^2 \overset{*}{\mathcal{E}}) \\
& - \varepsilon^2(\mathcal{B} \times \varepsilon \overset{*}{\mathcal{E}})] + \kappa_{18}[\varepsilon(\mathcal{B} \times \varepsilon^2 \mathcal{E}) - \varepsilon^2(\mathcal{B} \times \varepsilon \mathcal{E})] + \kappa_{19} \mathcal{B} + \kappa_{13}\varepsilon(\mathcal{B} \times \mathcal{E}) \\
& + \kappa_{20} \varepsilon \mathcal{B} + \kappa_{21} \varepsilon^2 \mathcal{B} + \kappa_{22} \mathcal{B} \times \varepsilon \mathcal{B} + \kappa_{23} \mathcal{B} \times \varepsilon^2 \mathcal{B}
\end{aligned}
\tag{6.39}
$$

where $\sigma_1, \ldots, \sigma_{18}$ and $\kappa_1, \ldots, \kappa_{18}$ are functions of the invariants listed in (6.32) and the following invariants:

$$\overset{*}{E}\cdot\overset{*}{E}\,,\;\overset{*}{E}\,\varepsilon\,\overset{*}{E}\,,\;\overset{*}{E}\,\varepsilon^2\,\overset{*}{E}\,,\;(\overset{*}{E}\,\varepsilon\,B)^2\,,\;(\overset{*}{E}\,\varepsilon\,B)(B\,\varepsilon^2\,\overset{*}{E})\,,$$

$$(\overset{*}{E}\,\varepsilon\,B)(\overset{*}{E}\cdot B),\overset{*}{E}\cdot B\times\varepsilon\,\overset{*}{E}\,,\;\overset{*}{E}\cdot B\times\varepsilon^2\,\overset{*}{E}\,,\;\overset{*}{E}\,\varepsilon(E\times\varepsilon^2\,\overset{*}{E})\,,$$

$$(6.40)\qquad (\overset{*}{E}\,\varepsilon\,B)(B\times\varepsilon\,B)\cdot\overset{*}{E}\,,\;(\overset{*}{E}\cdot B\times\varepsilon^2\,B)(B\,\varepsilon\,\overset{*}{E})\,,\;E\cdot\overset{*}{E}\,,$$

$$E\,\varepsilon\,\overset{*}{E}\,,\;E\,\varepsilon^2\,\overset{*}{E}\,,\;(B\cdot E)(\overset{*}{E}\,\varepsilon\,B)\,,\;(B\cdot\overset{*}{E})(E\,\varepsilon\,B)\,,$$

$$(E\,\varepsilon\,B)(B\,\varepsilon\,\overset{*}{E})\,,\;(\overset{*}{E}\,\varepsilon\,B)(B\,\varepsilon^2\,B)\,,\;(E\,\varepsilon\,B)(B\,\varepsilon^2\,\overset{*}{E})\,,$$

$$(\hat{E}\cdot B)(B\,\varepsilon^2\,\overset{*}{E})-(\overset{*}{E}\cdot B)(B\,\varepsilon^2\,E)\,,\;\overset{*}{E}\cdot B\times\varepsilon\,E\,,\;\overset{*}{E}\cdot B\times\varepsilon\,E\,,$$

$$E\cdot B\times\varepsilon\,\overset{*}{E}\,,\;\overset{*}{E}\cdot B\times\varepsilon^2\,E\,,\;E\cdot B\times\varepsilon^2\,\overset{*}{E}\,,\;\overset{*}{E}\,\varepsilon(B\times\varepsilon^2\,E)$$

$$+E\,\varepsilon(B\times\varepsilon^2\,\overset{*}{E})\,,\;(\overset{*}{E}\,\varepsilon\,B)B\,\varepsilon(B\times E)+(E\,\varepsilon\,B)B\,\varepsilon(B\times\overset{*}{E})\,,$$

$$(\overset{*}{E}\cdot B\times\varepsilon^2\,B)(B\,\varepsilon\,E)+(E\cdot B\times\varepsilon^2\,B)B\,\varepsilon\,\overset{*}{E}$$

and $\sigma_{19},\ \ldots,\ \sigma_{23}$ have the form

$$\sigma_{19}=(\overset{*}{E}\,\varepsilon\,B)\,\sigma_{24}+(E\,\varepsilon\,B)\,\sigma_{25}-(B\,\varepsilon^2\,E)\,\sigma_{26}-(B\,\varepsilon^2\,\overset{*}{E})\sigma_{27}$$

$$\sigma_{20}=(B\cdot E)\,\sigma_{28}+(B\cdot\overset{*}{E})\,\sigma_{29}+(E\,\varepsilon\,B)\,\sigma_{30}+(\overset{*}{E}\,\varepsilon\,B)\,\sigma_{31}$$

$$(6.41)\qquad\qquad +(B\,\varepsilon^2\,E)\,\sigma_{32}+(B\,\varepsilon^2\,\overset{*}{E})\,\sigma_{33}+B\,\varepsilon(B\times\overset{*}{E})\,\sigma_{34}$$

$$\qquad\qquad +B\,\varepsilon(B\times E)\,\sigma_{35}+(E\cdot B\times\varepsilon^2\,B)\,\sigma_{36}+(\overset{*}{E}\cdot B\times\varepsilon^2\,B)\,\sigma_{37}$$

$$\sigma_{21}=(\overset{*}{E}\,\varepsilon\,B)\,\sigma_{38}+(E\,\varepsilon\,B)\,\sigma_{39}+(E\cdot B)\,\sigma_{26}+(\overset{*}{E}\cdot B)\,\sigma_{27}$$

$$\sigma_{22}=-(\overset{*}{E}\,\varepsilon\,B)\,\sigma_{34}-(E\,\varepsilon\,B)\,\sigma_{35}$$

$$\sigma_{23}=(B\,\varepsilon\,E)\,\sigma_{36}+(B\,\varepsilon\,\overset{*}{E})\,\sigma_{37}$$

where $\sigma_{24},\ \ldots,\ \sigma_{37}$ are functions of the invariants (6.32) and (6.40). By replacing the σ_{rs} by κ_{rs} in (6.41) one obtains the functional forms of $\kappa_{19},\ \ldots,\ \kappa_{23}$.

For an isotropic material, from (6.32), (6.33) and (6.34) to (6.39) the set of dependent quantities ϵ , η_{oo} , $\underset{\sim}{t}$, $\underset{\sim}{\mathcal{P}}$, $\underset{\sim}{\mathcal{M}}$ are covariant tensor functions of $\underset{\sim}{c}$, $\underset{\sim}{\mathcal{E}}$, $\underset{\sim}{\mathcal{B}}$ and $\underset{\sim}{u}$ under the full Lorentz group, and $\underset{\sim}{j}$ and $\underset{\sim}{q}$ have the forms (6.38) and (6.39). (For materials with no center of symmetry, they are covariant tensor functions under the proper Lorentz group.) While the constitutive equations for isotropic materials are deducible from the general forms (6.18) to (6.27), in many cases it is more convenient to rederive these equations from the second law of thermodynamics.

The second law (5.26) for an isotropic material becomes (using (1.55) and (1.65)):

$$- n_o(\eta_{oo} + \frac{\partial \Psi_o}{\partial \theta}) D\theta - \frac{1}{\theta} \{t^{(\alpha\beta)} + \mathcal{P}^{[\alpha} \mathcal{E}^{\beta]} + \mathcal{M}^{[\alpha} \mathcal{B}^{\beta]}$$

$$+ 2n_o \frac{\partial \Psi_o}{\partial c_{\gamma\beta}} c^{\alpha}{}_{\gamma} - n_o \frac{\partial \Psi_o}{\partial u_\alpha} u^\beta - n_o \frac{\partial \Psi_o}{\partial \mathcal{E}^\gamma} u^\gamma \mathcal{E}^\alpha u^\beta - n_o \frac{\partial \Psi_o}{\partial \mathcal{B}^\gamma} u^\gamma \mathcal{B}^\alpha u^\beta$$

(6.42) $\quad + (\mathcal{P}^\alpha + \frac{\partial \Psi_o}{\partial \mathcal{E}_\alpha}) \mathcal{E}^\beta + (\mathcal{M}^\alpha + \frac{\partial \Psi_o}{\partial \mathcal{B}_\alpha}) \mathcal{B}^\beta\} \gamma_{\alpha\sigma} x^\sigma_K DX^K{}_{,\beta}$

$$- \frac{1}{\theta} (\mathcal{P}_\alpha + n_o \frac{\partial \Psi_o}{\partial \mathcal{E}^\alpha}) x^\alpha_K D\mathcal{E}^K - \frac{1}{\theta} (\mathcal{M}_\alpha + n_o \frac{\partial \Psi_o}{\partial \mathcal{B}^\alpha}) x^\alpha_K D\mathcal{B}^K$$

$$- \frac{q^\beta \overset{*}{\theta}_\beta}{\theta^2} + \frac{j^\alpha \mathcal{E}_\alpha}{\theta} \geq 0$$

A necessary and sufficient condition for (6.42) is

(6.43) $\quad \eta_{oo} = - \frac{\partial \Psi_o}{\partial \theta}$

(6.44) $\quad t^{(\alpha\beta)} = - 2n_o s^\alpha{}_\tau s^\beta{}_\sigma \frac{\partial \Psi_o}{\partial c_{\gamma(\sigma}} c^{\tau)}{}_\gamma$

$$(6.45) \qquad s^{\alpha}{}_{\tau} \, s^{\beta}{}_{\sigma} \, [2n_o \frac{\partial \psi_o}{\partial c_{\gamma[\tau}} \, c^{\sigma]}{}_{\gamma} - \wp^{[\tau} \mathcal{E}^{\sigma]} - \mathcal{M}^{[\tau} \mathcal{B}^{\sigma]}) = 0$$

$$(6.46) \qquad \wp^{\alpha} = -n_o \, s^{\alpha}{}_{\gamma} \frac{\partial \psi_o}{\partial \mathcal{E}_{\gamma}}$$

$$(6.47) \qquad \mathcal{M}^{\alpha} = -n_o \, s^{\alpha}{}_{\gamma} \frac{\partial \psi_o}{\partial \mathcal{B}_{\gamma}}$$

$$(6.48) \qquad 2n_o \frac{\partial \psi_o}{\partial c_{\gamma\beta}} c_{\gamma\alpha} u_{\beta} + n_o \frac{\partial \psi_o}{\partial \mathcal{E}_{\beta}} u_{\beta} \mathcal{E}_{\alpha} + n_o \frac{\partial \psi_o}{\partial \mathcal{B}_{\beta}} u_{\beta} \mathcal{B}_{\alpha} + n_o \frac{\partial \psi_o}{\partial u^{\gamma}} s^{\gamma}{}_{\alpha} = 0$$

$$(6.49) \qquad j^{\alpha} \mathcal{E}_{\alpha} - \frac{q^{\beta} \overset{*}{\theta}_{\beta}}{\theta} \geq 0$$

The requirement that ψ_o is an invariant function of θ , $c_{\alpha\beta}$, $\mathcal{E}_{\alpha}$, $\mathcal{B}_{\alpha}$ and u_{α} under Lorentz transformations gives:

$$(6.50) \qquad 2 \frac{\partial \psi_o}{\partial c_{\gamma[\alpha}} c^{\beta]}{}_{\gamma} + \frac{\partial \psi_o}{\partial \mathcal{E}_{[\alpha}} \mathcal{E}^{\beta]} + \frac{\partial \psi_o}{\partial \mathcal{B}_{[\alpha}} \mathcal{B}^{\beta]} + \frac{\partial \psi_o}{\partial u_{[\alpha}} u^{\beta]} = 0$$

By substituting (6.46) and (6.47) into (6.50) one sees that (6.45) and (6.48) are satisfied. By using (5.19) in conjunction with (6.50) it can be established that a necessary and sufficient condition that the constitutive equations for an isotropic material are Lorentz invariant and satisfy the second law of thermodynamics is:

    i.   The free energy ψ_o is a scalar invariant function of θ , c , $\mathcal{E}$, $\mathcal{B}$, u under the full Lorentz group.

    ii.   The entropy η_{oo} , the stress tensor t , the polarization vector $\wp$, and the magnetization vector $\mathcal{M}$ are determined from the free energy by

$$(6.51) \qquad \eta_{oo} = -\frac{\partial \psi_o}{\partial \theta}$$

$$(6.52) \qquad \underset{\sim}{t} = -2n_o \underset{\sim}{c} \frac{\partial \psi_o}{\partial \underset{\sim}{c}}$$

$$(6.53) \qquad \underset{\sim}{\mathcal{P}} = -n_o \frac{\partial \psi_o}{\partial \underset{\sim}{\mathcal{E}}}$$

$$(6.54) \qquad \underset{\sim}{\mathcal{M}} = -n_o \frac{\partial \psi_o}{\partial \underset{\sim}{\mathcal{B}}}$$

iii. The nonmechanical $\underset{\sim}{p}$ is determined by (6.23).

iv. The conduction current $\underset{\sim}{j}$ and heat flow vector $\underset{\sim}{q}$ have the forms (6.38) and (6.39) and satisfy the inequality

$$(6.55) \qquad \underset{\sim}{j} \cdot \underset{\sim}{\mathcal{E}} - \frac{\underset{\sim}{q} \cdot \overset{*}{\underset{\sim}{\theta}}}{\theta} \geq 0$$

It should be noted that (5.19) is satisfied identically if i. and ii. hold. If ψ_o is a scalar invariant of $\underset{\sim}{c}$, $\underset{\sim}{\mathcal{E}}$, $\underset{\sim}{\mathcal{B}}$, $\underset{\sim}{u}$ under the full Lorentz group, it must have the form

$$(6.56) \qquad \psi = \psi(\theta, I_a, e_b, b_c) \qquad \left(\begin{array}{l} a = 1,2,3 \\ b = 0,1,2 \\ c = 0,1,\ldots,11 \end{array} \right)$$

where

$$I_1 = \text{tr}\,\varepsilon \quad , \quad I_2 = \text{tr}\,\varepsilon^2 \quad , \quad I_3 = \text{tr}\,\varepsilon^3$$

$$(6.57) \quad e_0 = \mathcal{E}\cdot\mathcal{E} \quad , \quad e_1 = \mathcal{E}\varepsilon\mathcal{E} \quad , \quad e_2 = \mathcal{E}\varepsilon^2\mathcal{E}$$

$$b_0 = \mathcal{B}\cdot\mathcal{B} \quad , \quad b_1 = \mathcal{B}\varepsilon\mathcal{B} \quad , \quad b_2 = \mathcal{B}\varepsilon^2\mathcal{B}$$

$$b_3 = \mathcal{B}\cdot(\varepsilon\mathcal{B})\times(\varepsilon^2\mathcal{B}) \quad , \quad b_4 = (\mathcal{E}\cdot\mathcal{B})^2$$

$$b_5 = (\mathcal{E}\cdot\mathcal{B})(\mathcal{B}\varepsilon\mathcal{E}) \quad , \quad b_6 = (\mathcal{E}\cdot\mathcal{B})(\mathcal{B}\varepsilon^2\mathcal{E})$$

$$b_7 = \mathcal{E}\varepsilon(\mathcal{B}\times\mathcal{E}) \quad , \quad b_8 = \mathcal{E}\varepsilon^2(\mathcal{B}\times\mathcal{E}) \quad , \quad b_9 = (\mathcal{B}\times\varepsilon\mathcal{E})\varepsilon^2\mathcal{E}$$

$$b_{10} = (\mathcal{E}\cdot\mathcal{B})\,\mathcal{B}\varepsilon(\mathcal{E}\times\mathcal{E}) \quad , \quad b_{11} = (\mathcal{E}\cdot\mathcal{B})\,\mathcal{B}\varepsilon^2(\mathcal{E}\times\mathcal{B})$$

The theory presented in this section provides a theoretical justification for the nonrelativistic theories of Toupin [5] and Dixon and Eringen [6] for moving electromagnetic materials. The application of the second law of thermodynamics complements the researches of Jordan and Eringen [3], [4], Toupin [5] and Dixon and Eringen [6]. The constitutive equations for a polar able material ($\mathcal{M} = 0$, $\underline{j} = 0$, $\sigma_0 = 0$) are equivalent (under the appropriate change of variables) to those of Toupin [5] if terms of the order v^2/c^2 are neglected. The general form of the constitutive equations of Dixon and Eringen [6] are valid if one replaces $\underline{D}$ by $\underline{D} + \dfrac{v \times H}{c^2}$ and $\underline{B}$ by $\underline{B} - \dfrac{v \times E}{c^2}$.[1]

---

[1] We are here using mks units. In the mks system of units the ratio $|\underline{D}|/|\dfrac{v \times H}{c^2}|$ is of the same order of magnitude as $|\underline{E}|/|\underline{v} \times \underline{B}|$. The same is true for the ratios $|\underline{E}|/|\underline{v} \times \underline{D}|$ and $|\underline{B}|/|\dfrac{v \times E}{c^2}|$.

The specific forms of the constitutive equations of Jordan and Eringen [3], [4] and Dixon and Eringen [6] are significantly simplified by the second law. For example, the stress tensor, polarization and magnetization are independent of the temperature gradient. For an isotropic material the second law does not allow a term in the polarization similar to the Hall effect in the current.

Electromagnetic Fluid

For a viscous fluid which possesses electrical properties an appropriate set of independent variables is[1]:

$$(6.58) \qquad \theta \;,\; n_o \;,\; \overset{*}{d}_{\alpha\beta} \;,\; \overset{*}{\theta}_{\alpha} \;,\; \mathscr{E}_{\alpha} \;,\; \mathscr{B}_{\alpha} \;,\; u_{\alpha}$$

The dependent variables in this case are:

$$(6.59) \qquad \epsilon \;,\; \eta_{oo} \;,\; q^{\beta} \;,\; t^{(\alpha\beta)} \;,\; \mathscr{P}^{\alpha} \;,\; \mathscr{M}^{\alpha} \;,\; j^{\alpha}$$

with p^{α} and $t^{[\alpha\beta]}$ determined from (5.18) and (5.19). The constitutive equations for the set (6.59) must satisfy the principles enunciated in Chapter III. In particular, they must satisfy the second law of thermodynamics, the appropriate form of which is now (5.26). Using (6.58) and (6.59), (5.26) reduces to:

---

[1]See footnotes p. 60 and p. 100 on the independence of these variables.

$$-n_o(\eta_{oc} + \frac{\partial \psi}{\partial \theta})D\theta + \frac{1}{\theta}[t^{(\alpha\beta)} + n_o^2 \frac{\partial \psi_o}{\partial n_o} s^{\alpha\beta}] \overset{*}{d}_{\alpha\beta} - \frac{n_o}{\theta}\frac{\partial \psi_o}{\partial u_\beta} Du_\beta$$

$$(6.60) \quad - \frac{n_o}{\theta}\frac{\partial \psi_o}{\partial \overset{*}{\theta}_\beta} D\overset{*}{\theta}_\beta - \frac{n_o}{\theta}\frac{\partial \psi_o}{\partial \overset{*}{d}_{\alpha\beta}} D\overset{*}{d}_{\alpha\beta} + \frac{\mathcal{P}^{[\alpha}\mathcal{E}^{\beta]}}{\theta}\overset{*}{\omega}_{\alpha\beta} + \frac{\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]}}{\theta}\overset{*}{\omega}_{\alpha\beta}$$

$$- [\mathcal{P}_\alpha + n_o\frac{\partial \psi_o}{\partial \mathcal{E}^\alpha}]\frac{D\mathcal{E}^\alpha}{\theta} - [\mathcal{M}_\alpha + n_o\frac{\partial \psi_o}{\partial \mathcal{B}^\alpha}]\frac{D\mathcal{B}^\alpha}{\theta} + \frac{J^\alpha \mathcal{E}_\alpha}{\theta} - \frac{q^\alpha \overset{*}{\theta}_\alpha}{\theta^2} \geq 0$$

At any point, $\underline{x}$, in space-time the following quantities can be assigned arbitrarily:

$$(6.61) \quad \theta \, , \, n_o \, , \, \overset{*}{\theta}{}^i \, , \, \mathcal{E}^i \, , \, \mathcal{B}^i \, , \, u^i \, , \, \overset{*}{d}{}^{ij} \, , \, \overset{*}{\omega}{}^{ij} \, , \, D\theta^i \, , \, D\mathcal{E}^i \, ,$$

$$D\mathcal{B}^i \, , \, Du^i \, , \, D\overset{*}{d}{}^{ij} \qquad (i,j = 1,2,3)$$

By the argument leading from (3.13) to (3.14), a necessary and sufficient condition that the entropy production (6.60) is non-negative is that the following relations hold:[1]

$$(6.62) \quad \eta_{oo} = -\frac{\partial \psi_o}{\partial \theta}$$

$$(6.63) \quad \frac{\partial \psi_o}{\partial u^\gamma} s^\gamma{}_\alpha + \frac{\partial \psi_o}{\partial \mathcal{B}_\gamma} u^\gamma \mathcal{B}_\alpha + \frac{\partial \psi_o}{\partial \mathcal{E}_\gamma} u^\gamma \mathcal{E}_\alpha = 0$$

$$(6.64) \quad \frac{\partial \psi_o}{\partial \overset{*}{\theta}_\beta} = 0$$

$$(6.65) \quad \frac{\partial \psi_o}{\partial \overset{*}{d}_{\beta\alpha}} = 0$$

---

[1] Since not all components of the tensors in set (6.58) are independent, in deducing (6.62) to (6.67) one should consider ψ_o as a function of $\theta \, , \, n_o \, , \, \overset{*}{\theta}{}^i \, , \, \mathcal{E}^i \, , \, \mathcal{B}^i \, , \, \overset{*}{d}{}^{ij} \, , \, u^i$.

$$(6.66) \qquad \mathcal{P}_\alpha = n_0 \frac{\partial \Psi_0}{\partial \mathcal{E}^\gamma} s^\gamma{}_\alpha$$

$$(6.67) \qquad \mathcal{M}_\alpha = -n_0 \frac{\partial \Psi_0}{\partial \mathcal{B}^\gamma} s^\gamma{}_\alpha$$

$$(6.68) \qquad \mathcal{P}^{[\alpha} \mathcal{E}^{\beta]} + \mathcal{M}^{[\alpha} \mathcal{B}^{\beta]} = 0$$

$$(6.69) \qquad t^{(\alpha\beta)} = -n_0{}^2 \frac{\partial \Psi_0}{\partial n_0} s^{\alpha\beta} + {}_D t^{\alpha\beta}$$

$$_D t^{\alpha\beta} = {}_D t^{\beta\alpha} \quad , \quad {}_D t_{\alpha\beta} u^\beta = 0$$

$$(6.70) \qquad _D t^{\alpha\beta} \overset{*}{d}_{\alpha\beta} - \frac{q^\alpha \overset{*}{\theta}_\alpha}{\theta} + j^\alpha \mathcal{E}_\alpha \geq 0$$

The following general statement can be made about the form of the constitutive equations: A necessary and sufficient condition that the constitutive equations satisfy the second law of thermodynamics, the balance of moment of energy-momentum, and are covariant under proper orthochronous Lorentz transformations is:

1. The free energy assumes the form:

$$(6.71) \qquad \Psi_0 = \Psi_0(\theta, n_0, J_1, J_2, J_3)$$

where

$$(6.72) \qquad J_1 = \mathcal{E} \cdot \mathcal{E} \quad , \quad J_2 = \mathcal{B} \cdot \mathcal{B} \quad , \quad J_3 = \mathcal{E} \cdot \mathcal{B}$$

ii.   The entropy η_{∞} , the stress tensor $\underset{\sim}{t}$, the polarization vector $\underset{\sim}{\mathscr{P}}$, and the magnetization vector $\underset{\sim}{\mathscr{M}}$ are related to the free energy by:

(6.73) $\eta_{oo} \doteq - \dfrac{\partial \Psi_o}{\partial \theta}$

(6.74) $\underset{\sim}{t} = - n_o^2 \dfrac{\partial \Psi_o}{\partial n_o} \underset{\sim}{S} + {}_D\underset{\sim}{t}$

(6.75) $\underset{\sim}{\mathscr{P}} = -2n_o \dfrac{\partial \Psi_o}{\partial J_1} \underset{\sim}{\mathscr{E}} - n_o \dfrac{\partial \Psi_o}{\partial J_3} \underset{\sim}{\mathscr{B}}$

(6.76) $\underset{\sim}{\mathscr{M}} = -2n_o \dfrac{\partial \Psi_o}{\partial J_2} \underset{\sim}{\mathscr{B}} - n_o \dfrac{\partial \Psi_o}{\partial J_3} \underset{\sim}{\mathscr{E}}$

iii.   The nonmechanical momentum $\underset{\sim}{p}$ is determined by:

(6.77) $\underset{\sim}{p} = \underset{\sim}{q} - \underset{\sim}{\mathscr{P}} \times \underset{\sim}{\mathscr{B}} - \underset{\sim}{\mathscr{E}} \times \underset{\sim}{\mathscr{M}}$

iv.   The conduction current $\underset{\sim}{j}$, the heat flow vector $\underset{\sim}{q}$, and the dissipative part of the stress tensor ${}_D\underset{\sim}{t}$ have the form:

(6.78) $\underset{\sim}{q} = \kappa_1 \overset{*}{\underset{\sim}{g}} + \kappa_2 \underset{\sim}{\mathscr{E}} + \kappa_3 \underset{\sim}{\mathscr{B}} + \kappa_4 \overset{*}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}} + \kappa_5 \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + \kappa_6 \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} + \kappa_7 \overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + \kappa_8 \overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + {}_9\overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}}$

$\qquad + \kappa_{10} \overset{*}{\underset{\sim}{g}} \times \underset{\sim}{\mathscr{E}} + \kappa_{11} \overset{*}{\underset{\sim}{g}} \times \underset{\sim}{\mathscr{B}} + \kappa_{12} \underset{\sim}{\mathscr{E}} \times \underset{\sim}{\mathscr{B}} + \kappa_{13} \overset{*}{\underset{\sim}{g}} \times \overset{*}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}} + \kappa_{14} \underset{\sim}{\mathscr{E}} \times \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}}$

$\qquad + \kappa_{15} \underset{\sim}{\mathscr{B}} \times \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} + \kappa_{16} \overset{*}{\underset{\sim}{g}} \times \overset{*2}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}} + \kappa_{17} \underset{\sim}{\mathscr{E}} \times \overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + \kappa_{18} \underset{\sim}{\mathscr{B}} \times \overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} + \kappa_{19} \overset{*}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}} \times \overset{*2}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}}$

$\qquad + \kappa_{20} \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} \times \overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + \kappa_{21} \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} \times \overset{*2}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} + \kappa_{22} \underset{\sim}{\mathscr{E}} \times \overset{*}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}} + \kappa_{23} \underset{\sim}{\mathscr{B}} \times \overset{*}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}}$

$\qquad + \kappa_{24} \underset{\sim}{\mathscr{B}} \times \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + \kappa_{25} \overset{*}{\underset{\sim}{g}} \times \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{E}} + \kappa_{26} \overset{*}{\underset{\sim}{g}} \times \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} + \kappa_{27} \underset{\sim}{\mathscr{E}} \times \overset{*}{\underset{\sim}{d}} \underset{\sim}{\mathscr{B}} + \kappa_{28} \underset{\sim}{\mathscr{E}} \times \overset{*2}{\underset{\sim}{d}} \overset{*}{\underset{\sim}{g}}$

$$+ \kappa_{29}\mathcal{B} \times \overset{*2*}{d}\ell + \kappa_{30}\mathcal{B} \times \overset{*2}{d}\mathcal{E} + \kappa_{31}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{E} + \kappa_{32}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{B} + \kappa_{33}\mathcal{E} \times \overset{*2}{d}\mathcal{B}$$

$$+ \kappa_{34}\overset{*}{d}\mathcal{E} \times \overset{*2*}{d}\ell + \kappa_{35}\overset{*}{d}\mathcal{B} \times \overset{*2*}{d}\ell + \kappa_{36}\overset{*}{d}\mathcal{B} \times \overset{*2}{d}\mathcal{E} + \kappa_{37}\overset{*}{d}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{E}$$

$$+ \kappa_{38}\overset{*}{d}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{B} + \kappa_{39}\overset{*}{d}\mathcal{E} \times \overset{*2}{d}\mathcal{B}$$

$$
\begin{aligned}
\underline{J} =\ & \sigma_1\mathcal{E} + \sigma_2\overset{*}{\ell} + \sigma_3\mathcal{B} + \sigma_4\overset{*}{d}\overset{*}{\ell} + \sigma_5\overset{*}{d}\mathcal{E} + \sigma_6\overset{*}{d}\mathcal{B} + \sigma_7\overset{*2*}{d}\ell + \sigma_8\overset{*2}{d}\mathcal{E} \\
& + \sigma_9\overset{*2}{d}\mathcal{B} + \sigma_{10}\overset{*}{\ell} \times \mathcal{E} + \sigma_{11}\overset{*}{\ell} \times \mathcal{B} + \sigma_{12}\mathcal{E} \times \mathcal{B} + \sigma_{13}\overset{*}{\ell} \times \overset{*}{d}\ell \\
& + \sigma_{14}\mathcal{E} \times \overset{*}{d}\mathcal{E} + \sigma_{15}\mathcal{B} \times \overset{*}{d}\mathcal{B} + \sigma_{16}\overset{*}{\ell} \times \overset{*2*}{d}\ell + \sigma_{17}\mathcal{E} \times \overset{*2}{d}\mathcal{E} \\
& + \sigma_{18}\mathcal{B} \times \overset{*2}{d}\mathcal{B} + \sigma_{19}\overset{*}{d}\overset{*}{\ell} \times \overset{*2*}{d}\ell + \sigma_{20}\overset{*}{d}\mathcal{E} \times \overset{*2}{d}\mathcal{E} + \sigma_{21}\overset{*}{d}\mathcal{B} \times \overset{*2}{d}\mathcal{B} \\
& + \sigma_{22}\mathcal{E} \times \overset{*}{d}\overset{*}{\ell} + \sigma_{23}\mathcal{B} \times \overset{*}{d}\overset{*}{\ell} + \sigma_{24}\mathcal{B} \times \overset{*}{d}\mathcal{E} + \sigma_{25}\overset{*}{\ell} \times \overset{*}{d}\mathcal{E} \\
& + \sigma_{26}\overset{*}{\ell} \times \overset{*}{d}\mathcal{B} + \sigma_{27}\mathcal{E} \times \overset{*}{d}\mathcal{B} + \sigma_{28}\mathcal{E} \times \overset{*2*}{d}\ell + \sigma_{29}\mathcal{B} \times \overset{*2*}{d}\ell \\
& + \sigma_{30}\mathcal{B} \times \overset{*2}{d}\mathcal{E} + \sigma_{31}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{E} + \sigma_{32}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{B} + \sigma_{33}\ \mathcal{E} \times \overset{*2}{d}\mathcal{B} \\
& + \sigma_{34}\overset{*}{d}\mathcal{E} \times \overset{*2*}{d}\ell + \sigma_{35}\overset{*}{d}\mathcal{B} \times \overset{*2*}{d}\ell + \sigma_{36}\overset{*}{d}\mathcal{B} \times \overset{*2}{d}\mathcal{E} \\
& + \sigma_{37}\overset{*}{d}\overset{*}{\ell} \times \overset{*2}{d}\mathcal{E} + \sigma_{38}\overset{**}{d}\ell \times \overset{*2}{d}\mathcal{B} + \sigma_{39}\overset{*}{d}\mathcal{E} \times \overset{*2}{d}\mathcal{B}
\end{aligned}
\tag{6.79}
$$

$$
\begin{aligned}
D^t =\ & \lambda_1\underline{\underline{I}} + \lambda_2\overset{*}{d} + \lambda_3\overset{*2}{d} + \lambda_4\overset{*}{\ell} \otimes \overset{*}{\ell} + \lambda_5\mathcal{E} \otimes \mathcal{E} + \lambda_6\mathcal{B} \otimes \mathcal{B} + \lambda_7[\ell \otimes \mathcal{E} + \mathcal{E} \otimes \ell] \\
& + \lambda_8[\ell \otimes \mathcal{B} + \mathcal{B} \otimes \ell] + \lambda_9[\mathcal{E} \otimes \mathcal{B} + \mathcal{B} \otimes \mathcal{E}] + \lambda_{10}[\ell \otimes (\ell \times \mathcal{E}) + (\ell \times \mathcal{E}) \otimes \ell] \\
& + \lambda_{11}[\ell \otimes (\ell \times \mathcal{B}) + (\ell \times \mathcal{B}) \otimes \ell] + \lambda_{12}[\mathcal{E} \otimes (\mathcal{E} \times \mathcal{B}) + (\mathcal{E} \times \mathcal{B}) \otimes \mathcal{E}] \\
& + \lambda_{13}[\mathcal{E} \otimes (\mathcal{E} \times \ell) + (\mathcal{E} \times \ell) \otimes \mathcal{E}] + \lambda_{14}[\mathcal{B} \otimes (\mathcal{B} \times \ell) + (\mathcal{B} \times \ell) \otimes \mathcal{B}] \\
& + \lambda_{15}[\mathcal{B} \otimes (\mathcal{B} \times \mathcal{E}) + (\mathcal{B} \times \mathcal{E}) \otimes \mathcal{B}] + \lambda_{16}[\mathcal{E} \otimes (\ell \times \mathcal{B}) + (\ell \times \mathcal{B}) \otimes \mathcal{E}] \\
& + \lambda_{17}[\mathcal{B} \otimes (\ell \times \mathcal{E}) + (\ell \times \mathcal{E}) \otimes \mathcal{B}] + \lambda_{18}[\ell \otimes \overset{*}{d}\ell + \overset{*}{d}\ell \otimes \ell] \\
& + \lambda_{19}[\mathcal{E} \otimes \overset{*}{d}\mathcal{E} + \overset{*}{d}\mathcal{E} \otimes \mathcal{E}] + \lambda_{20}[\mathcal{B} \otimes \overset{*}{d}\mathcal{B} + \overset{*}{d}\mathcal{B} \otimes \mathcal{B}] \\
& + \lambda_{21}[\ell \otimes \overset{*2*}{d}\ell + \overset{*2*}{d}\ell \otimes \ell] + \lambda_{22}[\mathcal{E} \otimes \overset{*2}{d}\mathcal{E} + \overset{*2}{d}\mathcal{E} \otimes \mathcal{E}] \\
& + \lambda_{23}[\mathcal{B} \otimes \overset{*2}{d}\mathcal{B} + \overset{*2}{d}\mathcal{B} \otimes \mathcal{B}] + \lambda_{24}[\ell \otimes \overset{*}{d}\mathcal{E} + \overset{*}{d}\mathcal{E} \otimes \ell]
\end{aligned}
\tag{6.80}
$$

$$+ \lambda_{25}[\overset{*}{\varrho} \otimes \overset{*}{\overset{.}{a}} \varnothing + \overset{*}{\overset{.}{a}} \varnothing \otimes \overset{*}{\varrho}] + \lambda_{26}[\varnothing \otimes \overset{*}{\overset{.}{a}} \varnothing + \overset{*}{\overset{.}{a}} \varnothing \otimes \varnothing] + \lambda_{27}[\varnothing \otimes \overset{*}{\overset{.}{a}} \overset{*}{\varrho} + \overset{*}{\overset{.}{a}} \overset{*}{\varrho} \otimes \varnothing]$$

$$+ \lambda_{28}[\varnothing \otimes \overset{*}{\overset{.}{a}} \overset{*}{\varrho} + \overset{*}{\overset{.}{a}} \overset{*}{\varrho} \otimes \varnothing] + \lambda_{29}[\varnothing \otimes \overset{*}{\overset{.}{a}} \varnothing + \overset{*}{\overset{.}{a}} \varnothing \otimes \varnothing] + \lambda_{30}[\overset{*}{\varrho} \otimes \overset{*2}{\overset{.}{a}} \varnothing + \overset{*2}{\overset{.}{a}} \varnothing \otimes \overset{*}{\varrho}]$$

$$+ \lambda_{31}[\overset{*}{\varrho} \otimes \overset{*2}{\overset{.}{a}} \varnothing + \overset{*2}{\overset{.}{a}} \varnothing \otimes \overset{*}{\varrho}] + \lambda_{32}[\varnothing \otimes \overset{*2}{\overset{.}{a}} \varnothing + \overset{*2}{\overset{.}{a}} \varnothing \otimes \varnothing] + \lambda_{33}[\varnothing \otimes \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho} + \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho} \otimes \varnothing]$$

$$+ \lambda_{34}[\varnothing \otimes \overset{*2}{\overset{.}{a}} \varnothing + \overset{*2}{\overset{.}{a}} \varnothing \otimes \varnothing] + \lambda_{35}[\varnothing \otimes \overset{*2*}{\overset{.}{a}} \varnothing + \overset{*2*}{\overset{.}{a}} \varnothing \otimes \varnothing] + \lambda_{36}[\overset{*}{\overset{.}{a}} \varnothing \otimes \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho} + \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \varnothing]$$

$$+ \lambda_{37}[\overset{*}{\overset{.}{a}} \varnothing \otimes \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho} + \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho} \otimes \overset{*}{\overset{.}{a}} \varnothing] + \lambda_{38}[\overset{*}{\overset{.}{a}} \varnothing \otimes \overset{*2}{\overset{.}{a}} \varnothing + \overset{*2}{\overset{.}{a}} \varnothing \otimes \overset{*}{\overset{.}{a}} \varnothing]$$

$$+ \lambda_{39}[\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} - \overset{*}{\overset{.}{a}} \times \overset{*}{\varrho}] + \lambda_{40}[\varnothing \times \overset{*}{\overset{.}{a}} - \overset{*}{\overset{.}{a}} \times \varnothing] + \lambda_{41}[\varnothing \times \overset{*}{\overset{.}{a}} - \overset{*}{\overset{.}{a}} \times \varnothing]$$

$$+ \lambda_{42}[\overset{*}{\varrho} \times \overset{.}{a}^2 - \overset{.}{a}^2 \times \overset{*}{\varrho}] + \lambda_{43}[\varnothing \times \overset{*2}{\overset{.}{a}} - \overset{*2}{\overset{.}{a}} \times \varnothing] + \lambda_{44}[\varnothing \times \overset{*2}{\overset{.}{a}} - \overset{*2}{\overset{.}{a}} \times \varnothing]$$

$$+ \lambda_{45}[\overset{*}{\overset{.}{a}}(\overset{*}{\varrho} \times \overset{*2}{\overset{.}{a}}) - \overset{*2}{\overset{.}{a}}(\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}})] + \lambda_{46}[\overset{*}{\overset{.}{a}}(\varnothing \times \overset{*2}{\overset{.}{a}}) - \overset{*2}{\overset{.}{a}}(\varnothing \times \overset{*}{\overset{.}{a}})]$$

$$+ \lambda_{47}[\overset{*}{\overset{.}{a}}(\varnothing \times \overset{*2}{\overset{.}{a}}) - \overset{*2}{\overset{.}{a}}(\varnothing \times \overset{*}{\overset{.}{a}})] + \lambda_{48}[\overset{*}{\varrho} \otimes (\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) + (\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \overset{*}{\varrho}]$$

$$+ \lambda_{49}[\varnothing \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) + (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \varnothing] + \lambda_{50}[\varnothing \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*}{\overset{.}{a}} \varnothing) \otimes \varnothing]$$

$$+ \lambda_{51}[\overset{*}{\varrho} \otimes (\overset{*}{\varrho} \times \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho}) + (\overset{*}{\varrho} \times \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \overset{*}{\varrho}] + \lambda_{52}[\varnothing \otimes (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \varnothing]$$

$$+ \lambda_{53}[\varnothing \otimes (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \varnothing] + \lambda_{54}[\overset{*}{\varrho} \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) + \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) \otimes \overset{*}{\varrho}]$$

$$+ \lambda_{55}[\overset{*}{\varrho} \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) + \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) \otimes \overset{*}{\varrho}] + \lambda_{56}[\varnothing \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \varnothing) + \overset{*}{\overset{.}{a}}(\varnothing \times \varnothing) \otimes \varnothing]$$

$$+ \lambda_{57}[\varnothing \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) + \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) \otimes \varnothing] + \lambda_{58}[\varnothing \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) + \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) \otimes \varnothing]$$

$$+ \lambda_{59}[\varnothing \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \varnothing) + \overset{*}{\overset{.}{a}}(\varnothing \times \varnothing) \otimes \varnothing] + \lambda_{60}[\overset{*}{\varrho} \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) + (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \overset{*}{\varrho}]$$

$$+ \lambda_{61}[\overset{*}{\varrho} \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) + (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \overset{*}{\varrho}] + \lambda_{62}[\varnothing \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*}{\overset{.}{a}} \varnothing) \otimes \varnothing]$$

$$+ \lambda_{63}[\varnothing \otimes (\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \varnothing) + (\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \varnothing) \otimes \varnothing] + \lambda_{64}[\varnothing \otimes (\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \varnothing) + (\overset{*}{\varrho} \times \overset{*}{\overset{.}{a}} \varnothing) \otimes \varnothing]$$

$$+ \lambda_{65}[\varnothing \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*}{\overset{.}{a}} \varnothing) \otimes \varnothing] + \lambda_{66}[\overset{*}{\varrho} \otimes (\varnothing \times \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho}) + (\varnothing \times \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \overset{*}{\varrho}]$$

$$+ \lambda_{67}[\overset{*}{\varrho} \otimes (\varnothing \times \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho}) + (\varnothing \times \overset{*2*}{\overset{.}{a}} \overset{*}{\varrho}) \times \overset{*}{\varrho}] + \lambda_{68}[\varnothing \otimes (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \varnothing]$$

$$+ \lambda_{69}[\varnothing \otimes (\overset{*}{\varrho} \times \overset{*2}{\overset{.}{a}} \varnothing) + (\overset{*}{\varrho} \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \varnothing] + \lambda_{70}[\varnothing \otimes (\overset{*}{\varrho} \times \overset{*2}{\overset{.}{a}} \varnothing) + (\overset{*}{\varrho} \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \varnothing]$$

$$+ \lambda_{71}[\varnothing \otimes (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \varnothing] + \lambda_{72}[\overset{*}{\overset{.}{a}}(\overset{*}{\varrho} \times \varnothing) \otimes \varnothing + \varnothing \otimes \overset{*}{\overset{.}{a}}(\overset{*}{\varrho} \times \varnothing)]$$

$$+ \lambda_{73}[\overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho}) \otimes \varnothing + \varnothing \otimes \overset{*}{\overset{.}{a}}(\varnothing \times \overset{*}{\varrho})] + \lambda_{74}[\varnothing \otimes (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) + (\varnothing \times \overset{*}{\overset{.}{a}} \overset{*}{\varrho}) \otimes \varnothing]$$

$$+ \lambda_{75}[\overset{*}{\varrho} \otimes (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) + (\varnothing \times \overset{*2}{\overset{.}{a}} \varnothing) \otimes \overset{*}{\varrho}]$$

where $\kappa_1, \ldots, \kappa_{34}$; $\sigma_1, \ldots, \sigma_{34}$; $\lambda_1, \ldots, \lambda_{75}$ <u>are functions</u> <u>of the following invariants:</u>

$$
\begin{aligned}
& I_d \ , \ II_d \ , \ III_d \ , \ J_1 \ , \ J_2 \ , \ J_3 \ , \ \overset{*}{\theta}\cdot\overset{*}{\theta} \\
& \mathcal{E}\cdot\overset{*}{\theta} \ , \ \mathcal{B}\cdot\overset{*}{\theta} \ , \ \mathcal{B}\cdot\mathcal{E}\times\overset{*}{\theta} \ , \\
& \overset{*}{\theta}\overset{*}{d}\overset{*}{\theta} \ , \ \mathcal{E}\overset{*}{d}\mathcal{B} \ , \ \mathcal{B}\overset{*}{d}\mathcal{B} \ , \ \overset{*}{\theta}\overset{*}{d}\mathcal{E} \ , \ \overset{*}{\theta}\overset{*}{d}\mathcal{B} \ , \ \mathcal{E}\overset{*}{d}\mathcal{B} \ , \\
& \overset{*}{\theta}\overset{*}{d}{}^2\overset{*}{\theta} \ , \ \mathcal{E}\overset{*}{d}{}^2\mathcal{E} \ , \ \mathcal{B}\overset{*}{d}{}^2\mathcal{B} \ , \ \overset{*}{\theta}\overset{*}{d}{}^2\mathcal{E} \ , \ \overset{*}{\theta}\overset{*}{d}{}^2\mathcal{B} \ , \ \mathcal{E}\overset{*}{d}{}^2\mathcal{B} \ , \\
& \overset{*}{\theta}\overset{*}{d}(\mathcal{E}\times\overset{*}{\theta}) \ , \ \mathcal{E}\overset{*}{d}(\mathcal{E}\times\overset{*}{\theta}) \ , \ \overset{*}{\theta}\overset{*}{d}(\mathcal{E}\times\mathcal{B}) \ , \ \mathcal{B}\overset{*}{d}(\mathcal{E}\times\mathcal{B}) \ , \ \mathcal{E}\overset{*}{d}(\mathcal{E}\times\mathcal{B}) \ , \\
& \mathcal{B}\overset{*}{d}(\mathcal{E}\times\mathcal{B}) \ , \ \mathcal{E}\overset{*}{d}(\overset{*}{\theta}\times\mathcal{B}) \ , \ \mathcal{B}\overset{*}{d}(\overset{*}{\theta}\times\mathcal{E}) \ , \\
& \overset{*}{\theta}\overset{*}{d}{}^2(\mathcal{E}\times\overset{*}{\theta}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\mathcal{E}\times\overset{*}{\theta}) \ , \ \overset{*}{\theta}\overset{*}{d}{}^2(\mathcal{E}\times\mathcal{B}) \ , \ \mathcal{B}\overset{*}{d}{}^2(\overset{*}{\theta}\times\mathcal{B}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\mathcal{E}\times\mathcal{B}) \ , \\
& \mathcal{B}\overset{*}{d}{}^2(\mathcal{E}\times\mathcal{B}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\overset{*}{\theta}\times\mathcal{B}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\overset{*}{\theta}\times\mathcal{B}) \ , \\
& \overset{*}{\theta}\overset{*}{d}{}^2(\overset{*}{\theta}\times\overset{*}{d}\overset{*}{\theta}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\mathcal{E}\times\overset{*}{d}\mathcal{E}) \ , \ \mathcal{B}\overset{*}{d}{}^2(\mathcal{B}\times\overset{*}{d}\mathcal{B}) \ , \\
& \overset{*}{\theta}\overset{*}{d}{}^2(\mathcal{E}\times\overset{*}{d}\overset{*}{\theta}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\overset{*}{\theta}\times\overset{*}{d}\mathcal{E}) \ , \ \overset{*}{\theta}\overset{*}{d}{}^2(\mathcal{B}\times\overset{*}{d}\overset{*}{\theta}) \ , \ \mathcal{B}\overset{*}{d}{}^2(\overset{*}{\theta}\times\overset{*}{d}\mathcal{B}) \ , \\
& \mathcal{E}\overset{*}{d}{}^2(\mathcal{B}\times\overset{*}{d}\mathcal{E}) \ , \ \mathcal{B}\overset{*}{d}{}^2(\mathcal{E}\times\overset{*}{d}\mathcal{B}) \ , \ \mathcal{E}\overset{*}{d}{}^2(\mathcal{B}\times\overset{*}{d}\overset{*}{\theta})
\end{aligned}
$$
(6.81)

v. <u>The heat flux</u> $\underset{\sim}{q}$ <u>, the conduction current</u> $\underset{\sim}{j}$ <u>, and</u> <u>the dissipative stress tensor</u> $_D\underline{t}$ <u>satisfy the</u> <u>inequality:</u>

(6.82)
$$
_D t^{\alpha\beta}\overset{*}{d}_{\alpha\beta} - \frac{q^\alpha \theta_\alpha}{\theta} + j^\alpha \mathcal{E}_\alpha \ge 0
$$

<u>which in particular implies that when</u> $\overset{*}{d}_{\alpha\beta} = 0$, $\overset{*}{\theta}_\alpha = 0$, $\mathcal{E}_\alpha = 0$, <u>then</u>

$$(6.83) \qquad {}_D t^{\alpha\beta} = 0 \ , \quad q^{\alpha} = 0 \ , \quad j^{\alpha} = 0$$

These results follow directly from equations (6.62) to (6.70) and invariance requirements. In particular, (6.64) and (6.65) imply that

$$\psi_o = \psi_o(\theta \ , \ n_o \ , \ \mathcal{E}_{\alpha} \ , \ \mathcal{B}_{\alpha} \ , \ u_{\alpha})$$

and in order that ψ_o be an invariant under the proper orghochronous Lorentz group, it must have the functional form (6.71). Equations (6.73) to (6.76) follow directly from (6.62), (6.69), (6.66), (6.67) and (6.71). Equations (6.63) and (6.68) are identically satisfied (cf. (6.71), (6.75) and (6.76)). The lengthy expressions (6.78) to (6.80) are a straightforward application of the remarks in the appendix.

The foregoing theory of viscous fluids is so complicated that in the generality presented in this section it has little practical use. However, it provides a foundation for several approximate theories. In particular, the class of perfect fluids is a special case of this theory.

Perfect Fluids

For a perfect fluid, the entropy production vanishes. That is:

$$(6.84) \qquad \underset{\sim}{q} = 0 \ , \quad {}_D \underset{\sim}{t} = 0 \ , \quad \underset{\sim}{j} = 0$$

That is, for a perfect fluid, the materials are described by:

i.   The free energy has the form (6.71).

ii.   The entropy η_{00} , the polarization vector $\mathcal{P}$,
      and the magnetization vector $\mathcal{M}$ are given by
      (6.73), (6.75) and (6.76) respectively and the
      stress tensor is determined from:

$$(6.85) \quad \underset{\sim}{t} = - n_o^2 \, \frac{\partial \Psi_o}{\partial n_o} \, \underset{\sim}{\varepsilon}$$

iii. The nonmechanical momentum $\underset{\sim}{p}$ is determined by:

$$(6.86) \quad \underset{\sim}{p} = - \mathcal{P} \times \underset{\sim}{\mathcal{C}} - \underset{\sim}{\mathcal{E}} \times \mathcal{M}$$

The theory of perfect electromagnetic fluids presented in this
section is comparable to the theory of Penfield and Haus [36]. That
the polarization and magnetization are derived from the free energy is
a consequence of the second law of thermodynamics in the theory of
this section. In the theory of Penfield and Haus the polarization
and magnetization are deduced from a variational principle. The major
difference between the work of this article and that of Penfield
and Haus is in the difference in the Amperian model and the magnetic
dipole model for the magnetic term in the interaction of electro-
magnetic fields on matter.

Concluding Remarks

The relativistic theory of electromagnetic materials presented in this paper provides an insight into the continuum behavior of electromagnetic interactions with matter. The coupling of mechanical, electromagnetic and thermal phenomena into one theory has resulted in a complicated system of partial differential equations. The complete physical and mathematical nature of these equations in the generality treated in the previous sections is beyond the scope of our present knowledge. The nonlinearity of the equations makes any solution to boundary-value problems difficult.

Considerable insight into the nature of a system of partial differential equations can be obtained from an investigation of the propagation of waves and singular surfaces. This seems to be a fruitful class of problems which can be treated relativistically. Some progress in this direction has already been made and will appear in a later work.

Though the present theory is capable of describing many physical phenomena (polarization, magnetization, heat conduction, piezoelectricity, thermoelectric effects, to name a few) such physical phenomena as viscoelasticity, optical activity and gyrotropic effects are definitely excluded. Simple theories of viscoelasticity (the Kelvin-Voigt, Maxwell and other higher order rate theories) can be formulated by an appropriate change in constitutive equations. The treatment of

heredity requires the more difficult study of functional constitutive equations. A description of optical activity and gyrotropic effects will probably entail a reformulation of the balance equations to include spin, couple stresses, and quadrupoles. Whether ferromagnets and electrets are described by the theory presented in this article cannot be answered until a deeper physical study of these phenomena is undertaken.

Finally it must be mentioned that for   accelerating frames and curved spaces where the special theory fails new unified theories are needed employing the fundamental ideas of the general theory of relativity. Such a grandiose plan presently is out of our reach.

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APPENDIX ON POLYNOMIAL INVARIANTS OF VECTORS
AND TENSORS FOR THE LORENTZ GROUP

$\underline{N\text{-Vectors}}$ $V_\alpha^{(i)}$ $(i = 1,\ldots,N)$

The polynomial invariants for the Lorentz group can be deduced
from the polynomial invariants of the four dimensional orthogonal
group (see Weyl [39, p.65]). If f is an even invariant polynomial
of N-vectors $V_\alpha^{(i)}$ $(i = 1,\ldots,N)$, then f is a polynomial in the
scalar products

(A.1) $\gamma^{\alpha\beta} V_\alpha^{(i)} V_\beta^{(j)}$ $(i \leq j ; i,j = 1,\ldots,N)$

If f is an odd invariant polynomial of N-vectors $V_\alpha^{(i)}$, f is
a sum of terms of the form:

(A.2) $\epsilon^{\alpha\beta\gamma\delta} V_\alpha^{(i_1)} V_\beta^{(i_2)} V_\gamma^{(i_3)} V_\delta^{(i_4)}$ g

where g is an even polynomial and

$i_1 < i_2 < i_3 < i_4$ $(i_1, i_2, i_3, i_4 = 1,\ldots,N)$

The following special case is often encountered in physical situations.

$$\underline{N\text{-Vectors}} \quad V_\alpha^{(i)} \quad \underline{\text{and a}} \ \underline{\text{Time-Like}} \ \underline{\text{Unit}} \ \underline{\text{Vector}} \ u_\alpha$$

$$(u_\alpha u^\alpha = -1)(i = 1,\ldots,N)$$

In many physical situations the world velocity vector u_α is included in the group of vectors considered in Case 1. The world velocity is time-like

$$(A.3) \qquad u_\alpha u^\alpha = -1$$

Any polynomial in $V_\alpha^{(i)}$ and u_α is a polynomial in the vectors $\overset{*}{V}_\alpha^{(i)}$ and u_α and the scalar products

$$(A.4) \qquad v^{(i)} = V_\alpha^{(i)} u^\alpha \qquad (i = 1,\ldots,N)$$

where $\overset{*}{V}_\alpha^{(i)}$ is defined as

$$(A.5) \qquad \overset{*}{V}_\alpha^{(i)} = S_\alpha{}^\beta V_\beta^{(i)} \ ; \quad S_\alpha{}^\beta = \delta_\alpha{}^\beta + u_\alpha u^\beta \ ; \quad \overset{*}{V}_\alpha^{(i)} u^\alpha = 0$$

This follows from the decomposition

$$(A.6) \qquad V_\alpha^{(i)} = \overset{*}{V}_\alpha^{(i)} - v^{(i)} u_\alpha$$

Since the $v^{(i)}$ are scalar invariants, one need only consider a polynomial invariant in

(A.7) $\quad \overset{*}{V}_\alpha (i) \quad , \quad u_\alpha$

By applying the results of Case 1 to the set (A.7) and observing

(A.8) $\quad u^\alpha \overset{*}{V}_\alpha (i) = 0 \quad ; \quad u^\alpha u_\alpha = -1$

and

(A.9) $\quad \epsilon^{\alpha\beta\gamma\delta} \overset{*}{V}_\alpha (i_1) \; \overset{*}{V}_\beta (i_2) \overset{*}{V}_\gamma (i_3) \; \overset{*}{V}_\delta (i_4) = 0$

one obtains: An even invariant polynomial f in the vectors $\overset{*}{V}_\alpha (i)$, u_α is a polynomial in the invariants

(A.10) $\quad \gamma^{\alpha\beta} \overset{*}{V}_\alpha (i) \overset{*}{V}_\beta (j) \qquad (i \leq j \; ; \; i,j = 1,\ldots,N)$

and an odd invariant polynomial f is a sum of terms of the form:

(A.11) $\quad \epsilon^{\alpha\beta\gamma\delta} \overset{*}{V}_\alpha (i_1) \; \overset{*}{V}_\beta (i_2) \; \overset{*}{V}_\gamma (i_3) \; u_\delta \; g$

$$(i_1 < i_2 < i_3 \; ; \; i_1, i_2, i_3 = 1,\ldots N)$$

where g is an even invariant polynomial.

The identity (A.9) follows from the observation that from (A.8) the fourth row of the determinant on the left hand side of (A.9) is a linear combination of the first three rows.

The procedure of decomposing tensors into spatial tensors and scalars will be used in the next section. It is the key to using the results of the three dimensional orthogonal group for the integrity bases of the Lorentz group.

$\underline{M\text{-Symmetric}}$ $\underline{\text{Tensors}}$ $T_{\alpha\beta}^{(R)}$, $\underline{N\text{-Vectors}}$ $V_\alpha^{(i)}$

$\underline{\text{and}}$ $\underline{a}$ $\underline{\text{Time-Like}}$ $\underline{\text{Unit}}$ $\underline{\text{Vector}}$ u_α $(u_\alpha u^\alpha = -1)$

Any polynomial in $T^{(R)}$, $V^{(i)}$, and u is a polynomial in the tensors $\overset{*}{T}^{(R)}$, the vectors $\overset{*}{V}^{(i)}$, $\overset{*}{A}^{(R)}$, u and the scalars

(A.12) $T^{(R)} = T_{\alpha\beta}^{(R)} u^\alpha u^\beta$ $(R = 1,\ldots,M)$

(A.13) $v^{(i)} = V_\alpha^{(i)} u^\alpha$ $(i = 1,\ldots,N)$

where

(A.14) $\overset{*}{T}_{\alpha\beta}^{(R)} = s_\alpha{}^\gamma s_\beta{}^\delta T_{\gamma\delta}^{(R)} = \overset{*}{T}_{\beta\alpha}^{(R)}$

(A.15) $\overset{*}{V}_\alpha^{(i)} = s_\alpha{}^\beta v_\beta^{(i)}$

(A.16) $\overset{*}{A}_\alpha^{(R)} = s_\alpha{}^\gamma T_{\gamma\delta}^{(R)} u^\delta$

This follows from (A.6) and the identity

$$(A.17) \qquad T^{(R)}_{\alpha\beta} = \overset{*}{T}^{(R)}_{\alpha\beta} - \overset{*}{A}^{(R)}_{\alpha} u_{\beta} - \overset{*}{A}^{(R)}_{\beta} u_{\alpha} + T^{(R)} n_{\alpha} u_{\beta}$$

It should be noted that

$$(A.18) \qquad \overset{*}{T}^{(R)}_{\alpha\beta} u^{\beta} = 0$$

$$(A.19) \qquad \overset{*}{A}^{(R)}_{\alpha} u^{\alpha} = 0$$

$$(A.20) \qquad \overset{*}{V}^{(1)}_{\alpha} u^{\alpha} = 0$$

Thus one can consider a polynomial in the tensors and vectors

$$(A.21) \qquad \overset{*}{T}^{(R)}_{\alpha\beta} \quad , \quad \overset{*}{V}^{(k)}_{\alpha} \quad , \quad u_{\alpha} \qquad (R = 1,\ldots,M \; ; \; k = 1,\ldots,M + N)$$

where

$$(A.22) \qquad \overset{*}{V}^{(R+M)}_{\alpha} = A^{(R)}_{\alpha}$$

with the coefficients polynomials in $T^{(R)}$ and $v^{(1)}$.

The coefficients of an invariant polynomial must be isotropic tensors for the Lorentz group. This implies that they are sums of terms which are products of $\gamma^{\alpha\beta}$ and $\varepsilon^{\alpha\beta\gamma\delta}$. By noting that

(A.23) $\quad (\overset{*}{\pi}_K)^{\alpha\beta} u_\beta = 0 \quad , \quad (\overset{*}{\pi}_K)^{\alpha\beta} u_\alpha = 0$

(A.24) $\quad \overset{*}{V}_\alpha(i) u^\alpha = 0$

(A.25) $\quad \epsilon_{\alpha\beta\gamma\delta} (\overset{*}{\pi}_L)^{\alpha\rho} \overset{*}{V}_\rho(i) (\overset{*}{\pi}_M)^{\beta\sigma} \overset{*}{V}_\sigma(j) (\overset{*}{\pi}_N)^{\gamma\tau} \overset{*}{V}_\tau(k) (\overset{*}{\pi}_P)^{\delta\mu} \overset{*}{V}_\mu(\ell) = 0$

(A.26) $\quad \epsilon_{\alpha\beta\gamma\delta} (\overset{*}{\pi}_L)^{\alpha\beta} (\overset{*}{\pi}_M)^{\gamma\sigma} \overset{*}{V}_\sigma(k) (\overset{*}{\pi}_P)^{\delta\tau} \overset{*}{V}_\tau(\ell) = 0$

(A.27) $\quad \epsilon_{\alpha\beta\gamma\delta} (\overset{*}{\pi}_L)^{\alpha\beta} (\overset{*}{\pi}_M)^{\gamma\delta} = 0$

an invariant polynomial in $\overset{*}{\underline{\mathcal{T}}}(R)$, $\overset{*}{\underline{V}}(j)$ and $\underline{u}$ must be a polynomial in

(A.28) $\quad \overset{*}{V}_\alpha(i) (\overset{*}{\pi}_S)^{\alpha\beta} \overset{*}{V}_\beta(j) \quad , \quad \mathrm{tr}\,(\overset{*}{\underline{\mathcal{L}}}_K)$

(A.29) $\quad \overset{*}{\epsilon}_{\alpha\beta\gamma} (\overset{*}{\pi}_L)^{\alpha\delta} \overset{*}{V}_\delta(i) (\overset{*}{\pi}_M)^{\beta\rho} \overset{*}{V}_\rho(j) (\overset{*}{\pi}_N)^{\gamma\tau} \overset{*}{V}_\tau(k)$

(A.30) $\quad \overset{*}{\epsilon}_{\alpha\beta\gamma} (\overset{*}{\pi}_K)^{\alpha\beta} (\overset{*}{\pi}_M)^{\gamma\delta} \overset{*}{V}_\delta(i)$

where $\overset{*}{\underline{\mathcal{L}}}_L$, $\overset{*}{\underline{\mathcal{L}}}_M$, $\overset{*}{\underline{\mathcal{L}}}_N$, $\overset{*}{\underline{\mathcal{L}}}_P$, and $\overset{*}{\underline{\mathcal{L}}}_S$ are matrix products in $\overset{*}{\underline{\mathcal{T}}}(R)$ and χ; $\overset{*}{\underline{\mathcal{L}}}_K$ are matrix products in $\overset{*}{\underline{\mathcal{T}}}(R)$. The proof of (A.23) and (A.24) follows from the definitions of $\overset{*}{\underline{V}}(i)$ and $\overset{*}{\underline{\mathcal{L}}}_K$. Identities (A.25), (A.26) and (A.27) are deduced by observing that the fourth row of the determinants is a linear combination of the first three (cf. (A.18) to (A.20)). Here $\overset{*}{\epsilon}_{\alpha\beta\gamma}$ is defined by:

(A.31) $\quad \overset{*}{\epsilon}_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta} u^{\delta}$

The following identities are useful to reduce (A.28) to (A.30)

$$(A.32) \quad \epsilon_{\alpha_1\beta_1\gamma_1\delta_1} \, \epsilon_{\alpha_2\beta_2\gamma_2\delta_2} = - \begin{vmatrix} \gamma_{\alpha_1\alpha_2} & \gamma_{\alpha_1\beta_2} & \gamma_{\alpha_1\gamma_2} & \gamma_{\alpha_1\delta_2} \\ \gamma_{\beta_1\alpha_2} & \gamma_{\beta_1\beta_2} & \gamma_{\beta_1\gamma_2} & \gamma_{\beta_1\delta_2} \\ \gamma_{\gamma_1\alpha_2} & \gamma_{\gamma_1\beta_2} & \gamma_{\gamma_1\gamma_2} & \gamma_{\gamma_1\delta_2} \\ \gamma_{\delta_1\alpha_2} & \gamma_{\delta_1\beta_2} & \gamma_{\delta_1\gamma_2} & \gamma_{\delta_1\delta_2} \end{vmatrix}$$

From (A.32) one can show that

$$(A.33) \quad \epsilon^{\alpha_1\beta_1\gamma_1\delta_1} \, \epsilon_{\alpha_1\beta_2\gamma_2\delta_2} = - \begin{vmatrix} \delta^{\beta_1}_{\beta_2} & \delta^{\beta_1}_{\gamma_2} & \delta^{\beta_1}_{\delta_2} \\ \delta^{\gamma_1}_{\beta_2} & \delta^{\gamma_1}_{\gamma_2} & \delta^{\gamma_1}_{\delta_2} \\ \delta^{\delta_1}_{\beta_2} & \delta^{\delta_1}_{\gamma_2} & \delta^{\delta_1}_{\delta_2} \end{vmatrix}$$

By dotting (A.32) with $u^{\delta_1} u^{\delta_2}$ and adding u_{α_1} times the fourth row to the first row, u_{β_1} times the fourth row to the second row, u_{γ_1} times the fourth row to the third row, one obtains

$$(A.34) \quad \overset{*}{\epsilon}_{\alpha_1\beta_1\gamma_1}\, \overset{*}{\epsilon}_{\alpha_2\beta_2\gamma_2} = \begin{vmatrix} s_{\alpha_1\alpha_2} & s_{\alpha_1\beta_2} & s_{\alpha_1\gamma_2} \\[6pt] s_{\beta_1\alpha_2} & s_{\beta_1\beta_2} & s_{\beta_1\gamma_2} \\[6pt] s_{\gamma_1\alpha_2} & s_{\gamma_1\beta_2} & s_{\gamma_1\gamma_2} \end{vmatrix}$$

By dotting the identity

$$(A.35) \quad \begin{vmatrix} \gamma_{\alpha_1\alpha_2} & \gamma_{\alpha_1\beta_2} & \gamma_{\alpha_1\gamma_2} & \gamma_{\alpha_1\delta_2} & \gamma_{\alpha_1\epsilon_2} \\[6pt] \gamma_{\beta_1\alpha_2} & \gamma_{\beta_1\beta_2} & \gamma_{\beta_1\gamma_2} & \gamma_{\beta_1\delta_2} & \gamma_{\beta_1\epsilon_2} \\[6pt] \gamma_{\gamma_1\alpha_2} & \gamma_{\gamma_1\beta_2} & \gamma_{\gamma_1\gamma_2} & \gamma_{\gamma_1\delta_2} & \gamma_{\gamma_1\epsilon_2} \\[6pt] \gamma_{\delta_1\alpha_2} & \gamma_{\delta_1\beta_2} & \gamma_{\delta_1\gamma_2} & \gamma_{\delta_1\delta_2} & \gamma_{\delta_1\epsilon_2} \\[6pt] \gamma_{\epsilon_1\alpha_2} & \gamma_{\epsilon_1\beta_2} & \gamma_{\epsilon_1\gamma_2} & \gamma_{\epsilon_1\delta_2} & \gamma_{\epsilon_1\epsilon_2} \end{vmatrix} = 0$$

with $u^{\epsilon_1} u^{\epsilon_2}$ and multiplying the fourth row by u_{α_1} and adding it to the first row, etc., one can deduce that

$$(A.36) \quad \begin{vmatrix} s_{\alpha_1\alpha_2} & s_{\alpha_1\beta_2} & s_{\alpha_1\gamma_2} & s_{\alpha_1\delta_2} \\ s_{\beta_1\alpha_2} & s_{\beta_1\beta_2} & s_{\beta_1\gamma_2} & s_{\beta_1\delta_2} \\ s_{\gamma_1\alpha_2} & s_{\gamma_1\beta_2} & s_{\gamma_1\gamma_2} & s_{\gamma_1\delta_2} \\ s_{\delta_1\alpha_2} & s_{\delta_1\beta_2} & s_{\delta_1\gamma_2} & s_{\delta_1\delta_2} \end{vmatrix} = 0$$

From (A.34) it follows that

$$\overset{*}{\epsilon}_{\alpha_1\beta_1\gamma_1} \overset{*}{\epsilon}^{\alpha_1\beta_2\gamma_2} = s_{\beta_1}{}^{\beta_2} s_{\gamma_1}{}^{\gamma_2} - s_{\gamma_1}{}^{\beta_2} \epsilon_{\beta_1}{}^{\gamma_2}$$

$$(A.37)$$

$$\overset{*}{\epsilon}_{\alpha_1\beta_1\gamma_1} \overset{*}{\epsilon}^{\alpha_1\beta_1\gamma_2} = 2 s^{\gamma_2}{}_{\gamma_1}$$

If one defines the skew-symmetric tensor by

$$(A.38) \quad \overset{*(1)}{v}_{\alpha\beta} = \overset{*}{\epsilon}_{\alpha\beta\gamma} \overset{*(1)\gamma}{v} = - \overset{*(1)}{v}_{\beta\alpha}$$

$$(A.39) \quad \overset{*(1)}{v}_{\alpha\beta} u^\beta = 0$$

From (A.38) it follows that

$$(A.40) \quad \overset{*}{v}_\alpha^{(1)} = \frac{1}{2} \overset{*}{\epsilon}_{\alpha\beta\gamma} \overset{*(1)\beta\gamma}{v}$$

By substituting (A.40) into (A.28) to (A.30) it is seen that the polynomial invariants of M symmetric tensor, N-vectors and a time-like unit vector reduces to the set of invariants (A.12) and (A.13) and the invariants of M symmetric and M + N skew-symmetric space-like tensors which are polynomials of degree three or less in the skew-symmetric tensors. This is the starting point of the work of Rivlin, Spencer and Smith (cf. [40], [41], [42], [33]) for three dimensional matrices.

From (A.36) space-like tensors satisfy an equation which in matrix notation is identical to the three dimensional Clayley-Hamilton theorem. If one multiplies (A.36) by $\overset{*\beta_1\beta_2}{a} \overset{*\gamma_1\gamma_2}{a} \overset{*\delta_1\delta_2}{a}$ and expands the determinant the following result is obtained:

$$
\overset{*3}{\underline{a}} - (\operatorname{tr} \overset{*}{\underline{a}}) \overset{*2}{\underline{a}} + \frac{1}{2} [(\operatorname{tr} \overset{*}{\underline{a}})^2 - \operatorname{tr} \overset{*2}{\underline{a}}] \overset{*}{\underline{a}} - \frac{1}{6} [\operatorname{tr} \overset{*3}{\underline{a}}
$$

(A.41)

$$
+ 2\operatorname{tr} \overset{*3}{\underline{a}} - 3(\operatorname{tr} \overset{*2}{\underline{a}})(\operatorname{tr} \overset{*}{\underline{a}})] \underline{S} = 0
$$

where $\overset{*\alpha\beta}{a}$ satisfies

(A.42) $S^\alpha_{\ \gamma} S^\beta_{\ \delta} \overset{*\gamma\delta}{a} = \overset{*\alpha\beta}{a}$

Since it is only this theorem and several properties of the trace of products of matrices that are employed by Rivlin, Spencer and Smith in their series of papers on the reduction of polynomials in 3×3 matrices, their results (cf. [40], [41], [42], [33]) hold for space-

like tensors and space-like vectors under the Lorentz group. (In particular all relations listed by Spencer [42, p.54] are valid for space-like tensors. Only these relations are employed by these authors for the reduction of polynomial invariant of 3×3 matrices under the orthogonal group.)

The integrity bases listed by Spencer [42] and Smith [33] are minimal in the sense that no invariant listed by them is a polynomial in the invariants in their list. The integrity bases for space-like tensor derived from their results is also minimal in the above sense. If it were not, it would imply that their result was not minimal. Smith has proven that their results (cf. [36]) are minimal.

The results of Wineman and Pipkin [37] for continuous tensor functions of tensors for the orthogonal group in three dimensions hold for a space-like tensor function of u_α and four-tensors (even though the Lorentz group is not compact). The fact that a polynomial basis in u_α and four tensors is a functional basis can be proven by their method or by observing that if it were not a contradiction would arise with their results. By transforming to a frame in which $u_\alpha = (0, 0, 0, -1)$ and observing that the resulting tensor must be invariant under the three dimensional orthogonal group one can see that the above remark as to the form of a continuous invariant function must hold for space-like tensors.

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The basic laws of a special relativistic theory of continuous media suitable for the treatment of electromagnetic interactions with materials are formulated. The kinematics, dynamics and thermodynamics of a continuum are discussed from a relativistic viewpoint. Constitutive equations are deduced for thermoelastic solids, thermoviscous fluids and electromagnetic materials.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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