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PREPARED BY	D. Harder	11/4/65
SUPERVISED BY	R. B. Martindale	11/4/65
APPROVED BY	W. R. Archibald	11/4/65
APPROVED BY	for M. L. Reeves	11/5/65
APPROVED BY	I. J. Stampalia	11/10/65

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ABSTRACT

The equations for a simulation of planet artificial satellite orbit parameter estimation are defined. The estimation process consists of sequential linear differential correction based on Earth-based radar doppler measurements and/or on-board measurements of the direction of planet local vertical. The parameter set includes the Keplerian elements of the satellite orbit relative to the planet, planet and Sun gravitational constants, and components of the position and velocity of the planet and Earth relative to the Sun.

KEY WORD LIST

Satellite orbit

System model

Observations

Parameter Set, Parameter Estimation

Filtering: sequential, linear, least squares

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EQUATIONS FOR THE SIMULATION OF PLANET SATELLITE ORBIT DETERMINATION

1.0 INTRODUCTION

The equations defined herein are for a simulation of the process of planet artificial satellite orbit parameter estimation based on DSIF doppler measurements and/or measurements of the direction of local vertical from the satellite to the planet. The simulation was designed for analysis of the problem of the determination of satellite orbits about planets other than Earth, but is applicable with minor modifications to Lunar satellites as well. The simulation consists of two parts. First a set of observations are computed which are without error in the physical model assumed. Then the model parameters are disturbed in the form of an initial estimate and the process of improving the estimate in a specified subset of the parameters from a set of disturbed observations is simulated. The method of estimation consists of sequential differential corrections of the estimate of the parameter subset with each observation, using the Kalman formulation of least squares filtering. The total set of parameters which may be estimated are:

- Satellite Keplerian orbital elements
- Mars and Sun primary gravitational constants
- Planet oblateness constant
- Solar radiation pressure
- Mars velocity and position components
- Earth velocity and position components

The physical model is considerably simpler than that of the real world because the objective was to design an efficient tool for analyzing the effects of the most significant error sources. The principal simplification of the model is the propagation of Earth and planet orbits as heliocentric conics, starting with an arbitrary point out of the table of ephemerides in the neighborhood of run start time. Since the program allows estimation of Earth and planet ephemerides, this is not as much of a simplification as it would at first appear. For a planet such as Mars with no large satellites, this method might well be applied without modification in an operational orbit determination program. For Earth's orbit, the Moon-Earth system must be propagated in an operational program. The second model simplification is the neglect of all terms in the expansion of the planet gravitational potential except those for mass and oblateness with the assumption that the planet axis of symmetry is known. The third important simplification is that the DSIF measurements are taken to be range rate measurements, thus by-passing a number of steps in the process of reducing the doppler data.

The orbit of the satellite relative to the planet is propagated by the method of variation of Keplerian parameters. The time rate of change of each parameter due to the assumed disturbing functions is integrated to obtain the elements of the osculating ellipse versus time.

An important advantage gained by estimating Keplerian parameters instead of state vector is the simplification of the problem of isolating the initial ambiguity in orbit plane orientation about the line-of-sight from Earth in the case when the estimation of parameters is based on DSIF doppler data only. The method also has an advantage in the estimation process in that the matrix of first order coefficients of error in the Keplerian elements at one point in time with respect to error in the elements at another point in time is the identity matrix.

Table 1 lists the nomenclature for coordinate systems and variables as used in the derivation of equations. To simplify the notation the time-dependent variables are generally written without an explicit reference to time. When used in this way, it is understood that the symbol for the variable denotes its value at problem time t. When it is necessary to relate the variables at two or more problem times, the time symbol is added as a subscript or in normal functional notation, e.g., C_t or $C(t)$. Subscripts other than t denote the element of a vector or a matrix. The equations for the satellite, planet, and Earth ephemerides and for the observations apply to the propagation of numbers both in the model and in the estimate of the model. A distinction is made between model and model estimate only when necessary to clarify the nature of the estimation process. The estimate of a model parameter is denoted by a caret above the symbol, e.g., the estimate of $\overline{X}2$ is represented by $\hat{X}2$.

The simulation program is documented in D2-84082-1.

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TABLE 1

DEFINITION OF SYSTEM VARIABLES AND SYMBOLS

VARIABLE	SYMBOL	
. Earth Equatorial Cartesian Reference Axes	x_1	
. Direction of Vernal Equinox, E1950	$\bar{x}_{11}$	
. Earth Axis of Rotation	$\bar{x}_{13}$	Unit vectors
. $\bar{x}_{13}$ x $\bar{x}_{11}$	$\bar{x}_{12}$	
. Planet Equatorial Cartesian Reference Axes	x_2	
. Intersection of Equatorial Plane and and Plane of the Reference Meridian	$\bar{x}_{21}$	
. Axis of Symmetry	$\bar{x}_{23}$	Unit vectors
. $\bar{x}_{23}$ x $\bar{x}_{21}$	$\bar{x}_{22}$	
. Planet-Centered Plane-of-the-Sky Cartesian Reference Axes	x_3	
. Planet-Earth Line at Arbitrary Epoch	$\bar{x}_{33}$	
. Intersection of Planet Equatorial Plane and Plane of the Meridian through $\bar{x}_{31}$	$\bar{x}_{31}$	Unit vectors
. $\bar{x}_{33}$ x $\bar{x}_{31}$	$\bar{x}_{32}$	
. Unit Vector in Direction of Canopus	$\overline{CX}$	Unit vectors
. Unit Vector Along Vehicle - Sun Line	$\overline{SX}$	
. Vehicle-Centered Cartesian Reference Axes	x_5	
. $\overline{SX}$	$\bar{x}_{51}$	
. $\overline{SX}$ x $\overline{CX}$	$\bar{x}_{52}$	Unit vectors
. $\bar{x}_{52}$ x $\overline{SX}$	$\bar{x}_{53}$	

TABLE 1 (Cont'd)

VARIABLE	SYMBOL	DIMENSION
. Satellite Position - Velocity Vector		
. Relative to Earth in x_1 reference system	$\overline{X1}$	6
. Relative to Planet in x_2 reference system	$\overline{X2}$	6
. Relative to Planet in x_3 reference system	$\overline{X3}$	6
. Relative to Planet in x_5 reference system	$\overline{X5}$	6
e.g., $(X2)_i$, $i = 1, 3$ are the position components in x_2		
$(X2)_i$, $i = 4, 6$ are the velocity components in x_2		
. Satellite Keplerian Orbit Parameters		
Semi-major axis	a	
Eccentricity	e	
*Argument of Periapsis	ω_1 or ω_2	
Longitude of Ascending Node	Ω_1 or Ω_2	
Inclination	i_1 or i_2	
Time from Last Periapsis Passage	t_p	
. True Anomaly, Radius Vector at time t_p	v, r	
. Alternate Notation for Keplerian Orbit Parameters:		
$\overline{Y1} = (a, e, \omega_1, \Omega_1, i_1, t_p)$	$\overline{Y1}$	6
$\overline{Y2} = (a, e, \omega_2, \Omega_2, i_2, t_p)$	$\overline{Y2}$	6

*Subscript denotes cartesian reference system: 1, Planet Eq, system; 2, Plane-of-the-sky system

TABLE 1 (Cont'd)

VARIABLE	SYMBOL	DIMENSION
. Time Rate of Change of Satellite Orbital Elements, $\overline{Y1}$		
. Due to Planet Oblateness	$\overline{DYM}$	6
. Due to Sun Differential Grav. Effect	$\overline{DYS}$	6
. Due to Solar Radiation Pressure	$\overline{DYR}$	6
. Planet Heliocentric Orbit		
. Position and Velocity Relative to Sun in x_1 System	$\overline{XB1}$	6
. Keplerian Orbital Elements	$\overline{YB1}$	6
. Earth Heliocentric Orbit		
. Position & Velocity to Sun in x_1 System	$\overline{XB2}$	6
. Keplerian Orbital Elements	$\overline{YB2}$	6
. Radar Site Location & Velocity		
. Geocentric Inertial Latitude and Longitude	λ_s, Γ_s	1, 1
. Relative to Earth Center in x_1 System	$\overline{XS}$	6
. Relative to Mars Center in x_2 System	$\overline{XS2}$	6
. Range from Radar Site to Vehicle	R	Scalar
. Range Rate	$\dot{R}$	Scalar

TABLE 1 (Cont'd)

VARIABLE	SYMBOL	DIMENSION
• Direction of Planet Local Vertical in x_5 System		
• Azimuth	AZ	1
• Elevation	EL	1
• Earth Rotation Rate	W_E	Scalar
• Earth Radius	R_E	Scalar
• Mars Radius	R_M	Scalar
• Gravitational Constants		
Earth	U_1	Scalar
Planet	U_2	Scalar
Sun	U_3	Scalar
• Planet Oblateness Constant	J_2	Scalar
• Error Sensitivity Matrices		
• Keplerian Elements $\overline{Y1}$ to State Vector Elements $\overline{X2}$	S1	6 x 6
• State Vector Elements $\overline{X2}$ to Keplerian Elements $\overline{Y1}$	S1I	6 x 6
• Keplerian Elements $\overline{Y2}$ to State Vector Elements $\overline{X3}$	S2	6 x 6
• State Vector Elements $\overline{X3}$ to Keplerian Elements $\overline{Y2}$	S2I	6 x 6
• Solution Parameter Set at Step i to Solution Parameter Set at Step $i-1$	S	22 x 22

TABLE 1 (Cont'd)

VARIABLE	SYMBOL	DIMENSION
. Parameter Error Covariance Matrix	C	22 x 22
. Geometric Transformation Matrices		
. $x_1 \longrightarrow x_2$	T3	6 x 6
. $x_2 \longrightarrow x_3$	T4	6 x 6
. $x_2 \longrightarrow x_5$	T5	3 x 3
. Derivatives of Observations with Respect to System Parameters	$D_{i,j}$	
. Observation Type Subscript		
. Range Rate	$i = 1$	
. Azimuth of Planet Local Vertical	$i = 2$	
. Elevation of Planet Local Vertical	$i = 3$	
. Parameter Type Subscript		
. Satellite Orbital Elements		
$\overline{Y2}$: $(Y2)_1 = a$	$j = 1$	
$(Y2)_2 = e$	$j = 2$	
$(Y2)_3 = \omega_2$	$j = 3$	
$(Y2)_4 = \Omega_2$	$j = 4$	
$(Y2)_5 = i_2$	$j = 5$	
$(Y2)_6 = t_p$	$j = 6$	
. Mars Ephemeris Elements $\overline{XB1}$	$j = 7,12$	
. Earth Ephermeris Elements $\overline{XB2}$	$j = 13,18$	
. Solar Radiation Pressure	$j = 19$	
. Planet Oblateness	$j = 20$	
. Sun Gravitational Constant	$j = 21$	
. Planet Gravitational Constant	$j = 22$	

2.0 COORDINATE SYSTEMS

Four cartesian coordinate systems are involved in the system model. Earth and planet initial position and velocity relative to the sun are given in Earth equatorial E.1950 coordinates (x_1). The planet position and velocity vector relative to Earth and the radar site loci are computed in the Earth equatorial coordinate system and transformed to planet equatorial coordinates (x_2). The satellite orbit relative to Mars is propagated in the planet equatorial system. Radar range and range rate are then computed in this system. The direction of planet local vertical from the satellite is computed in the Sun-Canopus reference system (x_5). The satellite state and orbit parameters are estimated by differential correction of the Keplerian orbital elements relative to a coordinate system called the plane-of-the-sky system (x_3), the equator of which is normal to the planet-Earth line at epoch. The first epoch is at run start time and may remain fixed or be advanced periodically. Figures 1-3 show the relationships of the four cartesian systems. The 3 x 3 transformation matrix (TT3) between x_1 and x_2 is fixed and is a program input. Since both x_2 and x_3 are fixed inertial systems, the 6 x 6 position-velocity transformation matrix is given by $T3 = \begin{pmatrix} TT3 & 0 \\ 0 & TT3 \end{pmatrix}$. The transformation matrices between x_2, x_3, and x_5 are derived below in terms of Earth, Sun, planet, and satellite position vectors.

2.1 Planet Equatorial - Plane-of-the-Sky Transform (T4)

At epoch, let:

$\overline{XB1}$	:	Planet state vector relative to Sun in x_1
$\overline{XB2}$	:	Earth state vector relative to Sun in x_1
$\overline{X1E} = \overline{XB2} - \overline{XB1}$	:	Earth state vector relative to planet in x_1

FIGURE 1

THE EARTH EQUATORIAL CARTESIAN REFERENCE SYSTEM (x_1)

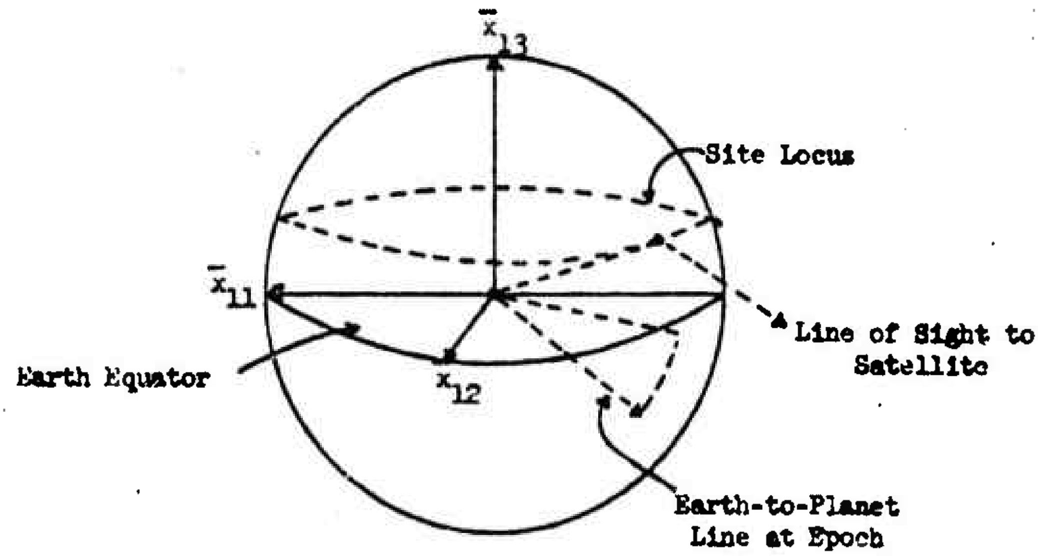

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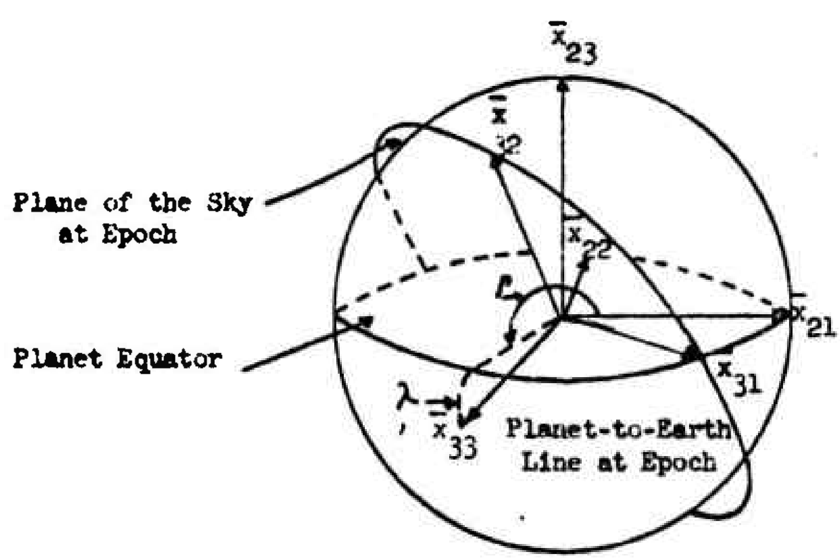
FIGURE 2

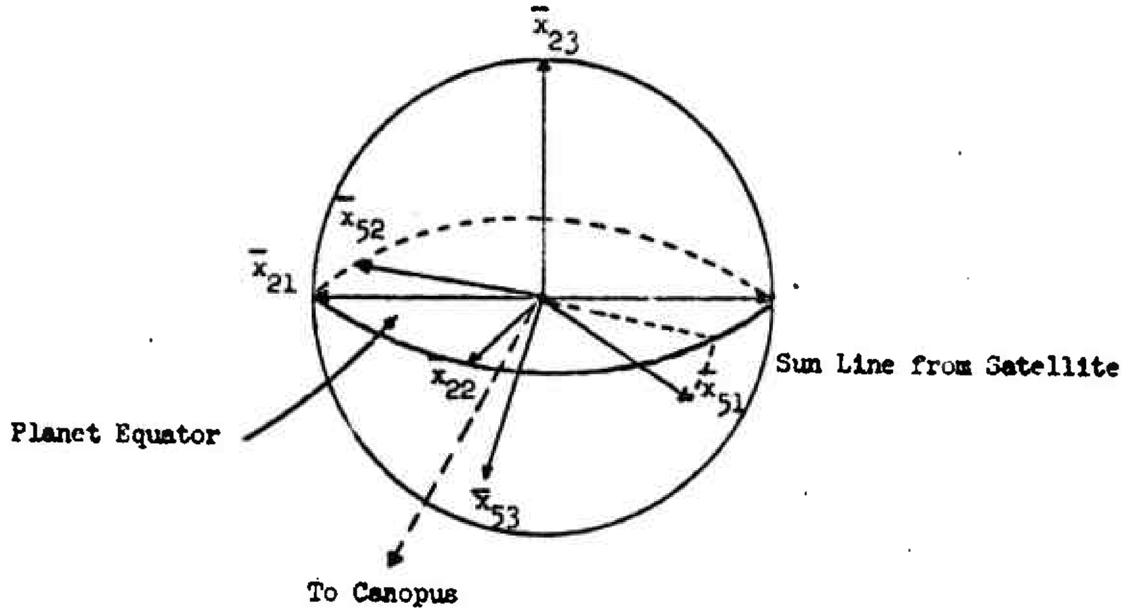
PLANET EQUATORIAL AND PLANE OF THE SKY CARTESIAN REFERENCE SYSTEMS (x_2 and x_3)



Γ, λ = angles in transformation T4

FIGURE 3

PLANET EQUATORIAL AND SUN-CANOPUS CARTESIAN REFERENCE SYSTEMS (x_2 and x_5)



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$\overline{X2E} = T3 \cdot \overline{X1E}$ Earth state vector relative to Mars in x_2

$$\Gamma = \arctan \frac{(X2E)_2}{(X2E)_1}$$

$$\lambda = \arcsin \frac{(X2E)_3}{|\overline{X2E}|}$$

$$TT4 = \begin{bmatrix} -\sin\Gamma & \cos\Gamma & 0 \\ -\cos\Gamma \cdot \sin\lambda & -\sin\Gamma \cdot \sin\lambda & \cos\lambda \\ \cos\Gamma \cdot \cos\lambda & \sin\Gamma \cdot \cos\lambda & \sin\lambda \end{bmatrix}$$

$$T4 = \begin{bmatrix} TT4 & 0 \\ 0 & TT4 \end{bmatrix}$$

2.2 Planet Equatorial - Sun-Canopus Transform (T5)

$\overline{SX2} = -\overline{X2} - T3 \cdot \overline{XB1}$ Sun line vector from vehicle in x_2

$\overline{SX} = \overline{SX2}/|\overline{SX2}|$ Sun line unit vector direction cosines in x_2

$$(T5)_{1,i} = (SX)_i \quad ; \quad i = 1,2,3$$

$$\sin\theta = \sqrt{1. - (\overline{SX} \cdot \overline{CX})^2}$$

$$(T5)_{2,1} = \left[(SX)_2 \cdot (CX)_3 - (SX)_3 \cdot (CX)_2\right] / \sin\theta$$

$$(T5)_{2,2} = \left[(SX)_3 \cdot (CX)_1 - (SX)_1 \cdot (CX)_3\right] / \sin\theta$$

$$(T5)_{2,3} = \left[(SX)_1 \cdot (CX)_2 - (SX)_2 \cdot (CX)_1\right] / \sin\theta$$

$$(T5)_{3,1} = (T5)_{2,2} \cdot (SX)_3 - (T5)_{2,3} \cdot (SX)_2$$

$$(T5)_{3,2} = (T5)_{2,3} \cdot (SX)_1 - (T5)_{2,1} \cdot (SX)_3$$

$$(T5)_{3,3} = (T5)_{2,1} \cdot (SX)_2 - (T5)_{2,2} \cdot (SX)_1$$

3.0 TRANSFORMATIONS AMONG SATELLITE ORBIT PARAMETER SETS

The satellite orbit is represented at various points in the program by the position-velocity vector in x_2 and x_3 coordinate systems, as well as by the Keplerian elements relative to either x_2 or x_3. Figures 4 and 5 show the orbit geometry in the two coordinate systems. The equations for the transformations from position-velocity at a specified time to Keplerian elements and vice versa are given in Reference 2 and will not be discussed here. Let their outputs be represented by the mnemonics CARCON and CONCAR. For example,

$$\overline{Y1} = \text{CARCON}\ (\overline{X2}, \text{time})$$

$$\overline{X2} = \text{CONCAR}\ (\overline{Y1})$$

The output of CONCAR also includes true anomaly, ν, and distance to planet center, r.

The Keplerian orbit parameter variations are propagated in the planet equatorial system in which the pole is assumed to be the axis of symmetry. Since the parameters are estimated in the plane of the sky system, it is necessary to convert from $\overline{Y1}$ to $\overline{Y2}$ before filtering the observations at a given time, and then to convert again to $\overline{Y1}$ to propagate the orbit to the next observation time. This can be accomplished with the transformations CARCON and CONCAR as follows:

$$\overline{\hat{X2}}\ (t_i) = \text{CONCAR}\ (\overline{\hat{Y1}}\ (t_i),\ t_i\ )$$

$$\overline{\hat{X3}}\ (t_i) = T4 \cdot \overline{\hat{X2}}\ (t_i)$$

$$\overline{\hat{Y2}}\ (t_i) = \text{CARCON}\ (\ \overline{\hat{X3}}\ (t_i),\ t_i\ )$$

The carot above the symbols denotes estimated rather than model parameter sets. After filtering the observations, a new estimate replaces the old estimate in $\overline{Y2}$ and this is converted to a new estimate in $\overline{Y1}$ by reversing the order of trans-

FIGURE 4

ORBIT GEOMETRY IN MARS EQUATORIAL SYSTEM

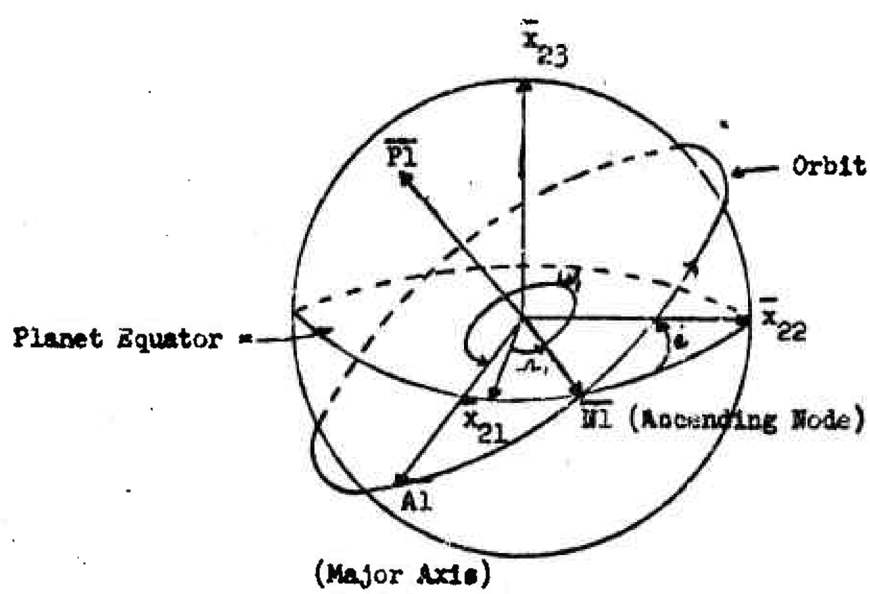

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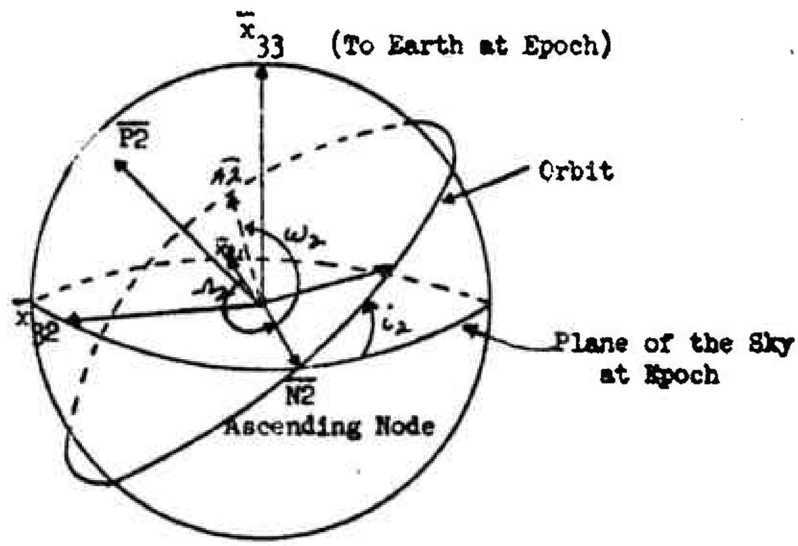
FIGURE 5

ORBIT GEOMETRY IN PLANE OF THE SKY SYSTEM



formation. This series of transformations contains redundant operations since in going from $\overline{Y1}$ to $\overline{Y2}$ and vice versa only the angles which define the orientation of the orbit plane and the major axis to the reference axes change. The equations for a direct conversion of the angles ω, Ω, and i follows. (NOTE: The transformation from Keplerian parameters to state vector is still required in the observation equations.)

Let $\overline{A1}$ and $\overline{P1}$ represent the unit vectors of the major axis and pole of the orbit in the x_2 coordinate system and let $\overline{A2}$ and $\overline{P2}$ represent the same vectors in the x_3 coordinate system. (See Figures 4 and 5). The direction cosines of $\overline{A1}$ and $\overline{P1}$ are:

$$(A1)_1 = \cos \omega_1 \cdot \cos \Omega_1 - \sin \omega_1 \cdot \cos i_1 \cdot \sin \Omega_1$$

$$(A1)_2 = \cos \omega_1 \cdot \cos \Omega_1 + \sin \omega_1 \cdot \cos i_1 \quad \cos \Omega_1$$

$$(A1)_3 = \sin \Omega_1 \cdot \sin i_1$$

$$(P1)_1 = \sin \Omega_1 \cdot \sin i_1$$

$$(P1)_2 = -\cos \Omega_1 \cdot \sin i_1$$

$$(P1)_3 = \cos i_1$$

$$\overline{A2} = TT4 \cdot \overline{A1}$$

$$\overline{P2} = TT4 \cdot \overline{P1}$$

Let $\overline{N2}$ represent the unit vector in the direction of the ascending nods in the x_3 system and let $\overline{N}$ be the vector which forms a right-handed system with $\overline{A2}$ and $\overline{P2}$. That is, $\overline{N} = \overline{A2} \times \overline{P2}$

$$(NN)_1 = (A2)_2 \cdot (P2)_3 - (A2)_3 \cdot (P2)_2$$

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$$(NN)_2 = -(A2)_1 \cdot (P2)_3 + (A2)_3 \cdot (P2)_1$$

$$(NN)_3 = (A2)_1 \cdot (P2)_2 - (A2)_2 \cdot (P2)_1$$

We can write four equations involving the components of $\overline{N2}$ and the argument of periapsis, ω_2, as follows:

$$N2_3 = 0$$

$$\overline{P2} \cdot \overline{N2} = (P2)_1 \cdot (N2)_1 + (P2)_2 \cdot (N2)_2 = 0$$

$$\overline{N2} \cdot \overline{NN} = \sin\omega_2 = (N2)_1 \cdot (NN)_1 + (N2)_2 \cdot (NN)_2$$

$$\overline{N2} \cdot \overline{A2} = \cos\omega_2 = (N2)_1 \cdot (A2)_1 + (N2)_2 \cdot (A2)_2$$

Solving for $(N2)_1$, $(N2)_2$, $\cos\omega_2$ and $\sin\omega_2$, we have:

$$\omega_2 = \arctan\left[\frac{\sin\omega_2}{\cos\omega_2}\right]$$

$$\Omega_2 = \arctan\left[\frac{(N2)_2}{(N2)_1}\right]$$

From the definition of inclination, we have:

$$i_2 = \text{arc cos}\ ((\overline{P2})_3)$$

Let the mnemonic for this transformation be CONCON. For example:

$$\overline{Y2} = \text{CONCON}\ (\overline{Y1}, T4)$$

or $$\overline{Y1} = \text{CONCON}\ (\overline{Y2}, T4^T)$$

It is necessary later to calculate the time rate of change in $\overline{Y2}$ as a function of the time rate of change in $\overline{Y1}$. For this we may either write explicit equations for the partial derivatives of $\overline{Y2}$ components with respect to $\overline{Y1}$ components, or we may calculate finite differences. The later method was used as follows:

$$\frac{\partial\,\overline{Y2}}{\partial(Y1)_i} = (\text{CONCON}\ (\overline{Y1} + \Delta(Y1)_i\ ) - \text{CONCON}\ (\overline{Y1})\ )/\ \Delta(Y1)_i$$

$$i = 3, 4, 5$$

$$\frac{\partial\,\overline{Y2}}{\partial(Y1)_i} = 1 \qquad\qquad i = 1, 2, 6$$

Then:

$$\frac{d(\overline{Y2})}{dt} = \frac{d\ (\overline{Y1})}{dt} \cdot \frac{\partial\,\overline{Y2}}{\partial\,\overline{Y1}}$$

We will also find it necessary later to calculate the partial derivatives of the satellite position and velocity components with respect to Keplerian elements and vice versa. Again, the method of finite differences was used: for instance,

$$\frac{\partial\,\overline{X2}}{\partial(\overline{Y1})_i} = (\text{CONCAR}\ (\overline{Y1} + \Delta(Y1)_i\ ) - \text{CONCAR}\ (\overline{Y1})\ )/\ \Delta(Y1)_i$$

$$i = 1, 6$$

4.0 EARTH AND PLANET EPHEMERIDES

The ephemerides of the planet and Earth are propagated as conic sections. The initial conditions are in x_1 coordinates taken from a table of ephemerides. These are converted to heliocentric orbit parameters by CARCON:

$\overline{YB1}$ = CARCON ($\overline{XB1}$, TIME 1) planet

$\overline{YB2}$ = CARCON ($\overline{XB2}$, TIME 2) sun

Position and velocity at observation times t_i are obtained by:

$$\overline{XB1}(t_i) = \text{CONCAR}\ (\overline{YB1}, t_i)$$

$$\overline{XB2}\ (t_i) = \text{CONCAR}\ (\overline{YB2}, t_i)$$

When the position and velocity components are included in the set of parameters to be solved for the estimates in the parameter sets are denoted by placing a caret above the vector bar. After filtering in cartesian coordinates, the new position-velocity estimates are converted to new estimates of conic parameters with CARCON:

$$\overline{\hat{YB1}}\,(t_i) = \text{CARCON}\left(\overline{\hat{XB1}}\,(t_i),\ t_i\right)$$
$$\overline{\hat{XB2}}\,(t_i) = \text{CARCON}\left(\overline{\hat{XB2}}\,(t_i),\ t_i\right)$$

5.0 EPHEMERIS OF SATELLITE

The ephemeris of the satellite is propagated in the planet equatorial system, the polar axis of which is the axis of symmetry. The variation in Keplerian elements, $\frac{d\overline{Y1}}{dt}$, are integrated to produce the elements of the osculating ellipse. At specified observation times the $\overline{Y1}$ elements are converted to position and velocity components $\overline{X2}$ with the CONCAR transformation and to plane of the sky elements $\overline{Y2}$ with the CONCON transformation.

Simple trapezoidal integration was used as follows:

$$Y_i\,(t+\Delta t) = Y_i\,(t) + \left.\frac{d\,Y_i\,(t)}{dt}\right|_{\text{oblate}} \cdot \Delta t + \left.\frac{d\,Y_i\,(t)}{dt}\right|_{\text{sun mass}} \cdot \Delta t + \left.\frac{d\,Y_i\,(t)}{dt}\right|_{\text{solar radiation}} \cdot \Delta t$$

J. M. Danby in reference 2 develops a method for obtaining the time derivatives of the conventional orbit elements in the frame of planet symmetry from partials of the perturbing function with respect to these elements. His analysis is applicable to any perturbation expressible in terms of radial, cross-radial and normal force components or a disturbing function F, F being a function of the orbit elements.

5.1 Orbit Parameter Variation vs. a Disturbing Function F

Assign the following notation:

$a, e, \omega, \Omega, i, T$	conventional orbit elements in reference of symmetry
r, ν	radius vector and true anomaly at time t
μ	planet gravitational constant
F	disturbing function
T	time at periapsis passage
t_p	$t_p = t - T$, where t is present time
P	orbit period
n	mean motion, $2\pi/P$
R,B,N	Radial, cross-radial, and normal components of disturbing force at r and ν

and define the intermediate variables

$$\omega^* = \omega + \Omega$$

$$\varepsilon = -nT + \omega + \Omega$$

$$\varepsilon_1 = -n(t-T) + \omega + \Omega$$

$$u = \omega + \nu$$

From equations 11.9.9 we select expressions for the time derivatives for five of our six orbit elements.

1) $$\frac{da}{dt} = \frac{2na^2}{\mu} \cdot \frac{\partial F}{\partial \varepsilon}$$

2) $$\frac{de}{dt} = \frac{na(1-e^2)}{\mu e} \cdot \frac{\partial F}{\partial \varepsilon} - \frac{na\sqrt{1-e^2}}{\mu e} \left\{ \frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \omega^*} \right\}$$

3) $$\frac{d\omega}{dt} = \frac{na\sqrt{1-e^2}}{\mu e} \cdot \frac{\partial F}{\partial e} - \frac{na}{\mu\sqrt{1-e^2}} \cdot \cot i \, \frac{\partial F}{\partial i}$$

4) $$\frac{d\Omega}{dt} = \frac{na}{\mu\sqrt{1-e^2}} \csc \frac{\partial F}{\partial i}$$

5) $$\frac{di}{dt} = \frac{-na}{\mu\sqrt{1-e^2}} \left\{ \frac{\partial F}{\partial \Omega} \csc i + \tan \frac{1}{2} i \left\{ \frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial \omega^*} \right\} \right\}$$

He also includes an expression for the time rate of change of ε_1.

6) $$\frac{d\varepsilon_1}{dt} = \frac{-2na^2}{\mu} \cdot \frac{\partial F}{\partial a} + \frac{na\sqrt{1-e^2}}{\mu e} \left\{ 1 - \sqrt{1-e^2} \right\} \frac{\partial F}{\partial e}$$

$$+ \frac{na}{\mu\sqrt{1-e^2}} \tan \frac{1}{2} i \cdot \frac{\partial F}{\partial i}$$

An expression for rate of change of time from pericenter is obtained by differentiating the equation for ε_1.

$$\frac{dt_p}{dt} = -\frac{1}{n} \cdot \frac{d\varepsilon_1}{dt} + \frac{1}{n} \cdot \frac{d\omega^*}{dt} - \frac{3}{2} \cdot \frac{t_p}{a} \cdot \frac{da}{dt}$$

$$= -\frac{1}{n} \cdot \frac{d\varepsilon_1}{dt} + \frac{1}{n} \cdot \frac{d\omega}{dt} + \frac{1}{n} \frac{d\Omega}{dt} - \frac{3}{2} \cdot \frac{t_p}{a} \cdot \frac{da}{dt}$$

Substituting equation 6 yields :

7) $$\frac{dt_p}{dt} = \frac{2na^2}{n\mu} \cdot \frac{\partial F}{\partial a} - \frac{na\sqrt{1-e^2}}{n\mu e} \left\{ 1 - \sqrt{1-e^2} \right\} \frac{\partial F}{\partial e}$$

$$- \frac{na}{n\mu\sqrt{1-e^2}} \cdot \tan \frac{1}{2} i \cdot \frac{\partial F}{\partial i} + \frac{1}{n} \frac{d\omega}{dt} + \frac{1}{n} \frac{d\Omega}{dt} - \frac{3}{2} \cdot \frac{t_p}{a} \cdot \frac{da}{dt}$$

Danby's equations 11.9.8 gives the required derivatives of F:

8) $$\frac{\partial F}{\partial a} = R \frac{r}{a}$$

9) $$\frac{\partial F}{\partial e} = -R\, a \cos\nu + B\, a \left\{ \frac{r}{p} + 1 \right\} \sin\nu$$

10) $$\frac{\partial F}{\partial i} = N\, r \sin u$$

11) $$\frac{\partial F}{\partial \omega^*} = \frac{-R\, ae \sin\nu}{\sqrt{1-e^2}} - B \frac{a^2}{r} \cdot \sqrt{1-e^2} + Br$$

12) $\frac{\partial F}{\partial \Omega} = - B y \sin^2 \frac{1}{2} i - N r \sin i \cos u$

13) $\frac{\partial F}{\partial \mathcal{E}} = \frac{R\, a\, e}{\sqrt{1-e^2}} \cdot \sin\nu - \frac{B a^2}{r} \cdot \sqrt{1-e^2}$

Substituting expressions 8)to 13) in expressions 1) to 6) yields the time rate of change of orbit elements due to each disturbing function F.

5.2 The Disturbing Function Due to Oblateness

The oblateness disturbing function is

$$F = -\frac{\mu}{2} J_2 \left(\frac{r_o^2}{r^3} \cdot 3 \cdot \sin^2 i \cdot \sin^2 (\omega + \nu) - 1 \right)$$

where r_o is the equatorial radius of planet and J_2 is the oblateness constant. The partials of F with respect to i and ω can be found by straight forward differentiation.

$$\frac{\partial F}{\partial i} = -3 \cdot \mu \cdot J_2 \cdot \frac{r_o^2}{r^3} \cdot \sin^2 (\omega + \nu) \cdot \sin i \cdot \cos i$$

and

$$\frac{\partial F}{\partial \omega} = -3 \mu J_2 \cdot \frac{r_o^2}{r^3} \cdot \sin^2 i \cdot \sin (\omega + \nu) \cdot \cos (\omega + \nu)$$

Since symmetry about the axis of rotation is assumed (inclusion of J_2 only)

$$\frac{\partial F}{\partial \Omega} = 0$$

The force components R, N, and B can now be defined in terms of these derivatives. Equation 10) gives an expression for N :

$$N = \frac{\partial F}{\partial i} \cdot \frac{1}{r \sin (\omega + \nu)}$$

Equation 12) gives an expression for B :

$$B = -\frac{\partial F}{\partial i} \cdot \frac{1}{r} \cdot \frac{\sin i \cos(\omega+\nu)}{\sin^2 \frac{1}{2} i \sin(\omega+\nu)}$$

Finally, equation 11) gives an expression for R :

$$R = \frac{-\sqrt{1-e^2)}}{ae \sin \nu} \left\{ \frac{\partial F}{\partial \omega} - \left\{ \frac{1}{r} \cdot \frac{\sin i \cos(\omega+\nu)}{\sin^2 \frac{1}{2} i \sin(\omega+\nu)} \cdot \frac{\partial F}{\partial i} \right\} \left\{ \frac{a^2}{r} \sqrt{1-e^2} - r \right\} \right\}$$

since

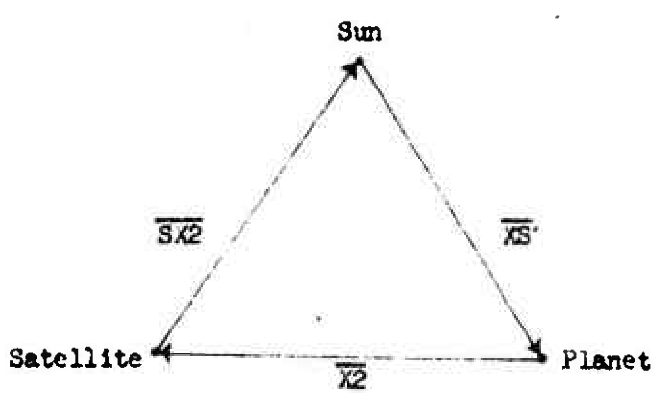
$$\frac{\partial F}{\partial \Omega} = 0 \quad \text{and} \quad \frac{\partial F}{\partial \omega} = \frac{\partial F}{\partial \omega^*} .$$

These values of R, B, and N by direct substitution into 8), 9), and 13) give immediate values for the remaining partials of F(oblate) with respect to the parameters.

5.3 The Disturbing Functions Due to Sun Mass and Solar Radiation Pressure

The position and velocity of mars in sun centered earth equatorial cartesian coordinates is denoted by $\overline{XB1}$. These are transformed to sun-centered, mars-equatorial, cartesian coordinates, $\overline{XS}$.

$$\overline{XS} = T3 \cdot \overline{XB1}$$



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The position and velocity of the sun in mars centered mars equatorial cartesian coordinates given by

$$\overline{SX2} = -\overline{X2} - \overline{XS}$$

The disturbing function duc to sun mass in satellite centered mars equatorial coordinates is then

$$SF = GM_{sun}\left\{\frac{\overline{XS}}{|\overline{XS}|^3} + \frac{\overline{SX2}}{|\overline{SX2}|^3}\right\}$$

The rotation matrix from mars equatorial to radial, cross-radial, and normal is denoted by TT6.

$$TT6(1,1) = \cos(\omega+\nu)\cdot\cos\Omega - \sin i\cdot\sin\Omega\cdot\sin(\omega+\nu)$$
$$TT6(1,2) = \cos(\omega+\nu)\cdot\sin\Omega + \sin(\omega+\nu)\cdot\cos i\cdot\cos\Omega$$
$$TT6(1,3) = \sin(\omega+\nu)\cdot\sin i$$

$$TT6(2,1) = -\cos\Omega\cdot\sin(\omega+\nu) - \cos(\omega+\nu)\cdot\cos i\cdot\sin\Omega$$
$$TT6(2,2) = -\sin(\omega+\nu)\cdot\sin\Omega + \cos(\omega+\nu)\cdot\cos i\cdot\cos\Omega$$
$$TT6(2,3) = \cos(\omega+\nu)\cdot\sin i$$

$$TT6(3,1) = \sin i\cdot\sin\Omega$$
$$TT6(3,2) = -\sin i\cdot\cos\Omega$$
$$TT6(3,3) = \cos i$$

Then the radial, cross-radial, and normal force components are

$$\begin{bmatrix} R \\ B \\ N \end{bmatrix}_{SF} = (TT6)\cdot\begin{bmatrix} SF(1) \\ SF(2) \\ SF(3) \end{bmatrix}$$

These can now be substituted directly into equations 8) to 13) to give partials of F(sun mass) with respect to the orbit parameters.

The components of the force due to solar radiation pressure are given by:

$$\overline{RF} = - RCON \cdot TT6 \cdot \overline{SX}$$

where RCON is a constant depending on vehicle characteristics. Again, substitution in equations 8) to 13) gives the partials of the disturbing function with respect to the parameters.

5.4 Extension of the Model

The model for variation of parameters is extended by defining each new disturbing function and deriving its partial derivatives with respect to the parameters in one of the two ways illustrated above. If the disturbance is represented by one or more terms in the expansion of the planet gravitational potential, the partials are obtained by differentiation of the terms with respect to the parameters as was done for the oblateness function. If the disturbing function is expressed as a force vector as in the case of sun mass and radiation pressure, the vector is resolved into components R, N, B, which are substituted in equations 8) to 13) to obtain the partials of the disturbing function with respect to the parameters. Equations 1) to 6) apply for all disturbing forces to obtain the time rate of change in the parameters.

6.0 THE OBSERVATION EQUATIONS

6.1 DSIF Range Rate

The range of the satellite from a radar site is given by

$$R = \sqrt{\sum_{i=1}^{3} \left[(X2)_i - (XS2)_i\right]^2}$$

where $(X2)$ and $(XS2)$, $(i = 1,3)$ are the components of the satellite and site positions relative to the planet in the x_2 coordinate system. The site position-velocity vector is calculated first in the x_1 system, then transformed to x_2, as follows:

$$\Gamma_s = (\Gamma s)_o + WE\,(t - t_o)$$

$$(X1S)_1 = RE \cdot \cos\lambda_s \cdot \cos\Gamma_s + (X1E)_1$$

$$(X1S)_2 = RE \cdot \cos\lambda_s \quad \sin\Gamma_s + (X1E)_2$$

$$(X1S)_3 = RE \cdot \sin\lambda_s - + (X1E)_3$$

$$(X1S)_4 = -WE \cdot RE\, \cos\lambda_s \cdot \sin\Gamma_s + (X1E)_4$$

$$(X1S)_5 = WE \cdot RE\, \cos\lambda_s \cdot \cos\Gamma_s + (X1E)_5$$

$$(X1S)_6 = (X1E)_6$$

$$\overline{XS2} = T3 \cdot \overline{X1S}$$

Differentiating the expression for range, we have:

$$\dot{R} = \frac{1}{R} \cdot \left\{ \sum_{i=1}^{3} \left((X2)_i - (XS2)_i \right) \cdot \left((X2)_{i+3} - (XS2)_{i+3} \right) \right\}$$

The filter equations require the formulation of the partial derivatives of the observations with respect to the parameters. In all of these formulations, to be derived later, the set of partial derivatives of the observations with respect to the elements of the position-velocity vector $\overline{X2}$ appears as a term. Differentiating the expression for range rate with respect to the elements of $\overline{X2}$, we have:

$$\frac{\partial \dot{R}}{(X2)_i} = \frac{(X2)_{i+3} - (XS2)_{i+3}}{R} - \frac{\dot{R} \cdot \left((X2)_i - (XS2)_i \right)}{R^2}$$

$$i = 1, 2, 3$$

$$\frac{\partial \dot{R}}{(X2)_i} = \frac{(X2)_{i-3} - (XS2)_{i-3}}{R}$$

$$i = 4, 5, 6$$

For future reference, let $D1_{1,j} = \frac{\partial \dot{R}}{(X2)_j}$, $j = 1, 2, \cdots, 6$

6.2 Planet Local Vertical

The position vector of the vehicle relative to the planet in the Sun-Canopus reference system is given by:

$$\overline{X5} = T5 \cdot \overline{X2}$$

where T5 is the transformation matrix derived in paragraph 2.2.

The direction of planet local vertical is defined in terms of two angles in the Sun-Canopus reference system as follows:

$$AZ = \arctan \left[\frac{(X5)_2}{(X5)_1} \right]$$

$$EL = \arcsin \left[\frac{(X5)_3}{|X5|} \right]$$

Differentiating, we have the partial derivatives of AZ and El with respect to the satellite position vector $\overline{X5}$:

$$\frac{\partial AZ}{\partial (X5)_1} = \left(-(X5)_2 \cdot \cos^2 (AZ) \right) / \, (X5)_1^{\;2}$$

$$\frac{\partial AZ}{\partial (X5)_2} = \frac{\cos (AZ)}{(X5)_1}$$

$$\frac{\partial(AZ)}{\partial(X5)_3} = 0$$

$$\frac{\partial(EL)}{\partial(X5)_1} = (X5)_3 \cdot (X5)_1 / |X5| \cdot \cos EL$$

$$\frac{\partial(EL)}{\partial(X5)_2} = (X5)_3 \cdot (X5)_2 / |X5| \cdot \cos EL$$

$$\frac{\partial(EL)}{\partial(X5)_3} = (1 + ((X5)_3)^2) / |X5| \cdot \cos EL$$

Applying the chain rule:

$$\frac{\partial(AZ)}{\partial(\overline{X2})} = \frac{\partial(AZ)}{\partial(\overline{X5})} \cdot \frac{\partial(\overline{X5})}{\partial(\overline{X2})} = \frac{\partial AZ}{\partial(\overline{X5})} \cdot T5$$

$$\frac{\partial(EL)}{\partial(\overline{X2})} = \frac{\partial(EL)}{\partial(\overline{X5})} \cdot \frac{(\overline{X5})}{\overline{X2}} = \frac{\partial(AZ)}{\partial(\overline{X5})} \cdot T5$$

For future reference, let

$$D1_{2,j} = \frac{\partial(AZ)}{\partial(X2)_j}$$

$$D1_{3,j} = \frac{\partial(EL)}{(X2)_j}$$

7.0 THE PARTIAL DERIVATIVES OF OBSERVATIONS WITH RESPECT TO THE PARAMETERS

The matrix of partial derivatives of the observations with respect to the parameters to be solved for is denoted by $D_{i,j}$, with the subscript definitions as outlined on the last page of Table 1. The equations for the partial derivatives of the observations with respect to vehicle position and velocity elements, $\overline{X2}$, were derived in paragraph 6.1 and 6.2. This matrix of partials is represented by $D1_{i,j}$ $i = 1,2,3$ and $j = 1,\cdots,6$. D1 and D are related to the matrix of partial derivatives of $\overline{X2}$ elements with respect to the parameters by the chain rule of differentiation:

$$D = D1 \cdot \begin{bmatrix} \frac{\partial (X2)_1}{\partial Y_1} & \frac{\partial (X2)_1}{\partial Y_2} & \cdots & \frac{\partial (X2)_1}{\partial Y_{22}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial (X2)_6}{\partial Y_1} & \frac{(X2)_6}{Y_2} & \cdots & \frac{(X2)_6}{Y_{22}} \end{bmatrix}$$

The remainder of the problem reduces to deriving the equations for the derivatives of $\overline{X2}$ with respect to the complete parameter set Y.

The first group of parameters consists of the Keplerian elements relative to the plane of the sky, $\overline{Y2}$ · $(Y_i = (Y2)_i\ ,\ i = 1,..6)$

By the method of finite differences:

$$D2 = \frac{\partial(\overline{X2})}{\partial(Y1)_i} = \left[\text{CONCAR}\ (\overline{Y1} + \Delta(Y1)_i\ ) - \text{CONCAR}\ (\overline{Y1}) \right] / \ \Delta(Y1)_i$$

$$= 1,..6$$

$$D3 = \frac{\partial(\overline{Y1})}{\partial(\overline{Y2})} = \left[\text{CONCON}\ (\overline{Y2} + \Delta(Y2)_i - \text{CONCON}\ (\overline{Y2}) \right] / \ \Delta(Y2)_i$$

$$i = 1,..6$$

$$D_{i,j} = D1 \cdot D2 \cdot D3$$

$$i = 1,.3$$

$$j = 1,..6$$

The second group of parameters consists of elements of planet and Earth velocity and position vectors. The direction of planet local vertical is insensitive to planet and Earth ephemeris deviations, so:

$$D_{i,j} = 0 \qquad i = 1, 2\ ,\ j = 7,..18$$

The derivative of range rate with respect to planet ephemeris deviations is given by:

$$\frac{\partial \dot{R}}{\partial(\overline{XB1})} = \frac{\partial \dot{R}}{\partial(\overline{X2})} \cdot \frac{\partial(\overline{X2})}{\partial(\overline{X1})} \cdot \frac{\partial(\overline{X1})}{\partial(\overline{XB1})}$$

or $D_{i,j} = D1 \cdot T3$

$i = 1; \quad j = 7,..12$

The derivative of range rate with respect to Earth ephemeris is given by:

$$\frac{\partial \dot{R}}{\partial(\overline{XB2})} = \frac{\partial \dot{R}}{\partial(\overline{XS2})} \cdot \frac{\partial(\overline{XS2})}{\partial(\overline{X1S})} \cdot \frac{\partial(\overline{X1S})}{\partial(\overline{XB2})}$$

where $\overline{XS2}$ and $\overline{X1S}$ are radar site position and velocity vectors in x_2 and x_1, coordinate systems, respectively. In matrix notation:

$$D_{i,j} = -D1 \cdot T3$$

$$= 1; \quad j = 13,..18$$

The final group of parameters are the physical constants: the planet and Sun primary gravitational constants, planet oblateness, and solar radiation pressure. The derivative of $\overline{X2}$ elements with respect to planet gravitational constant is found by the method of finite difference as follows:

$$\frac{\partial \overline{X2}}{\partial U_2} = \frac{\text{CONCAR}(\overline{Y1}, U_2 + \Delta U_2) - \text{CONCAR}(\overline{Y1}, U_2)}{\Delta U_2}$$

Then

$$\frac{\partial \dot{R}}{\partial U_2} = \frac{\partial \dot{R}}{\partial(\overline{X2})} \cdot \frac{\partial(\overline{X2})}{\partial U_2}$$

or $D(1,22) = D1 \cdot \dfrac{\partial(\overline{X2})}{\partial U_2}$

The elements of $D_{i,j}$ for the remaining physical constants are functions of the variation in the Keplerian elements associated with each constant. Consider the effect of oblateness over an observation interval Δt:

$$\overline{Y1}(t) = \overline{Y1}(t - \Delta t) + \int_{t}^{t+\Delta t} \frac{d(\overline{Y1})}{dt} \, dt_{\text{oblate}}$$

$$= \overline{Y1}(t - \Delta t) + J2 \cdot \left[\frac{(\Delta \overline{Y1})\ \text{Oblate}}{J2} \right]_{t,\ t+\Delta t}$$

Let:

$$\frac{\partial \overline{Y1}(t)}{J2} \sim \left. \frac{(\Delta \overline{Y1})\ \text{Oblate}}{J2} \right|_{t,t+\Delta t} = \frac{\overline{\Delta YM}}{J2}$$

Applying the chain rule of differentiation:

$$\frac{\partial\,(\text{OBSERVATIONS})}{\partial J2} = D_{i,j} = \frac{\partial(\text{OBS})}{\partial(\overline{X2})} \cdot \frac{\partial(\overline{X2})}{\partial(\overline{Y1})} \cdot \frac{\overline{\Delta YM}}{J2}$$

$$= D1 \cdot D2 \cdot \overline{\Delta YM}/J2$$

$$i = 1, 2, 3$$

$$j = 20$$

Similarly for solar radiation pressure and Sun mass:

$$D_{i,j} = D1 \cdot D2 \cdot \left[\overline{\Delta YR}/\text{RCON} \quad \overline{\Delta YS}/U_3 \right]$$

$$= 1, 2, 3; \quad j = 19 \text{ and } 21$$

8.0 THE KALMAN FILTER

The general theory of estimation which justifies the process of least squares filtering is treated in references 3 through 6. Particularly, the equations for the Kalman filter used in this simulation are developed in reference 6. The equations as implemented produce a differential correction in the current estimate of parameters with every observation set taken sequentially in time. The correction is a function of the following variables:

- the matrix of derivatives at the observation time of the observations with respect to the parameters, D, where D is of dimension IXM, I being the number of observation types, and M being the number of parameters in the set being estimated;
- the estimate of error in the parameters before the correction is made, represented by the parameter error covariance matrix C of dimension MXM.
- the observation error covariance matrix EN of dimension IXI.
- the vector of residuals in the observations before the correction is made, represented by the vector $\overline{F}$ of dimension I.

The estimate of observations and the associated elements of the D matrix at time t are calculated as a function of the estimate of parameters at time t before the correction is made, using the equations developed in paragraphs 2 through 7. The residual vector $\overline{F}$ at time $t + \Delta t$ is the difference between the simulated observations (Model observations plus random error) and predicted observations based on the parameter estimate at time t. The matrix EN is a program input which is constant throughout the run. The matrix C at time t, before the correction is made, is calculated as a function of its value after the last correction at time $t - \Delta t$, and as a function of the first order deviations in the parameters at time t with respect to deviations in the parameters at time

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$t - \Delta t$. Let the matrix of coefficient deviations at time t with respect to deviations at time $t - \Delta t$ be denoted by S. In the parameter set we have chosen, all of the diagonal elements of S are one and most of the off-diagonal elements are zero. The exceptions are:

- the coefficients of deviations in the Keplerian elements at time t with respect to the oblateness, Sun mass, and solar radiation pressure constants are respectively $\overline{\Delta YM}$/J2, $\overline{\Delta YS}/U_3$, and $\overline{\Delta YR}$/RCOM. These are the elements $S_{i,j}$, i = 1 through 6, and j = 19, 20, and 22.
- the coefficients of deviations in planet and Earth position components at time t with respect to velocity component deviations at time $t - \Delta t$ are all equal to Δt.

The parameter error covariance matrix at time t before correction is given by:

$$C_t = S \cdot C_{t-\Delta t} \cdot S^T$$

The differential correction in the parameters at time t is given by:

$$\overline{DY}_t = W \cdot \overline{F}_t$$

where $W = C_t \cdot D_t^T \left[D_t \cdot C_t \cdot D_t^T + EN \right]^{-1}$

The new estimate in the parameters and parameter error covariance are:

$$\text{NEW } \overline{Y}_t = \overline{Y}_t + \overline{DY}_t$$

$$\text{NEW } C_t = C_t - W \cdot D_t \cdot C_t$$

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13 ABSTRACT

The equations for a simulation of planet artificial satellite orbit parameter estimation are defined. The estimation process consists of sequential linear differential correction based on Earth-based radar doppler measurements and/or on-board measurements of the direction of planet local vertical. The parameter set includes the Keplerian elements of the satellite orbit relative to the planet, planet and Sun gravitational constants, and components of the position and velocity of the planet and Earth relative to the Sun.

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SUPPLEMENTARY

INFORMATION

AD-473967

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NUMBER D2-84073-1
REV LTR A

ACTIVE SHEET RECORD

SHEET NUMBER	REV LTR	ADDED SHEETS: SHEET NUMBER	REV LTR	SHEET NUMBER	REV LTR	SHEET NUMBER	REV LTR	ADDED SHEETS: SHEET NUMBER	REV LTR	SHEET NUMBER	REV LTR
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REVISIONS

LTR	DESCRIPTION	DATE	APPROVAL
A	Revision (A) consists of the following corrections and additions to the text: 1) Equation (7) on page 30 is revised to drop the last term in the equation. Replace page 30 with revised page 30. 2) The first half of page 32 is revised to include equations which eliminate a problem of discontinuity in the original equations at true anomaly equal to zero. Replace page 32 with the revised page 32 and a new page 32a. 3) A typographical error is corrected in the last line of page 36. Replace page 36 with the revised page 36. 4) Errors in the second, third, and fourth equations on page 37 are corrected. Replace page 37 with revised page 37.	2-8-66	RGM

5) $$\frac{di}{dt} = \frac{-na}{\mu\sqrt{1-e^2}}\left\{\frac{\partial F}{\partial \iota}\csc i + \tan\frac{1}{2}i\left(\frac{\partial F}{\partial \Sigma} + \frac{F}{\partial \omega^*}\right)\right\}$$

He also includes an expression for the time rate of change of Σ_1.

6) $$\frac{d\Sigma_1}{dt} = \frac{-2na^2}{\mu}\cdot\frac{\partial F}{\partial a} + \frac{na\sqrt{1-e^2}}{\mu e}\left\{1-\sqrt{1-e^2}\right\}\frac{\partial F}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\frac{1}{2}i\frac{\partial F}{\partial i}$$

An expression for rate of change of time from pericenter is obtained by differentiating the equation for Σ_1.

$$\frac{dt_p}{dt} = -\frac{1}{n}\cdot\frac{d\Sigma_1}{dt} + \frac{1}{n}\cdot\frac{d\omega^*}{dt} - \frac{3}{2}\cdot\frac{t_p}{a}\cdot\frac{da}{dt}$$

$$= -\frac{1}{n}\cdot\frac{d\Sigma_1}{dt} + \cdot\frac{1}{n}\cdot\frac{d\omega}{dt} + \frac{1}{n}\frac{d\Omega}{dt}$$

Substituting equation 6 yields:

7) $$\frac{dt_p}{dt} = \frac{2na^2}{n\mu}\cdot\frac{\partial F}{\partial a} - \frac{na\sqrt{1-e^2}}{n\mu e}\left\{1-\sqrt{1-e^2}\right\}\frac{\partial F}{\partial e} - \frac{na}{n\mu\sqrt{1-e^2}}\cdot\tan\frac{1}{2}i\cdot\frac{\partial F}{\partial i} + \frac{1}{n}\cdot\frac{d\omega}{dt} + \frac{1}{n}\cdot\frac{d\Omega}{dt}$$

Danby's equations 11.9.8 gives the required derivatives of F:

8) $$\frac{\partial F}{\partial a} = R\frac{r}{a}$$

9) $$\frac{\partial F}{\partial e} = -R\,a\cos\nu + B\,a\left\{\frac{r}{p}+1\right\}\sin\nu$$

10) $$\frac{\partial F}{\partial i} = N\,r\sin u$$

11) $$\frac{\partial F}{\partial \omega^*} = \frac{-R\,ae\sin\nu}{\sqrt{1-e^2}} - B\frac{a^2}{r}\cdot\sqrt{1-e^2} + Br$$

Substituting the expressions for dE from Equation (17) successively in Equation (16) yields:

$$(18)\qquad \frac{\partial r}{\partial a} = \frac{-3}{2} \cdot \frac{M \cdot e \cdot \sin \nu}{\sqrt{1 - e^2}} + \frac{r}{a}$$

$$(19)\qquad \frac{\partial \nu}{\partial a} = \frac{-3}{2} \cdot \frac{M \cdot a \cdot \sqrt{1 - e^2}}{r^2}$$ where M is the mean anomaly at time t.

Finally,

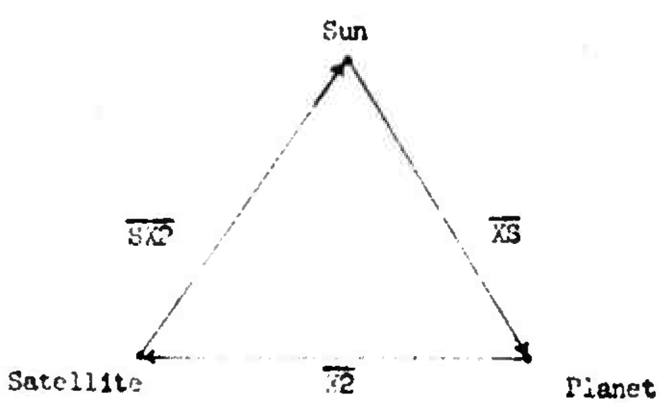
$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial a} + \frac{\partial F}{\partial \nu} \cdot \frac{\partial \nu}{\partial a}$$

and $R = \left(\frac{a}{r}\right)\left(\frac{\partial F}{\partial a}\right)$

5.3 The Disturbing Functions due to Sun Mass and Solar Radiation Pressure

The position and velocity of Mars in sun centered earth equatorial cartesian coordinate is denoted by $\overline{XBI}$. These are transformed to sun-centered, Mars-equatorial, cartesian coordinates, $\overline{XS}$.

$$\overline{XS} = T3 \cdot \overline{XBI}$$



Equation (12) yeilds an expression for B:

$$B = \frac{-\partial F}{\partial i} \cdot \frac{1}{r} \cdot \frac{\sin i \cos (\omega + \nu)}{\sin^2\left(\frac{i}{2}\right) \sin (\omega + \nu)}$$

The apparent discontinuity at $(\omega + \nu) = 0$ due to the $\sin (\omega + \nu)$ term in the denominators of N and B does not actually exist because $\partial F/\partial i$ has a factor $\sin^2 (\omega + \nu)$. Equation (11) yields a solution for the component R at all points except at $\nu = 0$. To avoid this discontinuity,the partial of F with respect to the semi-major axis is derived and substituted in Equation (8) to yield in expression for R. Equations (9) and (13) then yield the remaining partials of F with respect to the elements.

The partials of F with respect to r and ν are:

$$\frac{\partial F}{\partial r} = \frac{3}{2} \mu J_2 \frac{ro^2}{r^4} (\sin^2 i)\left(\sin^2 (\omega + \nu)\right)$$

$$\frac{\partial F}{\partial \nu} = -3 \mu J_2 \frac{ro^2}{r^3} (\sin^2 i)\left(\sin (\omega + \nu)\right)\left(\cos (\omega + \nu)\right)$$

The partials of r and ν with respect to (a) are derived from the ellipse time and position equations as follows:

(14) $$(t - T) = \pm \sqrt{\frac{a^3}{\mu}} (E - e \sin E)$$

(15) $$\cos E = \frac{a - r}{ae} = \frac{e + \cos \nu}{1 + e \cos \nu}$$

Holding time constant and differentiating (14) and (15) with respect to (a) yields:

(16) $$\frac{3}{2} (E - e \sin E) \, da = -a (1 - e \cos E) \, dE$$

(17) $$dE = \frac{1}{ae \sin E} \cdot \left(dr - \frac{r}{a} da\right) \quad \text{or} \quad dE = \frac{\sqrt{1 - e^2}}{1 + e \cos \nu} d\nu$$

$$i = 1, 2, 3$$

$$\frac{\partial \dot{R}}{(X2)_i} = \frac{(X2)_{i-3} - (SX2)_{i-3}}{R}$$

$$i = 4, 5, 6$$

For future reference, let $D1_{1,j} = \frac{\partial \dot{R}}{(X2)_j}$, $j = 1, 2, \ldots 6$

6.2 Planet Local Vertical

The position vector of the vehicle relative to the planet in the Sun-Canopus reference system is given by:

$$\overline{X5} = T5 \cdot \overline{X2}$$

where T5 is the transformation matrix derived in paragraph 2.2.

The direction of planet local vertical is defined in terms of two angles in the Sun-Canopus reference system as follows:

$$AZ = \arctan \left[\frac{(X5)_2}{(X5)_1} \right]$$

$$EL = \arcsin \left[\frac{(X5)_3}{|X5|} \right]$$

Differentiating, we have the partial derivations of AZ and EL with respect to the satellite position vector $\overline{X5}$:

$$\frac{\partial AZ}{\partial (X5)_1} = \left[-(X5)_2 \cdot \cos^2 (AZ) \right] / (X5)_1^{\ 2}$$

$$\frac{\partial (AZ)}{\partial (X5)_2} = \frac{\cos^2 (AZ)}{(X5)_1}$$

$$\frac{\partial(AZ)}{\partial(X5)_3} = 0$$

$$\frac{\partial(EL)}{\partial(X5)_1} = -(X5)_3 \cdot (X5)_1 / |X5|^3 \cdot \cos EL$$

$$\frac{\partial(EL)}{\partial(X5)_2} = -(X5)_3 \cdot (X5)_2 / |X5|^3 \cos EL$$

$$\frac{\partial(EL)}{\partial(X5)_3} = \left(\frac{1}{|X5|} - \frac{(X5_3)^2}{|X5|^3} \right) \cdot \frac{1}{\cos EL}$$

Applying the chain rule:

$$\frac{\partial(AZ)}{\partial(\overline{X2})} = \frac{\partial(AZ)}{\partial(\overline{X5})} \cdot \frac{\partial(\overline{X5})}{\partial(\overline{X2})} = \frac{\partial AZ}{\partial(\overline{X5})} \cdot T5$$

$$\frac{\partial(EL)}{\partial(\overline{X2})} = \frac{\partial(EL)}{\partial(\overline{X5})} \cdot \frac{\partial(\overline{X5})}{\partial \overline{X2}} = \frac{\partial(AZ)}{\partial(\overline{X5})} \cdot T5$$

For future reference, let

$$D1_{2,j} = \frac{\partial(AZ)}{\partial(X2)_j}$$

$$D1_{3,j} = \frac{\partial(EL)}{(X2)_j}$$

7.0 THE PARTIAL DERIVATIVES OF OBSERVATIONS WITH RESPECT TO THE PARAMETERS

The matrix of partial derivatives of the observations with respect to the parameters to be solved for is denoted by $D_{i,j}$, with the subscript definitions as outlined on the last page of Table 1. The equations for the partial derivatives of the observations with respect to vehicle position and velocity elements, $\overline{X2}$, were derived in paragraph 6.1 and 6.2. This matrix of partials is represented by $D1_{i,j}$ $i = 1,2,3$ and $j = 1,..6$. D1 and D are related to the matrix of partial