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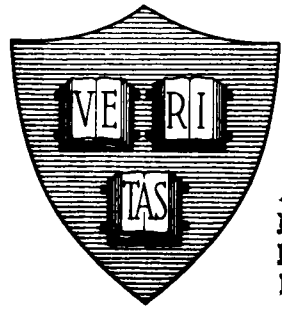
AN ESTIMATION THEORY APPROACH TO PULSE AMPLITUDE MODULATION RECEIVERS



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Technical Report No. 438

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Technical Report No. 438

Cruft Laboratory

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AN ESTIMATION THEORY APPROACH TO PULSE AMPLITUDE MODULATION RECEIVERS

by

Toby Berger

Division of Engineering and Applied Physics

Harvard University, Cambridge, Massachusetts

ABSTRACT

By generalizing a trivial problem in estimation theory, some interesting results have been obtained for a class of pulse amplitude modulation (PAM) communication channels. In particular, we consider a linear channel with memory perturbed by additive noise which is uncorrelated with the signal statistics. It is shown that, for specified second-order statistics and a mean square error criterion, the Bayes estimation rule for the specific case of zero-mean Gaussian signal and noise statistics is also both the best linear estimation rule and the minimax estimation rule. This fact is used in the theoretical design of sampled-data and continuous time receivers for such PAM systems. In the continuous time case with an infinite observation interval and stationary noise statistics, the receiver thus designed agrees with the optimum linear receiver previously derived for this case by Tufts [2] using a very different approach; the present analysis shows that Tufts's linear receiver is, therefore, also the best nonlinear receiver in the minimax sense.

In this report we consider the consequences of the following trivial estimation problem.

Problem 1:

The real random variable a has the Gaussian probability density

$$p_1(a) = \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{a^2}{2b^2}} \quad ; \quad -\infty < a < \infty . \tag{1}$$

The real random variable y is statistically independent of a and has the Gaussian probability density

$$p_2(y) = \frac{1}{\sqrt{2\pi}\, \sigma} e^{-\frac{y^2}{2\sigma^2}} \quad , \quad -\infty < y < \infty . \tag{2}$$

The sum random variable $x = a + y$ is observed. It is desired to find that estimation rule $\hat{a}(x)$ which minimizes the mean square error D between $\hat{a}(x)$ and a. This mean square error is defined by:

$$D = \overline{(\hat{a}(x) - a)^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\hat{a}(a+y)-a]^2 p_1(a)\, p_2(y)\, da\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\hat{a}(x)-a]^2 p_1(a)\, p_2(x-a) . \tag{3}$$

Solution:

It is well known ([1], p. 190) that the optimum estimate $\hat{a}_1(x)$ is the mean of a, conditional upon having observed x, namely

$$\hat{a}_1(x) = \int_{-\infty}^{\infty} da\, p(a|x) = \frac{\int_{-\infty}^{\infty} a\, p_2(x-a)\, p_1(a)\, da}{\int_{-\infty}^{\infty} p_2(x-a)\, p_1(a)\, da} . \tag{4}$$

Substituting (1) and (2) into (4) and performing the indicated computations (See Appendix I) yields the result:

$$\hat{a}_1(x) = \left(\frac{b^2}{b^2 + \sigma^2}\right) x \,. \tag{5}$$

The resultant minimum value of D found upon substitution of $\hat{a}_1(x)$ from (5) into (3) is (See Appendix I):

$$D_1 = \frac{b^2 \sigma^2}{b^2 + \sigma^2} \,. \tag{6}$$

We now introduce some terminology and notation which will be employed below in discussing other estimation problems related to Problem 1.

The expected value of a function $f(a, y)$ of two real random variables with joint density $p(a, y)$ will be denoted by

$$E_p[f(a, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a, y)\, p(a, y)\, da\, dy \,. \tag{7}$$

We shall denote by W the class of all joint probability densities $p(a, y)$ which satisfy the following requirements

$$E_p(a^2) = b^2 \tag{8a}$$

$$E_p(ay) = 0 \tag{8b}$$

$$E_p(y^2) = \sigma^2 \,. \tag{8c}$$

Problem 2:

It is known that the joint density of a and y is an element of W. The sum random variable $x = a+y$ is observed, and it is required that the estimation rule $\hat{a}(x)$ be linear, i.e., the only permissible estimation rules are of the form $\hat{a}(x) = kx$. We seek that estimation rule $\hat{a}_2(x)$ of this form which minimizes the mean square error D defined by

$$D(\hat{a}, p) = E_p \left\{[\hat{a}(x) - a]^2\right\} . \tag{9}$$

Solution:

First note that if $\hat{a}(x)$ is linear, then $D(\hat{a}, p)$ is independent of which $p \in W$ is used to compute it, because

$$D(kx, p) = E_p \left\{[k(a+y)-a]^2\right\} = E_p[(k-1)^2a^2] + E_p[2k(k-1)ay] + E_p[k^2y^2] ,$$

from which by use of (8) we obtain

$$D(kx, p) = (k-1)^2 b^2 + k^2\sigma^2 = f(k) . \tag{10}$$

Setting $f'(k) = 0$ yields $k = b^2/b^2+\sigma^2$, while $f''(k) = 2\sigma^2 > 0$ insures a minimum. Accordingly, $\hat{a}_2(x) = \hat{a}_1(x)$.

Alternatively, one could argue as follows. Since the value of $D(\hat{a}, p)$ is independent of $p \in W$ when $\hat{a}$ is linear, we can compute it using any $p \in W$ whatsoever. In particular, suppose we were to use

$$\tilde{p}(a, y) = p_1(a)\, p_2(y) \tag{11}$$

where $p_1(a)$ and $p_2(y)$ are given by Eqs. (1) and (2), respectively; one easily verifies that $\tilde{p}(a, y) \in W$. Now Problem 1 tells us that for any $\hat{a}(x)$ and, hence, a fortiori, for any linear $\hat{a}(x)$, we have

$$D(\hat{a}, \tilde{p}) \geq D(\hat{a}_1, \tilde{p}) = D_1 \ . \tag{12}$$

Since $\hat{a}_1$ is itself linear, we may conclude that $\hat{a}_2(x) = \hat{a}_1(x)$.

The alternative method of solution of Problem 2 proves useful in solving the following minimax estimation problem.

Problem 3:

It is desired to find the estimation rule $\hat{a}(x)$ which minimizes the maximum mean square error that occurs when $p(a, y)$ is varied over W, i.e., our optimum estimate $\hat{a}_3(x)$ is defined by the requirement

$$\left(\forall \hat{a}(x)\right)\left(\sup_{p \in W} D(\hat{a}, p) \geq \sup_{p \in W} D(\hat{a}_3, p)\right) . \tag{13}$$

Solution:

The following argument establishes that $\hat{a}_3(x) = \hat{a}_1(x)$. Since $\hat{a}_1(x)$ is linear, we know that $D(\hat{a}_1, p) = \text{const.} = D_1$ for all $p \in W$ and, therefore,

$$\sup_{p \in W} D(\hat{a}_1, p) = D_1 \ . \tag{14}$$

On the other hand, we know from Problem 1 that $\hat{a}_1(x)$ yields the least value of D when $p = \tilde{p}$. Accordingly, for all $\hat{a}(x)$ we have

$$\sup_{p \in W} D(\hat{a}, p) \geq D(\hat{a}, \tilde{p}) \geq D(\hat{a}_1, \tilde{p}) = D_1 = \sup_{p \in W} (\hat{a}_1, p) \ . \tag{15}$$

Comparing (15) with (13) shows that $\hat{a}_3(x) = \hat{a}_1(x)$.

In summary then, we see that in the space W of joint probability densities as defined by (8), the least favorable distribution with respect

to a mean square error criterion is $\tilde{p}(a, y)$ of equation (11) in which a and y are statistically independent, zero mean and Gaussian. Since $\hat{a}_1(x)$, the optimum estimation rule for $\tilde{p}(a, y)$ and a mean square error criterion, is linear in x, we have seen that it is in fact the solution to all three of the estimation problems we have considered.*

We now employ the knowledge gained from the above problems in order to solve some estimation problems regarding the pulse amplitude modulation system depicted in Fig. 1.

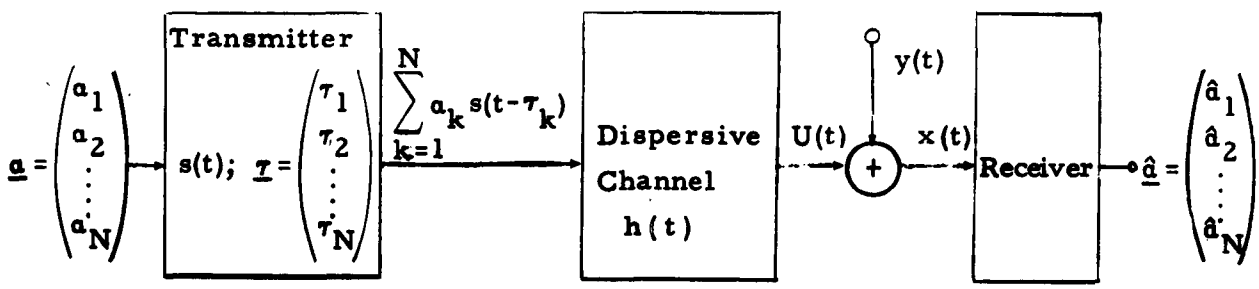

Figure 1: Block Diagram of PAM Communication System

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*We could have defined W somewhat more broadly and still had $\hat{a}_1(x)$ as the minimax solution. In particular, we could have replaced (8a) by $E_p(a^2) \leq b$ and (8c) by $E_p(y^2) \leq \sigma^2$, and we could have permitted probability distributions $P(a, y)$ which, because of discrete concentrations of probability, possess no density $p(a, y)$.

The transmitter forms the sum $\sum_{k=1}^{N} a_k s(t-\tau_k)$ from the random vector $\underline{a}$, the known vector $\underline{\tau}$ and the known function $s(t)$. This transmitter output is then sent over a linear, dispersive channel characterized by an impulse response $h(t)$. The receiver input $x(t)$ is an additive combination of the channel response $u(t)$ to the transmitter output and a sample function $y(t)$ of a random process. The receiver operates on the noisy waveform $x(t)$ to produce the vector $\hat{\underline{a}}$ of estimates of the respective components of $\underline{a}$. The design of the receiver may, therefore, be viewed as a problem in statistical estimation theory. We start with a relatively trivial problem analogous to Problem 1.

Problem 4:

Suppose that the message vector $\underline{a}$ has a zero-mean Gaussian probability density of the form

$$p(\underline{a}) = \frac{e^{-\frac{1}{2}\underline{a}^T \underline{\Phi}^{-1} \underline{a}}}{(2\pi)^{N/2} |\underline{\Phi}|^{1/2}} = \frac{e^{-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_{ij}^{-1} a_i a_j}}{(2\pi)^{N/2} |\underline{\Phi}|^{1/2}} . \qquad (16)$$

In Eq. (16), ϕ_{ij}^{-1} represents the $i\text{-}j^{th}$ element of the inverse of the autocorrelation matrix $\underline{\Phi}$. Further suppose that $y(t)$ is a sample function of a zero mean, possibly nonstationary Gaussian random process statistically independent of $\underline{a}$ and having the autocorrelation function $\gamma(t,s) = \overline{y(t)y(s)}$. The receiver input $x(t)$ is observed at M distinct time instants $t_1, t_2, \ldots, t_M$; the observed random variables, therefore, constitute

an M-dimensional column vector $\underline{x}$ with components $x_m = x(t_m)$; $m = 1, 2, \ldots, M$. We seek that estimation rule $\hat{\underline{a}}(\underline{x})$ which minimizes the mean square error between $\hat{\underline{a}}$ and $\underline{a}$ defined as

$$D = \overline{[\hat{\underline{a}}(\underline{x}) - \underline{a}]^2} = \sum_{i=1}^{N} \int \int d\underline{a}\, d\underline{x}\, [\hat{a}_i(\underline{x}) - a_i]^2 p(\underline{a}) q(\underline{x} \mid \underline{a}) \,. \qquad (17)$$

The integration in Eq. (17) extends over the entire N+M dimensional space of which $d\underline{a}\, d\underline{x} = da_1 \ldots da_N dx_1 \ldots dx_M$ is a differential element of volume, and $q(\underline{x} \mid \underline{a})$ is the M-dimensional probability density of $\underline{x}$ for a given message vector $\underline{a}$.

Solution:

The signal component $u(t)$ of the receiver input waveform $x(t)$ is given by

$$u(t) = \sum_{j=1}^{N} a_j r(t - \tau_j), \quad \text{where} \quad r(t) = \int_{-\infty}^{\infty} s(t-\xi)\, h(\xi)\, d\xi \,. \qquad (18)$$

In order to write down $q(\underline{x} \mid \underline{a})$ explicitly, we introduce the following notation. Let $\underline{\Gamma}$ be the M x M autocorrelation matrix whose m-nth element is $\gamma(t_m, t_n) = \gamma_{mn}$, and let γ_{mn}^{-1} be the m-nth element of $\underline{\Gamma}^{-1}$. Let $\underline{u}(\underline{a})$ be the column vector with components $u(t_m) = \sum_{j=1}^{N} a_j r(t_m - \tau_j) = \sum_{j=1}^{N} c_{mj} a_j = u_m(\underline{a})$; $m = 1, 2, \ldots, M$. We then may write

$$q(\underline{x}|\underline{a}) = \frac{e^{-\frac{1}{2}(\underline{x}^T-\underline{u}^T(\underline{a}))\underline{\Gamma}^{-1}(\underline{x}-\underline{u}(\underline{a}))}}{(2\pi)^{N/2}\,|\underline{\Gamma}|^{1/2}} = \frac{e^{-\frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}\gamma_{mn}^{-1}[x_m-u_m(\underline{a})][x_n-u_n(\underline{a})]}}{(2\pi)^{M/2}\,|\underline{\Gamma}|^{1/2}} \tag{19}$$

It is a standard result of estimation theory ([1],pp. 188-190) that the optimum estimation rule is the mean of $\underline{a}$ conditional upon having observed $\underline{x}$, the i^{th} component of which is

$$\hat{a}_i(\underline{x}) = \frac{\int a_i\, p(\underline{a})\, q(\underline{x}|\underline{a})\, d\underline{a}}{\int q(\underline{x}|\underline{a})\, p(\underline{a})\, d\underline{a}}\,. \tag{20}$$

The integrations indicated in Eq. (20) are performed in Appendix II and yield the result

$$\hat{a}_i(\underline{x}) = \sum_{m=1}^{M} \theta_{im}\, x_m\,; \quad i = 1,\ 2,\ \ldots,\ N\,. \tag{21}$$

The constants θ_{im} appearing in (21) are defined in Appendix II.

The most important property of the estimation rule specified by (21) is that, like the optimum estimate $\hat{a}_1(x)$ of Problems 1, 2, and 3, it is linear in the observed random variables x_m.

We now introduce some terminology and notation which will be employed below in discussing other estimation problems related to Problem 4.

The expected value of a function $f(\underline{a}, \underline{y})$ of two real random vectors with joint probability density $p(\underline{a}, \underline{y})$ will be denoted by

$$E_p[f(\underline{a}, \underline{y})] = \int \int f(\underline{a}, \underline{y})\, p(\underline{a}, \underline{y})\, d\underline{a}\, d\underline{y} . \tag{22}$$

We shall denote by V the class of $(M+N)$ dimensional joint probability densities $p(\underline{a}, \underline{y})$ which satisfy the following requirements

$$\left.\begin{aligned} E_p(a_j a_k) &= \phi_{jk} \qquad (22a)\\ E_p(a_j y_m) &= 0 \qquad (22b)\\ E_p(y_m y_n) &= \gamma_{mn} \qquad (22c)\end{aligned}\right\} \quad j,k = 1,2,\ldots,N;\ \ m,n = 1,2,\ldots,M.$$

Problem 5:

Let $\underline{a}$ be the N-component random message vector and let $\underline{y}$ be M-component random vector whose elements are the noise samples $y(t_m) = y_m$. It is known that $p(\underline{a}, \underline{y}) \in V$. The M-component random vector $\underline{x}$ whose elements are the receiver input samples $x(t_m) = x_m$ is observed, and it is required that the estimation rule $\hat{\underline{a}}(\underline{x})$ be linear, i.e., the only permissible estimation rules are of the form $\hat{\underline{a}}(\underline{x}) = \underline{K}\,\underline{x}$, where $\underline{K}$ is an $N \times M$ rectangular matrix. We seek that estimation rule of this form which minimizes the mean square error D defined by

$$D(\hat{\underline{a}}, p) = E_p\left\{[\hat{\underline{a}}(\underline{x}) - \underline{a}]^2\right\} = \sum_{i=1}^{N} E_p\left\{[\hat{a}_i(\underline{x}) - a_i]^2\right\} \tag{23}$$

Solution:

For linear $\hat{\underline{a}}(\underline{x})$ we find that $D(\hat{\underline{a}}, p)$ is independent of which $p \in V$ is used to compute it, since we have

$$D(\underline{K}\underline{x}, p) = \sum_{i=1}^{N} E_p\left\{\left[\sum_{m=1}^{M} k_{im} x_m - a_i\right]^2\right\} = \sum_{i=1}^{N} E_p\left\{\left[\sum_{m=1}^{M} k_{im}[y_m + u_m(\underline{a})] - a_i\right]^2\right\}$$

$$D(\underline{K}\underline{x}, p) = \sum_{i=1}^{N} E_p\left\{\left[\sum_{m=1}^{M} k_{im}\left(y_m + \sum_{j=1}^{N} c_{mj} a_j\right) - a_i\right]^2\right\}$$

$$D(\underline{K}\underline{x}, p) = \sum_{i=1}^{N} E_p\left\{\left[\sum_{m=1}^{M} k_{im}\, y_m + \sum_{j=1}^{N}\left(\sum_{m=1}^{M} k_{im}\, c_{mj}\right) a_j - a_i\right]^2\right\} \quad . \qquad (24)$$

When the squaring operation indicated in (24) is carried out and the expectation operation is commuted with the summations, it becomes clear from (22) that $D(\underline{K}\underline{x}, p)$ is independent of $p \in V$. It then follows by reasoning directly analogous to the alternative method of solution of Problem 2 that the desired estimation rule is that already specified by Eq. (21) as the solution to Problem 4, i.e., the optimum estimation rule has $\underline{K} = \underline{\Theta} = (\theta_{im})$; the distribution analogous to $\tilde{p}(a, y)$ of Eq. (11) is

$$\tilde{p}(\underline{a}, \underline{y}) = \frac{e^{-\frac{1}{2}\left[\underline{a}^T \underline{\Phi}^{-1} \underline{a} + \underline{y}^T \underline{\Gamma}^{-1} \underline{y}\right]}}{(2\pi)^{\frac{M+N}{2}} |\underline{\Phi}|^{\frac{1}{2}} |\underline{\Gamma}|^{\frac{1}{2}}} \quad . \qquad (25)$$

Problem 6:

It is desired to find that estimation rule $\hat{\underline{a}}(\underline{x})$ which minimizes the maximum assumed by $D(\hat{\underline{a}}, p)$ as p is varied over V.

Solution:

In complete analogy to Problem 3, we see that Eq. (21) specifies the desired estimation rule.

In summary, the common solution of Problems 4, 5, and 6 is the optimum sampled-data mean square error receiver for the PAM system of Fig. 1, where the term "optimum" is, of course, to be construed with regard to the statistical knowledge assumed and the performance criteria adopted in these problems.

Of particular interest is the limit of infinitely large M which corresponds to the case of continuous observation of the receiver input. Although it is possible to perform the limiting operations directly upon the form of the θ_{km} as derived in Appendix II, it proves somewhat more convenient to revise our method of attack slightly. (For background material regarding the analysis that follows, the reader is referred to [1], pp. 98-106.)

Suppose the receiver input $x(t)$ is observed over the interval $T_1 < t < T_2$. The revision in our method consists of replacing the vector $\underline{x}$ of receiver input samples with a vector of expansion coefficients of $x(t)$ in terms of a set of functions $\{f_m(t)\}$ orthonormal over (T_1, T_2). In particular, we choose for our functions $f_m(t)$ the orthonormal solutions of the Karhunen-Loéve integral equation

$$\lambda_m f_m(t) = \int_{T_1}^{T_2} \gamma(t, s) f_m(s) ds \quad ; \quad T_1 < t < T_2 . \tag{26}$$

Thus, the elements of $\underline{x}$ are

$$x_m = \int_{T_1}^{T_2} f_m(t) x(t) dt . \tag{27}$$

Under the assumptions of Problem 4, the expansion coefficients x_m for a fixed message vector $\underline{a}$ would be statistically independent Gaussian random variables with variances λ_m and means $\mu_m(\underline{a})$ defined by

$$\mu_m(\underline{a}) = \int_{T_1}^{T_2} f_m(t) u(t) dt = \sum_{j=1}^{N} \left(\int_{T_1}^{T_2} f_m(t) r(t-\tau_j) dt \right) a_j = \sum_{j=1}^{N} \xi_{mj} a_j . \tag{28}$$

Accordingly, in place of Eq. (19) we obtain

$$q(\underline{x}|\underline{a}) = \frac{e^{-\frac{1}{2}\sum_{m=1}^{M} \frac{[x_m - \mu_m(\underline{a})]^2}{\lambda_m}}}{(2\pi\lambda_m)^{\frac{1}{2}}} . \tag{29}$$

Therefore, the computations done in Appendix II remain applicable if we replace c_{mj} by ξ_{mj} and γ_{mn}^{-1} by $\frac{\delta_{mn}}{\lambda_m}$. In the limit of infinite M, the vector $\underline{x}$ of expansion coefficients provides a representation of the observed receiver input voltage $x(t)$, $T_1 < t < T_2$, which is sufficiently accurate for most practical purposes. The components $\hat{a}_i(\underline{x})$ of our optimum estimate approach linear functionals of $x(t)$, $T_1 < t < T_2$, which

will be denoted by $\hat{a}_i[x(t), T_1 < t < T_2] = g_i$; $i = 1, 2, \ldots, N$. We now characterize the g_i more completely by examining the limiting behavior of Eq. (21).

Let $\theta_i(t)$ be the time function whose m^{th} Karhunen-Loéve expansion coefficient is $\theta_{im}\lambda_m$, i.e., let

$$\theta_i(t) = \sum_{m=1}^{\infty} \theta_{im}\lambda_m f_m(t) ; \qquad \theta_{im}\lambda_m = \int_{T_1}^{T_2} f_m(t)\,\theta_i(t)\,dt . \tag{30}$$

Then, let $z_i(t)$ be the solution of the integral equation

$$\theta_i(t) = \int_{T_1}^{T_2} \gamma(t, s)\, z_i(s)\, ds ; \quad T_1 < t < T_2 . \tag{31}$$

It follows that the i^{th} component of our optimum estimate may be expressed in the form

$$g_i = \sum_{m=1}^{\infty} \theta_{im} x_m = \sum_{m=1}^{\infty} \frac{x_m}{\lambda_m} \int_{T_1}^{T_2} f_m(t)\,\theta_i(t)\,dt = \sum_{m=1}^{\infty} \frac{x_m}{\lambda_m} \int_{T_1}^{T_2} dt\, f_m(t) \int_{T_1}^{T_2} \gamma(t, s)\, z_i(s)\, ds$$

$$g_i = \sum_{m=1}^{\infty} \frac{x_m}{\lambda_m} \int_{T_1}^{T_2} ds\, z_i(s) \int_{T_1}^{T_2} dt\, \gamma(t, s)\, f_m(t) = \int_{T_1}^{T_2} ds\, z_i(s) \sum_{m=1}^{\infty} \frac{x_m}{\lambda_m} \int_{T_1}^{T_2} \gamma(s, t)\, f_m(t)\, dt$$

$$g_i = \int_{T_1}^{T_2} ds\, z_i(s) \sum_{m=1}^{\infty} x_m f_m(s) = \int_{T_1}^{T_2} z_i(s)\, x(s)\, ds . \tag{32}$$

In deriving Eq. (32), use has been made of the fact that $\gamma(t, s)$ is a symmetric kernel. The equation indicates that g_i may be formed by passing $x(t)$ through a linear, time-invariant filter with impulse response $w_i(t)$ given by

$$w_i(t) = \begin{cases} z_i(T-t) \; ; & -T_2+T<t<-T_1+T \\ 0 \; ; & \text{elsewhere} \end{cases} . \tag{33}$$

The filter output at time T then will be

$$\int_{-\infty}^{\infty} x(T-\tau)\, w_i(\tau)\, d\tau = \int_{-T_2+T}^{-T_1+T} x(T-\tau) z_i(T-\tau) d\tau = \int_{T_1}^{T_2} z_i(s)\, x(s)\, ds = g_i . \tag{34}$$

It appears difficult to extend the general analysis beyond this point. In the specific case in which $T_1 \to -\infty$ and $T_2 \to \infty$, an explicit solution for $w_i(t)$ can be obtained if the noise process is wide sense stationary. The solution for this case was found by Tufts [2] using a very different approach. Tufts derived the mean square error optimum linear receiver under the assumption that the only statistical knowledge available to the receiver designer was the message vector autocorrelation matrix $\underline{\Phi}$ and the noise autocorrelation function $\gamma(t, s) = \psi(t-s)$; signal and noise were assumed additive and independent. Calculus of variations yielded an integral equation the solution of which was the transfer function $W_i(f)$ of the optimum receiver filter, namely

$$W_i(f) = \frac{R^*(f)}{\Psi(f)} \sum_{n=1}^{N} d(i,k)\, e^{j2\pi f(\tau_k - T)} \quad , \tag{35}$$

where $W_i(f) = \int_{-\infty}^{\infty} w_i(t)\, e^{-j2\pi ft}\, dt$, $R(f) = \int_{-\infty}^{\infty} r(t)\, e^{-j2\pi ft}\, dt$,

and $\Psi(f) = \int_{-\infty}^{\infty} \psi(\tau)\, e^{-j2\pi f\tau}\, d\tau$. Equation (35) represents a "matched filter" $\frac{R^*(f)}{\Psi(f)}$ independent of i followed by a delay line with N taps spaced like the transmission instants τ_k and weighted by gain coefficients which vary with i . The optimum estimate g_i is the sum of the weighted tap outputs of time T (see Fig. 2).

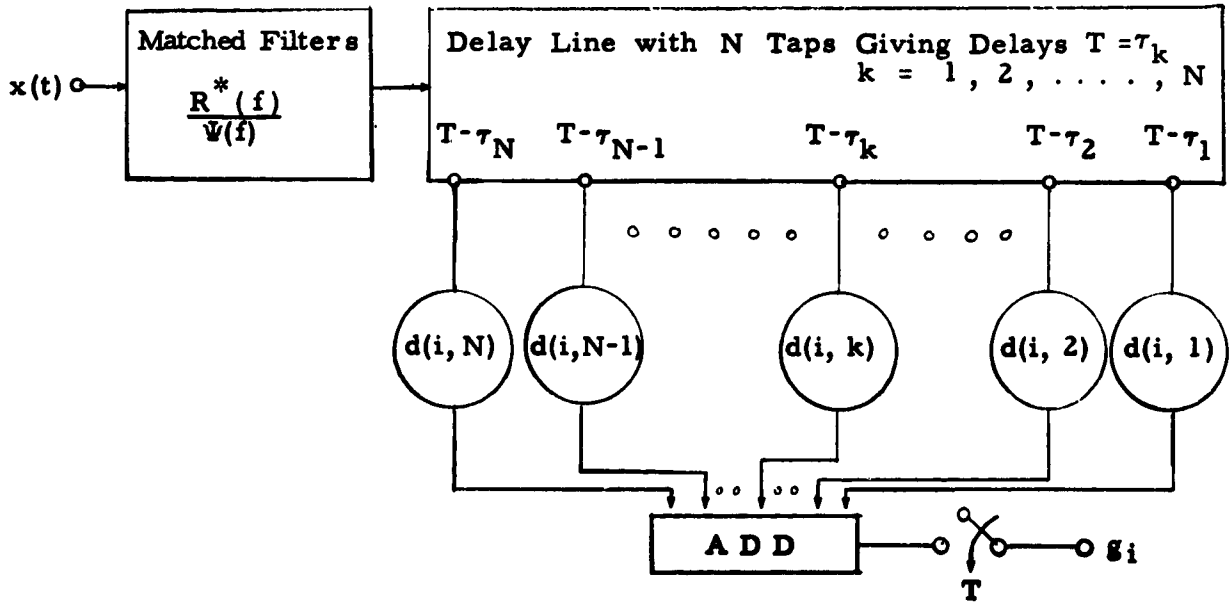

Fig.2-Optimum Receiver for Stationary Noise and an Infinite Observation Interval

In Appendix III, Eq. (30) is solved for $T_1 \to -\infty$, $T_2 \to +\infty$ and stationary noise, and the solution is shown to be equivalent to Tufts' solution as given by (35) and Fig. 2. The present analysis shows that the resultant receiver is not only the best linear one, but also the best nonlinear receiver in the minimax sense. For either of these interpretations of the receiver, the joint distribution $p[\underline{a}, y(t)]$ of signal and noise has been assumed to belong to the continuous analog of V, i.e., it is known that for all j, k, = 1, 2, . . . , N and for all t_1, $t_2 \in (-\infty, \infty)$

$$E_p (a_j a_k) = \phi_{jk} \tag{36a}$$

$$E_p [a_j y(t_1)] = 0 \tag{36b}$$

$$E_p [y(t_1) y(t_2)] = \gamma(t_1, t_2) \quad . \tag{36c}$$

Conclusion:

It has been shown that in mean square error estimation problems with additive, uncorrelated noise and known second-order statistics, the Bayes estimation rule for the specific case of statistically independent, zero-mean Gaussian signal, and noise statistics is also both the best linear estimation rule and the best minimax estimation rule. This fact has been used to design optimum sampled-data and continuous time receivers for a practical PAM communication system. In the continuous time case with an infinite observation interval and stationary noise statistics, the receiver thus designed agrees with the optimum linear receiver previously derived for this case

by Tufts [2] using a very different approach; the present analysis shows that Tufts's linear receiver is, therefore, also the best nonlinear minimax receiver.

The above theory could be employed as the mathematical basis for an investigation of the extent of degradation in system performance suffered by adopting a sampled data as opposed to a continuous time receiver. It is also hoped that the above constitutes a useful contribution to the complex field of the interrelations between different system performance criteria.

APPENDIX I

$$\hat{a}_1(x) = \frac{\int_{-\infty}^{\infty} a\, p_2(x-a)p_1(a)da}{\int_{-\infty}^{\infty} p_2(x-a)p_1(a)da} = \frac{\int_{-\infty}^{\infty} a e^{-\frac{1}{2}\left[\frac{(x-a)^2}{\sigma^2}+\frac{a^2}{b^2}\right]} da}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{(x-a)^2}{\sigma^2}+\frac{a^2}{b^2}\right]} da} = \sigma^2 \frac{d}{dx} \frac{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{a^2-2ax}{\sigma^2}+\frac{a^2}{b^2}\right]} da}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{a^2-2ax}{\sigma^2}+\frac{a^2}{b^2}\right]} da}$$

$$\hat{a}_1(x) = \sigma^2 \frac{d}{dx} \ln\left[\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{a^2-2ax}{\sigma^2}+\frac{a^2}{b^2}\right]} da\right] = \sigma^2 \frac{d}{dx} \ln\left[e^{\frac{x^2}{2\sigma^2}} \sqrt{2\pi(b^2+\sigma^2)}\, e^{-\frac{x^2}{2(b^2+\sigma^2)}}\right]$$

$$\hat{a}_1(x) = \sigma^2 \frac{d}{dx}\left[\frac{x^2}{2\sigma^2} - \frac{x^2}{2(b^2+\sigma^2)}\right] = \left(\frac{b^2}{b^2+\sigma^2}\right) x \ .$$

$$D = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left(\frac{b^2}{b^2+\sigma^2}\right)(a+y)-a\right]^2 p_1(a)p_2(y)da\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left(\frac{\sigma^2}{b^2+\sigma^2}\right)^2 a^2 + \left(\frac{b^2}{b^2+\sigma^2}\right)^2 y^2 - 2\left(\frac{b\sigma}{b^2+\sigma^2}\right)^2 ay\right] e^{-\frac{1}{2}\left(\frac{a^2}{b^2}+\frac{y^2}{\sigma^2}\right)} da\,dy$$

$$D = \left(\frac{\sigma^2}{b^2+\sigma^2}\right)^2 b^2 + \left(\frac{b^2}{b^2+\sigma^2}\right)^2 \sigma^2 + 0 = \frac{b^2\sigma^2}{b^2+\sigma^2} \ .$$

APPENDIX II

In what follows, omitted exponents in the denominator equal those in the numerator.

$$\hat{a}_k(\underline{x}) = \frac{\int a_k q(\underline{x}|\underline{a})p(\underline{a})d\underline{a}}{\int q(\underline{x}|\underline{a})p(\underline{a})d\underline{a}} = \frac{\int d\underline{a}\, a_k \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^{N}\sum_{k=1}^{N}\phi_{jk}^{-1}a_j a_k + \sum_{m=1}^{M}\sum_{n=1}^{M}\gamma_{mn}^{-1}(x_m - \sum_{j=1}^{N}c_{mj}a_j)(x_n - \sum_{k=1}^{N}c_{nk}x_k)\right]\right\}}{\int d\underline{a}\, \exp\left\{\right\}}$$

$$\hat{a}_k(\underline{x}) = \frac{\int d\underline{a}\, a_k \exp\left\{\frac{1}{2}\left[2\sum_{m=1}^{M}x_m\sum_{n=1}^{M}\gamma_{mn}^{-1}\sum_{k=1}^{N}c_{nk}a_k - \sum_{j=1}^{N}\sum_{k=1}^{N}\left(\phi_{jk}^{-1} + \sum_{m=1}^{M}\sum_{n=1}^{M}c_{mj}\gamma_{mn}^{-1}c_{nk}\right)a_j a_k\right]\right\}}{\int d\underline{a}\, \exp\left\{\right\}}$$

Let $\underline{C}$ be the M x N rectangular matrix whose n-kth element is c_{nk}, and consider the matrix $\underline{\Phi}^{-1} + \underline{C}^T\underline{\Gamma}^{-1}\underline{C} = \underline{\Lambda}$. Since both $\underline{\Phi}$ and $\underline{\Gamma}$ are symmetric and strictly positive definite, one easily verifies that $\underline{\Lambda}$ is, also. We perform a transformation to the principal axes of the quadratic form in the exponent by defining the new coordinates $\underline{\beta}$ according to the matrix relation $\underline{a} = \underline{A}\,\underline{\beta}$, where $\underline{A} = (a_{jk})$ is the matrix of normalized eigenvectors of $\underline{\Lambda}$. Denoting the eigenvalues of $\underline{\Lambda}$ by ν_ℓ; $\ell = 1, 2, \ldots, N$, we obtain

$$\hat{a}_i(\underline{x}) = \frac{\int d\underline{\beta}\left(\sum_{\ell=1} a_{i\ell}\beta_\ell\right) \quad \exp\left\{\underline{x}^T \underline{\Gamma}^{-1}\underline{C}\underline{A}\underline{\beta} - \frac{1}{2}\sum_{\ell=1}^{N} \nu_\ell \beta_\ell^2\right\}}{\int d\underline{\beta} \exp\left\{\right\}}$$

Defining $\underline{\zeta}^T = \underline{x}^T \underline{\Gamma}^{-1} \underline{C}\,\underline{A}$, we can express $\hat{a}_i(\underline{x})$ in the form

$$\hat{a}_i(\underline{x}) = \sum_{\ell=1}^{N} a_{i\ell} \left(\frac{\int \beta_\ell e^{\zeta_\ell \beta_\ell - \nu_\ell \beta_\ell^2} d\beta_\ell}{\int e^{\zeta_\ell \beta_\ell - \nu_\ell \beta_\ell^2} d\beta_\ell} \right) = \sum_{\ell=1}^{N} \frac{a_{i\ell}\zeta_\ell}{\nu_\ell} \quad .$$

Since $\zeta_\ell = \sum_{m=1}^{M} x_m \sum_{n=1}^{M} \gamma_{mn}^{-1} \sum_{k=1}^{N} c_{nk} a_{k\ell}$, this becomes

$$\hat{a}_i(\underline{x}) = \sum_{m=1}^{M} \theta_{im} x_m \quad , \qquad \text{where } \theta_{im} = \sum_{n=1}^{M} \gamma_{mn}^{-1} \sum_{k=1}^{N} c_{nk} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \quad .$$

APPENDIX III

If we make the replacements $c_{mj} = \xi_{mj}$ and $\gamma_{mn}^{-1} = \frac{\delta_{mn}}{\lambda_m}$ as discussed after Eq. (29), the expression for θ_{im} as given at the end of Appendix II becomes

$$\theta_{im} = \sum_{n=1}^{M} \frac{\delta_{mn}}{\lambda_m} \sum_{k=1}^{N} \xi_{nk} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} = \frac{1}{\lambda_m} \sum_{k=1}^{N} \xi_{mk} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \quad .$$

Combining this with Eq. (30) and the definition of ξ_{mk} from Eq. (28), we obtain

$$\lambda_m \theta_{im} = \int_{T_1}^{T_2} f_m(t)\, \theta_i(t)\, dt = \sum_{k=1}^{N} \xi_{mk} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} = \sum_{k=1}^{N} \int_{T_1}^{T_2} f_m(t)\, r(t-\tau_k)dt \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \, .$$

This may be rearrange in the form

$$\int_{T_1}^{T_2} f_m(t) \left[\theta_i(t) - \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \; r(t-\tau_k) \right] dt = 0 \, .$$

From the completeness of the $f_m(t)$ on (T_1, T_2) we may conclude that

$$\theta_i(t) = \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \; r(t-\tau_k) \; ; \quad T_1 < t < T_2 \, .$$

Writing $\gamma(t, s) = \psi(t - s)$ because the noise is stationary, and letting $T_1 \to -\infty$ and $T_2 \to \infty$, we may write the integral equation (31) in the form

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \; r(t-\tau_k) = \int_{-\infty}^{\infty} \psi(t - s)\, z_i(s)\, ds \; ; \quad t \in (-\infty, \infty) \, .$$

Fourier transforming then yields

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} R(f)\, e^{-j2\pi f \tau_k} = \Psi(f)\ Z_i(f)$$

Since $W_i(f) = Z_i^*(f)\ e^{-j2\pi fT}$ from (33), and since $\Psi(f) = \Psi^*(f)$, the solution for the transfer function of the filter to be sampled at $t = T$ in order to obtain g_i is

$$W_i(f) = \frac{R^*(f)}{\Psi(f)} \sum_{k=1}^{N} \left(\sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell} \right) e^{j2\pi f(\tau_k - T)} .$$

Accordingly, the optimum receiver is a matched filter followed by a tapped delay line, agreeing with Tufts's result as given in Eq. (35). The tap gain coefficients are given by $\sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell}$, and it remains to establish that these are equivalent to the $d(i,k)$ given by Tufts. Toward this end, let us recall from Appendix II that $\underline{A}$ is the matrix of normalized eigenvectors of $\underline{\Lambda}$, so that we have $\underline{A}^{-1} = \underline{A}^T$, and we may write

$$\underline{\Lambda} = \underline{A} \begin{pmatrix} \nu_1 & & 0 \\ & \nu_2 & \\ & & \ddots \\ 0 & & & \nu_N \end{pmatrix} \underline{A}^T \ ; \quad \underline{\Lambda}^{-1} = \underline{A} \begin{pmatrix} 1/\nu_1 & & & \\ & 1/\nu_2 & & \\ & & \ddots & \\ & & & 1/\nu_N \end{pmatrix} \underline{A}^T .$$

Accordingly, the i-kth element of $\underline{\Lambda}^{-1}$ is $\sum_{\ell=1}^{N} \frac{a_{i\ell} a_{k\ell}}{\nu_\ell}$, which shows that the tap gains for the optimum receiver prescribed by the present theory are the elements of the inverse of $\underline{\Lambda} = \underline{\Phi}^{-1} + \underline{C}^T \underline{\Gamma}^{-1} \underline{C}$. On the other hand, temporarily adopting Tufts's notation, we obtain by combining his Eqs. (7) and (15) the following representation for the matrix $\underline{D}$ of his tap gain coefficients d(i,k)

$$\underline{D} = \underline{M} - \underline{C} = \underline{M} - \underline{M}\,\underline{Q}\,(\underline{M}^{-1}+\underline{Q})^{-1} = \underline{M} - [\underline{M} - (\underline{M}^{-1}+\underline{Q})^{-1}] = (\underline{M}^{-1}+\underline{Q})^{-1}.$$

Accordingly, Tufts's tap gains are the elements of the inverse of $\underline{M}^{-1}+\underline{Q}$. Since his $\underline{M}$ is our $\underline{\Phi}$, it only remains to show that his matrix $\underline{Q}$ is the same as our matrix $\underline{C}^T \underline{\Gamma}^{-1} \underline{C}$. Now the i-kth element of $\underline{Q}$ is defined, in our notation, as

$$q_{ik} = q(\tau_i - \tau_k) = \int_{-\infty}^{\infty} \frac{R(f)\,R^*(f)}{\Psi(f)}\, e^{j2\pi f(\tau_i - \tau_k)}\, df .$$

We now study the behavior of $\underline{C}^T \underline{\Gamma}^{-1} \underline{C}$ in the limit $T_1 \to -\infty$, $T_2 \to \infty$, and $M \to \infty$. First, we have $c_{mj} = \xi_{mj}$, and $\gamma_{mn}^{-1} = \frac{\delta_{mn}}{\lambda_m}$ because we are using Karhunen-Loéve expansion coefficients, so that the i-kth element of $\underline{C}^T \underline{\Gamma}^{-1} \underline{C}$ is

$$\sum_{m=1}^{M} \sum_{n=1}^{M} \xi_{mi} \frac{\delta_{mn}}{\lambda_m} \xi_{nk} = \sum_{m=1}^{M} \frac{\xi_{mi}\xi_{mk}}{\lambda_m} = \sum_{m=1}^{M} \frac{\left(\int_{T_1}^{T_2} f_m(t)\, r(t-\tau_i)\,dt\right)\left(\int_{T_1}^{T_2} f_m(t)\, r(t-\tau_k)\,dt\right)}{\lambda_m} .$$

It is well known ([3], p. 93 or [4], p. 167) that the Fourier coefficients of a stationary, Gaussian process become statistically independent in the limit of an infinitely long expansion interval. Accordingly, in this limit we may use as our orthonormal functions $f_m(t)$ the trigonometric functions

$$f_m(t) = \frac{e^{-\frac{j2\pi mt}{T_2-T_1}}}{(T_2-T_1)^{1/2}} \quad ; \quad m = 0, \pm 1, \pm 2, \ldots \quad .$$

Because these functions are complex, some minor complications arise. In particular, defining $b_{mi} = \frac{\xi_{mi}}{\lambda_m^{1/2}}$, we had in the real case that $\underline{C}^T \underline{\Gamma}^{-1} \underline{C} = \underline{B}^T \underline{B}$; in the complex case, this becomes $\underline{C}^T \underline{\Gamma}^{-1} \underline{C} = \underline{B}^\dagger \underline{B}$, where the "$\dagger$" indicates conjugate transpose. Moreover, summations over the index m now run from -M to M rather than from 1 to M. Our expression for the i-kth element of $\underline{C}^T \underline{\Gamma}^{-1} \underline{C}$, therefore, becomes

$$\sum_{m=-M}^{M} \frac{\left(\int_{T_1}^{T_2} f_m^*(t)\, r(t-\tau_i)dt\right)\left(\int_{T_1}^{T_2} f_m(t)\, r(t-\tau_k)\, dt\right)}{\lambda_k} .$$

Substituting the trigonometric functions, this assumes the form

$$\sum_{m=-M}^{M} \frac{1}{(T_2-T_1)} \frac{e^{j2\pi(\frac{m}{T_2-T_1})(\tau_i-\tau_k)} \left(\int_{T_1}^{T_2} r(t)\, e^{+j2\pi(\frac{m}{T_2-T_1})t}\, dt \right) \left(\int_{T_1}^{T_2} r(t)\, e^{-j2\pi(\frac{m}{T_2-T_1})t}\, dt \right)}{\lambda_m} .$$

We now take our limit in such a way that as (T_2-T_1) and m approach infinity, their ratio f, representing frequency in cps, remains finite. In this limit, $\frac{1}{T_2-T_1}$ becomes the frequency differential df and λ_m becomes $\Psi(f)$ ([4], p. 168). The above sum, therefore, tends to the integral

$$\int_{-\infty}^{\infty} \frac{df\, e^{j2\pi f(\tau_i-\tau_k)} \left(\int_{-\infty}^{\infty} r(t)\, e^{j2\pi ft}\, dt \right) \left(\int_{-\infty}^{\infty} r(t)\, e^{-j2\pi ft}\, dt \right)}{\Psi(f)}$$

$$= \int_{-\infty}^{\infty} \frac{R(f)R^*(f)}{\Psi(f)}\, e^{j2\pi f(\tau_i-\tau_k)}\, df = q_{ik} .$$

This completes the demonstration of the fact that the present theory is in accord with Tufts's earlier work.

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By generalizing a trivial problem in estimation theory, some interesting results have been obtained for a class of pulse amplitude modulation (PAM) communication channels. In particular, we consider a linear channel with memory perturbed by additive noise which is uncorrelated with the signal statistics. It is shown that, for specified second-order statistics and a mean square error criterion, the Bayes estimation rule for the specific case of zero-mean Gaussian signal and noise statistics is also both the best linear estimation rule and the minimax estimation rule. This fact is used in the theoretical design of sampled-data and continuous time receivers for such PAM systems. In the continuous time case with an infinite observation interval and stationary noise statistics, the receiver thus designed agrees with the optimum linear receiver previously derived for this case by Tufts using a very different approach; the present analysis shows that Tufts's linear receiver is, therefore, also the best nonlinear receiver in the minimax sense.

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