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ON THE COMPUTATIONAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. In this Memorandum the authors discuss a method for solving large systems of differential equations where the solution is subject to certain boundary conditions.
SUMMARY

Two-point boundary-value problems for second-order systems of linear differential equations are usually solved by a process involving the inversion of a certain matrix. If the system is too large, it may be difficult to compute this inverse to a high degree of accuracy. The purpose of this paper is to demonstrate that this difficulty can in some cases be circumvented by applying a method like that of Bodewig and Hotelling.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>PREFACE</td>
<td>iii</td>
</tr>
<tr>
<td>SUMMARY</td>
<td>v</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2. AN ITERATIVE TECHNIQUE</td>
<td>2</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>5</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

Consider (as in [1]) the n-dimensional vector differential equation

\[ x'' + A(t)x = 0, \]

where the solution is subject to the boundary conditions

\[ x(0) = c, \quad x(1) = d. \]

The problem is generally solved as follows. Let \( X_1 \) and \( X_2 \) denote the matrix solution of

\[ x'' + A(t)x = 0 \]

satisfying the initial conditions

\[ x_1(0) = I, \quad x_1'(0) = 0, \]
\[ x_2(0) = 0, \quad x_2'(0) = I. \]

If \( g \) represents the (unknown) value of \( x'(0) \), where \( x(t) \) is the solution to the problem, then

\[ g = X_2(1)\begin{bmatrix} d - X_1(1)c \end{bmatrix}. \]

If \( X_2(1) \) is singular, then there may be many solutions, or none, and (1.5), of course, makes no sense.

If \( n \) is large, it may be difficult to compute \( X_2^{-1}(1) \) to a high degree of accuracy. The purpose of this paper is to discuss a method of overcoming this difficulty.
2. AN ITERATIVE TECHNIQUE

Let $X^*_2(1)$ be some approximation to $X^{-1}_2(1)$.

Define

\[
g_1 = X^*_2(1)[d - X_1(1)c],
\]

\[
g_n = X^*_2(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1}.
\]

Then we have the following theorem:

**Theorem.** If the spectral radius of $I - X^*_2(1)X_2(1)$ is less than one, then the sequence \( \{g_n\} \) defined by (2.1) converges to \( g \), the unique solution of (1.5).

**Proof.** First note that if $I - X^*_2(1)X_2(1)$ has spectral radius less than one, then $X^*_2(1)X_2(1)$ must be nonsingular. Thus $X^*_2(1)$ and $X_2(1)$ are nonsingular, which means that (1.5) has a unique solution. If $g$ is the unique solution of (1.5), then

\[
g_n - g = X^*_2(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1} - g
\]

\[
= X^*_2(1)[d - X_1(1)c - X_2(1)g_{n-1}]
\]

\[
- X^*_2(1)[d - X_1(1)c - X_2(1)g] + g_{n-1} - g
\]

\[
= (I - X^*_2(1)X_2(1))(g_{n-1} - g).
\]

If the spectral radius of $I - X^*_2(1)X_2(1)$ is less than one, this shows that \( \{g_n - g\} \) goes to zero as \( n \) goes to infinity, and this concludes the proof. This
Theorem may be viewed as an application of a method of matrix inversion like that of Bodewig and Hotelling (see [3], [4] for additional references).

**Corollary.** If \( A(t) = B^2 \), a constant positive-definite matrix, then taking \( X^2(l) = X_2(1) \) makes \( \{g_n\} \) converge to the solution.

**Proof.** Since \( X_2(1) = B^{-1} \sin B \), it follows that the eigenvalues of \( X_2(1) \) all have absolute value less than one, and thus all the eigenvalues of \( X_2^2(1) \) are between 0 and one.

**Corollary.** If each element of \( I - X_2^2(1)X_2(1) \) is less in absolute value than \( 1/n \), then \( \{g_n\} \) converges to the solution.

**Corollary.** If \( A(t) = -B^2 \), where \( B \) is a matrix with only real eigenvalues each of which is greater than zero, then taking \( X_2(1) = 2Be^{-B} \) makes \( \{g_n\} \) converge to the solution.

**Proof.** \( X_2(t) = B^{-1}\left(\frac{e^{3t} - e^{-3t}}{2}\right) \), whence
\[
X_2^2(1)X_2(1) \text{ equals } I - e^{-2B}.
\]

**Corollary.** If \( Y_1(t), Y_2(t) \) are solutions to \( Y'' + A(1-t)Y = 0 \) satisfying initial conditions like (1.4), then taking \( X_1^2(1) = Y_1(1) \) will make \( \{g_n\} \) converge to the solution if \( Y_2(1)X_2(1) \) has spectral radius less than one.
Proof. \( Y_1^2(1)X_2^2(1) = I - Y_1^1(1)X_2(1) \).

Corollary. If \( X_2^*(1) = \alpha A \), where \( A \) is the transpose of \( X_2(1) \) and \( \alpha \) is a positive constant chosen to be less than twice the reciprocal of the sum of the absolute values of each row of \( AX_2(1) \), then \( \{x_n\} \) converges to the solution.

Note that this last corollary is not apt to be computationally useful, however, since if \( X_2(1) \) has some very small eigenvalues (and thus is hard to invert), under the above procedure \( I - X_2^*(1)X_2(1) \) will have spectral radius very close to one, so that convergence will be slow.
REFERENCES


