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IDENTIFIABILITY OF MIXTURES OF EXPONENTIAL FAMILIES

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IDENTIFIABILITY OF MIXTURES OF EXPONENTIAL FAMILIES

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Let \( \mathcal{F}_0 = \{F(\cdot | \tau) : \tau \in T\} \) be a family of \( n \)-dimensional distribution functions (d.f.s.) depending on an \( m \)-dimensional parameter \( \tau \) which ranges over a Borel set \( T \) in \( \mathbb{R}^m \), the \( m \)-dimensional Euclidian space. We assume that for each fixed \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) the function \( F(x|\cdot) \) is Borel measurable. Let \( \mathcal{H}(\mathcal{G}) \) denote the set of all probability measures (p.m.s.) on the Borel field \( \mathcal{B}^n \) of \( \mathbb{R}^n \) \((\mathcal{B}^m \text{ of } \mathbb{R}^m) \) and let \( \mathcal{H}_T \) denote the set of those \( \gamma \in \mathcal{G} \) for which \( \gamma(T) = 1 \). The family \( \mathcal{F}_0 \) determines a mapping \( \psi : \mathcal{H}_T \rightarrow \mathcal{H} \) by the relation

\[
(1) \quad \psi(\gamma) = \int F(\cdot | \tau) \, d\gamma(\tau)
\]

We speak of the d.f. \( \psi(\gamma) \) as a mixture of \( \mathcal{F}_0 \) (w.r.t. \( \gamma \)). The mapping \( \psi \) is said to be identifiable if it is one to one. In certain connections (e.g. statistical estimation of \( \gamma \)) it is important to know whether \( \psi \) is identifiable. Various conditions for identifiability and nonidentifiability are known, see Teicher [4] and the references therein. Here we want to prove that, under mild restrictions, mixtures of exponential families \( \mathcal{F}_0 \) are identifiable. \( \mathcal{F}_0 \) is exponential (or of the Darmois-Koopman type) if for some \( \sigma \)-finite measure \( \mu \).
\[ dP(x|\tau) = a(\tau) \cdot b(x) \sum_{j=1}^{m} \tau_j h_j(x) \quad du(x) \]

for \( x \in \mathbb{R}^n, \tau = (\tau_1, \tau_2, \ldots, \tau_m) \in T, \) where \( a(\tau) > 0, \ b(x) \geq 0 \) and \( a, b, h_j, j = 1, \ldots, m \) are all measurable.

Let \( \gamma_1, \gamma_2 \in \mathcal{P}_T \) and let

\[ f_{\nu}(x) = \frac{d\nu(y)}{du} = b(x) \int_T a(\tau) e^{\sum_{j=1}^{m} \tau_j h_j(x)} \, d\gamma_{\nu}(\tau), \quad \nu = 1, 2. \]

Furthermore, let \( S = \{x : f_1(x) = f_2(x) \neq 0\}, \) let \( \eta = \{y = (h_1(x), \ldots, h_m(x)) : x \in S\} \) and let

\[ f_{\nu}^{\ast}(y) = \int_T a(\tau) e^{\langle \tau, y \rangle} \, d\gamma_{\nu}(\tau), \quad \nu = 1, 2. \]

where \( \langle \tau, y \rangle \) denotes the inner product of \( \tau \in T \) and \( y \in \mathbb{R}^m. \) Then \( f_1^{\ast}(y) = f_2^{\ast}(y) \) if \( y \in \eta; \) our aim is to show that under certain further restrictions this implies \( \gamma_1 = \gamma_2. \) Let \( c(\eta) \) denote the convex hull of \( \eta. \) We shall distinguish between four cases.

(i) \( \eta \) is finite.

(ii) \( \eta \) is infinite, \( c(\eta) \) is bounded and \( \eta \) does not have an accumulation point in the interior of \( c(\eta). \)

(iii) As (ii) except that \( c(\eta) \) is assumed unbounded.
(iv) \( \eta \) is infinite and \( \eta \) has an accumulation point in the interior of \( c(\eta) \).

Case (i). The important example of this case is the binomial distribution. An analysis of the identifiability problem for that distribution can be found in [4].

Case (ii). From the viewpoint of statistics (ii) is the case of least interest. We have obtained no general results. The problem is essentially this: \( (n = m = 1) \). Let \( \gamma_1 \) and \( \gamma_2 \) be two p.m.'s on \( (R, \mathfrak{G}) \) whose Laplace transforms \( \varphi_1(z) \) and \( \varphi_2(z) \) both exist in a strip \( 0 < \Re z < \rho, \rho > 0 \). Let \( \{x_n\} \) be a sequence of real numbers such that \( 0 < x_n < \rho \) for all \( n \) and \( x_n \to 0 \) as \( n \to \infty \). Find conditions under which \( \varphi_1(x_n) = \varphi_2(x_n) \) for all \( n \) implies \( \varphi_1(it) = \varphi_2(it) \) for all real \( t \) (i.e., identity of the Fourier transforms of \( \gamma_1 \) and \( \gamma_2 \) and hence identity of \( \gamma_1 \) and \( \gamma_2 \)).

Case (iii). We shall treat the subcase:

(iii). \( \eta \) contains the set \( I^+ \) of all lattice points in \( R^m \) with nonnegative components, i.e., \( I^+ = \{k = (k_1, \ldots, k_m) : k_j \text{ is a non-negative integer}, j = 1, \ldots, m\} \).

We have, since \( 0 = (0, \ldots, 0) \in I^+ \)

\[
(5) \quad f_1^*(0) = \int_T a(\tau) \, d\gamma_1(\tau) = \int_T a(\tau) \, d\gamma_2(\tau) = f_2^*(0).
\]
Let us denote the common (positive) value in (5) by $c$ and let us introduce the p.m.'s $\gamma^*_\nu$, $\nu = 1, 2$, by $d\gamma^*_\nu(\tau) = c^{-1} a(\tau) d\gamma(\tau)$. Thus

$$f^*_1(k) = \int_{T} e^{(r,k)} d\gamma^*_1(\tau) = \int_{T} e^{(r,k)} d\gamma^*_2(\tau) = f^*_2(k) \quad \forall \ k \in I^+.$$

Let $w$ be the transformation $\tau \rightarrow \lambda = w(\tau)$ where $\lambda = (\lambda_1, \ldots, \lambda_m) = (e_{\tau_1}, \ldots, e_{\tau_m})$; let $\Lambda = w(T)$ and $\pi_\nu = \gamma^*_\nu w^{-1}$, $\nu = 1, 2$. We obtain from (6)

$$\mu_{k_1 \cdots k_m} = \int_{\Lambda} \lambda_1 \cdots \lambda_m d\pi_1(\lambda) = \int_{\Lambda} \lambda_1 \cdots \lambda_m d\pi_2(\lambda)$$

$$\quad \forall \ k = (k_1, \ldots, k_m) \in I^+.$$

We can draw the following conclusion.

**Proposition 1.** Suppose that assumption (iii)' is satisfied and suppose that $\pi_1$ and $\pi_2$ are uniquely determined by their moments (7). Then $\pi_1 = \pi_2$ and consequently $\gamma_1 = \gamma_2$.

In order to derive a sufficient condition for $\gamma_1 = \gamma_2$ which is more useful than that of Proposition 1 we state the following lemma.

**Lemma 1.** Let $\pi$ be an arbitrary p.m. on $(R^m, C^m)$ with $\pi(R^+m) = 1$ where $R^+$ is the set of nonnegative reals and with all moments

$$\mu_{k_1 \cdots k_m} = \int_{R^m} \lambda_1 \cdots \lambda_m d\pi(\lambda), \quad k \in I^+$$

$(8)$
finite. If there exists a positive number \( \rho \) such that the series

\[
\sum_{k \in I^+} \frac{\mu_{k_1} \cdots k_m}{k_1! \cdots k_m!} \frac{k_1 + \cdots + k_m}{\rho}
\]

is convergent then \( \pi \) is the unique p.m. with these moments.

The lemma and its proof are straightforward generalizations of a result in the book of Cramer [2; 176].

Let us apply the lemma to (7). We find (dropping the subscript \( \nu \))

\[
0 \leq \sum_{k} \frac{k_1 + \cdots + k_m}{\rho k_1! \cdots k_m!}
\]

\[
= \int \sum_{k} \prod_{j=1}^{m} \frac{(\lambda \rho)^j}{k_j!} \, d\pi
\]

\[
= \int \prod_{j=1}^{m} \left( \sum_{z=1}^{\infty} \frac{(\lambda \rho)^z}{z!} \right) \, d\pi
\]

\[
= \frac{1}{c} \int_{T} a(\tau) \, e^\rho \sum_{j} e^{\tau^j} \, d\gamma(\tau)
\]

\[
\leq \frac{1}{c} \sup_{\tau \in T} a(\tau) \, e^\rho \sum_{j} e^{\tau^j}
\]

Therefore
Proposition 2. Suppose that assumption (iii)' is satisfied and suppose that

\[ \sup_{\tau \in T} a(\tau) e^{\rho \sum \tau_j} < \infty \]  

for some \( \rho > 0 \). Then \( \gamma_1 = \gamma_2 \).

As an application, let us consider the instance where \( n = m \), \( h_j(x) = x_j \) (\( j \)-th coordinate of \( x \); \( j = 1, 2, \ldots, m \)) and where the measure \( \mu \) in (2) is concentrated on \( I^+ \); then without loss of generality we can and will assume \( \mu \) to be counting measure on \( I^+ \). Hence the family \( \mathcal{F}_0 \) is given by

\[
F(x|\tau) = \begin{cases} 
\sum_{k=0}^{[x]} a(\tau) b(k) e^{(\tau, k)} & \text{if } x \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

in an obvious notation. Assumption (iii)' becomes: \( b(k) > 0 \forall k \in I^+ \) and we have

Corollary 1. If the family \( \mathcal{F}_0 \) given by (11) satisfies \( b(k) > 0 \forall k \in I^+ \) and

\[ \sup_{\tau \in T} a(\tau) e^{\rho \sum \tau_j} < \infty \]  

for some \( \rho > 0 \) then \( \psi \) is identifiable.

Specializing still further we obtain (Feller [3])

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Corollary 2. The mapping $\psi$ determined by the Poisson family
$\mathcal{F}_0 = \{F(\cdot | \tau) : -\infty < \tau < \infty\}$, where

$$F(x | \tau) = \sum_{k=0}^{[x]} e^{-\lambda} \frac{\lambda^k}{k!}, \quad x \geq 0, \quad \lambda = e^\tau$$

is identifiable.

Case (iv). We shall prove that $y_1 = y_2$ provided

(iv)' There exists an accumulation point $y^{(0)} = (y_1^{(0)}, \ldots, y_m^{(0)})$ of $\eta$ in the interior of $c(\eta)$ with the following property. If two arbitrary complex power series

$$\sum a_j^{(\nu)} (z_1 - y_1^{(0)})^j_1 (z_2 - y_2^{(0)})^j_2 \ldots (z_m - y_m^{(0)})^j_m,$$

$\nu = 1, 2$

coincide for all $z = (z_1, \ldots, z_m) \in \eta \cap V$ for some neighborhood $V$ of $y^{(0)}$, then they have identical coefficients.

We note that assumption (iv)' is equal to (iv) if $m = 1$. A sufficient condition for (iv)' is that $\eta$ be dense in some open subset of $\mathbb{R}^m$.

Proposition 3. Suppose that assumption (iv)' is satisfied. Then $y_1 = y_2$. 

Proof. Without loss of generality we can and will assume that the origin 0 is in η and that there is a neighborhood 
K = \{y : \|y_j\| < \rho, j = 1, \ldots, m\} of 0 for which K \subset c(η) and K contains y(0). Then

\begin{equation}
\label{13}
 f_1^*(0) = \int T a(\tau) \, d\gamma_1(\tau) = \int T a(\tau) \, d\gamma_2(\tau) = f_2^*(0).
\end{equation}

Let us denote the common value in (13) by c and let us define the p.m.s. \( \gamma_v^* \), \( v = 1, 2 \) by 
\( d\gamma_v^*(\tau) = \frac{1}{c} a(\tau) \, d\gamma_v(\tau) \). Furthermore, let \( \varphi_v \), \( v = 1, 2 \) denote the Laplace transform of \( \gamma_v \)

\[ \varphi_v(z) = \int_T e^{(\tau, z)} \, d\gamma_v^*(\tau) \]

where \( z = (z_1, \ldots, z_m) \), \( z_j = u_j + iv_j \) (\( j = 1, \ldots, m \)). \( \varphi_v \) exists for all \( z \in K' = (z|u = (u_1, \ldots, u_m) \in K) \). In fact, for any such \( z \), \( |\exp((\tau, z))| \leq \exp((\tau, u)) \) and a moments reflection shows that there exists a \( y \in \eta \) with \( (\tau, u) \leq (\tau, y) \); thus

\[ \int_T |e^{(\tau, z)}| \, d\gamma_v^*(\tau) \leq \frac{1}{c} \int T a(\tau) e^{(\tau, y)} \, d\gamma_v(\tau) < \infty. \]

More is true: \( \varphi_v \) is an analytic function of \( z = (z_1, \ldots, z_m) \) in the domain \( K' \). To prove this it suffices to show that \( \varphi_v \) is analytic in each of the variables \( z_j, j = 1, \ldots, m \) (see [1]). Hence let us consider
where $z = u + iv \in K'$, $e_j$ denotes the $j$-th unit vector in $\mathbb{R}^m$ and $h$ is an arbitrary complex number. Let $\delta > 0$ be so small that $z + he_j \in K'$ for all $h$ such that $|h| \leq \delta$. Using the (well known) inequality

$$|\frac{e^{\tau_j h}}{h} - 1| \leq \frac{|\tau_j| \delta}{h}$$

for $|h| \leq \delta$

we find that the integrand in (14) is dominated by

$$\frac{1}{\delta} \left( e^{(\tau, u + \delta e_j)} + e^{(\tau, u - \delta e_j)} \right)$$

and since the integral of this quantity is finite we may pass to the limit $h \to 0$ under the integration sign in (14) to obtain

$$\varphi(z + he_j) - \varphi(z) \to \int \tau_j e^{(\tau, z)} \, d\nu^*(\tau)$$

as $h \to 0$.

We have thus shown that $\varphi_v$ is analytic in $K'$. Consequently $\varphi_v$ can be expanded in a power series around $z(0) = y(0)$

$$\varphi_v(z) = \sum a_j^{(v)} (z_1 - y_1(0))^j_1 (z_2 - y_2(0))^j_2 \cdots (z_m - y_m(0))^j_m$$

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the expansion being valid in some neighborhood $V$ of $y^{(a)}$. We have $\varphi_1(z) = \varphi_2(z) \quad \forall z \in \eta'$ and hence, by assumption (iv)' and uniqueness of analytic continuation, $\varphi_1(z) = \varphi_2(z) \quad \forall z \in K'$. In particular $\varphi_1(z) = \varphi_2(z)$ for all purely imaginary $z = iv = (iv_1, \ldots, iv_m)$, i.e., the characteristic functions of $\gamma_1^*$ and $\gamma_2^*$ coincide, hence $\gamma_1^* = \gamma_2^*$ or, equivalently, $\gamma_1 = \gamma_2$.

By the remark preceding Proposition 3, we obtain

**Corollary 3.** Suppose that in the representation (2): (a) $\mu$ is $n$-dimensional Lebesgue measure, (b) the functions $h_j, j = 1, \ldots, m$ are all continuous, (c) the set \( \{ y : y = (h_1(x), \ldots, h_m(x)), \quad b(x) > 0, \quad x \in \mathbb{R}^n \} \) contains a (nonempty) open set. Then $\psi$ is identifiable.

Specializing still further we get

**Corollary 4.** Suppose that $\mathcal{F}_0$ is the Gaussian family

\[ \mathcal{F}_0 = \{ F(\cdot | \tau) | \tau = (\tau_1, \tau_2), \quad -\infty < \tau_1 < \infty, \quad 0 < \tau_2 < \infty \}, \]

\begin{equation}
\frac{dF(x_1, \ldots, x_n | \tau_1, \tau_2)}{du} = (2\pi \sigma^2)^{-\frac{n}{2}} \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^{n} (x_j - \mu_j)^2 \right)
= \left( \frac{\tau_2}{2\pi} \right)^{\frac{n}{2}} \exp \left( -\frac{\tau_1}{2\tau_2} \right) e^{h_1(x_1)e h_2(x_2) \tau_2}
\end{equation}

where $\mu$ is $n$-dimensional Lebesgue measure, $\tau_1 = \frac{\mu}{2}, \tau_2 = \sigma^{-2}$, $h_1(x) = \frac{1}{2} \Sigma x_j^2$ and $h_2(x) = -\frac{1}{2} \Sigma x_j^2$. If $n > 1$, then $\psi$ is identifiable (Teicher has shown, see [5], that $\psi$ is not identifiable if $n = 1$).
REFERENCES


