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Minimal Interchanges of (0,1)-Matrices and Disjoint Circuits in a Graph

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Mathematics Research
September 1963
MINIMAL INTERCHANGES OF (0,1)-MATRICES AND DISJOINT CIRCUITS IN A GRAPH

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Mathematical Note No. 323
Mathematics Research Laboratory
Boeing Scientific Research Laboratories
September 1963
SUMMARY

It is shown that the minimal number of interchanges necessary to transform one (0,1)-matrix into another equivalent one may be computed from the maximal number of edge-disjoint circuits in a bipartite graph derived from the difference of the matrices. This partially answers a question raised by Ryser. Two (0,1)-matrices are said to be equivalent if their difference has zero row and column sums. They are said to differ by an interchange if they are equivalent and their difference is zero except for a $2 \times 2$ minor.
1. Introduction. In this paper we obtain a partial answer in graph-theoretic form to a question raised by Ryser [2, page 68] concerning the minimal number of interchanges required to transform equivalent (0,1)-matrices into each other.

For given positive integers $m$ and $n$ we consider the collection of $m \times n$, (0,1)-matrices $A = \{a_{ij}\}$, 

$$a_{ij} = 0 \text{ or } 1, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$ 

We say the (0,1)-matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ are equivalent and write $A \sim B$ if and only if they have the same row and column sums, that is, if and only if

$$r_i = \sum_j a_{ij} = \sum_j b_{ij},$$

$$s_j = \sum_i a_{ij} = \sum_i b_{ij}.$$ 

We note immediately that $A \sim B$ if and only if $B - A \sim 0$, where 0 designates the $m \times n$ matrix of zeros.

Given a (0,1)-matrix $A$, we can obtain an equivalent one, $A'$, by finding a $2 \times 2$ minor of $A$ of the form

$$\begin{array}{ccc}
0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 0
\end{array} \quad \text{or} \quad \begin{array}{ccc}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{array}$$

and replacing it by the other. Ryser calls this transformation from $A$ to $A'$ an interchange and shows [1; 2, page 68] that any matrix equivalent to $A$ may be obtained from it by a suitable sequence of interchanges. We will show the following:
THEOREM 1. If $A$ and $B$ are equivalent $(0,1)$-matrices, then $B$
can be obtained from $A$ by a sequence of

$\frac{1}{2} \alpha(A,B) - \beta(G)$

and no fewer interchanges, where $\alpha(A,B)$ is the number of positions at
which $A$ and $B$ disagree, $G$ is the directed, bipartite graph derived
from $B - A = 0$, and $\beta(G)$ is the maximum number of edge disjoint
circuits in $G$.

Experimentation with a number of reasonably small examples has shown
that determination of the maximum number of interchanges by evaluating
$\beta(G)$ is considerably easier than by a direct examination of the matrices
$A$ and $B$. However, no simple algorithm for computing $\beta(G)$ has been
found.

In §2, we develop some convenient methods and notations concerning
graphs and matrices. In §3, we reprove Ryser's result that a sequence
of interchanges exists, showing, in fact, that a sequence of length (1.1)
extists. In §4, we prove a general result on graphs and show it implies
(1.1) is a lower bound for the number of interchanges.

2. Preliminaries.

DEFINITION 1. By a graph $G$ with multiplicities, or graph for short,
we mean a set $V = \{v_1, v_2, \ldots, v_t\}$ of vertices and an integer-valued
function $F$ on the ordered pairs of $V \times V$ satisfying $F(v_i, v_j) = - F(v_j, v_i)$
so that in particular $F(v_i, v_i) = 0$. We designate by $\mathcal{E}$ the collection of
ordered pairs $(v_i, v_j)$ of $V \times V$ for which $F(v_i, v_j) > 0$. We choose to
write the elements of $\mathcal{E}$ in the form $E(v_i, v_j)$ and say $E(v_i, v_j)$ is an
arc of $G$ directed from $v_i$ to $v_j$ of multiplicity $F(v_i, v_j)$. 
A graph with multiplicities may be thought of, if desired, as an undirected loopless graph where \( F(v_i, v_j) \neq 0 \) is a flux from \( v_i \) to \( v_j \) through the only edge connecting \( v_i \) and \( v_j \).

The class of all graphs with given vertex set \( V \) we designate \( \mathcal{G} = \mathcal{G}(V) \). Throughout we will suppose \( V \) is fixed but arbitrary. Of special interest is the subclass \( \mathcal{G}^* = \mathcal{G}(V) \) consisting of basic graphs — graphs with arcs of multiplicity 1 only. Basic graphs may be thought of as directed graphs with at most one arc, regardless of direction, connecting any distinct vertices. Given any basic graph \( G^* \) from \( \mathcal{G}^* \) we define a subset \( \mathcal{G}(G^*) \) of \( \mathcal{G}(V) \) as follows: \( G \) is in \( \mathcal{G}(G^*) \) if and only if for each arc \( E(v_i, v_j) \) of \( G \) either \( E(v_i, v_j) \) or \( E(v_j, v_i) \) is an arc of \( G^* \).

**Proposition 1.** If \( G_1 \) and \( G_2 \) are graphs in \( \mathcal{G} \) with functions \( F_1 \) and \( F_2 \), then the function \( F \) given by

\[
F(v_i, v_j) = F_1(v_i, v_j) + F_2(v_i, v_j)
\]

is the function of a graph \( G \) which we may call the sum \( G_1 + G_2 \). \( \mathcal{G}(V) \) is an additive group under this composition and each \( \mathcal{G}(G^*) \) is a subgroup.

We will say a sum \( \Sigma G_i \) of graphs in a class \( \mathcal{G}(G^*) \) is conjoint if for each arc \( E \) of \( G^* \) the nonzero integers \( F_i(E) \) have the same sign, that is, if there is no cancellation in forming the sum \( F = \Sigma F_i \) for \( G \), or in the undirected graph interpretation all fluxes reinforce. If, in fact, for each \( E \) in \( G^* \) at most one \( F_i(E) \) is nonzero we will say the sum \( \Sigma G_i \) is disjoint. It will be seen that conjointness in a sum of graphs is thus a generalization of the usual concept of edge disjointness. By a circuit of length \( r \) (an \( r \)-circuit) we mean a graph in \( \mathcal{G}^* \) having exactly \( r \geq 3 \) distinct arcs
joining $r$ distinct vertices $P_1, P_2, \ldots, P_r$ of $V$.

We say a graph is conservative if the sum of multiplicities of arcs leaving each vertex equals the sum of multiplicities of entering arcs. Any circuit is conservative, but also:

PROPOSITION 2. If the graph $G$ of $\mathcal{G}(G^*)$ is conservative, it can be written as a conjoint sum of circuits in $\mathcal{G}(G^*)$.

If we wish to consider bipartite graphs we can suppose the vertex set $V$ is the disjoint union of sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, $m + n = t$, and restrict attention to the subclass $\mathcal{G}^* \subset \mathcal{G}(V)$ containing those graphs which have no arcs connecting two points in $X$ or two points in $Y$. As before we will suppose the integers $m$ and $n$ and the sets $X$ and $Y$ understood when considering a class $\mathcal{G}^*$. The definitions and results on circuits, conservative graphs and subgroups $\mathcal{G}(G^*)$ will carry over to the bipartite case.

If $\mathcal{G}^*$ is the class of bipartite graphs on vertex sets $X$ and $Y$ of $m$ and $n$ elements respectively, we can define for each $m \times n$ matrix $A = [a_{ij}]$ the graph $G(A)$ in $\mathcal{G}^*$ whose function $F$ is given by $F(x_i, y_j) = -F(y_j, x_i) = a_{ij}$. The correspondence $A \mapsto G(A)$ is an isomorphism between the additive group of $m \times n$ matrices and the group $\mathcal{G}^*$. Accordingly, we will speak of these matrices and graphs interchangeably when convenient.

PROPOSITION 3. An $m \times n$ matrix $A$ is equivalent to zero if and only if $G(A)$ is conservative.
We note that if the graph \( G(C) \) in \( \mathcal{G} \) corresponding to the matrix \( C \) is an \( r \)-circuit we may permute the rows and columns of \( C \) to obtain an \( m \times n \) matrix

\[
\begin{bmatrix}
T & 0 \\
0 & 0
\end{bmatrix}
\]

where \( T \) is an \( r \times r \) matrix with \( r \) 1's on the diagonal, \( r-1 \) -1's on the superdiagonal and a -1 in the lower left. Propositions 2 and 3 combine to give:

**Proposition 4.** Every \( m \times n \) matrix \( A \) equivalent to zero is the conjoint sum of bipartite circuits. If the only entries of \( A \) are 0, 1 and -1, the sum is disjoint.

3. Proof that (1.1) can be attained. For any conservative graph \( G \) in some class \( (\mathcal{G}(G^*)) \) let \( \alpha = \alpha(G) \) be the sum of multiplicities of \( G \) and let \( \beta = \beta(G) \) be the largest integer for which \( G \) can be written as a conjoint sum of \( \beta \) circuits. It is easily seen that \( \alpha(G) \) and \( \beta(G) \) are independent of the particular choice of \( G^* \). In the remainder of this section only we direct our attention exclusively to a class \( (\mathcal{G}) \) of bipartite graphs. We note that if \( A \) and \( B \) are equivalent \((0,1)\)-matrices \( \alpha(A,B) \) the number of disagreements between \( A \) and \( B \), equals \( \alpha(G(B-A)) \).

**Lemma 1.** If \( A \) and \( B \) are equivalent \((0,1)\)-matrices then there exists a sequence

\[
A=A_0, A_1, A_2, \ldots, A_\beta = B \quad \beta = \beta(G(B-A)),
\]

of equivalent \((0,1)\)-matrices such that each difference

\[
C_i = A_i - A_{i-1}
\]
is a circuit (of length $r_i$) and

$$ (3.1) \quad B - A = \sum_{j=1}^{\beta} C_j $$

is a disjoint sum. Moreover,

$$ \beta \quad \alpha(A,B) = \sum_{i=1}^{r_i}. $$

Proof. $B - A$ satisfies the stronger conditions of Proposition 4, hence the disjoint sum (3.1) exists. The partial sums

$$ A_i = A + \sum_{j=1}^{i} C_j $$

are all equivalent to $A$ since the $C_i$ are equivalent to zero. The disjointness in (3.1) and the fact $A$ and $B$ are $(0,1)$-matrices imply the $A_i$ are also $(0,1)$-matrices.

**Lemma 2.** If $A$ and $B$ are equivalent $(0,1)$-matrices and $C = B - A$ is an $r$-circuit, then $r = 2s$ and there exists a sequence

$$ (3.2) \quad A = A_0, A_1, A_2, \ldots, A_{s-1} = B $$

of equivalent $(0,1)$-matrices for which the differences

$$ D_i = A_i - A_{i-1} $$

are circuits of length $4$, i.e. $A_i$ and $A_{i-1}$ differ only by an interchange.

Proof. All graphs in $\mathcal{G}^*$ are bipartite, hence the circuit $B - A$ has even length $r = 2s$. A weak result

$$ B = A + \sum_{j=1}^{s-1} D'_j $$

for certain $4$-circuits $D'_j$ follows easily upon examination of Figure 1.
Note that the $D'_1$ will necessarily visit vertices in $X$ and $Y$ alternately and hence are indeed elements of $\mathcal{S}'$. We seek a reordering $D'_1$ of $D'_1$ so that

$$A_i = A + \sum_{j=1}^{i} D'_j$$

are $(0,1)$-matrices. Clearly the $A_i$ will be equivalent. The sum

$$H = \sum_{j=2}^{s-1} D'_j$$

is a $(2s-2)$-circuit in $\mathcal{S}'$, and a matrix of zeros and ones. We assert either

(i) $A, A + D'_1, A + D'_1 + H = B$

or

(ii) $A, A + H, A + H + D'_1 = B$

is a sequence of equivalent $(0,1)$-matrices. The only possible difficulty is the value of the middle terms for the ordered pair $(x_i, y_j)$ corresponding to $E$ in Figure 1. But $D'_1$ and $H$ take opposite values for this pair, hence exactly one of $A + D'_1$ and $A + H$ is a $(0,1)$-matrix. By applying the same argument to the circuit $H$ instead of $C$ we may place additional terms between $A + D'_1$ and $B$ if (i) holds or between $A$ and $A + H$ if (ii) holds. Repeating this process a sufficient number of times
we will reach simultaneously the sequence (3.2) and the proper reordering of the $D_1$.

**Lemma 3.** If $A$ and $B$ are equivalent $(0,1)$-matrices there exists a sequence

$$(3.3) \quad A = A_0, A_1, A_2, \ldots, A_k = B$$

of equivalent $(0,1)$-matrices for which the differences $A_i - A_{i-1}$ are $k$-circuits and

$$k = \frac{1}{2} \alpha(A,B) - \beta(G(B-A)).$$

**Proof.** The existence of the sequence (3.3) follows from lemmas 1 and 2. The value of $k$ derives from the computation

$$\frac{1}{2} \sum \left( \frac{1}{2} r_i - 1 \right) = \frac{1}{2} \sum r_i - \sum 1 = \frac{1}{2} \alpha(A,B) - \beta.$$

4. **Proof that (1.1) is a lower bound.** Let $G$ be any graph in a subgroup $\mathbb{W}(G^*) \subset \mathbb{G}$. We have defined $\alpha(G)$ and $\beta(G)$. For any positive integer $\delta \geq 3$ let $\gamma = \gamma(G,\delta)$ be the smallest integer for which $G$ can be written as the sum of $\gamma$ circuits from $\mathbb{W}(G^*)$ of length $\delta$ or less. If $G$ cannot be so written set $\gamma = \infty$.

**Theorem 2.** If $G$ is a finite, conservative graph in $\mathbb{W}(G^*)$, then

$$\gamma(G,\delta) \geq \frac{\alpha(G) - 2\beta(G)}{\delta - 2}.$$

**Proof.** We need consider the case $\gamma < \infty$ only. We fix $\delta$ and define the function $\phi(G)$,

$$\phi(G) = \alpha(G) - 2\beta(G) - (\delta - 2) \cdot \gamma(G).$$
We must show

\[ \phi(G) \leq 0 \quad \text{for all } G \text{ in } \mathfrak{G}(G^*) \]

Suppose \((4.1)\) false. Choose a graph \(G_0\) from \(\mathfrak{G}(G^*)\) for which \(a(G_0)\) is as small as possible subject to

\[ \phi(G_0) > 0. \]

Since the empty graph satisfies \((4.1)\), we have

\[ a(G_0) > 0, \quad \beta(G_0) > 0, \quad \gamma(G_0) > 0. \]

Let

\[ G_0 = \sum_{i=1}^{N} D_i \]

be some expression for \(G_0\) as a sum of a minimum number of circuits of \(\mathfrak{G}(G^*)\) of length \(\delta\) or less. For each \(D_i\) let \(q(D_i)\) be the number of arcs of \(D_i\) which coincide (with proper orientation) with an arc of \(G_0\). There must exist a \(D_k\) for which \(q(D_k) \geq \delta - 1\) for otherwise we would have

\[ a(G_0) \leq \sum_{i=1}^{N} q(D_i) \leq (\delta - 2) \cdot \gamma(G_0) \]

in violation of \((4.2)\).

We suppose first, that \(q(D_k) = \delta\). Consider

\[ G' = \sum_{i=1}^{N} D_i \]
\[ i \neq k \]

By exhibiting a specific sum for \(G'\), \((4.4)\) shows that

\[ \gamma(G_0) \geq \gamma(G') + 1. \]
Further let
\[ G' = \sum_{i=1}^{\beta(G')} C_i \]
be a representation of \( G' \) as a conjoint sum of a maximal number of circuits.
Then, because \( q(D_k) = \delta \),
\[ G_0 = \sum_{i=1}^{\beta(G')} C_i + D_k \]
is a conjoint sum for \( G_0 \), implying
\[ (4.7) \quad \beta(G_0) \geq \beta(G') + 1. \]
Combining (4.5), (4.7) and \( \alpha(G_0) = \alpha(G') + \delta \) we conclude
\[ (4.8) \quad \phi(G_0) \leq \phi(G') \]
which contradicts the choice of \( G_0 \) as a smallest graph satisfying (4.2).
In the same way the assumption \( q(D_k) = \delta - 1 \) for a circuit \( D_k \) of length \( \delta - 1 \) leads to (4.8) with strict inequality.

As a third and last alternative, we assume there exists a circuit \( D_k \) in (4.3) of length \( \delta \) for which \( q(D_k) = \delta - 1 \). Let \( E \) be the only arc of \( D_k \) which does not coincide with an arc of \( G_0 \). Again we form \( G' \) as in (4.4) and find an expansion (4.6). We check that \( G' \) is in \( \Omega(G^*) \). In \( G' \), \( \delta - 1 \) multiplicities of \( G_0 \) have been decreased, and one corresponding to \( E^* \), the arc reverse to \( E \), has been increased (possibly from zero to one). Thus
\[ (4.9) \quad \alpha(G_0) = \alpha(G') + \delta - 2. \]
As before (4.5) must hold. Let \( C_h \) be any circuit in (4.6) which has an arc coinciding with \( E^* \). Now \( C_h + D_k \) may not be a circuit, but it is a nonvacuous, conservative graph which, by Proposition 2, is the conjoint sum.
of at least one circuit. Therefore, we have
\[ c_i + D_k = \sum_{i=1}^{s} \tilde{c}_i \]
and this sum is easily seen to be conjoint. Accordingly,
\[ \beta(G') \leq \sum_{i=1}^{s} c_i + \sum_{1 \leq i \neq k} \tilde{c}_i, \]
but (4.5), (4.9) and (4.10) imply (4.8) again, and we are forced to conclude (4.1) always holds. This concludes the proof of Theorem 2.

Theorem 1 now follows directly from Lemma 3 and Theorem 2 for $\delta = 4$, $\Omega(G^*) = \Omega^*$. 

A theorem similar to Theorem 1 can be proven for the case $\delta = 3$.

**THEOREM 3.** If $G$ is a conservative graph in $\Omega(V)$, $a(G)$ is the sum of multiplicities of arcs of $G$, and $\beta(G)$ is the largest integer for which $G$ can be written as a conjoint sum of circuits, then $G$ can be written as the sum of
\[ a(G) = 2\beta(G) \]
and no fewer 3-circuits from $\Omega(V)$.

**Proof.** Theorem 2 says (4.11) is a lower bound. The proof that (4.11) can be realized follows from Figure 2 in the same way Theorem 1 and Lemmas 1, 2 and 3 follow from Figure 1.
Figure 2.
REFERENCES
