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J. R. Sklar
F. C. Schweppe

The Angular Resolution of Multiple Targets

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LINCOLN LABORATORY

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J. R. SKLAR

Group 44

F. C. SCHWEPPE

Group 28

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by

J. R. Sklar

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ABSTRACT

The angular accuracy obtainable from an aperture of fixed size is considered, with emphasis on the multiple target case. By means of the Cramér-Rao Inequality, a lower bound to the measurement variance is computed. In addition, the degradation due to uncertainties in the number of targets present is considered. Loss of accuracy is small until the targets approach to within one beamwidth, at which point the degradation becomes severe.

This technical documentary report is approved for distribution.

Franklin C. Hudson

Franklin C. Hudson, Deputy Chief
Air Force Lincoln Laboratory Office

1. Introduction

Resolution of multiple targets has always been of some concern to the radar designer, but the recent interest in discrimination has given a new importance to be problem. In a multiple target environment, energy from several reflectors impinges on the receiving antenna and as a result, the data processing involves both the extraction of salient features of the received signal and the assignment of these features to their respective targets. Range and range rate resolution can be effected by the design of the radar signal in the time dimension and angular resolution can be achieved through design in the spacial dimensions (antenna design). However, there is at least an order of magnitude difference between the angular and range resolution capabilities of today's more sophisticated radars. The possibility that this discrepancy is at least partly due to a failure to utilize all available angular information in the return signal is indicated by the success of the monopulse lobe comparison technique for beam splitting when only a single target is present, and its abrupt failure when there are more than one in the same range resolution interval. Thus, the question of the existence of a middle ground in which a moderate degree of beam splitting in presence of multiple targets is possible arises.

This report investigates the amount of angular information inherent in a received signal by the use of the Cramér-Rao or Information Inequality. Bounds on the minimum obtainable angular accuracies are derived for both the one- and two-target case. The detection problem of deciding the number of targets present is also discussed and limited results given.

2. Preliminaries

The following two-dimensional radar model is used. The three-dimensional case is a direct generalization. Several targets, say Q, are reflecting energy continuously toward a linear aperture of fixed length, X. The incident plane wave is sampled at an arbitrarily large number of points on the aperture and noise is added to the samples. Since the transmission medium is linear, the contributions to the sample value from each target can be summed, and therefore the samples are a 2 component vector consisting of the quadrature components

$$z_1(x) = \sum_{j=1}^{Q} A_j \sin(\varphi_j + \Phi(x,\alpha_j)) + w_1(x) \qquad 0 \leq x \leq X$$

$$z_2(x) = \sum_{j=1}^{Q} A_j \cos(\varphi_j + \Phi(x,\alpha_j)) + w_2(x) \tag{1}$$

where A_j and φ_j are the magnitude and phase of the radiation from the j^{th} target, α_j is its angular position, x is the location of the sample point within the aperture, and Φ is a function of x and α_j which measures the phase difference between sample points due to the geometry. (See Fig. 1.) If λ is the wavelength of incident plane wave, then

$$\Phi(x,\alpha_j) = \frac{2\pi x}{\lambda} \cos \alpha_j$$

$w_1(x)$ and $w_2(x)$ are Gaussian random variables of zero mean representing the noise. Since $w_1(x)$ and $w_2(x)$ are orthogonal components of the rf noise, they are independent. In addition

$$E\left[w_i(x)w_i(y)\right] = \begin{cases} 0 & x \neq y \\ \sigma^2 & x = y \end{cases} \qquad i = 1,2$$

The amplitudes, A_j, the phases φ_j, and the angles α_j are considered to be unknown parameters; i.e., no a priori distribution for them is assumed known. Thus, there are 3Q unknown parameters and the problem is of the basic form

$$z_1(x) = f_1(x,\epsilon_1,\ldots,\epsilon_{3Q}) + w_1(x)$$
$$z_2(x) = f_2(x,\epsilon_1,\ldots,\epsilon_{3Q}) + w_2(x) \tag{2}$$

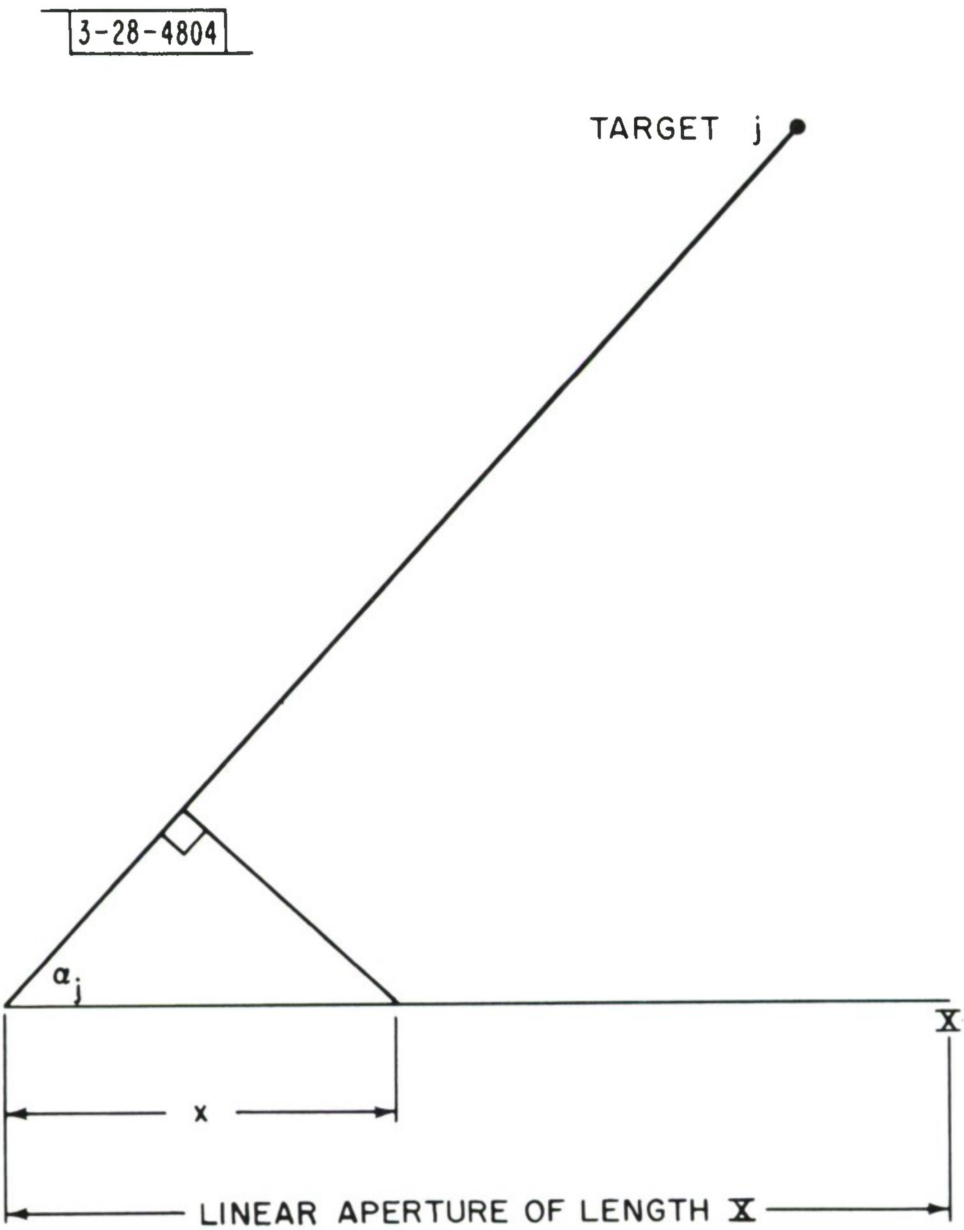

Fig. 1. Target Geometry.

where the ϵ's represent the unknown parameters, and the f_j's are known functions of x and the ϵ's. We will define $\hat{\epsilon}_j (j = 1,\ldots,3Q)$ as the estimate of the ϵ_j obtained from the observations $z_1(x)$, $z_2(x)$, $0 \leq x \leq X$.

We consider the case where the $z_i(x)$ are observed at N points of the aperture, equally spaced at intervals $\delta = X/N$.

Consider

$$p(\underline{z}(1),\ldots,\underline{z}(N) \mid \epsilon_1,\ldots\epsilon_{3Q})$$

where $\underline{z}(k)$ is a two-dimensional vector representing the k^{th} sample, and p is the probability of the observed sample conditional on the true values of the unknown parameters. If one desires to obtain unbiased estimates of a set of unknown quantities ϵ_j, based on observing a set of random variables $\underline{z}(k)$, one can place a lower bound on the variances of the estimators ϵ_j by means of the Cramér-Rao or Information Inequality. Two of many derivations of this fundamental result are found in Ref. 2, Chap. 32 and Ref. 3, Chap. 12 (the terms Carmér-Rao or Information Inequality are not employed; see instead the discussions on efficiency). In the case of multiple parameter estimation, this inequality can best be stated in matrix notation. Let

$$I_{ij} = -E\left[\frac{\partial^2}{\partial\epsilon_i \partial\epsilon_j} \log p\ (\underline{z}(1),\ldots\underline{z}(N)) \mid \epsilon_1,\ldots\epsilon_{3Q}\right]$$

be elements of a matrix $[I]$, and let

$$\sigma^2_{kj} = \left[E\ (\hat{\epsilon}_k - \epsilon_1)\ (\hat{\epsilon}_j - \epsilon_j)\right]$$

be the elements of a matrix $\left[\sigma^2\right]$. Then the Cramér-Rao Inequality states that

$$\left[\sigma^2\right] - \left[I\right]^{-1}$$

is positive definite, which implies that

$$\sigma^2_{kk} \geq \left[I\right]^{-1}_{kk} \tag{3}$$

This result is derived from the requirement that the estimates be unbiased and a generalized Schwartz inequality and is subject to the condition (usually present in practice) that [I] be positive definite. Thus the desired lower bound can be obtained by calculating the necessary derivatives and inverting the resulting matrix.

In the sequel it will be useful to employ the following alternate interpretation of $[I]^{-1}$. Consider Eq. (2). Expand the $f_i(x,\epsilon_1,\ldots,\epsilon_{3Q})$ in a Taylor series about some set of ϵ's, $\epsilon_1^o,\ldots,\epsilon_{3Q}^o$. Then

$$\Delta z_i(x) = \sum_{j=1}^{3Q} \frac{\partial f_i(x,\epsilon,\ldots,\epsilon_{3Q})}{\epsilon_j} \Delta\epsilon_j + w_i(x) + \mathcal{R}_i \quad , i = 1,2 \qquad (4)$$

where the partial derivative is evaluated at the ϵ^o's, $\mathcal{R}_i$ is the remainder, and

$$\Delta z_i(x) = z_i(x) - f_i(x,\epsilon_1^o,\ldots,\epsilon_{3Q}^o)$$

$$\Delta\epsilon_j = \epsilon_j^o - \epsilon_j$$

If the ϵ_j^o's are close enough to the true values to allow the remainder, $\mathcal{R}_i$, to be neglected. Eq. (4) can be considered as a linear regression problem with unknown coefficients, $\Delta\epsilon_j$. A minimum variance, unbiased estimate (or a maximum likelihood estimate) for this problem has a covariance matrix equal to $[I]^{-1}$. [References 2 and 3 give the linear regression equations which, when applied to Eq. (4), give $[I]^{-1}$ for the Gaussian case.] Therefore, $[I]^{-1}$ can be considered as either a bounding covariance matrix or the result of a linearized or small error, error analysis. Thus, for signal-to-noise ratios which are large enough to imply small errors, the inequality of Eq. (3) becomes equality.

3. Single Target

The first case to be considered is a single target at unknown angle α with unknown amplitude A and phase ϕ. The target is known to exist. When the noise is gaussian, with variance σ^2 and zero mean, the observation of N samples equally spaced at $\delta = \frac{X}{N}$ along the aperture leads to a conditional probability for the samples of

$$p(z(\delta), z(2\delta), \ldots \mid \alpha, A, \phi)$$

$$= \frac{1}{(2\pi)^N \sigma^{2N}} \exp \left\{ -\sum_{i=1}^{N} \left[z_1(i\delta) - A \sin\left(\phi + \frac{2\pi i\delta}{\lambda} \cos\alpha\right) \right]^2 \Big/ 2\sigma^2 \right.$$

$$\left. - \sum_{i=1}^{N} \left[z_2(i\delta) - A \cos\left(\phi + \frac{2\pi i\delta}{\lambda} \cos\alpha\right) \right]^2 \Big/ 2\sigma^2 \right\}$$

from Eq. (1).

After taking the proper derivatives and averages we obtain, for N large,

$$[I] = \begin{bmatrix} \frac{N}{3} \frac{A^2}{\sigma^2} \left(\frac{2\pi X}{\lambda}\right)^2 \sin^2\alpha & 0 & -\frac{N}{2} \frac{A^2}{\sigma^2} \left(\frac{2\pi X}{\lambda}\right) \sin\alpha \\ 0 & \frac{N}{\sigma^2} & 0 \\ -\frac{N}{2} \frac{A^2}{\sigma^2} \left(\frac{2\pi X}{\lambda}\right) \sin\alpha & 0 & \frac{NA^2}{\sigma^2} \end{bmatrix}$$

in which the indices 1, 2 and 3 correspond to α, A and ϕ respectively. From this we calculate

$$[I]^{-1} = \begin{bmatrix} \dfrac{12}{\dfrac{NA^2}{\sigma^2}\left(\dfrac{2\pi X}{\lambda}\right)^2 \sin^2\alpha} & 0 & \dfrac{6}{\dfrac{NA^2}{\sigma^2}\left(\dfrac{2\pi X}{\lambda}\right) \sin\alpha} \\ 0 & \dfrac{\sigma^2}{N} & 0 \\ \dfrac{6}{\dfrac{NA^2}{\sigma^2}\left(\dfrac{2\pi X}{\lambda}\right) \sin\alpha} & & \dfrac{4}{\dfrac{NA^2}{\sigma^2}} \end{bmatrix}$$

Consequently, the variance in the angle measurement is lower bounded by

$$\sigma_{\hat{\alpha}}^2 > \frac{12}{\dfrac{NA^2}{\sigma^2}\left(\dfrac{2\pi X}{\lambda}\right)^2 \sin^2\alpha} \qquad (5)$$

The fractional beamwidth error is therefore bounded by

$$\sigma_{\hat{\alpha}} \Big/ BW \geq \frac{1}{\pi}\sqrt{\frac{3}{R}}$$

where $R = \frac{NA^2}{\sigma^2}$ is the signal-to-noise ratio of the same aperture if all array element outputs are added and $BW = \lambda/X$ is the beamwidth for the aperture measured to the $\left(\frac{2}{\pi}\right)^2$ relative power level. We see that this is comparable to that obtained by monopulse lobe comparison techniques in Ref. 1.

4. Two Targets

The same procedure can be followed in the two-target case when exactly two targets are known to exist. The conditional probability expression is

$$P(\underline{z}(\delta),\underline{z}(2\delta),\ldots \mid \alpha_1,\alpha_2,A_1,A_2,\phi_1,\phi_2) =$$

$$\frac{1}{(2\pi)^N \sigma^{2N}} \exp\left\{ -\sum_{i=1}^{N} \left[z_1(i\delta) - A_1 \sin(\phi_1 + \frac{2\pi}{\lambda}(i\delta)\cos\alpha_1) - A_2 \sin(\phi_2 + \frac{2\pi}{\lambda}(i\delta)\cos\alpha_2) \right]^2 \Big/ 2\sigma^2 - \sum_{i=1}^{N} \left[z_2(i\delta) - A_1 \cos(\phi_1 + \frac{2\pi}{\lambda}(i\delta)\cos\alpha_1) - A_2 \sin(\phi_2 + \frac{2\pi}{\lambda}(i\delta)\cos\alpha_2) \right]^2 \Big/ 2\sigma^2 \right\}$$

Taking derivatives and computing averages the result is

$$I = [D]\,[\Gamma]\,[D] \qquad (6)$$

where $[\Gamma]$ is a 6 x 6 matrix given in Fig. 2 and $[D]$ is the 6 x 6 diagonal matrix,

$$[D] = \frac{N^{1/2}}{\delta} \begin{bmatrix} \frac{2\pi X}{\lambda} A_1 \sin\alpha_1 & & & & & \\ & \frac{2\pi X}{\lambda} A_2 \sin\alpha_2 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & A_1 & \\ & & & & & A_2 \end{bmatrix}$$

$$[\Gamma] = \begin{bmatrix} 1/3 & a_{21} & 0 & a_{41} & 1/2 & a_{61} \\ a_{21} & 1/3 & -a_{41} & 0 & a_{61} & 1/2 \\ 0 & -a_{41} & 1 & a_{43} & 0 & -a_{63} \\ a_{41} & 0 & a_{41} & 1 & a_{63} & 0 \\ 1/2 & a_{61} & 0 & a_{63} & 1 & a_{43} \\ a_{61} & 1/2 & -a_{63} & 0 & a_{43} & 1 \end{bmatrix}$$

$$a_{21} = \frac{2}{(aX)^2}\cos(aX + b) + \frac{1}{(aX)}\sin(aX + b) - \frac{2}{(aX)^3}\sin(aX + b) + \frac{2}{(aX)^3}\sin b$$

$$a_{41} = \frac{1}{(aX)}\cos(aX + b) - \frac{1}{(aX)^2}\sin(aX + b) + \frac{1}{(aX)^2}\sin b$$

$$a_{61} = \frac{1}{(aX)}\sin(aX + b) + \frac{1}{(aX)^2}\cos(aX + b) - \frac{1}{(aX)^2}\cos b$$

$$a_{43} = \frac{1}{(aX)}\sin(aX + b) - \frac{1}{(aX)}\sin b$$

$$a_{63} = \frac{1}{(aX)}\cos(aX + b) - \frac{1}{(aX)}\cos b$$

$$a = \frac{2\pi}{\lambda}(\cos \alpha_2 - \cos \alpha_1)$$

$$b = \varphi_1 - \varphi_2$$

Figure 2 -28-4805

$[\Gamma]$ Indices 1, 2, 3, 4, 5, and 6 correspond to parameters α_1, α_2, A_1, A_2, φ_1, and φ_2.

This factorization of the $[I]$ matrix is attractive since it places in evidence the dependence of the elements of $[I]^{-1}$ on the various parameters. (Recall the inverse of a diagonal matrix is a diagonal matrix with the elements inverted and also $[I]^{-1} = [D]^{-1} [\Gamma]^{-1} [D]^{-1}$.) Thus we may note that the element of $[I]^{-1}$ corresponding to the variance of amplitude is independent of the amplitude. This fact will be important in the detection discussion of Section 5.

Computing the inverse of the $[I]$ matrix is difficult, but when we are only concerned with the angular accuracy, only part of the inverted matrix is required. This can be obtained with a somewhat less tedious calculation by partitioning the $[\Gamma]$ matrix. Although it is then possible to obtain the variance of the angular measurement in closed form, the expression is too lengthy to be very useful. Therefore, the angular variance is plotted as a function of the target separation in Fig. 3 with the phase difference $b = \phi_1 - \phi_2$ as a parameter. From this normalized curve results for various values of a signal-to-noise ratio and angular position can be obtained.

In addition, the ratio of the variance in the two-target case to the variance in the one-target case is also plotted. One can observe that the results are essentially identical for target separations greater than one beamwidth ($BW = \frac{\lambda}{x}$), but that for smaller separations the measurement error in the two-target case becomes large very rapidly as the separation decreases. The phase difference b causes a wide variation in the form of the curves. However, for all values of this parameter, the variance is radically larger if the separation is less than about one beamwidth. Consequently, any attempt at beam splitting for separations much smaller than one beamwidth will prove futile.

5. Detection

The preceding two sections have considered estimating accuracy for multiple targets under the assumption that the exact number of targets present is known.

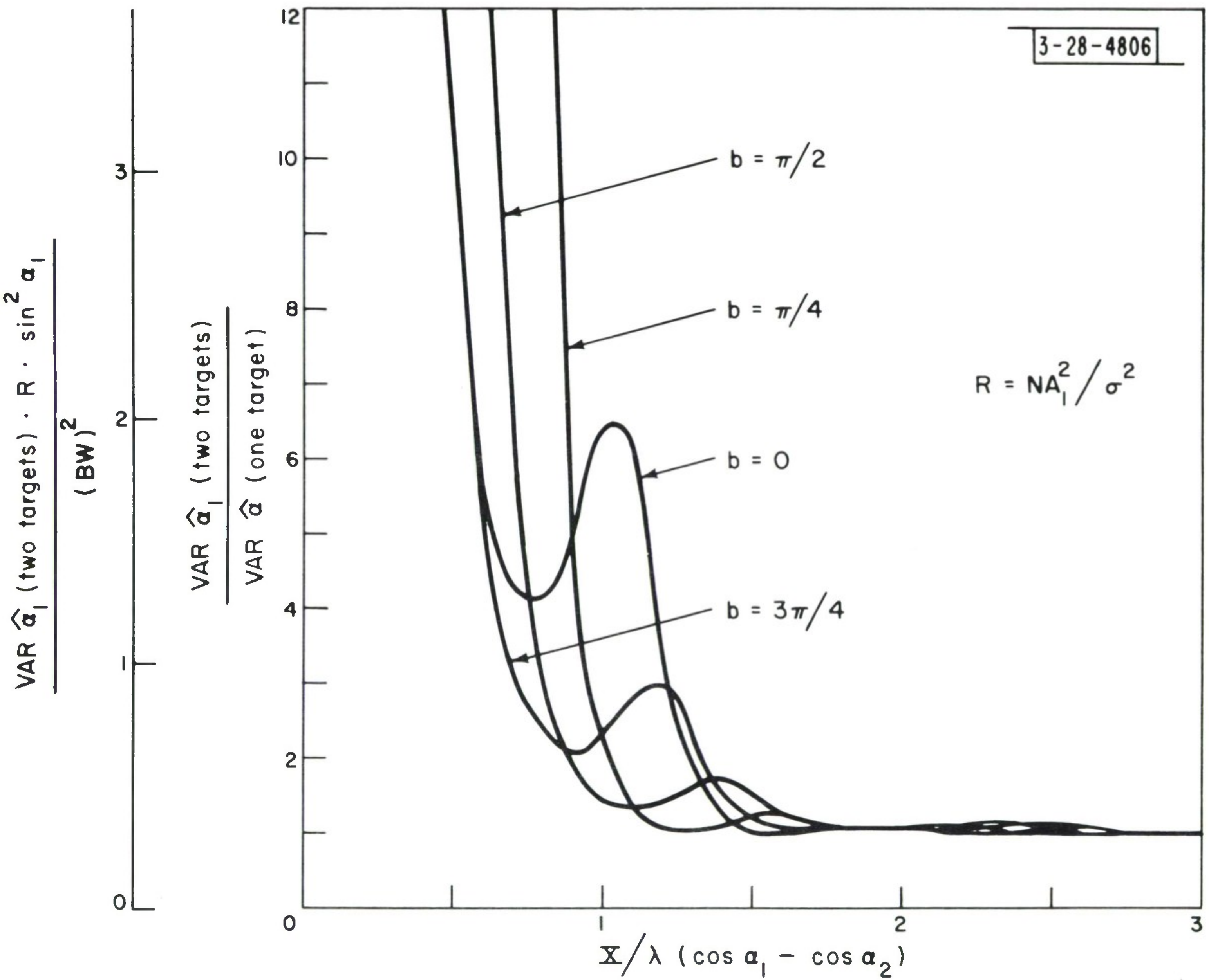

Fig. 3. Angular Accuracy in Presence of Two Targets.

Detection is the problem of deciding the number of objects present. Certain general aspects of multiple target detection are briefly reviewed, and a measure provided on the relative detection capability in the one- and two-target case.

A standard approach to the detection problem employs the theory of hypothesis testing.* In the case where at most one target may exist, it is hypothesized that no target exists (H_0 is the hypothesis). The alternate hypothesis is that a target does exist (H_1 is the alternate hypothesis). If A is the amplitude of the signal, these hypotheses are equivalent to

$$H_0: \quad A = 0$$

$$H_1: \quad A \neq 0$$

If up to two targets are possible, it is then necessary to check the possibilities of zero, one and two targets. Such a multi-level decision can be built out of two-level tests such as used for the one-target problem. For example, a first test might hypothesize that no targets are present vs the alternate hypothesis that at least one target is present. That is

$$H_0: \quad A_1 = 0 \ , \quad A_2 = 0$$

$$H_1: \quad \text{At least one } A_j \neq 0 \qquad j = 1,2 \tag{7}$$

where the A_j are the amplitudes of the signals. If the hypothesis that no targets are present is rejected, one could then hypothesize the existence of exactly one target vs the alternate that exactly two targets exist. That is

$$\begin{aligned} H_0: &\quad A_1 \neq 0 \ , \quad A_2 = 0 \\ H_1: &\quad A_1 \neq 0 \ , \quad A_2 \neq 0 \end{aligned} \tag{8}$$

*The following discussion is a very special application of the general statistical theory. References 2 and 3 are two of many which contain far more extensive discussions.

There are, of course, many other possible sequences, and the above is only an example.

The basic problem is thus the testing of some hypothesis H_0 vs some alternate H_1. The test is based on some test statistics, ξ, where ξ is some function of the observed data. There are many possible test statistics, two of which have particularly nice properties. The first of these is based directly on the estimated amplitudes $(\hat{A}_j)$ of the signals. For example, in the one-target case, the hypothesis that no target exists (H_0) is rejected if the magnitude of the estimated amplitude exceeds some chosen value. In the multiple-target case, similar tests are employed on the vector of the estimated amplitudes. The second test statistic is the likelihood ratio; i.e., the ratio of the maximum likelihood attainable under H_1 to the maximum likelihood attainable under H_0. (See, for example, Ref. 3) The amplitude statistic is more powerful, but the likelihood ratio test is independent of the signal-to-noise ratio. We shall confine discussions to the amplitude statistic.

To be explicit, assume the signal-to-noise ratio is high enough so that the variance given by the Cramér-Rao Inequality can actually be realized or, equivalently, that we are dealing with thesystem of linear equations given by Eq. (4).* The vector estimate $[\hat{A}_1, \hat{A}_2]$ is then a two-dimensional Gaussian random variable with covariance matrix given by the corresponding elements of the matrices of $[I]^{-1}$. Let $[I]_{AA}^{-1}$ denote this 2 x 2 sub-matrix of $[I]^{-1}$. If it is desired to test the hypothesis of Eq. (7), the test statistic ξ is, in matrix notation,

$$\xi^2 = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I \end{bmatrix}_{AA}^{-1} \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix}$$

*This approach is similar to the investigation of the asymptotic behavior of test statistics as employed in the statistical literature.

The equation, ξ^2 = constant, defines an ellipse of constant probability in $\hat{A}_1$, $\hat{A}_2$ space. Thus the hypothesis test, which rejects H_0 when $\xi^2 > \xi_0^2$, means the hypothesis is rejected when $(\hat{A}_1, \hat{A}_2)$ lies outside the constant probability ellipse specified by ξ_0^2. Let $\sigma_{\hat{A}_2}$ denote the variance of $\hat{A}_2$ as evaluated, using the corresponding diagonal term of $[I]^{-1}_{AA}$. For the case of Eq. (8), the test statistic is

$$\xi^2 = \left(\frac{\hat{A}_2}{\sigma_{\hat{A}_2}} \right)^2$$

which is the one-dimensional version of the preceding case.

The question of prime interest in this paper is how well detection can be performed. Unfortunately, a complete analysis requires the choice of explicit sequences of tests, and this choice depends on the particular problem of interest. In addition, the calculation of the resulting probabilities of the various types of error is somewhat laborious. Thus we shall merely indicate the relative degradation in detection capability which results from the presence of a second target. Consider the case of Eq. (8), where the existence of one target is definitely known ($A_1 \neq 0$), and the possible existence of a second is to be tested. The test statistic, ξ^2, is $(\hat{A}_2/\sigma_{\hat{A}_2})^2$. Thus

$$\xi = \frac{A_2}{\sigma_{\hat{A}_2}} + \eta$$

where η is a Gaussian, zero-mean random variable of unity variance. Now, consider the problem of deciding whether there are one or no targets. Exactly the same test is used, except that $\sigma_{\hat{A}_2}$ is replaced by the $\sigma_{\hat{A}}$ obtained from the corresponding element of $[I]^{-1}$ as evaluated in Sec. 3, Eq. (5). Thus, for a fixed false detection probability, the probability of detecting a second target of amplitude A_2 in the presence of another known to be present, equals the probability of detecting a single target of amplitude $\underline{A}$ when

$$\frac{A_2}{\sigma_{\hat{A}_2}} = \frac{A}{\sigma_{\hat{A}}}$$

The ratio, $\sigma_{\hat{A}_2}/\sigma_{\hat{A}}$, therefore measures the degradation of detection performance caused by the presence of a second target.

As with the case of the angular accuracies of Sec. 4, the closed form solutions are too complex to be very useful. Thus $\sigma^2_{\hat{A}_2}/\sigma^2_{\hat{A}}$ is plotted in Fig. 4 as a function of target separation with the phase difference b as a parameter.

6. Discussion

Figures 3 and 4 constitute the major results of this study and show that, as target separations increase beyond one beamwidth, the information loss caused by multiple targets rapidly disappears. Since these curves resulted from a limited analysis on a simplified model, we must discuss their limitations before drawing actual conclusions.

We considered only errors due to additive noise. Thus the antenna pattern is assumed to be known exactly. This assumption results in the angular accuracy of one target being independent of the cross sections (amplitude) of other targets. With an exactly known antenna pattern, this is not unreasonable, since the effect of other targets can be "subtracted out." In practice, however, uncertainties in antenna pattern can cause a large cross section target to sizeably degrade estimates from a nearby smaller target. In addition, an inequality was used in place of an exact error analysis. We can hope to approach the resulting bound only with a sufficiently high signal-to-noise ratio. Thus Figs. 3 and 4 indicate actual capabilities only for S/N ratios high enough to approach the bound but low enough to keep antenna uncertainties from predominating. The range of such S/N ratios depends on the particular problem under investigation. However, it is not unreasonable to expect some

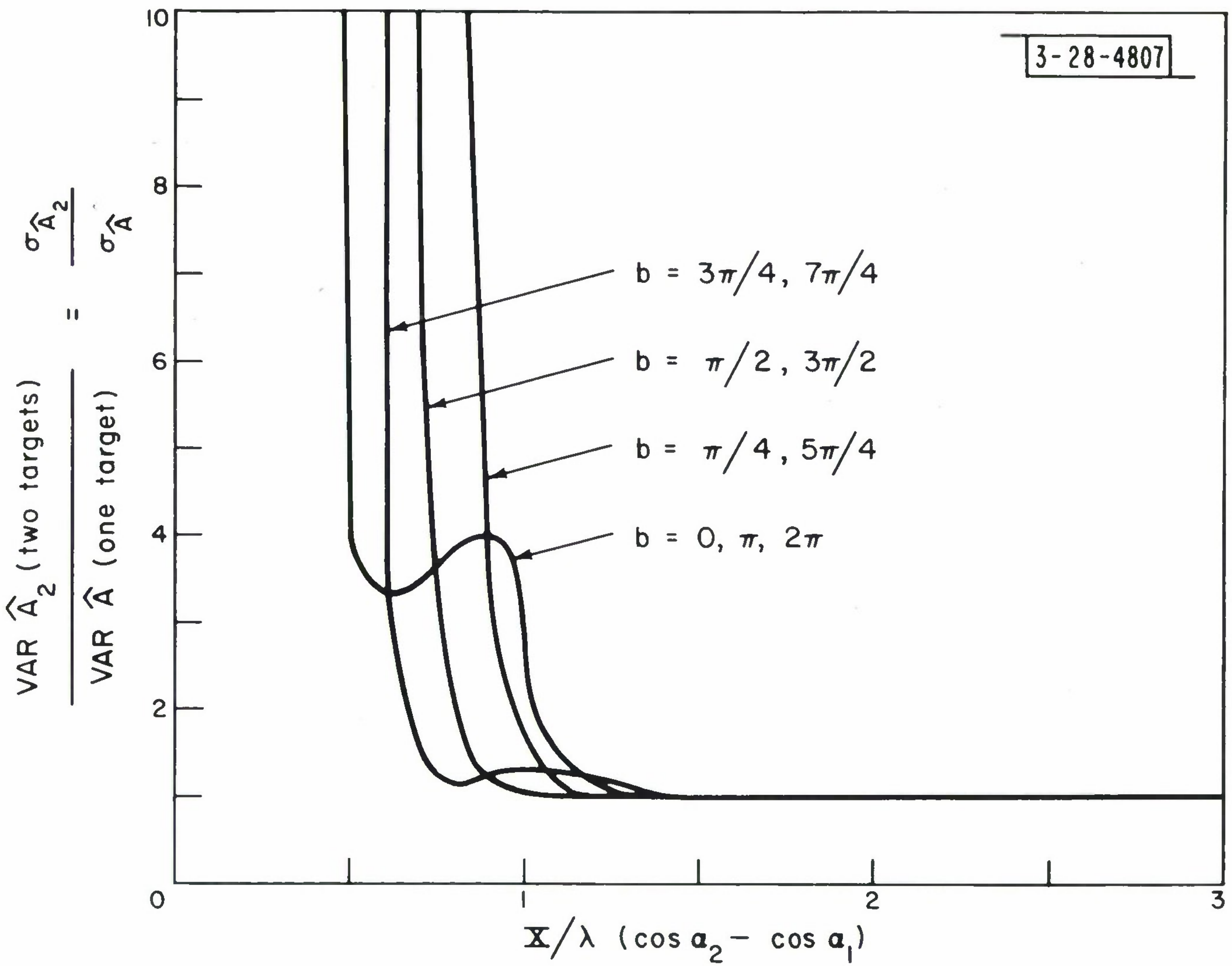

Fig. 4. Detection Degradation in Presence of Two Targets.

systems for which it is nonempty. For example, the same basic assumptions hold in both the one-target and two-target cases and the single target bound is often approached in practice.

We applied the basic theory to a very simple model: a one-dimensional aperture with two stationary targets and white noise. The two-dimensional aperture and more than two targets merely introduces more parameters into the problem. This is also true for moving targets as the unknown phase ϕ_j of the return signal from the j^{th} target becomes a parameterized function of time such as

$$\phi_j(t) = \phi_{j1} + \phi_{j2}t$$

to account for range and range rate and the angular position, α_j, becomes a similar parameterized function of time to account for the angular motion. A moving target also requires specification of the carrier modulation. The introduction of these parameters into the model is straightforward and would not change the basic method of analysis. However, it would greatly enhance the computational difficulties associated with explicit results. The simple model is valuable as it indicates the target separations for which the angular resolution problem can be ignored and either single target analysis used or range and range rate multiple target resolution studied by itself. Such a dichotomy provides a simplified but very useful picture of the over-all multiple target problem. The incorporation of n^{th} order Markovian observation noise changes the equation for information to a nonlinear differential equation, which can be numerically integrated on a digital computer. However, our basic problem is not well enough defined to justify a nonwhite noise model.

The most important limitation in the scope of the study is the fact that we have not considered an actual data processing system. If conditions are such that the bound can be approached, the classical maximum likelihood estimation technique provides the desired estimates. If we are not worried about computation time and cost, such estimates can be obtained using a large digital computer. However, for most situations, such a procedure is grossly impractical, as real time estimates are required and these imply primarily analog data processing. Unfortunately, a straightforward analog implementation of the maximum likelihood technique for several targets requires a large amoung of hardware, and the various avenues which might lead to a practical analog implementation have not yet been explored. Information is of no value if it cannot be extracted.

Let us now summarize what Figs. 3 and 4 actually imply. They provide a bound on possible system performance. More important, there is good reason to believe that some systems do or will exist for which this performance can be approached. Thus, for more than a beamwidth separation, multiple-target capability close to that of single target monopulse lobe compression is a distinct possibility, provided a practical data processing procedure can be developed and implemented. A prime purpose of this report is to motivate investigations on such data processing techniques.

ACKNOWLEDGEMENT

This study owes its existence to Dr. Eugene W. Pike. His conviction that effective "beam splitting" should be possible in the presence of multiple targets led to the following investigations.

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