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A "String Algorithm" for Shortest Paths in Directed Networks

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A "STRING ALGORITHM" FOR SHORTEST PATHS
IN DIRECTED NETWORKS

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The literature contains several algorithms for finding the shortest path between two nodes \( P \) and \( Q \) of a network, where the distances or arc-lengths are assumed to be positive. (For references, consult the review article by Pollack and Weibenson [3] and the book by Ford and Fulkerson [1].)

Some of the algorithms, and in particular some of the analogue devices, are applicable only when the distance matrix is symmetric. As was remarked in [1] and [3], this is true of the simplest of the analogue procedures—the "string algorithm" reported by Minty [2]. It consists of making an inelastic string model of the network, with knots corresponding to nodes and string-lengths proportional to the corresponding distances, and then stretching the knots \( P \) and \( Q \) as far apart as is possible without breaking the string; this produces at least one straight path from \( P \) to \( Q \), and each such straight path corresponds to a shortest path in the network.

Networks with asymmetric distance matrices are most conveniently represented by means of directed networks, in which every arc is regarded as a one-way street of the appropriate length. In the present note we describe a simple cutting procedure (related to one suggested by Thomas Seidman) which can be combined with any algorithm for undirected networks (symmetric distance matrix) so as to form a shortest-path algorithm for directed networks (asymmetric distance matrix). In particular, the cutting and stretching can be alternated to form a "string algorithm" for directed networks.

For each directed network \( N \), let \( N^u \) denote the corresponding undirected network.
THEOREM Suppose that \( P \) and \( Q \) are nodes of a finite directed network \( N_1 \) which has \( v \) nodes and \( e \) arcs, and that there is a path from \( P \) to \( Q \) in \( N_1 \). Suppose that \( A \) is an algorithm for finding shortest paths in undirected networks, and let the sequential procedure \( S_1, S_2, S_3, ... \) be as follows:

1. Apply \( A \) to the undirected network \( N^u_1 \) to find a shortest path \( \pi_1 \) from \( P \) to \( Q \) in \( N^u_1 \). Suppose \( \pi_1 \) is given by

   \[
   V_0^i V_1^i V_2^i ... V_h^i V_{h+1}^i V_{h+2}^i ...
   \]

   where the arcs \( a_j^i \) and the nodes \( V_j^i \) are listed as they appear in traversing \( \pi_1 \) from \( P = V_0^i \) to \( Q = V_h^i \).

2. If \( \pi_1 \) is also a path in \( N_1 \), terminate the procedure. If \( \pi_1 \) is not a path in \( N_1 \), there exists a smallest index \( r(1) \) and a largest index \( s(1) \) (possibly the same) such that the directions of \( a_{r(1)}^i \) and \( a_{s(1)}^i \) in \( \pi_1 \) are opposite to their directions in \( N_1 \). Let \( N_{1+1} \) be the directed network that is obtained from \( N_1 \) by deleting every arc of \( N_1 \) that (like \( a_r^i \)) ends in \( N_1 \) at \( V_{r+1}^i \) but is not \( a_{r+1}^i \), and deleting every arc of \( N_1 \) that (like \( a_s^i \)) starts in \( N_1 \) at \( V^i_s \) but is not \( a_{s+1}^i \). There is a path \( \Sigma \) from \( P \) to \( Q \) in \( N_{1+1} \) such that \( \Sigma \) is actually a shortest path from \( P \) to \( Q \) in \( N_1 \).

The procedure terminates at some stage \( G_t \) for which

\[
t \leq \min (v, (e+2)/2),
\]

and the path \( \pi_t \) is a shortest path from \( P \) to \( Q \) in the directed network \( N_1 \).
(The same conclusion holds if $G_2$ requires only the first of the
two deletions specified above, or if it requires only the second.)

Proof. Of course the algorithm itself does not involve the actual
construction of $\Sigma$, but we require the existence of $\Sigma$ (when $G_2$
does not specify termination) to show that the sequential procedure $S_1$, $S_1$, $S_2$, $S_2$, ...
can actually be followed and that each of the paths $\Sigma, \Sigma, ...$
is a shortest path from $P$ to $Q$ in $N_1$. Since $N_1$ is finite, the
procedure must terminate at some stage $\Sigma_t$ and then $\pi_t$ is a shortest
path from $P$ to $Q$ in $N_1$.

Suppose $G_1$ does not specify termination and let $\Sigma$ be a shortest
path from $P$ to $Q$ in $N_1$, given by

$W_0^1 W_1^1 W_2^1 ... W_{l(i)-1}^1$, $r(i)$ $W_{l(i)}^1 s(i)$,

where of course $W_0^1 = P$ and $W_{l(i)}^1 = Q$. In constructing $\Sigma$,
we consider the following three possibilities:

(i) no $W_j^1$ is equal to either $V_{r(i)-1}^i$ or $V_{s(i)}^i$;

(ii) there exists $j$ such that $W_j^i = V_{r(i)-1}^i$ and $j < k = W_k^i \neq V_{s(i)}^i$;

(iii) there exist $j$ and $k$ such that $j < k$, $W_j^i = V_{r(i)-1}^i$ and

$W_k^i = V_{s(i)}^i$.

When (i) holds, we define $\Sigma = \Sigma_t$ when (ii) holds, we obtain $\Sigma$
following $\pi_1$ from $P$ to $V_{m(i)-1}^i$ and then following $\Sigma$ from $V_{m(i)-1}^i$
to $Q$. When (iii) holds, we obtain $\Sigma$ by following $\pi_1$ from $P$ to
$V_{m(i)-1}^i$, next following $\Sigma$ from $V_{m(i)-1}^i$ to $\Sigma$, and then following
\[ n_i \text{ from } V_{n(i)}^i \text{ to } Q. \text{ In each case, it is easily verified that } \Sigma_i \text{ has the stated properties. Thus the existence of } t \text{ is established and it remains only to show that } t \leq \min(v,(e+2)/2). \]

Let us review the special properties of certain nodes and arcs of \( N_j \) relative to \( N_i \) itself and relative to \( N_j \) for \( j > i \).

(a) \( V_{r(i)}^i \neq Q \). If \( V_{r(i)}^i \neq P \), then at least one arc of \( N_i \) ends at \( V_{r(i)}^i \) but no arc of \( N_j \) ends there. If \( V_{r(i)}^i \neq P \), then at least two arcs of \( N_i \) end at \( V_{r(i)}^i \) but at most one arc of \( N_j \) ends there.

(b) \( \alpha_{r(i)}^i \) ends at \( P \) or is coterminal with another arc of \( N_i \); \( \alpha_{r(i)}^i \) does not end at \( Q \) and does not appear in \( N_j \).

(c) \( \alpha_{r(i)}^i \) does not end at \( P \) or \( Q \), and is nonexistent if \( \alpha_{r(i)}^i \) ends at \( P \). If \( \alpha_{r(i)}^i \) does not end at \( P \), then \( \alpha_{r(i)}^i \) is coterminal with another arc of \( N_i \) but not with another arc of \( N_j \).

We see from (a) that the \( t \) nodes \( Q, V_{r(1)}^i, V_{r(2)}^i, \ldots, V_{r(t-1)}^i \) are pairwise distinct, and consequently \( t \geq v \). From (b) and (c) it follows that the arcs \( \alpha_{r(1)}^i, \alpha_{r(2)}^i, \alpha_{r(2)}^2, \ldots, \alpha_{r(t-1)}^i, \alpha_{r(t-1)}^t \) are pairwise distinct. If \( \alpha_{r(i)}^i \) ends at \( P \), then \( \alpha_{r(i)}^i \) does not appear, but this happens for at most one value of \( i \), and since at least one arc of \( N_i \) ends at \( Q \) we conclude that \( e \geq 2t - 2 \).

The above reasoning completes the proof when \( G_4 \) is as originally described, and also when \( G_4 \) is replaced by \( G_4' \) which requires only the first of the specified deletions. Similar reasoning applied to \( G_4'' \), which
v nodes.  

2(v-1) arcs, all of the same length.

t = v = (e+2)/2

(πₐ follows upper arcs except at αᵣ(1))

v nodes.  

3v - 5 arcs, with lengths as indicated.

αᵣ(i) = αᵣ(i) indicated by (i)

t = v
Figure 3

3n - 1 arcs.

$y_n$ is uniquely determined.

$t = n + 1 - \frac{y + z}{2}$

$z = \frac{y + 2}{2}$

4 nodes.
REFERENCES

