<table>
<thead>
<tr>
<th>UNCLASSIFIED</th>
</tr>
</thead>
<tbody>
<tr>
<td>AD NUMBER</td>
</tr>
<tr>
<td>AD426174</td>
</tr>
<tr>
<td>LIMITATION CHANGES</td>
</tr>
<tr>
<td>TO:</td>
</tr>
<tr>
<td>Approved for public release; distribution is unlimited.</td>
</tr>
<tr>
<td>FROM:</td>
</tr>
<tr>
<td>Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; NOV 1963. Other requests shall be referred to Defense Advanced Research Projects Agency, 675 North Randolph Street, Arlinton, VA 22203-2114.</td>
</tr>
<tr>
<td>AUTHORITY</td>
</tr>
<tr>
<td>RAND ltr, 31 Mar 1966</td>
</tr>
</tbody>
</table>

THIS PAGE IS UNCLASSIFIED
NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
MEMORANDUM
RM-3821-ARPA
NOVEMBER 1963

CLOSURE TECHNIQUES FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

Richard Bellman and John M. Richardson

PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

The RAND Corporation
SANTA MONICA • CALIFORNIA
MEMORANDUM
RM-3821-ARPA
NOVEMBER 1963

CLOSURE TECHNIQUES FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS
Richard Bellman and John M. Richardson

This research is supported by the Advanced Research Projects Agency under Contract No. SD-79. Any views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of ARPA.

DDC AVAILABILITY NOTICE
Qualified requesters may obtain copies of this report from the Defense Documentation Center (DDC).
PREFACE

Part of the RAND research program consists of basic supporting studies in mathematics. The mathematical research in this Memorandum is concerned with techniques for obtaining approximate solutions to systems of nonlinear differential equations, which arise in a variety of physical problems.
SUMMARY

In some recent papers, [1], [2], [3], we have applied various closure techniques to the problem of obtaining approximate solutions of nonlinear differential equations. These questions fall within the area of differential approximation. Starting with a vector system

\[ \frac{dx}{dt} = g(x), \]

we replace it by another system

\[ \frac{dy}{dt} = h(y), \]

with simpler analytic and computation properties in such a way that \( \| x - y \| \) is small. In some cases we are content merely to have \( h(y) \) linear, i.e., of the form \( Ay + b \); in other cases, we want the dimension of \( y \) to be considerably less than that of \( x \).

In this paper we wish to consider the case where the original system is infinite dimensional and the approximating system is to be finite dimensional. Infinite dimensional systems arise in a natural fashion from the consideration of partial differential equations.
CONTENTS

PREFACE ................................................. ii
SUMMARY ............................................... iii

Section
1. INTRODUCTION. ............................... 1
2. EXTRAPOLATION AND SELF-CONSISTENCY. .... 2
3. NONLINEAR EQUATIONS ....................... 5

REFERENCES ......................................... 6
CLOSURE TECHNIQUES FOR INFINITE SYSTEMS
OF DIFFERENTIAL EQUATIONS

1. INTRODUCTION

In some recent papers, [1], [2], [3], we have applied various closure techniques to the problem of obtaining approximate solutions of nonlinear differential equations. These questions fall within the area of differential approximation. Starting with a vector system

(1.1) \[ \frac{dx}{dt} = g(x), \]

we replace it by another system

(1.2) \[ \frac{dy}{dt} = h(y), \]

with simpler analytic and computation properties, in such a way that \( \| x - y \| \) is small. In some cases we are content merely to have \( h(y) \) linear, i.e., of the form \( Ay + b \); in other cases, we want the dimension of \( y \) to be considerably less than that of \( x \).

In this paper we wish to consider the case where the original system is infinite dimensional and the approximating system is to be finite dimensional. Infinite dimensional systems arise in a natural fashion from the consideration of partial differential equations. For example, if we look for a solution of

(1.3) \[ u_t + u u_x = \varepsilon u_{xx} \]
in the form

\[(1.4) \quad u(x, t) = \sum_{k} u_k(t) e^{ikx},\]

we obtain an infinite system of ordinary differential equations for the components \(u_k(t)\); see [4].

The problem of closure is now more difficult since there is no immediate way of expressing the components corresponding to higher harmonies in terms of the components for smaller \(k\). In this paper we shall discuss two approaches.

2. EXTRAPOLATION AND SELF-CONSISTENCY

Let us suppose that we have an infinite linear system of the form

\[(2.1) \quad \frac{du_k}{dt} = \sum_{\ell=1}^{\infty} a_{k\ell} u_\ell, \quad u_k(0) = c_k, \quad k = 1, 2, \ldots \]

For a discussion of existence and uniqueness of solutions to equations of this type, and an examination of when the sections

\[(2.2) \quad \frac{du_k}{dt} = \sum_{\ell=1}^{N} a_{k\ell} u_\ell, \quad u_k(0) = c_k, \quad k = 1, 2, \ldots, N,\]

yield solutions which converge to the solutions of (2.1), see [5].

Instead of using the closure technique implicit in (2.2), namely, \(u_k = 0, \ k \geq N + 1\), we wish to approximate to the remainder by means of linear combinations of the preceding \(u_k\).
see [1], [2], [3].

The coefficients $b_{kl}$ are to be chosen so that

$$
(2.4) \quad \int_0^T \left[ \sum_{l \geq N+1} a_{kl} u_l - \sum_{l=1}^N b_{kl} u_l \right]^2 dt
$$

is a minimum. We obtain in this way the linear algebraic equations for the $b_{kl}$

$$
(2.5) \quad \sum_{l \geq N+1} a_{kl} \int_0^T u_l u_r dt = \sum_{l=1}^N b_{rl} \int_0^T u_l u_r dt,
$$

$$r = 1, 2, \ldots, N.
$$

The usual difficulty now confronts us. How do we compute the integrals $\int_0^T u_l u_r dt$ involving the unknown solution?

Consider first the case where $1 \leq r, l \leq N$. To obtain these integrals, we use the finite system in (2.2). Call the solutions $u_l^{(o)}$. The coefficients of $b_{rl}$ in (2.5) are then $\int_0^T u_l^{(o)} u_r^{(o)} dt$. Observe that these quantities can be computed directly in the course of obtaining the $u_k^{(o)}$ by adjoining to (2.2) the equations

$$
(2.6) \quad \frac{dw_{rl}}{dt} = u_r u_l, \quad w_{rl}^{(o)} = 0,
$$

and asking only for the values $w_{rl}(T)$. 
The more difficult problem is that of the calculation of \( \int_0^T u_r u_l \, dt \) for \( r = 1, 2, \ldots, w, \ l \geq N + 1 \). Here we use extrapolation techniques. Keep \( r \) fixed and let \( l \) vary over the integers, \( l = 1, 2, \ldots \). It is reasonable to suspect that \( f_{l,r} = \int_0^T u_r u_l \, dt \) will be a well-behaved sequence. Consequently, if we possess the values \( f_{1,r}, f_{2,r}, \ldots, f_{N,r} \), we can use any of a number of extrapolation techniques (see [6]) to obtain the values of \( f_{l,r} \) for \( l \geq N + 1 \).

The system in (2.5) then has the form

\[
(2.7) \quad M \geq l \geq N + 1 \quad a_{kl} f_{l,r} = \sum_{\ell=1}^{N} b_{r\ell} w_{r\ell}(T).
\]

Here \( M \) is a cut-off number, such as \( 2N \), which depends upon the size of the coefficients \( a_{kl} \) and the rapidity of convergence of the infinite series.

Solving (2.7) numerically, we obtain the coefficients \( b_{r\ell}^{(o)} \). We use the superscript to indicate the fact that these are the first approximations. To obtain higher approximations, we use self-consistency techniques.

In place of (2.2), let us now use

\[
(2.8) \quad \frac{du_k}{dt} = \sum_{\ell=1}^{N} a_{kl} u_l + \sum_{\ell=1}^{N} b_{k\ell}^{(o)} u_l, \quad u_k^{(o)} = c_k.
\]

Call the solutions of this equation \( u_k^{(1)}, k = 1, 2, \ldots, N \). We now proceed as before to calculate \( w_{r\ell}^{(1)}(T), f_{l,r}^{(1)} \) and
\( b_{kl}^{(1)} \). With the new coefficients \( b_{kl}^{(1)} \), we introduce the equation

\[
\frac{du_k}{dt} = \sum_{l=1}^{N} a_{kl} u_l + \sum_{l=1}^{N} b_{kl}^{(1)} u_l, \quad u_k(0) = c_k.
\]

This process is repeated until the values of the \( b_{kl} \) settle down.

3. **NONLINEAR EQUATIONS**

In the case where the infinite system arises from a nonlinear equation such as (1.3), we can obtain a finite system in a simple fashion. Write

\[
u_N = \sum_{|k| \leq N} u_k(t)e^{ikx},
\]

replace \( u_x \) by \( u_N(u_N)_x \), and in this fashion obtain the higher harmonics in terms of the lower harmonics.
REFERENCES


