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SOLUTION OF A SECOND ORDER AND FIRST DEGREE ORDINARY LINEAR DIFFERENTIAL EQUATION OF THE TYPE $x^2\ y'' = (y';\ y;\ x)$.

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FOREWORD

This report was prepared by the Polymer Branch of the Nonmetallic Materials Division. This work was initiated under Project No. 7340, "Nonmetallic and Composite Materials," Task No. 734004, "New Organic and Inorganic Polymers." It was administered under the direction of the AF Materials Laboratory, Deputy Cmdr/Research & Engineering, Aeronautical Systems Division, with Dr. W. E. Gibbs acting as project engineer.

The work reported here concerns the solution of a differential equation encountered when treating analytically the thermodynamics and hydrodynamics of polymers in solution. The present work represents one aspect of the theoretical and experimental program conducted internally on this topic. This program has as its general objective the correlation of polymer structure with physical properties.

This report covers work accomplished from July 1962 to July 1963.

ABSTRACT

The following Second Order and First Degree Ordinary Linear Differential Equation has been solved:

$$x^2 y'' + x\left(\tfrac{1}{2}x^2 + 2t - 4u\right) y' + \left(x^2 + 2u - 2t + 1\right) y = \frac{(u+1)!\,t!}{(2t)!} x^{2u+2},$$

where $u = 0,1,2,\ldots\infty$ and $t = 0,1,2,\ldots u$.

The general solution can be given by the following expression:

$$y = z^u \left\{ \frac{u!\,t!}{(2t)!} + \frac{d^u}{dz^u} \left[z^{\frac{1}{2}+u-t} e^{-\frac{1}{4}z} \left(k_1 + k_2 \int z^{t-u-\frac{3}{2}} e^{\frac{1}{4}z} \, dz \right) \right] \right\},$$

where $z = x^2$, and k_1, k_2 are constants.

This technical documentary report has been reviewed and is approved.

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DIFFERENTIAL EQUATION AND ITS SOLUTION

The following set of differential equations (DE) was originated in a treatment of hydrodynamic interaction of large spheres in a viscous medium:

$$L(X) = X^2 y'' + X\left(\tfrac{1}{2}X^2 + 2t - 4u\right)y' + \left(X^2 + 2u + 1 - 2t\right)y = \frac{(u+1)!\,t!}{(2t)!}\,X^{2u+2} \tag{1}$$

where $u = 0,1,2,\ldots\infty$ and $t = 0,1,2,\ldots u$.

Introducing a variable z, where $x = z^{\frac{1}{2}}$,

$$y' = 2z^{\frac{1}{2}}\frac{dy}{dz} \quad \text{and} \quad y'' = 4z\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$$

, equation (1) becomes:

$$L(z) = 4z^2\frac{d^2y}{dz^2} + z\left[z + 4t - 8u + 2\right]\frac{dy}{dz} + \left[z + 2u(2u + 1 - 2t)\right]y = \frac{(u+1)!\,t!}{(2t)!}\,z^{u+1} \tag{2}$$

The function y can be transformed in accordance to the following relationship:

$$y = v + w, \quad \text{where} \quad w = \frac{u!\,t!}{(2t)!}\,z^u \tag{3}$$

Due to the linearity of the DE under consideration, the DE of y can be expressed as a sum of differential equations of v and w:

$$L(y(z)) = L(v(z)) + L(w(z)) = \frac{(u+1)!\,t!}{(2t)!}\,z^{u+1} \tag{4}$$

Since:

$$L(w(z)) = 4z^2\frac{d^2w}{dz^2} + z\left[z + 4t - 8u + 2\right]\frac{dw}{dz} + \left[z + 2u(2u - 2t + 1)\right]w\,, \tag{5}$$

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it follows that:

$$L(w(z)) = \Big[4u(u-1) + u(z+4t-8u+2) + z + 2u(2u+1-2t)w =$$

$$(u+1)zw = \frac{(u+1)!\,t!}{(2t)!} z^{u+1} \tag{6}$$

Therefore, the DE of v is homogeneous:

$$L(v(z)) = 4z^2 \frac{d^2v}{dz^2} + z\Big[z+4t-8u+2\Big]\frac{dv}{dz} + \Big[z+2u(2u-2t+1)\Big]v = 0 \tag{7}$$

DE (7) can be simplified utilizing the following transformation:

$$v = z^u \eta, \tag{8}$$

where η is a new function.

Since

$$\frac{dv}{dz} = z^u \frac{d\eta}{dz} + u z^{u-1} \eta \quad \text{and}$$

$$\frac{d^2v}{dz^2} = z^u \frac{d^2\eta}{dz^2} + 2uz^{u-1}\frac{d\eta}{dz} + u(u-1)z^{u-2}\eta ,$$

the DE of the function η will acquire the following form:

$$L(\eta(z)) = 4z\frac{d^2\eta}{dz^2} + (z+4t+2)\frac{d\eta}{dz} + (u+1)\eta = 0 \tag{9}$$

Introducing a variable $\xi = \frac{1}{4}z$, equation (9) becomes:

$$L(\eta(\xi)) = \xi\frac{d^2\eta}{d\xi^2} + (\xi + t + \tfrac{1}{2})\frac{d\eta}{d\xi} + (u+1)\eta = 0 \tag{10}$$

For any specified value of u and t, zero or positive, a constant

$k = (t + \frac{1}{2}) - (u+1) = t - u - \frac{1}{2}$ can be inserted into equation (10):

$$L(\eta(\xi)) = \xi \frac{d^2\eta}{d\xi^2} + \left[\xi + k + (u+1)\right] \frac{d\eta}{d\xi} + (u+1)\,\eta = 0 \qquad (11)$$

Differential Equation (11) is similar to the equations treated by A. R. Forsyth and W. Jacobsthal (Lehrbuch der Differentialgleichungen, Braunschweig 1912, p. 759) ref (1). (See also reference (2)). An analogous treatment of equation (11) will lead to a special solution. However, an additional different solution will also be found.

Supposing η_{u+1} is the solution of DE (11), then it is obvious:

$$L_{u+1}(\eta_{u+1}) = \xi \eta''_{u+1} + \left[\xi + k + (u+1)\right]\eta'_{u+1} + (u+1)\eta_{u+1} = 0 \qquad (11\text{-a})$$

The differentiation of DE (11-a) leads to a new equation:

$$\frac{dL_{u+1}}{d\xi} = \xi\eta'''_{u+1} + \left[\xi + k + (u+2)\right]\eta''_{u+1} + (u+2)\eta'_{u+1} = 0 \qquad (12)$$

Hence:

$$\frac{dL_{u+1}(\eta_{u+1})}{d\xi} = L_{u+2}(\eta_{u+2}); \quad \eta'_{u+1} = \eta_{u+2} \quad \text{and:}$$

$$L_{u+2}(\eta_{u+2}) = \xi\eta''_{u+2} + \left[\xi + k + (u+2)\right]\eta'_{u+2} + (u+2)\eta_{u+2} = 0 \qquad (12\text{-a})$$

Therefore, knowing any solution η_S of DE $L_S(\eta_S) = 0$ with integer $S<u+1$, one can find a corresponding function η_{u+1} by consecutive $(u+1)-S$ differentiations of η_S, and such η_{u+1} is a solution of the DE $L_{u+1}(\eta_{u+1}) = 0$.

Supposing now $u = 0$, the corresponding DE will be:

$$L_1(\eta_1) = \xi\eta_1'' + \left[\xi + k + 1\right]\eta_1' + \eta_1 = 0 \qquad (13)$$

The differentiation of this particular case will lead to the following relationship:

$$\frac{dL_1(\eta_1)}{d\xi} = \xi\eta_1''' + \left[\xi + k + 2\right]\eta_1'' + 2\eta_1' = L_2(\eta_2) =$$

$$= \xi\eta_2'' + \left[\xi + k + 2\right]\eta_2' + 2\eta_2 = 0, \tag{14}$$

where $\eta_2 = \eta_1'$.

It is sufficient, therefore, to solve DE (13). If the solution η_1 is known, the function η_{u+1} can be found by u differentiations of η_1.

$$\eta_{u+1} = \frac{d^u\eta_1}{d\xi^u} \tag{15}$$

It is important to mention the relationship obtained between the differential equations of type $L_n(\eta_n)$ will be valid even if n = O. Assuming u = -1, the corresponding DE can be written as follows:

$$L_0(\eta_0) = \xi\eta_0'' + \left[\xi + k\right]\eta_0' = \xi\eta_1' + \left[\xi + k\right]\eta_1 = 0, \tag{16}$$

$$\text{if } \eta_0' = \eta_1 .$$

To prove this relationship for u = -1 it is sufficient to differentiate the DE $L_0(\eta_0)$:

$$\frac{dL_0(\eta_0)}{d\xi} = \xi\eta_1'' + \left[\xi + k + 1\right]\eta_1' + \eta = L_1(\eta_1) = 0 . \tag{17}$$

Equation (16) can be easily solved. Since:

$$\frac{d\eta_1}{\eta_1} = -(1 + \frac{k}{\xi})\,d\xi, \tag{18}$$

the first solution of η_1, ϕ , will be:

$$\phi_1 = C_1 \xi^{-k} e^{-\xi} , \tag{19}$$

where C_1 is a constant.

To prove that ϕ_1 also solves DE (13) ϕ_1' and ϕ_1'' can be evaluated and inserted into equation (13).

From equation (19) one can obtain:

$$\phi_1' = -\left(\frac{k}{\xi} + 1\right)\phi_1 \quad \text{and} \quad \phi_1'' = \left[\frac{k(k+1)}{\xi^2} + \frac{2k}{\xi} + 1\right]\phi_1 . \tag{20}$$

Due to equation (13), if $\phi_1 \neq 0$ the resulting expression will be valid:

$$\left[\frac{k(k+1)}{\xi} + 2k + \xi\right] - \left(\frac{k}{\xi} + 1\right)(\xi + k + 1) + 1 =$$

$$= \frac{k(k+1)}{\xi} + 2k + \xi - k - \xi - \frac{k(k+1)}{\xi} - k - 1 + 1 = 0 . \tag{21}$$

As already mentioned ϕ_1 is not the only solution of equation (13). The second solution ϕ_2 can be obtained if C_1 of equation (19) will be expressed as a function of ξ :

$$\phi_2 = \xi^{-k} e^{-\xi} C(\xi) \tag{22}$$

Since:

$$\frac{d\phi_2}{d\xi} = -\left(\frac{k}{\xi} + 1\right)\phi_2 + \xi^{-k} e^{-\xi} \frac{dC}{d\xi} , \tag{22-a}$$

and:

$$\frac{d^2\phi_2}{d\xi^2} = \left[\frac{k(k+1)}{\xi^2} + \frac{2k}{\xi} + 1\right]\phi_2 - 2\left(\frac{k}{\xi} + 1\right)\xi^{-k} e^{-\xi} \frac{dC}{d\xi} + \xi^{-k} e^{-\xi} \frac{d^2C}{d\xi^2} , \tag{22-b}$$

equation (13) becomes the DE of $C(\xi)$:

$$\xi^{-k} e^{-\xi}\left[\xi C'' + (1-k-\xi) C'\right] = 0 \tag{23}$$

Assuming $\xi^{-k} e^{-\xi} C' \neq 0$ one can obtain from equation (23):

$$\frac{dC'}{C'} = \left(\frac{k-1}{\xi} + 1\right) d\xi \tag{24}$$

Hence:

$$\frac{dC}{d\xi} = \xi^{k-1} e^{\xi} \cdot C_2 \tag{25}$$

and:

$$C(\xi) = C_2 \int \xi^{k-1} e^{\xi} \, d\xi . \tag{26}$$

where C_2 is a constant.

Finally:

$$\phi_2 = C_2 \, \xi^{-k} e^{-\xi} \int \xi^{k-1} e^{\xi} \, d\xi . \tag{27}$$

Since no other solutions exist, the general expression for η_1, is a linear combination of ϕ_1 and ϕ_2 :

$$\eta_1 = \xi^{-k} e^{-\xi} \left(C_1 + C_2 \int \xi^{k-1} e^{\xi} \, d\xi \right). \tag{28}$$

Remembering $\eta_{u+1} = \frac{d^u}{d\xi^u} \eta_1$, the general expression for η_{u+1} is:

$$\eta_{u+1} = \frac{d^u}{d\xi^u}\left[\xi^{-k} e^{-\xi} \left(C_1 + C_2 \int \xi^{k-1} e^{\xi} \, d\xi \right)\right]. \tag{29}$$

As $k = t-u-\frac{1}{2}$ this solution can be expressed by equation (30):

$$\eta_{u+1} = \frac{d^u}{d\xi^u}\left[\xi^{\frac{1}{2}+u-t} e^{-\xi}\left(C_1 + C_2 \int \xi^{t-u-\frac{3}{2}} e^{\xi} d\xi\right)\right]. \tag{30}$$

Utilizing the variable $z = 4\xi$ equation (30) becomes:

$$\eta_{u+1} = \frac{d^u}{dz^u}\left[z^{\frac{1}{2}+u-t} e^{-\frac{z}{4}}\left(k_1 + k_2 \int z^{t-u-\frac{3}{2}} e^{\frac{z}{4}} dz\right)\right]. \tag{31}$$

where k_1 and k_2 are new constants.

Combining equations (31), (8) and (3), the final expression for y appearing in DE (2) is given by:

$$y = z^u\left\{\frac{u!\,t!}{(2t)!} + \frac{d^u}{dz^u}\left[z^{\frac{1}{2}+u-t} e^{-\frac{z}{4}}\left(k_1 + k_2 \int z^{t-u-\frac{3}{2}} e^{\frac{z}{4}} dz\right)\right]\right\}, \tag{32}$$

where $z = x^2$.

TRANSFORMATION OF THE FUNCTION ϕ_2

The expression for η_{u+1} given by equation (30) can be generally written as

$$\eta_{u+1} = \frac{d^u}{d\xi^u}\phi_1 + \frac{d^u}{d\xi^u}\phi_2 . \tag{33}$$

It can be shown that the power of ξ appearing in the integral of the second term of equation (33) must not necessarily include u and t.

Since $t \leq u$, then $0 \leq u-t=m$.

Substituting m into the expression for ϕ_2 one can obtain:

$$\phi_2 = C_2 \xi^{\frac{1}{2}+m} e^{-\xi} \int \xi^{-\frac{3}{2}-m} e^{\xi} d\xi . \tag{34}$$

The integral appearing in equation (34) can be transformed as follows:

$$\int \xi^{-\frac{1}{2}-(m+1)} e^{\xi} d\xi = -\frac{1}{(m+\frac{1}{2})} \int e^{\xi} d\left[\xi^{-(m+\frac{1}{2})}\right] = -\frac{1}{(m+\frac{1}{2})} e^{\xi} \xi^{-(m+\frac{1}{2})} +$$

$$+ \frac{1}{(m+\frac{1}{2})} \int \xi^{-(m+\frac{1}{2})} e^{\xi} d\xi . \qquad (35)$$

Repeating this procedure m times, the integral under the consideration becomes:

$$\int \xi^{-\frac{1}{2}-(m+1)} e^{\xi} d\xi = \xi^{-(\frac{1}{2}+m)} e^{\xi} \left[\sum_{s=0}^{m} \frac{\xi^{s}}{(m+\frac{1}{2})(m-\frac{1}{2})\ldots(m+\frac{1}{2}-S)} \right] +$$

$$+ \frac{1}{(\frac{1}{2})(\frac{3}{2})\ldots(\frac{2m+1}{2})} \int \xi^{-\frac{1}{2}} e^{\xi} d\xi . \qquad (36)$$

Utilizing equation (36) the function ϕ_2 can be expressed by the following formula:

$$\phi_2 = C_2 \left\{ \sum_{s=0}^{m} \frac{\xi^{s}}{(m+\frac{1}{2})\ldots(m+\frac{1}{2}-S)} + \frac{\xi^{\frac{1}{2}+m} e^{-\xi}}{(\frac{1}{2})(\frac{3}{2})\ldots(\frac{2m+1}{2})} \int \xi^{-\frac{1}{2}} e^{\xi} d\xi \right\}. \qquad (37)$$

Since $s \leq m$ and $m \leq u$, the integer s appearing in equation (37) is always smaller than u with the exception of the last term in which $s = m$ cannot be greater than u. Therefore, differentiating the polynomial of equation (37) u times, all terms with $s < m$ will vanish and the term with $s = m$ can be zero or constant.

Hence:

$$\frac{d^u \phi_2}{d\xi^u} = A + B \frac{d^u}{d\xi^u} \left[\xi^{\frac{1}{2}+m} e^{-\xi} \int \xi^{-\frac{1}{2}} e^{\xi} d\xi \right]. \qquad (38)$$

where A and B are constants and A = O if $m < u$.

If $m = u$, then

$$A = C_2 \frac{u!}{(\frac{1}{2})(\frac{3}{2})\ldots(\frac{2u+1}{2})} \tag{39}$$

Finally:

$$\eta_{u+1} = A + \frac{d^u}{d\xi^u}\left[\xi^{\frac{1}{2}+u-t} e^{-\xi}\left(C_1 + B\int \xi^{-\frac{1}{2}} e^{\xi}\, d\xi\right)\right] \tag{40}$$

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