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THE EFFECT OF THE SURFACE LAYER OF A STAR UPON BLACK BODY RADIATION EMERGING FROM THE INTERIOR

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Brandeis University
Waltham, Massachusetts

Contract No. AF19(604)7283

Project No. 8608

Task No. 860805

Scientific Report No. ~~6~~ 1

March 1, 1963

Prepared
for

GEOPHYSICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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THE EFFECT OF THE SURFACE LAYER OF A STAR UPON BLACK BODY RADIATION EMERGING FROM THE INTERIOR

Murray Turoff
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ABSTRACT: A form of Burmann's Theorem is used to obtain an expansion of the radiation intensity at the surface of a star as an expansion explicitly dependent on the black body function, absorption coefficient, density, and the first few derivatives of these functions with respect to depth. Conditions for the convergence of the series are established. The expansion is modified for the case of organized outward motion of the surface layer and the effect of this motion on an absorption line is examined. Conditions for the shift of the line "center" to lower or higher frequencies due to the effects of the organized motion and the decay of temperature in the surface layer are also examined.

Introduction

In the outer layers of a star the marked decrease in the density and temperature leads to the formation of absorption lines. The actual shape of the absorption lines we observe is complicated considerably by such effects in the outer layers as pressure broadening, Doppler broadening, the presence of a temperature gradient, and organized motion. Although these effects can be taken into account through a suitable choice of the absorption coefficient and source function, the resulting integral expression for the radiation intensity is often difficult to treat analytically. In PART I of this paper we obtain a more convenient form of the radiation intensity by use of a lemma employing the property of marked decrease in the density which occurs in the outer layers. We also establish general conditions for the convergence of the resulting expansion. In PART II we treat a specific model which allows us to re-establish our conditions of convergence in terms of the commonly used scale-heights of density, temperature, and pressure.

In PART III we modify our original expression to take into account the presence of organized outward motion. The shift and shape of the line is examined in detail and an expression for the shift of the "center" of the line is derived. In PART IV the method employed to expand the radiation intensity is used to obtain a similar expansion for the radiation flux.

The lemma which is used several times in this approach is derived in the Appendix.

PART I: Expansion of the Radiative Intensity

Consider a stellar atmosphere under the following assumptions:

(1) The surface or outer layer of the star begins at x = 0 and extends to x = ∞. The density in this surface layer exhibits exponential decay as its dominant behavior, i.e.:

$$\rho(x) = \rho_0(x)\, e^{-x/\lambda_\rho} \tag{1}$$

where λ_ρ is essentially the scale height of the density and

$\frac{1}{\lambda_\rho} \gg \frac{1}{\rho_0}\frac{d\rho_0}{dx}$ for all x.

(2) The interior of the star is an approximate black-body; therefore the radiation emerging from the interior and incident on the outer layer at x = 0 is given by the Planck (black-body) function:[4,5,7]

$$B(T_0) = \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT_0} - 1} \tag{2}$$

where the temperature T_0 = T(x=0).

(3) The outer layer of the star is considered to be in approximate local thermodynamic equilibrium; however, the decrease in both temperature and density in this region of the star leads to the presence of absorption lines in the spectrum of the star.

Under these assumptions we may write the radiation intensity emerging from the star (plane-parallel atmosphere approximation) as:[1,2,3,4,5]

$$I^+(\infty) = B(T_0)\, e^{-\frac{\tau(0)}{\mu}} + \frac{1}{\mu}\int_0^\infty B(T(x'))\, K(x')\,\rho(x')\, e^{-\frac{\tau(x')}{\mu}}\, dx' \tag{3}$$

where

$$\tau(x) = \int_x^\infty K(x')\,\rho(x')\, dx' \tag{4}$$

is the optical depth of the surface layer and K(x) is the mass-absorption coefficient. The first term of (3) is the contribution to the luminosity from the interior. $\mu = \cos\theta$ where θ is the inclination to the outward normal. Both K and B are functions of the frequency (ν).

Because of the assumed exponential decay of the density, we may employ the relation (A11) derived in the Appendix to expand (4) and obtain:

$$\tau(x) = K(x)\,\rho(x)\,\lambda_1(x)\left[1 + \frac{\lambda_1^2(x)}{2}\left(\left(K_{xx} - K_x^2\right) + \left(\rho_{xx} - \rho_x^2\right)\right) + \ldots\right. \tag{5}$$

where for any function of x, i.e. A(x), we define

$$A_x = \frac{1}{A}\frac{dA}{dx} \qquad A_{xx} = \frac{1}{A}\frac{d^2A}{dx^2} \tag{6}$$

and

$$\lambda_1(x) = \frac{-1}{\rho_x + K_x} \tag{7}$$

The function $\lambda_1(x)$ has units of length and may be thought of as a 'quasi'-scale height.

We may also apply (A11) to (3) to obtain:

$$I^+(\infty) = \frac{B(T_0)\,e^{-\frac{\tau(0)}{\mu}}}{1 - \frac{K\rho\lambda_2}{\mu}}\left[1 + \frac{K\rho\lambda_2^3}{2\mu\left(1 - \frac{K\rho\lambda_2}{\mu}\right)^2}\left[\left(B_{xx} - B_x^2\right) + \left(K_{xx} - K_x^2\right) + \left(\rho_{xx} - \rho_x^2\right) - \frac{K\rho}{\mu\lambda_1}\right] + \ldots\right. \tag{8}$$

where

$$\lambda_2(x) = \frac{-1}{\rho_x + K_x + B_x} \tag{9}$$

and the right hand side of (8) is evaluated at x = 0. We note:

$$\lambda_2 = \frac{\lambda_1}{1 - \lambda_1 B_x} \tag{10}$$

If T(x) is a decreasing function of x as is expected in the photosphere and lower chromosphere, we note that:

$$\lambda_2 < \lambda_1 \qquad \text{as} \qquad B_x < 0 \tag{11}$$

Combining the expansions of I^+ and Υ, we obtain, in lowest order:

$$I^+(\infty) = \frac{B(\tau_0)\, e^{-\frac{K\rho\lambda_1}{\mu}}}{1 - \frac{K\rho\lambda_2}{\mu}} \tag{12}$$

The second order terms will be small if condition (11) holds and the following conditions are met:

$$\left(\frac{K\rho\lambda_1}{\mu}\right)^2 << 1 \tag{13}$$

$$\frac{1}{2}\lambda_1^2\left(K_{xx} - K_x^2\right) << 1 \tag{14}$$

$$\frac{1}{2}\lambda_1^2\left(\rho_{xx} - \rho_x^2\right) << 1 \tag{15}$$

$$\frac{1}{2}\lambda_2^2\left(B_{xx} - B_x^2\right) << 1 \tag{16}$$

PART II: Analysis of Convergence for a Specific Model

To examine the convergence of this expansion in more detail, we assume the absorption coefficient is the sum of a continuous part and a line part:[4,5]

$$K(x) = K^{c}(x) + K^{\ell}(x) \tag{17}$$

with

$$K^{c}(x) = k_c \left(\frac{\nu_c}{\nu}\right)^3 \left(1 - e^{-\beta}\right) \tag{18}$$

where $\beta = \dfrac{h\nu}{KT}$; ν_c and k_c are constants and the factor $(1 - e^{-\beta})$ is the correction for stimulated emission. For $K^{\ell}(x)$ we adopt the following forms which depend on the part of the line shape we are considering:[4]

$$K^{\ell_1}(x) = k_\ell \left(\frac{c}{V_T} e^{-\left(\frac{(\nu-\nu_0)c}{\nu_0 V_T}\right)^2}\right) \quad \text{for} \quad \left|\frac{\nu-\nu_0}{\nu_0}\right| < \frac{V_T}{c} \tag{19}$$

and

$$K^{\ell_2}(x) = \frac{k_\ell}{\nu_0\sqrt{\pi}} \left(\frac{\delta + \delta_p(x)}{\left(\frac{\nu-\nu_0}{\nu_0}\right)^2}\right) \quad \text{for} \quad \left|\frac{\nu-\nu_0}{\nu_0}\right| > \frac{V_T}{c} \tag{20}$$

where ν_0 is the central frequency, δ is the natural line width, c is the velocity of light, and k_ℓ is a constant. $\delta_p(x)$ is the pressure broadening width, and $V_T(x) = \left(\frac{2RT(x)}{M}\right)^{\frac{1}{2}}$ is the thermal velocity. Equations (19) and (20) give $K^{l}(x)$ for the center and the wings of the line respectively and are approximations to the more general convolution integral which is used to combine both the Doppler and Pressure Broadening effects.[4,5]

We further assume

$$\begin{aligned} T(x) &= T_0 \, e^{-x/\lambda_T} \\ \rho(x) &= \rho_0 \, e^{-x/\lambda_\rho} \\ \delta_p(x) &= \delta_p \, e^{-x/\lambda_\delta} \end{aligned} \tag{21}$$

where λ_T, λ_ρ, λ_δ are the associated scale heights for these quantities. While the assumption of a pure exponential behavior for the above quantities may only approximate a physically realizable situation, it does, however, allow us to obtain simplified conditions for establishing the validity of our approach in terms of the scale heights. We list a number of relations which will be of use:

$$-\frac{1}{T_x} = \lambda_T \quad -\frac{1}{\rho_x} = \lambda_\rho \quad -\frac{1}{\delta_{\rho x}} = \lambda_\delta \tag{22}$$

$$B_x = -\frac{\beta}{\lambda_T(1-e^{-\beta})} \tag{23}$$

$$K_x^c = \frac{1}{\lambda_T}\frac{\beta e^{-\beta}}{(1-e^{-\beta})} \tag{24}$$

$$K_x^{\ell_1} = \frac{1}{2\lambda_T}\left(1 - 2\left(\frac{\nu-\nu_0}{\nu_0}\frac{c}{V_T}\right)^2\right) \tag{25}$$

$$K_x^{\ell_2} = -\frac{1}{\lambda_\delta}\frac{\delta_\rho}{\delta+\delta_\rho} \tag{26}$$

Incorporating all these results, we obtain, after some algebra,

$$\lambda_1 = \frac{\lambda_\rho}{1+\lambda_\rho H(\nu)} \tag{27}$$

where

$$H(\nu) = \frac{1}{\lambda_T}\left(\frac{1}{K^{\ell_1}(\nu)+K^c(\nu)}\right)\left(K^{\ell_1}(\nu)\left(\left(\frac{\nu-\nu_0}{\nu_0}\frac{c}{V_T}\right)^2-\frac{1}{2}\right) - K^c(\nu)\frac{\beta e^{-\beta}}{1-e^{-\beta}}\right)$$

for

$$\left|\frac{\nu-\nu_0}{\nu_0}\right| < \frac{V_T}{c}$$

and

$$H(\nu) = \frac{1}{(K^{\ell_2}(\nu)+K^c(\nu))}\left(\frac{K^{\ell_2}(\nu)\ \delta_\rho}{\lambda_\delta\ (\delta+\delta_\rho)} - \frac{K^c(\nu)}{\lambda_T}\frac{e^{-\beta}\beta}{(1-e^{-\beta})}\right)$$

for

$$\left|\frac{\nu-\nu_0}{\nu_0}\right| > \frac{V_T}{c}$$

In order to satisfy (13), we wish $\lambda_\rho H(\nu)$ to be small when $H(\nu)$ is negative. A sufficient condition for this is that $(\lambda_\rho/\lambda_T)^2$ is small compared to one. We see that as either β increases from zero or $\left|\frac{\nu-\nu_0}{\nu_0}\right|$ increases from zero this condition is relaxed. We therefore require the following inequalities to be satisfied:

$$\left(K_\rho \lambda_\rho/\mu\right)^2 << 1 \tag{28}$$

$$\left(\frac{\lambda_\rho}{\lambda_T}\right)^2 << 1 \tag{29}$$

We now note

$$\lambda_2 = \frac{\lambda_1}{1 + \frac{\lambda_1}{\lambda_T}\left(\frac{\beta}{1-e^{-\beta}}\right)} < \lambda_1 \tag{30}$$

and

$$\frac{1}{2}\lambda_\rho^2\left(\rho_{xx} - \rho_x^2\right) = 0 \tag{31}$$

$$\frac{1}{2}\lambda_\rho^2\left(K_{xx}^c - K_x^{c\,2}\right) = \left(\frac{\lambda_\rho}{\lambda_T}\right)^2\left[\frac{e^\beta(\beta-\beta^2)-\beta}{(e^\beta-1)^2}\right] \tag{32}$$

$$\frac{1}{2}\lambda_2^2\left(B_{xx} - B_x^2\right) = \left(\frac{\lambda_1}{\lambda_T}\right)^2\left[\frac{\beta\left((\beta+1)e^{-\beta}-1\right)}{(1-e^{-\beta})^2\left(1+\left(\frac{\lambda_1}{\lambda_T}\right)\frac{\beta}{1-e^{-\beta}}\right)^2}\right] \tag{33}$$

$$\approx \frac{1}{2}\left(\frac{\lambda_\rho}{\lambda_T}\right)^2\frac{\beta}{\left(1+\frac{\lambda_\rho}{\lambda_T}\right)^2}, \quad \beta \text{ small}$$

$$\approx -\frac{1}{2}\left(\frac{\lambda_\rho}{\lambda_T}\right)^2\frac{\beta}{\left(1+\frac{\lambda_\rho}{\lambda_T}\beta\right)^2}, \quad \beta \text{ large}$$

$$\frac{1}{2}\lambda_\rho^2\left(K_{xx}^{\ell_1} - K_x^{\ell_1\,2}\right) = -\left(\frac{\lambda_\rho}{\lambda_T}\right)^2\frac{1}{4}\left(\frac{\nu-\nu_0}{\nu_0}\frac{c}{V_T}\right)^2\left(1+2\left(\frac{\nu-\nu_0}{\nu_0}\frac{c}{V_T}\right)^2\right) \tag{34}$$

$$\frac{1}{2}\lambda_\rho^2\left(K^{\ell_2}_{xx} - K^{\ell_2\,2}_{x}\right) = \frac{1}{2}\left(\frac{\lambda_\rho}{\lambda_\delta}\right)^2 \frac{\delta\,\delta_\rho}{(\delta+\delta_\rho)^2} \tag{35}$$

We see that condition (28) is sufficient to make the quantities (32) to (34) small. We also note that (15) is identically zero for a pure exponential decay of the density. We now require the following quantity is to be small compared to one:

$$\frac{1}{2}\left(\frac{\lambda_\rho}{\lambda_\delta}\right)^2 \frac{\delta\,\delta_\rho}{(\delta+\delta_\rho)^2} \tag{36}$$

It remains only to point out that:

$$\begin{aligned} \lambda_\rho^2\left(K_{xx} - K_x^2\right) &= \left(\frac{K^\ell\lambda_\rho}{K^\ell+K^c}\right)^2\left(K^\ell_{xx} - K_x^{\ell\,2}\right) + \left(\frac{K^c\lambda_\rho}{K^c+K^\ell}\right)^2\left(K^c_{xx} - K_x^{c\,2}\right) \\ &\quad + \frac{K_\ell K_c}{(K_\ell+K_c)^2}\lambda_\rho^2\left[K^\ell_{xx} + K^c_{xx} - 2K^\ell_x K^c_x\right] \end{aligned} \tag{37}$$

The first two terms are small because of (29) and (36); using this fact, we may write the last term as:

$$\approx \frac{\lambda_\rho^2 K_\ell K_c}{(K_\ell+K_c)^2}\left(K^\ell_x - K^c_x\right)^2$$

From (24), (25), and (26), we see this quantity is also small under conditions (29) and (36). Furthermore, it is also small when K^ℓ/K^c is either very small or very large compared to unity.

We may now summarize that sufficient, but not necessary, conditions for the expansion to converge for all ν are:

$$\begin{aligned} &(1) \quad \left(\frac{K_\rho\lambda_\rho}{\mu}\right)^2 \ll 1 \\ &(2) \quad \left(\frac{\lambda_\rho}{\lambda_T}\right)^2 \ll 1 \\ &(3) \quad \frac{1}{2}\left(\frac{\lambda_\rho}{\lambda_\delta}\right)^2 \frac{\delta\,\delta_\rho}{(\delta+\delta_\rho)^2} \ll 1 \end{aligned} \tag{38}$$

The first condition tends to restrict the result to the forward direction of the radiation ($\mu = 1$); otherwise it is only a statement that the outer layer of the star is optically thin to the black-body radiation from the interior. The second statement requires the density to decrease faster than the temperature in the outer layer, which is to be expected for most stellar photospheres. The third condition certainly holds when either one of the quantities δ (natural line width) or δ_ρ (pressure broadening line width) greatly exceeds the other. Because of the second condition, we can expect that $\lambda_\rho \gtrsim \lambda_\delta$ and that the greatest value expression (3) can therefore obtain is approximately 1/8.

Therefore, to a good approximation, under conditions (38) we may use equation (12) for $I^+(\infty)$. Expanding equation (12) for $\mu = 1$, we have:

$$I^+(\infty) \cong B(T_0)\left(1 - K\rho(\lambda_1 - \lambda_2)\right) \tag{39}$$

We may also define a deviation, r, from the black-body distribution:

$$r = \frac{I^+(\infty) - B(T_0)}{B(T_0)} = -K\rho(\lambda_1 - \lambda_2)$$

$$= \frac{K\rho(-B_x)}{(\rho_x + K_x)(\rho_x + K_x + B_x)} = K\rho(-B_x)\lambda_1\lambda_2 \tag{40}$$

using relations (7) and (9) for λ_1 and λ_2.

PART III: Effect of Organized Motion

We now wish to modify the original expansion for I^+ (8) to take into account organized outward motion of matter in the surface layer. The only change necessary is to replace $B(x,\nu)$ and $K(x,\nu)$ in (8) with $B_D(x, \nu(x))$ and $K_D(x, \nu(x))$ where

$$\nu(x) = \nu\left(1 + \frac{\nu}{c}\, u(x)\right)$$

and (41)

$$u(x) = u_\infty f(x) \qquad f(0) = 0 \qquad f(\infty) = 1 \qquad \frac{df}{dx} \geq 0$$

u(x) is the velocity of the organized outward motion and we define the boundary of the surface layer (x = 0) as the point where u = 0 and $\frac{du}{dx} > 0$.

For the steady state, the conservation of current implies ρu = constant; this can be made consistent with the assumptions on the density variation over any finite interval by choosing u large enough. In fact, since all physical effects take place in the first few scale heights, violations of ρu = constant are not important at distances greater than a few times λ_ρ.

We may expand B_D and K_D to obtain:

$$B_D(x,\nu(x)) = B(x,\nu) + \frac{\nu}{c}\, u(x)\, \nu \frac{dB}{d\nu} + \ldots$$

$$K_D(x,\nu(x)) = K(x,\nu) + \frac{\nu}{c}\, u(x)\, \nu \frac{dK}{d\nu} + \ldots \qquad (42)$$

From the boundary conditions we have established in (41), we note:

$$B_D(0, \nu(0)) = B(0,\nu)$$

$$K_D(0, \nu(0)) = K(0,\nu) \qquad (43)$$

Assuming $\frac{u}{c} << 1$, we note that in equation (8) for I^+ we need only modify the first order terms; a modification of the second order terms for the Doppler effect would give third order corrections. We further see that only the derivatives of B and K are modified which in turn implies that only λ_1 and λ_2 are modified. Defining:

$$B_\nu = \frac{\nu}{B}\frac{\partial B}{\partial \nu} \qquad K_\nu = \frac{\nu}{K}\frac{\partial K}{\partial \nu} \qquad \varepsilon = \frac{1}{c}\frac{\partial u}{\partial x} \tag{44}$$

we have

$$\lambda_1^D = \frac{\lambda_1}{1 - \lambda_1 \mu \varepsilon K_\nu} \tag{45}$$

$$\lambda_2^D = \frac{\lambda_2}{1 - \lambda_2 \mu \varepsilon (K_\nu + B_\nu)} \tag{46}$$

We may then rewrite our first order expression for I^+ (12) as

$$I_D^+(\infty) = \frac{B(T_0)\, e^{-\frac{K\rho\lambda_1}{\mu(1-\lambda_1\mu\varepsilon K_\nu)}}}{1 - \frac{K\rho\lambda_2}{\mu(1-\lambda_2\mu\varepsilon(K_\nu+B_\nu))}} \tag{47}$$

The Doppler modification of the first order terms for I^+ [equation (12)] is of the same order of magnitude as the second order terms for I^+ found in equation (8), and omitted in our expression (47) for I_D^+. However, for the purpose of attempting to isolate the effect of the outward motion from the other second order terms, we will neglect this error at present. A quantity which is correct to the second order is the deviation (γ_D) of the Doppler shifted intensity (I_D^+) from the normal intensity (I^+). By taking equation (8) both in the normal form and modified to second order for the Doppler effect and expanding the exponentials and denominators, we obtain to second order:

$$r_D = \frac{I_D^+(\infty) - I^+(\infty)}{I^+(\infty)} = -K\rho\,\varepsilon\left(\lambda_1^2 k_\nu - \lambda_2^2 (K_\nu + B_\nu)\right) \tag{48}$$

If we assume that B and K^c are constant with respect to frequency over the width of the spectral line, we find:

$$r_D \cong 2\varepsilon k_\ell \left(\frac{c}{V_T}\right)^3 \rho\lambda_\rho \left(1 - \frac{1}{1-\lambda_\rho B_x}\right)\left[\frac{\nu}{\nu_0}\left(\frac{\nu-\nu_0}{\nu_0}\right) e^{-\left(\frac{\nu-\nu_0}{\nu_0}\frac{c}{V_T}\right)^2}\right] \tag{49}$$

at the center

$$r_D \propto \frac{\nu/\nu_0}{\left(\frac{\nu-\nu_0}{\nu_0}\right)^3} \tag{50}$$

on the wings. We also find for r itself, from equation (40):

$$r \cong r_1 + r_2 \tag{51}$$

where

$$r_1 \cong -K\rho\lambda_\rho\left(1 - \frac{1}{1-\lambda_\rho B_x}\right)$$

$$r_2 \cong -\lambda_\rho^2 \rho \frac{dK}{dx}\left(1 - \left(\frac{1}{1-\lambda_\rho B_x}\right)^2\right)$$

In the diagram we have indicated roughly the behavior of r_D, r_1, and r_2 over the line width. There is no scale indicated as this may vary greatly with the respect choice of many different parameters. Although we have assumed B constant over the line we see from (23) that the factor $\frac{1}{1-\lambda_\rho B_x}$ decreases with increasing frequency. This has the tendency to shift all the deviations various amounts to the blue as ν increases. This shift is partly due to the temperature gradient, since it arises partly from the term in B_x; we see that it acts opposite to the organized Doppler shift. The diagram is only meant to show that if the Doppler effect were dominant, i.e.

$$\frac{1}{c}\frac{dU}{dx} >> \frac{1}{\lambda_T}$$

the maximum of the line would be shifted to a lower frequency ($\nu < \nu_0$). If the decrease in temperature is dominant, i.e.

$$\frac{1}{\lambda_T} >> \frac{1}{c}\frac{dU}{dx}$$

we would expect a shift to the blue ($\nu > \nu_0$). It is possible, under the right conditions, to obtain a double peak in the absorption line.

In order to try and obtain an equation for the shift of the absorption line, we may expand (47) under the conditions (38) to obtain:

$$I_D^+ \cong B\left(1 - K\rho(\lambda_1 - \lambda_2) - K\rho\varepsilon\left(\lambda_1^2 K_\nu - \lambda_2^2(K_D + B_D)\right)\right)$$

for $\mu = 1$ (52)

We may also expand λ_1 and λ_2 under the same conditions to obtain

$$\lambda_1 \approx \lambda_\rho(1 + \lambda_\rho K_x)$$

$$\lambda_2 \approx \frac{\lambda_\rho}{1 - \lambda_\rho B_x}\left(1 + \frac{\lambda_\rho K_x}{1 - \lambda_\rho B_x}\right) \quad (53)$$

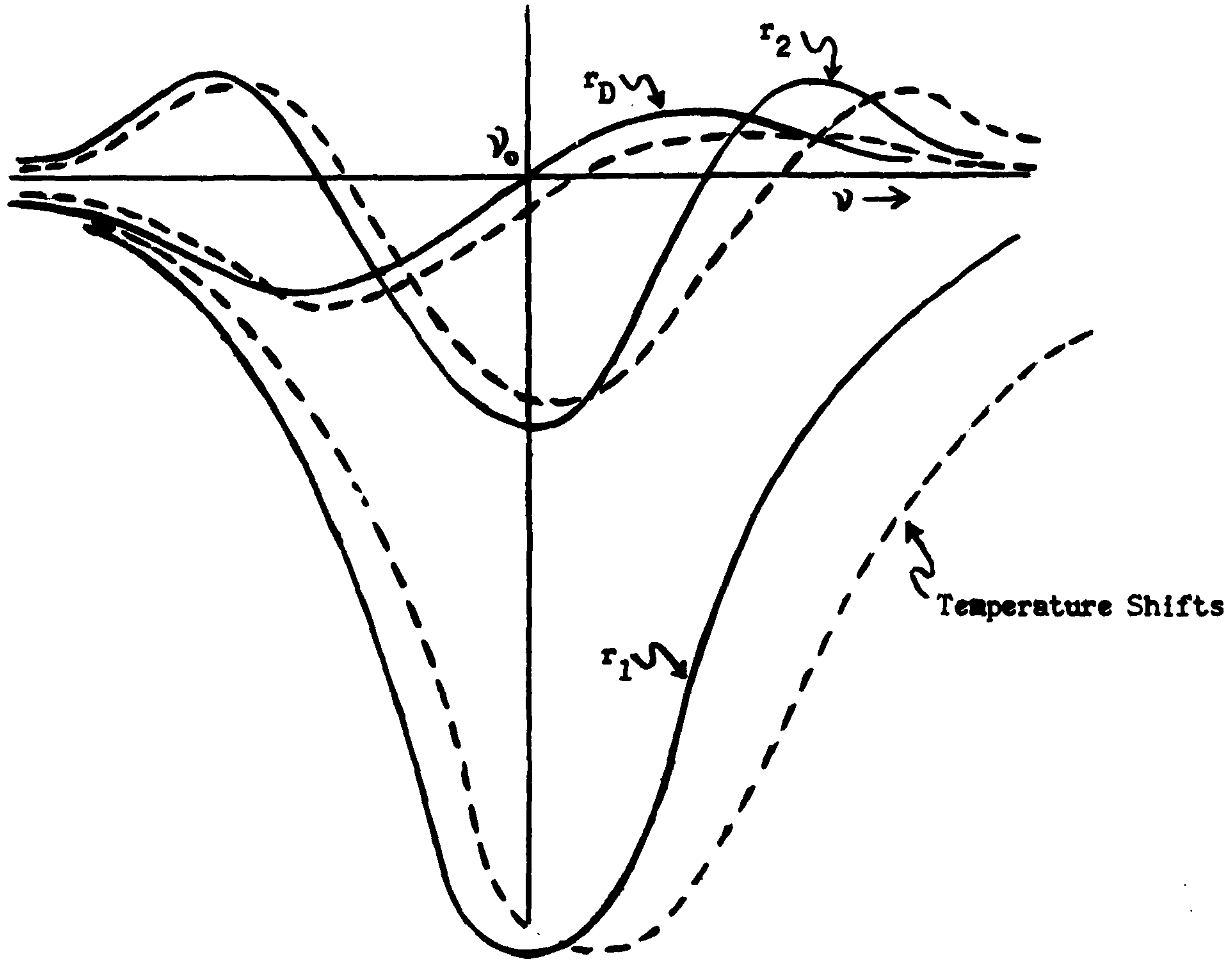
Using (23) and (18), we note that for $\beta \ll 1$ we have approximately:

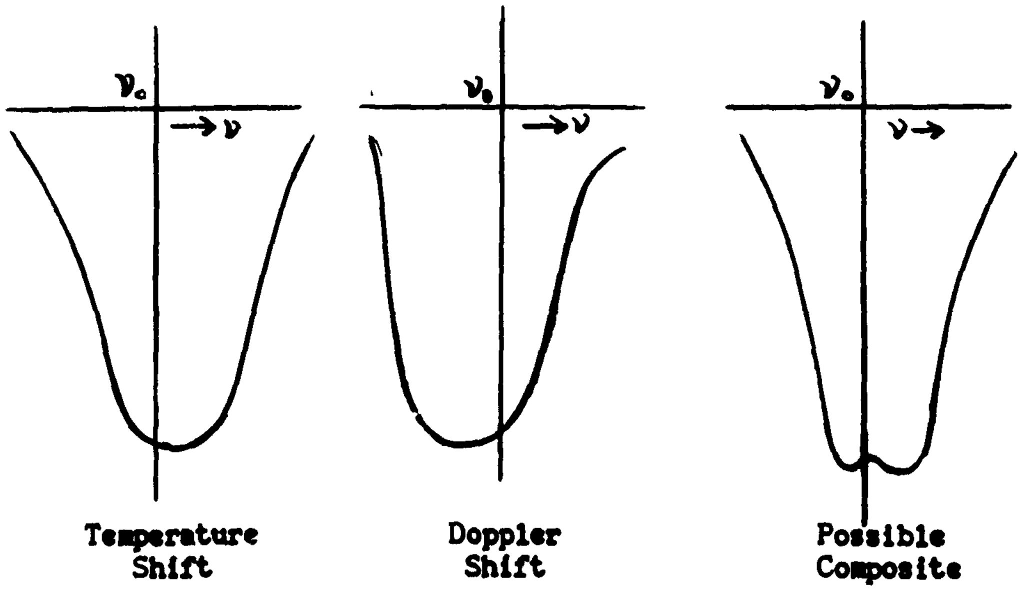
$$\frac{1}{1 - \lambda_\rho B_x} \approx 1 - \frac{\lambda_\rho}{\lambda_T} \qquad B_D \approx 2$$

$$K^c \approx \kappa_c\left(\frac{\nu_c}{\nu}\right)\beta \qquad K_\nu \cong -2 \quad (54)$$

Carrying only the first order term yields:

$$I_D^+ \approx B(1 - \gamma K) \quad (55)$$


r_2
r_D
ν_o
$\nu \rightarrow$
Temperature Shifts
r_1


ν_o
$\rightarrow \nu$
ν_o
$\rightarrow \nu$
ν_o
$\nu \rightarrow$
Temperature Shift
Doppler Shift
Possible Composite

where

$$\gamma = \rho \lambda_\rho \left(\frac{\lambda_\rho}{\lambda_T} - \frac{\lambda_\rho}{c} \frac{dU}{dX} \right) \tag{56}$$

Taking the derivative of (55) with respect to ν, we obtain the critical points in the line at the zeroes of this derivative:

$$B_\nu = \frac{\gamma \left(\nu \frac{dK}{d\nu} \right)}{1 - K\gamma} \tag{57}$$

For the center of the line we have, from equation (19)

$$K^\ell(\eta) \cong k_\ell \frac{c}{V_T} \left(1 - \eta^2 \right)$$

$$\frac{dK^\ell(\eta)}{d\nu} = -2 k_\ell \frac{c}{V_T} \left(\eta^2 + \eta \frac{c}{V_T} \right) \tag{58}$$

where

$$\eta = \left(\frac{\nu - \nu_0}{\nu_0} \frac{c}{V_T} \right) \tag{59}$$

when $\eta << \frac{c}{V_T}$ we find

$$\eta \approx \left(1 - \frac{V_T}{c \gamma k_\ell} \right) \frac{V_T}{c}$$

or

$$\nu_{CENTER} = \nu_0 \left(1 - \frac{k_\ell \rho \lambda_\rho \left(\frac{\lambda_\rho}{\lambda_T} - \frac{\lambda_\rho}{c} \frac{dU}{dX} \right) - \frac{V_T}{c}}{\left(\frac{c}{V_T} \right)^2 k_\ell \rho \lambda_\rho \left(\frac{\lambda_\rho}{\lambda_T} - \frac{\lambda_\rho}{c} \frac{dU}{dX} \right)} \right) \tag{60}$$

We see from (56) and (60) that under the condition:

$$\frac{\lambda_\rho}{c} \frac{dU}{dX} < \frac{\lambda_\rho}{\lambda_T} < \frac{\lambda_\rho}{c} \frac{dU}{dX} + \frac{V_T}{c k_\ell \rho \lambda_\rho} \tag{61}$$

a shift to increasing frequency will occur. If the condition is violated, we will have a shift to lower frequency. We note that since $k_{\ell}\rho\lambda_{\rho}$ and V_T/c are both small numbers under conditions (38) we would expect that $\dfrac{V_T}{c\,k_{\ell}\rho\lambda_{\rho}}$ is much larger than λ_{ρ}/λ_T or $\dfrac{\lambda_{\rho}}{c}\dfrac{dU}{dX}$. Therefore we may summarize that the following conditions hold:

$$\frac{1}{c}\frac{dU}{dX} < \frac{1}{\lambda_T} \qquad \text{for a shift to the blue}$$

$$\frac{1}{c}\frac{dU}{dX} > \frac{1}{\lambda_T} \qquad \text{for a shift to the red} \tag{62}$$

for β small.

These conditions reflect our earlier considerations, made on a more intuitive basis. The singularity when $\frac{1}{c}\frac{dU}{dX} = \frac{1}{\lambda_T}$ in (60) is spurious, for the expansion breaks down in that region.

For β large we find that $\lambda_2 \rightarrow 0$, which means that the second term on the right hand side of (52) will be an order of magnitude larger than the third term. Therefore we expect the Doppler effect to be much smaller for large β.

We note that K^{ℓ_2} (20) is proportional to $\left(\frac{\nu-\nu_0}{\nu_0}\right)^{-2}$ and its derivative proportional to $\left(\frac{\nu-\nu_0}{\nu_0}\right)^{-3}$; therefore, we expect that a similar analysis, using the pressure broadening coefficient of absorption, leads to a third degree equation for $\frac{dI_p^+}{d\nu} = 0$ in the first approximation. We therefore expect that the effect of pressure broadening increases the likelihood of having a double peak in the absorption line. This is due to the extension of the wings of the line by the pressure broadening. However, equation (55) is not accurate enough to examine this possibility.

PART IV: Expression for the Radiative Flux

To illustrate further the usefulness of the approach used here, we derive a simple expression for the flux at the surface of the star. Recalling:[1,2,3,4]

$$F_\nu^+(\infty) = 2\int_0^1 I_\nu^+(\infty,\mu)\,\mu\,d\mu \tag{63}$$

where $F_\nu^+(\infty)$ is the net outward flux, we find that

$$F_\nu^+(\infty) = 2\Big(B(T_0)E_3(\tau(0))\Big) + \int_0^\infty B(x')K(x')\rho(x')E_2\big(\tau(x')\big)\,dx'$$

where
$$E_N(t) = \int_1^\infty \frac{e^{-yt}}{y^N}\,dy$$

Employing the lemma derived in the Appendix plus the expansion for $E_n(x)$ also stated in the Appendix, we find:

$$F_\nu^+ = \frac{2B(T_0)e^{-K\rho\lambda_1}}{2+K\rho\lambda_1}\left[1 + 2K\rho\lambda_2 - \frac{1}{8}K\rho\lambda_1 + \ldots\right. \tag{65}$$

where the next terms in the series are of order $(K\rho\lambda_1)^2$, $(K\rho\lambda_2)^2$, and $(K\rho\lambda_1)(K\rho\lambda_2)$. Equation (65) is a relatively simple expression for the flux, which may be examined in much the same manner as we have treated the intensity of radiation.

Summary

The method discussed here treats the effect of the surface layer of the star on the black-body radiation emerging from the interior. If the density in the surface layer falls of rapidly, the expansion obtained for the intensity of the radiation converges. The advantage of the expansion is that it involves no integrals and does not seriously limit the form of the absorption coefficient. The equation for the intensity (8) may therefore be applied to the particular model or situation one wishes to treat. Furthermore, the results of the expansion are expressed as a function of 'x' rather than the optical depth. The equation for the intensity or the flux as a function of 'x' is perhaps more applicable to problems governing the general dynamics of the surface layer of the star where the radiative problem is only one part.

The results we have obtained for the shift of the absorption line in the case of organized motion is, of course, limited both in accuracy and the range of applicability. However, accurate results for particular lines in particular model atmospheres can be obtained quite easily by using our basic equation (8), modified by (45) and (46) for the Doppler correction, and a good computer.

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APPENDIX: A Special Form of Burmann's Theorem[6]

We wish to derive a form of Burmann's theorem which is useful in problems of the type considered in this paper.

Given an integral function of x:

$$\int_{x_0}^{x} g(x')\,dx'$$

where $g(x')$ is a monotone function from the point x_0 to the point $x > x_0$, the following relation may be derived:

$$\int_{x_0}^{x} g(x')\,dx' = \sum_{i=1}^{\infty} a_i(x_0)\left(Q(x) - Q(x_0)\right)^i \qquad \text{(A1)}$$

where $Q(x)$ is an arbitrary monotone function in the interval x_0 to x and the $a_i(x_0)$ may be expressed in terms of $Q(x)$, $g(x)$, and their derivatives, evaluated at x_0. Defining:

$$\mu(x) = \frac{1}{\frac{dQ}{dx}} \qquad \text{(A2)}$$

and taking the derivative of both sides of (A1) with respect to x, we have:

$$\mu(x)\,g(x) = \sum_{i=1}^{\infty} a_i(x_0)\frac{i!}{(i-1)!}\left(Q(x) - Q(x_0)\right)^{i-1} \qquad \text{(A3)}$$

Letting $x \to x_0$ in (A3), we obtain:

$$a_1(x_0) = \mu(x_0)\,g(x_0) \qquad \text{(A4)}$$

Now define

$$a_k(x) = \frac{1}{k!}\sum_{i=k}^{\infty} a_i(x_0)\frac{i!}{(i-k)!}\left(Q(x) - Q(x_0)\right)^{i-k} \qquad \text{(A5)}$$

Note the identity as $x \to x_0$. We take the derivative of both sides of (A5) with respect to x; using (A2) we obtain

$$a_{k+1}(x) = \frac{\mu(x)}{k+1}\frac{da_k}{dx}$$

$$a_1(x) = \mu(x) g(x) \tag{A6}$$

or equivalently

$$a_{k+1}(x) = \frac{1}{(k+1)!}\left(\mu(x)\frac{d}{dx}\right)^k \mu(x) g(x) \tag{A7}$$

The relations (A6) or (A7) provide a straightforward procedure for successively determining the a_i's. The only other stipulation necessary is that $a_i < \infty$ as $x \to x_0$ which in turn depends on the choice of Q(x).

If g(x) is exactly integrable a suitable choice of Q(x) is equivalent to doing the integration, as the series will terminate after the required terms are obtained. A detailed discussion of the convergence properties of this type of series may be found in reference (6).

As an example, consider the exponential integrals:

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^N}\,dt$$

Let

$$Q(t) = t\,g(t)$$

we obtain

$$E_n(x) = \frac{e^{-x}}{x+n-1}\left(1 - \frac{1}{2}\frac{x}{(x+n-1)^2} - \frac{x\left(x^2 + (2n-4)x + n(n-1)\right)}{6\,(x+n-1)^4} + \dots\right.$$

for n = 1 using

$$E_1(x) = \frac{e^{-x}}{x} - \frac{E_2(x)}{x}, \quad Q(t) = g(t)$$

we find

$$E_1(x) = \frac{e^{-x}}{x}\left(1 - \frac{1}{(x+2)} - \frac{1}{(x+2)^3} - \frac{(x^2+2x+6)}{6(x+2)^5} - \ldots\right.$$

The first three terms of these series expresses $E_n(x)$ to an accuracy of two places or better over the whole range of x from zero to infinity.

For our purposes we are interested in integrals of the type:

$$\int_x^\infty g(x')dx' \tag{A8}$$

where

$$\lim_{x\to\infty} g(x) = e^{-x/a} \tag{A9}$$

Defining

$$g_x = \frac{1}{g}\frac{dg}{dx} \qquad g_{xxx} = \frac{1}{g}\frac{d^3g}{dx^3}$$

$$g_{xx} = \frac{1}{g}\frac{d^2g}{dx^2} \qquad \lambda = \frac{1}{-\frac{1}{g}\frac{dg}{dx}} \tag{A10}$$

we have

$$\int_x^\infty g(x')dx' = g(x)\lambda(x)\Big[1 + \frac{\lambda^2}{2}\left(g_{xx} - g_x^2\right) +$$

$$\frac{\lambda^4}{6}\left(2g_{xx}(g_{xx} - g_x^2) + (g_{xx}^2 - g_x g_{xxx})\right) + \ldots\Big] \tag{A11}$$

In this case we chose Q(x) = g(x). We see that if g(x) uniquely equalled a decreasing exponential the expansion would terminate after the first term. Physically we may think of λ as a scale height. Even though g(x) may be a very complex function, if the dominant behavior was that of an exponential we expect the series to converge rather rapidly.

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