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TRANSLATION OF

THE CONDITIONALITY OF MATRICES

("Ob obuslovlennosti matrirts")

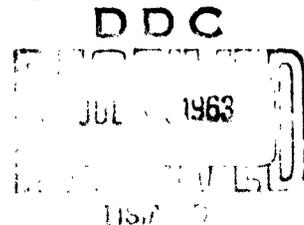
by

D. K. Faddeev

Matematicheskii institut Steklov, Trudy,  
No. 53: 387-391, 1959.

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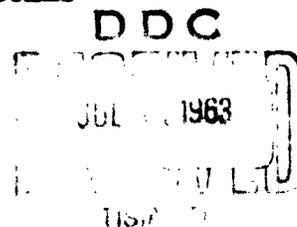
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## The Conditionality of Matrices

by

D. K. Faddeev

A non-singular matrix  $A$  is "well conditioned" if the solution of the system of linear equations  $Ax = b$  is stable, i.e., if it varies little with a small change in the elements of matrix  $A$  and in the column  $b$  of the free terms. Some quantitative features of conditionality are known in literature - the so-called condition numbers. A. Turing [1] proposed to use the  $N$ -number, equal to  $\frac{1}{n} N(A)N(A^{-1})$ , where  $N(A) = \sqrt{\text{Sp}A^*A}$ , as the degree of conditionality,  $A^*$  being a conjugate and transposed matrix from  $A$ , and  $n$  - the order of the matrix  $A$ . A stricter but easier computed number  $M$  equal to  $\frac{1}{n} M(A)M(A^{-1})$ , where  $M(A) = n \max |a_{ij}|$ , can be the degree of conditionality. It is known that  $N(A) < M(A)$  so that the condition number  $N$  does not exceed  $M$ . The least value of  $N$  is 1, and this is reached for the "most conditioned" single matrix. High values of the condition numbers characterizes a poorly conditioned matrix.

The  $N$ -number as well as the  $M$ -number are products of certain norms [ $N(A)$  and  $M(A)$ ] of the matrix  $A$  and its reciprocal, with the normalizing factor  $\frac{1}{n}$ . Starting from other norms we can construct several other condition numbers. The most natural is the  $H$ -number, equal to  $|A| |A^{-1}|$ , where  $|A| = \max_{|s|=1} |As| = \sqrt{\rho_1}$ ,  $\rho_1$  is the largest eigenvalue of the matrix  $A^*A$ .

It is easy to see that  $\|A^{-1}\| = \sqrt{\mu_n^{-1}}$ , where  $\mu_n$  is the smallest eigenvalue of the matrix  $A^*A$ . Thus the H-number of the conditionality is  $\sqrt{\frac{\mu_1}{\mu_n}}$ . For symmetrical matrices, it agrees with the P-number of Todd [2], which is equal to  $\frac{\max |\lambda_i|}{\min |\lambda_i|}$ , when  $\lambda_i$  - are eigenvalues of the matrix  $A$ .

We will explain the sense of the condition numbers  $N$  and  $H$  of the matrix  $A$ , by starting from the following roughly approximate probability scheme.

We will examine the effect of the change in the elements of matrix  $A$  on the change in the solution of the corresponding linear system  $Ax = b$ , where  $b$  is a vector, which will be considered accurate here.

Let us assume the elements of the matrix  $A$  are real independent random values with a standard deviation  $\sigma^2$  and average values  $a_{ij}$ .

Let  $(a_{ij})^{-1} = a_{ij}$ . Then

$$\frac{da_i}{da_{ij}} = -a_{ij}^2$$

and

$$dx_k = -\sum a_{ij}^2 da_{ij}$$

so that,  $dx_k$  can be considered linear combinations of the independent values of  $da_{ij}$ : Let us find the second moments for the vector

$$dx = \begin{pmatrix} dx_1 \\ \dots \\ dx_n \end{pmatrix}$$

i. e., let us compute  $\mathcal{E} \cdot dx_k dx_l$ . \*

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\*M. O. is Russian notation for the Western  $\mathcal{E}$ .

We have

$$\text{m. o. } dx_1 dx_2 = \text{m. o. } \sum_{ij} \sum_{pq} a_{ij} x_j da_{ij} x_i da_{pq}$$

Due to the independence of  $da_{ij}$  and  $da_{pq}$

$$\text{m. o. } da_{ij} da_{pq} = 0, \text{ if } i \neq p, j \neq q.$$

Furthermore, according to assumption

$$\text{m. o. } (da_{ij})^2 = \sigma^2,$$

therefore

$$\text{m. o. } dx_1 dx_2 = \sum_{ij} \sigma^2 a_{ij} x_j^2 = \sigma^2 \sum_{ij} a_{ij} x_j^2 = \sigma^2 M_{11} \sum_j x_j^2 = \sigma^2 M_{11} |s|^2,$$

where  $M_{kl} = \sum_j a_{kj} x_j$  is the element of the  $k$ th row and of column  $l$  of the matrix  $A^{-1} (A^{-1})^0 = (A^0 A)^{-1}$ . Thus, in particular,

$$\begin{aligned} \text{m. o. } |ds|^2 &= \text{m. o. } (dx_1^2 + \dots + dx_n^2) = \sigma^2 (M_{11} + \dots + M_{nn}) \times \\ &\times (x_1^2 + \dots + x_n^2) = \sigma^2 N^2(A^{-1}) |s|^2 = \frac{\sigma^2 \frac{1}{n} N^2(A) N^2(A^{-1}) |s|^2}{\frac{1}{n} N^2(A)} = \\ &= n^2(A) \frac{\sigma^2 |s|^2}{N^2(A)}, \end{aligned}$$

where  $n(A)$  is the N-number of conditionality. Extracting the root, we get

$$\sqrt{\frac{\text{m. o. } |ds|^2}{|s|^2}} = n(A) \frac{\sigma}{N(A)}.$$

By considering that  $\frac{1}{n} N(A)$  is the mean square value of the element of matrix  $A$ , and  $\sigma$  is the mean square of the errors in the elements of the matrix (equal to the mean square error of  $\sigma$  of each element of  $a_{ij}$ ), we find that the condition number  $N$  indicates how many times the ratio of the mean square error of the unknown to the mean square of the unknowns themselves exceeds the ratio of the mean square error of the coefficients of the system to the mean square of the coefficients themselves.

In [1], Turing gives (verbally) an estimate

$$\sqrt{\frac{\text{m. o. } |dx|^2}{|s|^2}} < N(A) N(A^{-1}) \frac{c}{\frac{1}{n} N(A)} = nN(A) \frac{c}{\frac{1}{n} N(A)},$$

where  $c^2 = \frac{1}{n} \sum_{ij} \sigma_{ij}^2$ ,  $\sigma_{ij}^2 = \text{m. o. } dx_{ij}^2$ , is valid without assuming the equality of all  $\sigma_{ij}$ .

Let us turn to the formula

$$\text{m. o. } dx_1 dx_2 = c^2 M_n |s|^2.$$

Assume the vector  $dx$ , for which the moments are given in the last formula, has the multivarying distribution density  $\frac{\Delta}{(2\pi)^n |s|^2} e^{-\frac{Q(t_1, \dots, t_n)}{2|s|^2}}$ , where  $\Delta$  is the absolute value of the determinant of matrix  $A$ ,

$Q(t_1, \dots, t_n)$  - the quadratic form with matrix  $A^*A$ .

Because of this, the condition number  $H$  of the matrix  $A$ , which is equal to  $\sqrt{\frac{\lambda_1}{\lambda_n}}$  gives the ratio between the major semi-axis of the ellipsoid of concentration  $Q(t_1, \dots, t_n) = c$  and its minor semi-axis; i. e., the  $H$ -number characterizes the lengthiness of the ellipsoid of concentration.

Therefore, for a poorly conditioned matrix, solution of the linear system is least specific toward the long semi-axes of the ellipsoid  $Q(x_1, \dots, x_n) = c$ , i. e., toward the eigenvectors of the matrix  $A^*A$  belonging to the small eigenvalues.

Let us note that when the free terms are also independent, random values having a uniform, small dispersion, the second moments of the components of the solution differ only by a multiplicative factor from the corresponding moments, which were calculated assuming constancy of the free terms.

Namely:

$$\text{n. o. } dx_i dx_i = (\sigma^2 |s_i^2 + \sigma_i^2) M_{ii}$$

where  $\sigma_i^2$  is the dispersion of each of the free terms of the system.

We now turn to an examination of the eigenvalues and eigenvectors. We will assume, as before, the matrix elements are random values with the mean values  $a_{ij}$  and uniform dispersions  $\sigma$ . The eigenvalues of the matrix  $(a_{ij})$  will be considered as natural and mutually different.

From the equality

$$A u_i = \lambda u_i$$

we conclude

$$(dA) u_i + A du_i = d\lambda u_i + \lambda du_i$$

By scalar multiplication of this equality by the eigenvector  $v_i$  of the transposed matrix, we will get

$$((dA) u_i, v_j) + (A du_i, v_j) = (d\lambda u_i, v_j) + \lambda (du_i, v_j)$$

whence

$$((dA) u_i, v_j) = d\lambda_i (u_i, v_j) + (\lambda_i - \lambda_j) (du_i, v_j)$$

Without disturbing the generality, we can consider  $(u_i, v_i) = 1$ ,

$$(u_i, v_j) = 0 \quad \text{when } i \neq j \quad \text{and} \quad (du_i, v_i) = 0.$$

When  $i = j$ , we get

$$d\lambda_i = ((dA) u_i, v_i) = \sum_{k,l} da_{kl} v_{ik} u_{il}$$

whence

$$d\lambda_i d\lambda_j = \sum_{k,l,m,n} a_{kl} a_{mn} v_{ik} v_{jn} da_{kl} da_{mn}$$

and

$$\text{n. o. } d\lambda_i d\lambda_j = \sigma^2 \sum_{k,l} a_{kl} a_{kl} v_{ik} v_{jk} = \sigma^2 (u_i, u_j) (v_i, v_j)$$

In particular,

$$\text{n. o. } d\lambda_i^2 = \sigma^2 (u_i, u_i) (v_i, v_i) = \frac{\sigma^2}{\cos^2(u_i, v_i)}.$$

The coefficient  $\frac{1}{\cos^2(u_i, v_i)}$  has a clear geometrical sense. It characterizes the deviation of the eigenvector  $u_i$  from the perpendicular  $v_i$  to the space spanned by the remaining eigenvalues. Therefore, the number  $\frac{1}{\cos^2(u_i, v_i)}$  can be called the bending coefficient of the eigenvector  $u_i$ .

Thus, the variance of the eigenvalue is  $\sigma^2$ , multiplied by the square of the bending coefficient.

If the matrix  $(a_{ij})$  is symmetric, the result is obtained quite simply i. e., n. o.  $d\lambda_i d\lambda_j = 0$  and n. o.  $d\lambda_i^2 = \sigma^2$  so that in this case, the eigenvalues will be independent random values with variance  $\sigma^2$ .

This conclusion is drawn under the unlikely assumption that all the matrix elements are independent as well as symmetric with respect to the main diagonal.

However, we can consider that the result is valid, when the matrix (and not only its average value) is considered symmetrical, when the variance of the diagonal elements is taken as  $\sigma^2$ , and that of the non-diagonal as  $\frac{1}{2} \sigma^2$ .

The law of concentration of the components of eigenvectors is, naturally, quite awkward, even in the examined simplified system. Without mentioning formulas, we note that the bending coefficient as well as the proximity of the eigenvalues are affected by the stability of the eigenvectors.

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