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TRANSLATION OF

THE CONDITIONALITY OF MATRICES

("Ob obuslovlennosti matrits")

by

D. K. Faddeev

Matematicheskii institut Steklov, Trudy,
No. 53: 387-391, 1959.

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The Conditionality of Matrices

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A non-singular matrix A is "well conditioned" if the solution of the system of linear equations $Ax = b$ is stable, i.e., if it varies little with a small change in the elements of matrix A and in the column b of the free terms. Some quantitative features of conditionality are known in literature - the so-called condition numbers. A. Tiuring [1] proposed to use the N-number, equal to $\frac{1}{n} N(A)N(A^{-1})$, where $N(A) = \sqrt{\mathrm{Sp}A^*A}$, as the degree of conditionality, A^* being a conjugate and transposed matrix from A, and n- the order of the matrix A. A stricter but easier computed number M equal to $\frac{1}{n} M(A)M(A^{-1})$, where $M(A) = n \max |a_{ij}|$, can be the degree of conditionality. It is known that $N(A) < M(A)$ so that the condition number N does not exceed M. The least value of N is 1, and this is reached for the "most conditioned" single matrix. High values of the condition numbers characterizes a poorly conditioned matrix.

The N-number as well as the M-number are products of certain norms [N(A) and M(A)] of the matrix A and its reciprocal, with the normalizing factor $\frac{1}{n}$. Starting from other norms we can construct several other condition numbers. The most natural is the H-number, equal to $\|A\|\,\|A^{-1}\|$, where $\|A\| = \max_{|x|=1} |Ax| = \sqrt{\mu_1}$, μ_1 is the largest eigenvalue of the matrix A^*A.

It is easy to see that $\|A^{-1}\| = \sqrt{\mu_n^{-1}}$, where μ_n is the smallest eigenvalue of the matrix A^*A. Thus the H-number of the conditionality is $\sqrt{\frac{\mu_1}{\mu_n}}$. For symmetrical matrices, it agrees with the P-number of Todd [2], which is equal to $\frac{\max|\lambda_i|}{\min|\lambda_i|}$, when λ_i -are eigenvalues of the matrix A.

We will explain the sense of the condition numbers N and H of the matrix A, by starting from the following roughly approximate probability scheme.

We will examine the effect of the change in the elements of matrix A on the change in the solution of the corresponding linear system Ax = b, where b is a vector, which will be considered accurate here.

Let us assume the elements of the matrix A are real independent random values with a standard deviation σ^2 and average values a_{ij}.

Let $(a_{ij})^{-1} = \alpha_{ij}$. Then

$$\frac{dx_k}{da_{ij}} = -\alpha_{ki}x_j$$

and

$$dx_k = -\sum \alpha_{ki}x_j da_{ij}$$

so that, dx_k can be considered linear combinations of the independent values of da_{ij}: Let us find the second moments for the vector

$$dx = \begin{pmatrix} dx_1 \\ \cdot\ \cdot \\ dx_n \end{pmatrix},$$

i.e., let us compute $\mathcal{E}\,.\ dx_k dx_l$. *

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*M. O. is Russian notation for the Western $\mathcal{E}$.

We have

$$\text{M. o. } dx_k dx_l = \text{M. o. } \sum_{ij}\sum_{pq} \alpha_{ki} x_j da_{ij} \alpha_{lp} x_q da_{pq}.$$

Due to the independence of da_{ij} and da_{pq}

$$\text{M. o. } da_{ij} da_{pq} = 0, \quad \text{if} \quad i \neq p,\ j \neq q.$$

Furthermore, according to assumption

$$\text{M. o. } (da_{ij})^2 = \sigma^2,$$

therefore

$$\text{M. o. } dx_k dx_l = \sum_{ij} \sigma^2 \alpha_{ki} \alpha_{li} x_j^2 = \sigma^2 \sum_i \alpha_{ki} \alpha_{li} \sum_j x_j^2 = \sigma^2 M_{kl} \sum_j x_j^2 = \sigma^2 M_{kl} |x|^2,$$

where $M_{kl} = \sum_i \alpha_{ki} \alpha_{li}$ is the element of the kth row and of column l of the matrix $A^{-1}(A^{-1})^* = (A^* A)^{-1}$. Thus, in particular.

$$\text{M. o. } |dx|^2 = \text{M. o. } (dx_1^2 + \cdots + dx_n^2) = \sigma^2 (M_{11} + \cdots + M_{nn}) \times$$
$$\times (x_1^2 + \cdots + x_n^2) = \sigma^2 N^2(A^{-1}) |x|^2 = \frac{\sigma^2 \frac{1}{n} N^2(A) N^2(A^{-1}) |x|^2}{\frac{1}{n} N^2(A)} =$$
$$= n^2(A) \frac{\sigma^2 |x|^2}{\frac{1}{n^2} N^2(A)},$$

where n (A) is the N-number of conditionality. Extracting the root, we get

$$\sqrt{\frac{\text{M. o. } |dx|^2}{|x|^2}} = n(A) \frac{\sigma}{\frac{1}{n} N(A)}.$$

By considering that $\frac{1}{n}$ N(A) is the mean square value of the element of matrix A, and σ is the mean square of the errors in the elements of the matrix (equal to the mean square error of σ of each element of a_{ij}), we find that the condition number N indicates how many times the ratio of the mean square error of the unknown to the mean square of the unknowns themselves exceeds the ratio of the mean square error of the coefficients of the system to the mean square of the coefficients themselves.

In [1], Tiuring gives (verbally) an estimate

$$\sqrt{\frac{\text{м. о. } |dx|^2}{|x|^2}} < N(A)\,N(A^{-1})\frac{\sigma}{\frac{1}{n}N(A)} = n\,n(A)\frac{\sigma}{\frac{1}{n}N(A)},$$

where $\sigma^2 = \frac{1}{n^2}\sum_{ij}\sigma_{ij}^2,\ \sigma_{ij}^2 = \text{м. о. } da_{ij}^2,$ is valid without assuming the equality of all σ_{ij}.

Let us turn to the formula

$$\text{м. о. } dx_k dx_l = \sigma^2 M_{kl}|x|^2.$$

Assume the vector dx, for which the moments are given in the last formula, has the multivarying distribution density $\frac{\Delta}{(2\pi\sigma^2|x|^2)^{\frac{n}{2}}} e^{-\frac{Q(t_1,\ldots,t_n)}{2\sigma^2|x|^2}}$, where Δ is the absolute value of the determinant of matrix A, $Q(t_1, \ldots, t_n)$ - the quadratic form with matrix A*A.

Because of this, the condition number H of the matrix A, which is equal to $\sqrt{\frac{\mu_1}{\mu_n}}$, gives the ratio between the major semi-axis of the ellipsoid of concentration $Q(t_1, \ldots, t_n) = c$ and its minor semi-axis; i.e., the H-number characterises the lengthiness of the ellipsoid of concentration.

Therefore, for a poorly conditioned matrix, solution of the linear system is least specific toward the long semi-axes of the ellipsoid $Q(x_1, \ldots, x_n) = c$, i.e., toward the eigenvectors of the matrix A*A belonging to the small eigenvalues.

Let us note that when the free terms are also independent, random values having a uniform, small dispersion, the second moments of the components of the solution differ only by a multiplicative factor from the corresponding moments, which were calculated assuming constancy of the free terms.

Namely:

$$\text{м. о. } dx_s dx_t = (\sigma^2 |x|^2 + \sigma_1^2) M_{st},$$

where σ_1^2 is the dispersion of each of the free terms of the system.

We now turn to an examination of the eigenvalues and eigenvectors. We will assume, as before, the matrix elements are random values with the mean values a_{ij} and uniform dispersions σ. The eigenvalues of the matrix (a_{ij}) will be considered as natural and mutually different.

From the equality

$$Au_i = \lambda_i u_i$$

we conclude

$$(dA)u_i + A\,du_i = d\lambda_i u_i + \lambda_i du_i.$$

By scalar multiplication of this equality by the eigenvector v_j of the transposed matrix, we will get

$$((dA)u_i, v_j) + (A(du_i), v_j) = (d\lambda_i u_i, v_j) + \lambda_i (du_i, v_j),$$

whence

$$((dA)u_i, v_j) = d\lambda_i (u_i, v_j) + (\lambda_i - \lambda_j)(du_i, v_j).$$

Without disturbing the generality, we can consider $(u_i\ v_i) = 1$,

$$(u_i, v_j) = 0 \quad \text{when} \quad i \neq j \quad \text{and} \quad (du_i, v_i) = 0.$$

When $i = j$, we get

$$d\lambda_i = ((dA)u_i, v_i) = \sum_{s,t} da_{st} v_{is} u_{it}$$

whence

$$d\lambda_i d\lambda_j = \sum_{s_1, t_1, s_2, t_2} u_{it_1} u_{jt_2} v_{is_1} v_{js_2} da_{s_1 t_1} da_{s_2 t_2}$$

and

$$\text{м. о. } d\lambda_i d\lambda_j = \sigma^2 \sum_{s,t} u_{it} u_{jt} v_{is} v_{js} = \sigma^2 (u_i, u_j)(v_i, v_j).$$

In particular,

$$\text{м. о. } d\lambda_i^2 = \sigma^2 (u_i, u_i)(v_i, v_i) = \frac{\sigma^2}{\cos^2(u_i, v_i)}.$$

The coefficient $\frac{1}{\cos^2(u_i, v_i)}$ has a clear geometrical sense. It characterizes the deviation of the eigenvector u_i from the perpendicular v_i to the space spanned by the remaining eigenvalues. Therefore, the number $\frac{1}{\cos(u_i, v_i)}$ can be called the bending coefficient of the eigenvector u_i.

Thus, the variance of the eigenvalue is σ^2, multiplied by the square of the bending coefficient.

If the matrix (a_{ij}) is symmetric, the result is obtained quite simply i. e., м. о. $d\lambda_i d\lambda_j = 0$ and м. о. $d\lambda_i^2 = \sigma^2$ so that in this case, the eigenvalues will be independent random values with variance σ^2.

This conclusion is drawn under the unlikely assumption that all the matrix elements are independent as well as symmetric with respect to the main diagonal.

However, we can consider that the result is valid, when the matrix (and not only its average value) is considered symmetrical, when the variance of the diagonal elements is taken as σ^2, and that of the non-diagonal as $\frac{1}{2}\sigma^2$.

The law of concentration of the components of eigenvectors is, naturally, quite awkward, even in the examined simplified system. Without mentioning formulas, we note that the bending coefficient as well as the proximity of the eigenvalues are affected by the stability of the eigenvectors.

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LITERATURE CITED

1. Tiuring, A. "Nakoplenie oshibki v matrichnykh protsessakh" (Accumalating errors in matrix processes), Uspekhi matematicheskikh nauk, 6, No. 1 (41): 138-162, 1951.

2. Todd, I. "The condition of certain matrices" Mechanics of Applied Mathematics, Quarterly Journal, 2: 469-472, 1949.

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