NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
INSTITUT DE PHYSIQUE THÉORIQUE

György I. Targonski,
Institut de Physique Théorique de l'Université de Genève

A Bibliography on Functional Equations.

First Version No. 7
Nov. 1958
Best Available Copy
A Bibliography on Functional Equations.

First Version

Annual Summary Report No. 1. - Contract No.
AF 61 (052) - 602 - 20.4. 1963.

György I. Targonski
Institut de Physique Théorique de l'Université de Genève

The research reported in this document has been sponsored by the OFFICE OF SCIENTIFIC RESEARCH, OAR, through the European Office, Aerospace Research, United States Air Force, under a personal contract with the author.
This is a bibliography on the literature of functional equations, up to approximately the middle of 1962. It consists of two parts: a list of papers and books up to the end of 1945, and a similar list from 1946 on, with comments on the contents.

This subdivision is due to the fact that the author was not able to devote more than a fraction of his time to this work. It is, averted that the first part shall be re-edited with comments to the second, and that the whole bibliography shall be streamlined into a single volume.

In order to compile such a bibliography, it was necessary first to define a Functional Equation. This has been done in the past in several ways, some of which appear too wide, others too narrow.

In the most general case, a Functional Equation is an equation which serves to determine one, or more unknown functions, or classes of such functions. In this sense, every differential, difference, and integral equation is also a Functional Equation; to compile a bibliography on such a wide class is almost certainly impossible and most certainly unnecessary.

Differential-difference and integral equations have been described in a very large number of excellent textbooks, easily accessible to everyone interested; they also have a vast literature of publications, growing at a speed difficult even to follow. Let us then describe as "functional equations proper" those functional equations which are neither differential, nor difference, nor integral equations, nor a mixture of these, nor do they contain all differential-, difference or integral operators. This is the present author's definition of a functional equation; a historical definition rather than an axiomatic one, and a definition he should be most eager to see replaced by a better one. It should be mentioned here that narrower definitions exist, which have an axiomatic character; the reader is referred to the remarks on Aczél's book, one of the very few ever to be published on functional equations. This narrower definition is well-founded, but one still feels that it is descriptive rather than essential, and the definition mentioned above rules out the famous Abel equation,

\[ f \left( \sum_{i} y_i \right) = f(x) + 1 \]  

the first functional equation to become known and probably the most important of them all. The reason for this exclusion is interesting. The current "axiomatic" definition mentioned above tends to rule out functional equations in which the number of variables is not higher than the number of variables in the unknown function. The reason is that such equations are rather difficult to solve, and their solution requires quite different methods. Let us, in contrast to (1), see a functional equation which is "admissible":

\[ f(x+y) = f \left( f(x), f(y) \right) \]
I.e. a so-called addition theorem for the unknown function $f(x)$. Here, the number of variables in the equation is 2, while the unknown function is of one variable. We have, so to speak, one degree of freedom we can utilize in many ways: we can put e.g. $x = y, y = 0, y = -x$ and obtain from (2) three equations:

\[(3)\]

\[
\begin{align*}
(a) & \quad f(2x) = F \left[ f(x), f(x) \right] \\
(b) & \quad f(x) = F \left[ f(x), f(0) \right] \\
(c) & \quad f(0) = F \left[ y, f(-x) \right] 
\end{align*}
\]

which can also be combined among themselves, e.g. (3) (b) and (3) (c); all of them are separate equations of the type of (1). It is no wonder that literature on the equations of the type of (2) is overwhelmingly larger than that of type (1) and that some definitions, as said, tend to rule out type (1); this latter, however, is the more fundamental and, in the long run, more important in any case, they are included in our "historical definition" and thus in this Bibliography. We excluded systems of functional equations, a limitation one has to admit to be a little arbitrary. Equally arbitrary was the way the line had to be drawn towards the fields of mathematics bordering on that of functional equations. Let us describe the way the line was drawn in each case.

The most close ties link the theory of functional equations to iteration theory; in fact, it is unnatural and even impossible to consider them as two different theories. Iteration methods are indispensable in solving equations of type (1); on the other hand, the solution of the problem of arbitrary iteration index, one of the most important problems of iteration theory, has to rely on the "translation equation"

\[(4)\]

\[F \left[ f(x, y), u \right] = f(x, u + v)\]

which, in its turn, leads to Abel's equation (1).

Still, in practice, it was not too difficult to find a dividing line; clearly practically all numerical approximation methods are iteration methods, the practical value of which has been immensely increased since digital electronic computers are available; there was no question of including any part of the vast literature of these methods. Iteration theory was included in those cases where functional equations are involved in an essential and explicit way.

Another field intimately related to functional equations is that of calculability and nomographability. To quote a simple example: in order to calculate a function $w = f(x, y, z)$ by a nomogram of points in alignment, one needs first a "simple" nomogram consisting of two curves, linking $x$ to $y$, and then a second such nomogram to link the result to $z$. A necessary condition for the nomographability is thus that solutions $g, h$ should exist to the functional equation

\[(5)\]

\[f(x, y, z) = g\left[ h(y, z), z \right]\]
Such topics have been included but only if the nomographic side is not predominant. It is planned to add to the final version of this bibliography a special list of works on what one could name "theoretical nomography".

As said, equations containing differential operators were excluded, so was the theory of geometrical objects, which forms a separate branch of mathematics. On the borderline between functional equations and algebra, the functional equations of distributivity, associativity etc., had to be included; in fact, the contributions of Abel to this subject were among the first results of the theory. Deeper-going investigations however, such as the theory of continuous groups, form a domain by themselves and had to be omitted. Similar consideration prompted us to leave out the theory of stochastic processes, primarily rooted in probability theory; one or two papers related to the common generalization of exponential and Poisson distributions were included due to their exclusive reliance on the appropriate functional equation.

Number theoretical functions, strongly multiplicative functions etc., were excluded on the ground that these are functions of positive integers only; again we are facing an established part of number theory. This is in line with our tendency to collect the "floating" material of the functional equations, not firmly or exclusively attached to any established branch of mathematics, and as one hopes, to be organized into one of the most interesting branches of our science.

Functional inequalities do not, as a matter of fact, enter this bibliography; due to their very general character, they determine properties of functions rather than functions or even "classes" of them.

Thus we have drawn the borderline separating the subject of this bibliography from other branches of mathematics; even this brief survey shows the central position of this discipline within mathematics—a view, one may add, perhaps correctly seen with the eyes of those interested mainly in functional equations.

The comment on the paper, given in part II, does not tend to describe in detail the contents; this should have multiplied by a factor of ten the size of this bibliography. The aim was not even to describe the main theorem, but rather to inform the reader about the problem which is solved, or treated in the paper in question, and, more often than not, to give the equation which is solved. This, it is hoped, shall be useful not so much to the mathematician, but to the physicist etc., facing a functional equation and trying to locate literature on it.

One last remark: this bibliography does not pretend to be complete; the author shall be grateful to colleagues pointing out errors or omissions, he shall equally welcome lists of publications, reprints, and other material which may render the final version of this bibliography more useful.
Some technical remarks.

The notation FE (s) stands for Functional Equation (s).

The title of a book is given in the language in which it is written, and an English translation is added. The title of a paper is always given in English, and an abbreviation points out the language in which it is written.

The abbreviations are the following:

| C | Czech     | I | Italian |
| D | Dutch     | J | Japanese |
| Da | Danish   | P | Polish   |
| E | English   | Po | Portuguese |
| Es | Esperanto | R | Romanian |
| F | French    | Ru | Russian |
| G | German    | S | Spanish  |
| H | Hungarian | Se | Serb, Croat |

The books and papers are given in the alphabetical order of the author, and, for one author, in the order of publication.

Within the alphabetical order, ö has been taken as oe, etc.

For languages using an alphabet other than the Latin, the usual phonetical transcription has been used, in case of doubt, the name figures in both forms, one with a reference to the other form, e.g. "Wilner, see Vilner".

In the case of papers written by several authors, the paper figures under the name of the author first in alphabetical order, in the rare case, however, where the alphabetical order was not followed in the title of the paper, the name figures under the name of the first author, irrespective of alphabetical order. The name of the other authors figures of course also in the list, with a reference to the first.

Some papers were added to the bibliography at the last moment and for these there is no comment, only the reference, they are denoted by an *.
Part 1

A list of publications until the end of the year 1945
Abel, N.H.

A general method to find a function of one variable if a property of this function is expressed by an equation in two variables (F).


Determination of a function by means of an equation which contains one variable only (F).

Oeuvres complètes II (1824) 35-39.

Investigation of functions z in two independent variables x and y such that $f(x, y)$ has the property that $f(z, f(x, y))$ is a symmetric function of z, x and y (G).


Investigation on the series $1 + a_1 x + \frac{m(m-1)}{2} x^2$. (C).

Oeuvres Complètes I (1826) 219-250.

Pseudo-homogeneous functions (R).

Rev. mat. Timisoara 3, no. 1 (1923) 3-4.

Pseudo-homogeneous functions and a new class of differential and partial differential equations (F).


The analytic solution of a functional system (F).


On two FEs (F).

Mathematica Cluj Timisoara 19 (1943) 23-25.

Alexiewicz, A. and Orlicz, W.

Remarks on the FE $(x+y)^2 = f(x) + f(y)$ (F).

Fundamenta math. 13 (1932) 314-315.

Alt, W.

On real functions of one real variable possessing a rational addition theorem (G).

Deutsche Math. 5 (1940) 1-12.

Amaldi, V.

(See Pincherle, S.)

Andrade, J.

On Poisson's FE (F).

Bull. Soc. math. France 28 (1900) 55-58

Andreoli, G.

On a simple and well-known FE (L)

Angelesco, A.

On a functional property of conics (F).

On a functional property common to the circle and the logarithmic spiral (F).

On a FE (R).

Angheluta, Th.

On a FE characterizing the polynomials (F).

On a FE (F).
C.R. Paris 194 (1932) 420-422.

On the integration of a FE (F).

On a FE defining polynomials in several variables (F).

On a FE (F).

Circular transformations characterized by a FE (R).

Appel, P.

Forming a function possessing the property $F[\phi(x)] = F(x)$. (F).
C.R. Acad. Sci. 88 (1879) 607-610.

On functions such that $F(\sin \frac{x}{2} \pm x) = F(x)$. (F).
C.R. Acad. Sci. 88 (1879) 1022-1024.

On linear differential equations the integrals of which satisfy relation of the form $F[\phi(x)] = \Psi(x) F(x)$. (F).

On linear differential equations which can be transformed into themselves by a change of function and variable (F).

On the integral $\int f(y) d f(x)$ where $x$ and $y$ are symmetrically related (F).
<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aumann, G.</td>
<td>Constructing mean values of several variables II (G).</td>
</tr>
<tr>
<td>Baer, R.</td>
<td>A Theory of Crossed Characters (E).</td>
</tr>
<tr>
<td></td>
<td>Trans. Amer. math. Soc. 54 (1943) 103-170.</td>
</tr>
<tr>
<td>Babbage, Ch.</td>
<td>Algebraical analysis of FEs (F).</td>
</tr>
<tr>
<td></td>
<td>Ann. de mat. pur appl. 12 (1821/22) 72-103.</td>
</tr>
<tr>
<td>Ballantine, J.P.</td>
<td>On a Certain Functional Condition (E).</td>
</tr>
<tr>
<td>Banach, S.</td>
<td>On the FE ( f(x + y) = f(x) + f(y) ). (F).</td>
</tr>
<tr>
<td></td>
<td>Fundamenta math. 1 (1920) 123-124.</td>
</tr>
<tr>
<td>Banach, S., Ruziewicz, S.</td>
<td>On the solutions of a FE by J. Cl. Maxwell (F).</td>
</tr>
<tr>
<td>Bettle, R.D.</td>
<td>On the Complete independence of Schimnack's Postulates for the Arithmetic Mean (E).</td>
</tr>
<tr>
<td>Behrbohm, H.</td>
<td>On the algebraicity of the meromorphisms of an elliptic function field (G).</td>
</tr>
<tr>
<td>Bell, E.T.</td>
<td>A Partial Isomorph of Trigonometry (E).</td>
</tr>
<tr>
<td></td>
<td>Algebraic Arithmetic (E).</td>
</tr>
<tr>
<td></td>
<td>Possible Types of Multiplication Sries (E).</td>
</tr>
<tr>
<td></td>
<td>FEs of Tentials (E).</td>
</tr>
<tr>
<td></td>
<td>Distributivity of Associative Polynomial Compositions (E).</td>
</tr>
<tr>
<td></td>
<td>A FE in Arithmetic (E).</td>
</tr>
<tr>
<td>Bemporad, G.</td>
<td>On the arithmetic mean (1).</td>
</tr>
<tr>
<td></td>
<td>R.C. Accad. Lincei (6) 3 (1926) 87-91.</td>
</tr>
<tr>
<td></td>
<td>The significance of the arithmetic mean (1).</td>
</tr>
<tr>
<td>Bernstein, B.A.</td>
<td>Postulates for Abelian Groups and Fields in Terms of Non-associative Operations (E).</td>
</tr>
</tbody>
</table>
Bieberbach, L. Remarks on the Thirteenth Problem of Hilbert (G).

Addendum to the paper Remarks on the Thirteenth Problem of Hilbert (G).

Blumberg, H. Non-measurable Functions Connected with Certain Functional Equations (E).

Bohnenblust, A. An axiomatic Characterization of $L_p$-Spaces (E).


Bourlet, C. On operations in general and on linear differential equations of infinite order (F).
Ann. sci. Ecole norm. sup. (3) 14 (1897) 133-150.

On certain equations analogous to differential equations (With a remark by P. Appel) (F).

On the problem of iteration (F).
Ann. Fac. Sci. Toulouse (1) 12 no. 3 (1898) 1-12.

Broggi, U. On the principle of arithmetic means (F).
Enseign. math 11 (1909) 14-17.

Brouwer, L.E.J. The theory of finite continuous groups independent of the Lie axioms (G).

Burstin, C. On a special class of real periodic functions (G).

Burstin, C., and Mayer, W. Distributive Groups (G).

Burstin, C. A contribution to the theory of functions of two variables (G).
Tôhoku math. J. 31 (1929) 300-311.

Caccioppoli, R. On the FE $f(x+y) = f(x) + f(y)$ (1).

The FE $F(x+y) = F[f(x), f(y)]$ (1).

Cantor, M. FE's with three independent variables (G).
Carmichael, R. D.  
On certain FEs (E).  

Della Casa, L.  
Relations of heterogeneous quantities (1).  

Cayley, A.  
On a Theorem of Abel. Note (F).  

On a FE.  
*Quart. J. pure appl. math.* 15 (1878) 319-325.  
Reprinted in *Coll. math. papers X, 278-306.*

Certone, J.  
The Ternary Operation $(abc) = ab^{-1}c$ of a Group (E).  

Chini, M.  
On a FE which gives rise to two remarkable formulae of 
induced methods (E).  

Cîrănescu, N.  
On the functional definition of polynomials and on some 
"two and three level formulas" (F).  

Some FEs characterizing the linear function (F).  

Colucci, A.  
On the FE $f(xy) = f(x) + f(y)$.  

van der Corput, J. G.  
Goniometric functions characterized by a FE (D).  
*Euclides* 17 (1940) 55-75.

Cremer, H.  
On Schröder's FE and the Schwarz problem of mapping "corners".  

Darboux, G.  
On the fundamental theorem of projective geometry (F).  

Deltheil, R.  
(see Borzl, E.)

Deslisle, A.  
Determination of the most general function satisfying the 
FE of the G function (G).  
Dickson, L. E.  
An extension of the Theory of Numbers by Means of Correspondences between Fields (E).  
Homogenous Polynomials with a Multiplication Theorem (E).  
Composition of polynomials (F).  
C.R. Paris 172 (1921) 636-640.  

Dienes, P.  
Reality and Mathematics (E).  
Budapest 1914.  

Dodd, E. L.  
The Chief Characteristic of Statistical Means (E).  
Colorado College Publ. 21 (1936) 89-92.  
Some Elementary Means and Their Properties (E).  
Colorado College Publ. 21 (1936) 35-59.  

Ermelowa, O. W.  
On the separation of variables in an equation of any number of variables (R).  
Ut chenie zapiski Mosk. Gos. Univ. nom. 28 (1939) 43-54.  

Falk, M.  
On the principal properties of analytic functions of one variable possessing addition theorems (G).  

Farkas, J.  
On iterative functions (F).  
J. de Math. 10 (1884) 101-106.  

Fatou, P.  
On the uniform solutions of certain FEs (F).  
On rational substitutions (F).  
On FEs and the Properties of certain boundaries (F).  
C.R. Acad. Sci. 166 (1918) 204-206.  
On FEs (second note) (F).  

Favre, A.  
On homogenous functions (F).  

de Finetti, B.  
On the notion of the mean (I).  

Fornenti, C.  
On problems of Abal (I).  
Reale Ist. Lomb. Rend. (2) 8 (1875) 276-282.
Franklin, P. Two FE's with Integral Arguments (E).

Frattini, G. (See Berel, E.)

Fréchet, M. A Functional Definition of Polynomials (F).

Every continuous functional can be developed into a series of functionals of integral order (F).

On the FE \( f(x+y) = f(x) + f(y) \). (Es).
Enseign. math. 15 (1913) 390-393.

On an article about the FE \( f(x+y) = f(x) + f(y) \). (F).
Enseign. math. 16 (1914) 136.

Nomographie

Fréchet, M. On the most general continuous solution of a FE in the theory of probability chains (F).

The most general continuous solution of a FE in the theory of probability chains (F).

The most general continuous solution of a FE in the theory of probability chains. Supplément (F).

Fridmen, A.A. On the question of the proof of the parallelogram of forces (G).

Galbura, G. On a certain functional equation (I).
Acta Pont. Acad. Sci. 5 (1941) 7-41.

Euclides 16 (1939) 92-99.

Ghermonescu, M. On some FE's of M.D. Pompeiu (F).

On a FE characterizing the polynomials (F).
Mathematica Cluj-Timisoara 19 (1943) 148-158.
(Germanese, W.A.)

On some FE s (F).

On a FE (F).

On a functional property common to circle and logarithmic spirals (R).
Gaz. mat. Buc. 50 (1945) 216-249.

On some extensions of the Cauchy (F).

Galab, S.

On homogeneous functions I. The equation of Euler (F).

On a FE in the theory of geometrical objects (G).
Wiadom. mat. 45 (1939) 97-137.

Grant, I.D.

Doubly Homogeneous Functional Equations (E).

Gravv, A.

Study on the FE s (F).

Gronwall, T.H.

On the equations in three variables which can be represented by point nomographs (F).
J. Math. pure appl. (6) 8 (1712) 59-112.

A FE in the Kinetic Theory of Gases (E).

Hadamard, J.

Two Works on Iteration and Related Questions (E).
Bull. Amer. math. Soc. 50 (1944) 67-75.

Halphen, G. H.

On certain series for the development of functions of one variable (F).
C.R. Acad. Sci. 23 (1881) 731-733.

On homogeneous functions (F).

Hamel, G.

A base of all numbers and the non-continuous solutions of the FE f(x+y) = f(x) + f(y) (G).

Hantzsche, W., and Wendt, H.

On the arithmetic, geometric and harmonic means (G).
Hartman, P., and Kershner, R.  
The Structure of Maccnical Functions (E).  

Haruki, H.  
On Certain Stnctoherent FE Concerning the  
Elliptic Functions (E).  

Haupt, O.  
On the Theory of the exponential function and the  
trigonometric functions (G).  

Hayashi, T.  
On a FE Treated by Abel (E).  
Tokyo sugaku-bunr. (1) 1911) 29-104.

On a FE Treated by Abel (E).  

On solution of FE: (E).  
Tôhoku math. J. 3 (1913) 42-63.

Hecke, E.  
A new kind of zeta functions and their relation to the  
distribution of prime numbers I (G).  

A new kind of zeta functions and their relation to the  
distribution of prime numbers II (G).  
Math. Z. 6 (1920) 11-21.

Herbrand, J.  
Investigation of bounded solutions of certain FE: (F).  

Herschfeld, A.  
On Bell's Functional Equations (E).  

Hille, E.  
A Pythagorean Functional Equation (E).  

A Class of FE: (E).  

Hestinsky, B.  
(FE: related to probability chains) (F)  
Paris 1939.

Huntington, E.V.  
Sets of Independent Postulates for the Arithmetic Mean,  
the Geometric Mean, the Harmonic Mean and the Root-  
Mean-Square (E).  

Hurwitz, W.A.  
Note on the Definition of an Abelian Group by  
Independent Postulates (E).  
Ann. Math. (2) 6 (1905) 24-96.
Ivolggi, E

On the notion of a complete function and of a periodic function (F).

On the substitutions in one variable and the functions invariant under it (F).

General properties of substitutions of one variable and of the functions invariant under them (F).

Ingraham, M. H.

Solutions of Certain FE's Related to a General Linear Set (E).

Ionescu, D. V.

Generalization of a FE of D. Pompeiu (F).

Some applications of certain FE's (F).
Mathematica Cluj Timisoara 19 (1943) 139-146.

Isenkerke, C.

(The method of iteration of functions).
Das Verfahren der Funktionenwiederholung (F).
Leipzig (1897).

Jacobstahl, E.

On the FE f(x+y) = f(x) * f(y).
Norske Vid. Selsk. forhandlinger 12 (1939) 74-75.

Jensen, J. L. W. V.

Solution of FE's with a minimum of conditions imposed (G).
Mat. Tidskr. (B) 8 (1897) 23-27.

On convexe functions and inequalities between means values (F).
Acta math. 30 (1906) 179-191.

Jessen, B.

On the generalization of the arithmetic mean (G).

Juckel, G.

Contribution to the theory of interest calculation I (H).
Mat. Fiz. Lapok 8 (1899) 283-284.

Julia, G.

Memoire on the interaction of rational functions (F).
J. Math. 8 1 (1918).

Kac, M.

A remark on FE's (F).

Kaczmarz, S.

On the FE f(x) + f(x+y) = F(y) f(x+y).
Fundamenta math. 6 (1924) 122-129.
Kellog, O.D. Nomograms with Points in Alignment (E).

Kershner, R. (See Hartman, P.).

Kirchhoff, B. (Lectures on Electricity and Magnetism.)
On the FE \( f(x) = \frac{ab}{c^2 - b^2 - cx} \) = \( A - B \frac{b}{a-x} \)
Vorlesungen über Elektrizität und Magnetismus (G).
Leipzig (1891) 66.

Kitagawa, T. On Some Class of Weighted Means (E).

Koebe, P. On those analytic functions of one variable which possess
an algebraic addition theorem (G).
Dissertation (Berlin 1905).

Koenigs, G. Investigations on FEs (F).
Ann. Fac. norm 3 1 (1924).
Investigation on uniform substitutions (F).
Bull. Sci. math. (2) 7 (1893) 243-257.

On the integrals of certain FEs (F).
C.R. Acad. Sci. 99 (1884) 1016-1017.

Investigations on the integrals of certain FEs (F).
Ann. Ec. Norm. (3) 1 Suppl. (1884) 1-41.
New Investigations on FEs (F).

Kolmogoroff, A.N. On the notion of a mean (F).

Analytic methods of the theory of probability (G).

Koopman, B.O. Hamiltonian systems and transformations in Hilbert space (E).

Karkin, A. On a problem of interpolation (F).
Karmes, M. On the Functional Equation \( f(2x) = f(x) + f(y) \). (E).

Krafft, M. Deduction of the trigonometric functions by means of their PEs (G).

Krull, W. General valuation theory (G).

Kürschék, J. Generalization of the notion of absolute value (H).

Lalou, V. On lines, formation and general theory of fields (G).

Lecornu, L. On the problem of collineatrices (F).
  C. R. Paris 122 (1896) 813-816.

Lecornu, L. On a PE (F).
  Bull. Soc. math. (2) 27 (1903) 37-44.

Lémeray, E. M. On the problem of iteration (F).
  C. R. Paris 125 (1897) 872-875.

On a new algorithm (F).
  C. R. Acad. Sci. 125 (1897) 524-525.

On the linear PEs (F).
  C. R. Acad. Sci. 125 (1897) 1160-1164.

On some general algorithms and on iteration (F).

On the PEs characterizing the associative and the distributive operations (F).

Levi-Civita, T. On functions admitting an addition formula of the type \( f(x+y) = \sum_{(i)} x_i(y) \).
  (G).

Lévy, P. (The Theory of Addition of random variables).
  Théorie de l'addition des variables aléatoires.
  Paris (1937).
Lindenbaum, A. On sets in which every equation of a given family has a number of solutions fixed in advance (F).
Fundamenta Math. 20 (1933) 1-29.

Loewy, A. Automatic foundation of the theory of interest (G).

Lattner, J. C. On functions satisfying the equation
\[ \phi(x) - \psi(y) = \psi \left[ \frac{f(y) \phi(x) + f(x) \psi(y)}{\phi(x) + \psi(y)} \right] \qquad \text{(G).} \]

Lunn, A. C. Some Foundations of Trigonometry (E).

Lyn van der, G. Abstract polynomials 1 (F).

On the equation \( f(x + y) = f(x) \cdot f(y) \).

Magnus, L. J. On relations of functions satisfying the equation
\[ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \frac{\partial h}{\partial z} \qquad \text{(G).} \]

Malchair, H. A theorem on functions of several variables (F).

Matsumara, S. On the exaltation of the mean (G).
Tohoku math. J. 36 (1933) 260-262.

Mayniew, R. On continuous functions of one real variable possessing an algebraical addition theorem (F).

Minetti, S. On the function \( f(x + y) = f(x) \cdot f(y) \).

Mineur, H. On the non continuous solutions of a class of FEs (F).
C. R. Paris 170 (1920) 793-796.

On functions admitting an algebraic addition theorem (F).
C. R. Paris 172 (1921) 1461-1463.
(Mineur, H.) On functions which admit an algebraic addition theorem (F).

On the analytic theory of continuous finite groups (F).

de Mira Fernandes, A. A multiplication theorem (P).
Portugaliae math. 1 (1940) 340-342.

Moisil, Gr.C. On a system of Feas (F).

Möllerup, J. A trigonometric-axiomatic investigation (Da).
Mat. Tidskr. (A) (1927) 24-33.

Montel, P. On the functions of one real variable admitting an algebraic addition theorem (F).

Moore, E.H. Concerning the definition by a system of functional properties of the function
\[ f(z) = \frac{\sin \frac{\pi z}{2}}{\frac{\pi}{2}} \]  
(\text{E}).
Ann. of math. 2 (1895) 43-49.


Myrberg, P.J. On systems of analytic functions possessing an addition theorem (G).
Preisschrift (Leipzig) (1922).

On systems of functions admitting an algebraic addition theorem (F).

Nagumo, M. On a class of means (G).
Japanese J. Math. 7 (1939) 71-79.

Some analytic investigations in linear metric rings (G).

Nakahara, I. Axioms for the Weighted Means (E).
Nakano, H. On a continuous matrix function (G).

Nanes, C. The equation of Cauchy genralized by
Ponette (F).
Combin. Acta 3 (1941) 141-144.

Narumi, S. Note on the Law of the Arithmetic Mean (E).

Nayar, V. Note on the function F: E → E (F).

Netto, E. Contributions to the theory of ordered functions (F).

Nikolaev, P.V. Linear models of ordered sets (F).

Mitsui, T. Multivariate functional transformations of
nonnegative data (F).

Nöther, E. The F2 class of normal curves (C).

d' Ocagne, M. On the equation y = x + 2.

Onicescu, O. Functions F: E → E (F).

D. On the F2 class of normal curves and multivariate (F).
Sv. mat. fys. Medd. 31 (1927) 6-10.

Orlicz, W. (See Musielak, W.)

Ostrowski, A. On some solutions of the FE
\( F(x) = F(y) = F(z) \) (G).
Acta math. 61 (1933-1934) 271-294.

Mathematical models of HIV on the FE of the
exponential function and related FE's (G).

Poincaré, P. On the function F: E → E (F).
Acta math. 27 (1943) 1-61.

Pepis, J. On a family of plane sets and the solutions of the FE
\( F(x, y) = F(y, z) + F(z, x) \) for \( x, y, z \in E \).
Application to the general theory of interest (F).
Perron, O.  
On addition and subtraction theorems (\( G \)).  

On a FE important in the theory of invariants (\( G \)).  
Math. Z. 48 (1943) 106-112.

A new construction of the non-Euclidean (hyperbolic)  
geometry (\( G \)).  

Petrini, H.  
On functions possessing an algebraic addition  
thereum (\( G \)).  

Pexider, H.W.  
Note on functional theorems (\( G \)).  

An application of a formula of Cauchy (\( F \)).  

Note on functional theorems (\( G \)).  

On symmetric functions of indeterminate variables (\( C \))  

Phragmén, E.  
On a theorem concerning elliptic functions (\( F \)).  
Acta math. 7 (1885) 33-42.

Picard, E.  
On a class of new transcendentals (\( F \)).  

On a class of new transcendentals (\( F \)).  
Acta math. 18 (1894) 133-154.

On a class of new transcendentals (\( F \)).  
Acta math. 19 (1900) 333-337.

Two lectures on certain FE's and on non-Euclidean  
geometry (\( F \)).  
Bull. Sci. math. (2) 46 (1922) 404-416.

(Lectures on some FE's with applications to various  
problems of analysis and mathematical physics).  
Leçons sur quelques équations fonctionnelles avec des  
applications à divers problèmes d'analyse et de physique  
mathématique (\( F \)).  

On certain FE's and a class of algebraic surfaces (\( F \)).  
Picard, E. On a FE (F).
On functions of one variable possessing an
addition theorem (F).

Pincherle, S. Mémoire on the linear functional calculus (F).

Pincherle, S. and Amaldi, U. (Distributive operations)
Le Operazioni Distributive (F).
Bologna (1901).

Pincherle, S. Solution of a class of FE's.
Rend. del cir. mat. di Palermo 13 (1904) 273-293.

Pincherlé, S. Functional operations and equations (F).
Encyclopédie der Mathematischen Wissenschaften mit

Poincaré, H. Functional equations and operations (F).
Encyclopaedia des sciences mathématiques pures et
appliquées. II, 1, II, 25 (1912).

Poincaré, H. On a new class of uniform transcendental (F).
J. Math. pure appl. (4) 6 (1890) 313-365.

Poisson, S.D. Mémoire on the distribution of electricity on
the surface of conducting bodies (F), p.43.

Pompeiu, D. On the FE \( f(x+y) + f(x) - y = f(x) f(y) \) \( \text{ (F).} \)
Traité de Mécanique, Vol. 1 2a edition, Paris
(1833) 47.

Pompeiu, D. On a FE arising in a problem of means (F).
C.R. Paris 190 (1930) 1107-1109.

Papoviciu, T. On certain FE's which define polynomials (F).
Mathematica Cluj 10 (1935) 194-208.

Papoviciu, T. On bounded solutions and measurable solutions of
some FE's (F).
Mathematica Cluj 14 (1938) 47-106.

Rabinow, D.G. Independent Sets of Postulates for Abelian Groups
and Fields in Terms of the Inverse Operations (F).
Rausenberger, O.
The theory of general periodicity (G).

(Definition of periodicity in the general sense) p.143.
Lehrbuch der Theorie der periodischen Funktionen (G).
Textbook on the theory of periodic functions.
Leipzig (1884).

Reisch, P.
New solution of the FE for matrices (G).
Math. Z. 49 (1944) 411-426.

Ridder, J.
On the Additive Functional Equation and an Additive
Functional congruence (E).
Euclides 18 (1941) 84-92.

Ritt, J.F.
Real Functions with Algebraic Addition Theorems (E).

Rogers, L.L.J.
On Function Sum Theorems Connected with the Series
\[ \sum \frac{x^n}{n^2} \] (E).

Roullet, H.
(See Fréchet, M.)

Ruziewicz, S.
(See Banach, S.)

Saint-Robert, P.
Solution of certain equations in three variables by means
of a slide rule (F).

Sato, R.
A Study of Functional Equations (E).

Schimmack, R.
Axiomatic Investigations on vector addition (G).

The theorem of the arithmetic mean on an axiomatic
foundation (G).

The theorem of the arithmetic mean on an axiomatic
foundation (G).
Schottky, F.  
On the addition theorem of the cotangent and of the function

\[ \zeta(u) = \frac{\mathcal{G}'(u)}{\mathcal{G}(u)} \quad (G) \]

J. reine angew. Math. 110 (1892) 324-337.

Schroeder, E.  
On infinitely many algorithms for the solution of equations (G).

On iterated functions (G).

Schröder, E.  
A note on the range of operation of the calculus of logic (G).

On a specific definition of a function by demanding formal properties (G).
J. reine angew. Math. 90 (1880) 189-220.

On algorithms and calculi (G).
Arch. Math. Phys. (2) 5 (1897) 225-278.

Schur, I.  
On the representation of finite groups by substitutions of linear fractions (G).

On the rational representation of the general linear group (G).

On the continuous representation of the general linear group (G).

Schweitzer, A.R.  
On a FE (E).

On a FE (E).

Remark on a FE (E).

Theorems on FEs (E).
Bull. Amer. math. Soc. 18 (1911-1912) 492.
Theorems on FEs (E).

Remarks on FEs (E).

Note on FEs (E).

On Quasi-Transitive and Symmetric Functions (E).

A Generalization of FEs (E).

On Certain FEs (E).

Generalization of Certain FEs (E).

An Extension of FEs (E).

Remarks on FEs (E).

On the Formal Properties of FEs (E).

On the Solution of a Class of FEs (E).

Generalized Quasi-Transitive Functional Relations (E).

On the Dependence of Algebraic Equations upon Quasi-Transitiveness (E).

A New Functional Characterization of the Arithmetic Mean (E).

On the Dependence of Algebraic Equations upon Quasi-Transitiveness, II (E).

A Bifurcative Generalization of a FE due to Cauchy (E).
(Schweitzer, A.R.)

On a New Representation of a Finite Group (E).

Definition of New Categories of FEs (E).

On a Type of Quasi-Transitive FEs (E).

On a Type of Quasi-Transitive FEs II (E).
Bull. Amer. math. Soc. 23 (1916-1917) 76-78.

A Problem on Quasi-Transitive FEs (E).
Bull. Amer. math. Soc. 23 (1916-1917) 78-79.

Some Theorems on Quasi-Transitive FEs (E).
Bull. Amer. math. Soc. 23 (1916-1917) 79-80.

On the Analogy between FEs and Geometric Order Relations (E).
Bull. Amer. math. Soc. 23 (1916-1917) 80.

Some remarks Concerning Quasi-Transitive FEs (E).

FEs Based on Iterative Compositions (E).

On the Implicit Corrolctives of Certain FEs (E).
Bull. Amer. math. Soc. 23 (1916-1917) 394.

On the Iterative Functional Compositions of Index (n, k), (n, k = 1, 2, ...), and Associated FEs (E).

On the Iterative Properties of an Abstract Group (E).

On Certain Articles on FEs (E).

On Iterative FEs of the Distributive Type (E).

On the Iterative Properties of an Abstract Group (E).

On Iterative FEs of the Distributive Type (E).
(Schweitzer, A.R.)

- On Iterative Properties of an Abstract Group IV (E.).
  Bull. Amer. math Soc. 25 (1919) 257.

- On the History of FEi (E).
  Bull. Amer. math Soc. 25 (1919) 489.

- On the Iterative Properties of the Abstract Field (E).
  Bull. Amer. math Soc. 26 (1920-1921) 440-444.

  Bull. Amer. math Soc. 26 (1920-1921) 296.

Seilinger, D.N.

- On a FE (Ru).

Serret, J.A.

- On a class of equations (F).
  J. de math. 15 (1869) 152-168.

Sibiriani, F.

- On the reversion functions (I).

Siegel, C.L.

- Iteration of analytic functions (E).

Sierpinski, W.

- On the FE $f(x+y) = f(x) + f(y)$.
  Fundamenta math. 1 (1920) 116-122.

- On a property of Hamel's functions (F).
  Fundamenta math. 6 (1924) 354-356.

- Remarks on functions of several real variables (F).

Soreau, R.

- Reduction of $F_{123} = 0$ to the form
  $f_1 f_2 + f_2 g_2 + h_3 = 0$.
  (F).


Spiess, O.

- (Basic notions of iteration calculus)

- Theory of the linear iterated equation with constant coefficients (G).
Schapira, H. Iteration as the fundamental process of mathematical operations (G).

Stückel, P. On a FE investigated by Abel (G).

On the FE
\[ f(x; y) = \sum_{i=1}^{n} i! \cdot \lambda_i \cdot y^i \]  
(R. C. Acad. Lincei 22 (1912) 322-333).

Stephanos, C. On a category of FEs (F).
Math.-Kongress, Heidelberg (1904) 29.

On a category of FEs (F).

On a characteristic property of determinants (F).
Annali Mat. 13 (1913) 233-235.

Stirling, J. Note on a Functional Equation E., layed by Sir George Stokes (E).

Suschewitsch, A. On a Generalization of the Associative Law (E).

Suto, O. On Some Classes of Functional Equations (E).
Tôhoku math. J. 3 (1913) 47-62.

Studies on Some FEs (E).

Studies on Some FEs (E).

Law of the Arithmetical Mean (E).
Tôhoku math. J. 6 (1914) 79-91.

Szász, P. On hyperbolic trigonometry (H).

Székefalvi-Nagy, B. On measurable representations of Lie groups (G).

Takasaki, M. Abstraction of Symmetric Transformations (J).
Tôhoku math. J. 49 (1943) 145-207.
<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tarski, A.</td>
<td>A Contribution to the axiomatics of Abel Groups (G)</td>
<td>Fundamenta math. 30 (1938) 253-256.</td>
</tr>
<tr>
<td>Terracini, A.</td>
<td>A remark on the FE of isomorphic mapping (G).</td>
<td>Math. Ann. 32 (1921) 141-144.</td>
</tr>
<tr>
<td>Veress, P.</td>
<td>On the notion of mean (H).</td>
<td>Mat. fiz. lapok 43 (1936) 46-60.</td>
</tr>
</tbody>
</table>
- 30 -

Ward, H., and Fuller, F. B.
The Continuous Iterations of Real Functions (E).

Wellstein, J.
Two FE's (G).

Wendt, H.
(See Hantzscha, W.)

Wiener, N.
The Isomorphisms of Complex Algebras (E).
Bull. Amer. Math. Soc. 27 (1921) 443-445.

Wilson, E. B.
Note on the Function Satisfying the Functional Relation \( f(u + v) = f(u)f(v) \) (E).
Ann. Math. (2) 1 (1929) 57-60.

Wilson, W. H.
On a Certain General Class of FE's (E).

On a Certain General Class of FE's (E).

On Certain Related FE's (E).

Two General FE's (E).
Bull. Amer. Math. Soc. 31 (1925) 328-334.

Yosida, K.
On the Groups Embedded in the Matrical Complete Ring (E).
Part II

A list of publications, with comments, from 1246 on.
Alaci, V.

On a class of FE (R). *

Altman, M.

FEs involving a paramater (E).

An iteration procedure based on Newton's method is established to solve the FE
\[ F(x, \mu) = 0 \]
where both \( x \) and \( \mu \) are elements of Banach spaces. Convergence and speed of convergence of the iteration process are investigated.

An iterative method of solving FE (E). *

Iterative methods of higher order (E). *

A generalization of a Legendre method for FE (E). *

Anastassiadis, J.

On the solutions of the FE
\[ f(x + t) = \varphi(x) f(x) \quad (F) \]

The above FE is solved under the condition \( f(1) = 1 \) and some mild restrictions.

Anghelut. Th.

The FE of bisymmetry (E). *

On the FE of Translation (R).

A proof is given for the fact already known that
\[ \mathcal{F}(x + t) = \mathcal{F}(x) \mathcal{F}(y + t) \]
is solved by
\[ F(x, y) = F^{-1}[F(x) + y] \]

Remarks on the FE of Poisson (R).

The author proves that every non-vanishing solution of the
Poisson FE \[ f(x+y) + f(x-y) = 2f(x)f(y) \]
satisfies the algebraic addition theorem
\[ f(x+y)^2 - 2f(x)f(y)f(x+y) + f(x)^2 + f(y)^2 - 1 = 0 \]
This is used to prove that every continuous solution of the original FE is analytic.

(Angheluta, Th.)

C "FE with three unknowns \( f \) \( z \in (\mathbb{R}), \star \)

Angheluta, T.

(See Angheluta, Th.)

Arrighi, G.

On the FE \( 2Q(x)q(y) = q(x+y) + q(x-y) \)

The solution of the equation, proved by Picard, to be \( \cos \lambda x \) and \( \cosh \lambda x \), assuming continuity, is given, under the weak condition of right continuity.

Aczél, J.

The Notion of Mean Values (E).
Norske Vid. Selsk. forhandlinger 19 (1946) 83-86.

The author defines a normal mean value \( M(x_1, x_2, \ldots, x_n) \) by the properties: symmetry in the variables; \( M(x_1, x_2, \ldots, x_n) = x_j \); \( M \) is monotone increasing in each variable; \( M \) is continuous function of the "vector" \( [x_1, x_2, \ldots, x_n] \); finally

\[ M = M(x_1, x_2, \ldots, x_n) \] is symmetric in all its \( n \) variables ("bisymmetry") under these conditions the Kolmogoroff-Negumo theorem is proved:

\[ M(x_1, \ldots, x_n) = F^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right] \]
where \( F \) is continuous and monotone.

On Mean Values and Operations Defined for two Variables (E).

The validity of the Kolmogoroff-Negumo theorem (see Aczél: The Notion of Mean Values) is proven, for the case of two variables, replacing the monotony condition by the weaker condition:

\[ x < y \implies M(x, y) < \frac{1}{2} \]
An analogue of the Kolmogoroff-Negumo theorem is also given.
On a FE (F).

The "generalized addition theorem"
\[ f(AX + BY + C) = \mathcal{D}(f(x), f(y)) \]
is treated.

On Mean Values (E).
Bull. Amer. math. Soc. 54 (1948) 272-400.

\[ M_n(x_1, x_2, \ldots, x_n) \]
be a sequence of mean value functions of \(1, 2, \ldots, n\) variables; a necessary and sufficient condition is searched for under which a continuous and increasing function \( f(x) \) exists, so that \( M_n(x_1, x_2, \ldots, x_n) = f^{-1}(f(x_1) + \cdots + f(x_n)) \)
if being independent of \(n\), the author proves that the bioriomatic condition:
\[ M[M(x_1, x_2, \ldots, x_{n+1}), \ldots, M(x_n, x_{n+1})] \]
is necessary and sufficient.

On a class of FEs (G).

The only continuous solution of
\[ f(ax_1 + bx_2, ax_3 + bx_4) = \beta_1 f(x_1) + \beta_2 f(x_2) + \beta_3 f(x_3) + \beta_4 f(x_4) + \beta_5 f(x_5) + \beta_6 f(x_6) \]
with \( p_1(0) = p_2(0) = 0 \)
is
\[ f(x) = Ax + B \]

On operations defined for real functions (F).

Let \( f(x, y) \) be such that \( A \leq f(x, y) \leq B \)
provided \( A \leq x, y \leq B \)
Then an operation \( x \ast y = f(x, y) \) is defined
if it is shown that
\[ x \ast y = f(x, y) = \mathcal{P} [\mathcal{Q}(x) + \mathcal{Q}(y)] \]
if and only if the operation is monotone, continuous and associative.

Aczél, J.,
Kalmar, L. and
Mikusiński, J. G.

On the translation equation (F).
Studia math. 12 (1951) 112-115.

The FE \( f[f(x, y), u] = f(x, u + v) \)
is dealt with. This is one of the most important FEs, since its
solution enables us to write iterated functions of arbitrary index; using the notation
\[ f_n(x) = f(f(x), n) \]

the FE expresses the relation
\[ f_n[f_m(x)] = f_{m+n}(x) \]
where \( f_n(x) \) is defined by
\[ f_n(x) = f(f(...f(x)...), n) \]

The authors prove the existence of solutions under various assumptions. Especially, under some monotony assumptions,
\[ f(x, u) = e^{-u}[f(x) + u] \]
using this formula, the generalization of the \( n \)-th iterated function of \( f \) is given by arbitrary real \( u \)
\[ f_u(x) = e^{-u}[f(x) + u] \]

FEs in applied mathematics (H). *

On FEs in several variables I. Elementary solution methods for FEs in several variables (H).
Matnapok 2 (1951) 99-117.

The continuous, increasing solutions of the "mean function" equation
\[ \int m(x, y) = \int m(x, z) \]
are
\[ \int m(x, y) = e^{-y}[f(x) + f(y) + p] \]
while the continuous and increasing solutions of
\[ \int f(x, y) = \int f(x, z) \]
are
\[ \int f(x, y) = e^{-y}[f(x) + f(y)] \]

Some FEs in connection with the theory of continuous groups (H). *
Axi Első Négyes Matematikai Kongresszus közleményei 1950
(Budapest 1952) 565-569.

On Composed Poisson-Distributions III (E).

A FE is set up for the probability distribution of the event that exactly \( k \) events occur in the time interval \( [t_1, t_2] \)
the assumption is made that the number of events in two non-overlapping time intervals are independent. The solution is constructed by induction, it contains the distributions of exponential decay and the Poisson distribution as special cases.

Reduction of FEs of several variables to the solution of partial differential equation.
Outlines of a general treatment of some FEs (G).

The classes considered contain certain well-known and important FEs as special cases; thus the "addition theorem"

$$ F(x+y) = F(F(x), F(y)) $$

the "generalized Jensen equation"

$$ F(F(x), y) = F(x, F(y)) $$

and so on. In the majority of cases existence and uniqueness of the solution is proved. Thus, the addition theorem has a strictly increasing and continuous solution if and only if the "addition function" $F(x, y)$ is strictly increasing in both variables, and the associative law

$$ F(F(x, y), z) = F(x, F(y, z)) $$

holds, where $F(x, y) = F(x, y)$. 

A Solution of Some Problems of K. Borsuk and L. Jánossy (E).

Associative equations of the type

$$ F(F(x, y), z) = F(F(x, y), z) $$

are treated in connection with L. Jánossy's work on an axiomatic foundation of probability theory.
Algebraical remarks on the Fréchet solution of the Kolmogoroff equation (F).

The general solution of the FE $P(s,t)P(t,u)=P(s,u)$ is given under more general conditions than the solution given earlier by Fréchet. This equation plays an important role in polarity theory.


Addition theorems:
\[
\begin{align*}
\ell(x_1,y_1) &= \ell \left( \ell(x_1,y_2), \ell(x_2,y_2) \right) \\
\ell(x_1-y_1) &= \ell \left( \ell(x_1,y_2), \ell(x_2,y_2) \right)
\end{align*}
\]

are investigated. The main results: the addition theorem has a non-constant continuous solution if and only if there exists an open interval on the real axis which is a group under the operation \( \ell \).

Furthermore, for any solution \( \ell(x) = \ell(cx) \) is also a solution. The subtraction theorem has a continuous solution (which is then strictly increasing) if and only if there exists an open interval on the real axis on which the operation \( \ell \) is continuous, transitive, involutory, and if there exists a right hand unit e such that \( \ell \circ e = \ell \).

Some general methods in the theory of FEs in one variable. New applications of FEs (Ru).
Uspechi mat. nauk. 11 (1956) 6 (69) 3-33.

Several classes of FEs are examined in view of possible applications. These vary as widely as scalar and vectorial multiplication of vectors, the Poisson distribution, and non-euclidean distance. In particular, the results lead to a characterization of the distance function
\[
\begin{align*}
\ell(X,Y) &= \ell \text{ (a) \cos} \left( \frac{x_1y_1 - x_2y_2 + x_3y_3}{\sqrt{(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)}} \right) \\
\ell(X,Y) &= \ell \text{ (b) \cosh} \left( \frac{x_1y_1 - x_2y_2 - x_3y_3}{\sqrt{(x_1^2 - x_2^2 - x_3^2)(y_1^2 - y_2^2 - y_3^2)}} \right)
\end{align*}
\]
in elliptic geometry and

in hyperbolic geometry.

Miscellaneous on FEs (G).

Several FEs and systems of FEs are treated, mostly addition theorems.
Aczél, J., and Kiesewetter, H.  
On the reduction of degree in a class of FEs (G). *  

Aczél, J.  
Some general methods in the theory of FEs and some recent applications I. (H). *  

On the differentiability of the integrable solutions of class of FEs (G).  

It is shown that the integrable solutions of the FEs

\[ f(x) + \sum_{k=1}^{n} C_k(x) g_k(x) + A(x, y) = 0 \]

and

\[ f(x) - g(x) + \sum_{k=1}^{n} C_k(x) g_k(x) + A(x, y) = 0 \]

are differentiable, provided the \( g_k(x) \) are continuous, the \( C_k(x) \) differentiable and \( A(x, y) \) integrable in

\[ \frac{\partial A}{\partial x} \]

continuous in both variables.

Aczél, J., Hasszu, M. and Straus, E. G.  
FEs for Products and Compositions of Functions (E). *  

Aczél, J.  
Some general methods in the theory of FEs, and some recent applications. *  

Aczél, J., Golab, J., Kuczma, M. and Siwek, E.  
The cross ratio is solution of a FE (G). *  

Aczél, J.  
(Lectures on FEs and their applications).  
Vorlesungen über Funktionalgleichungen und ihre Anwendungen (G).  

This is the first book ever to be published on FEs; a book by Picard, published in 1928, "Lesons sur quelques équations fonctionnelles," is a treatise on a small, but well-chosen number of FEs; Ghermanescu's book on FEs, published in 1960, is a treatise on a special class of FEs; in fact, it deals with one, very general equation which includes difference equations.

Aczél's book does not propose to give a general theory of FEs, since such a theory does not exist. The book consists of two parts: the first part...
deals with functions of one variable, the second with functions of several variables. Within the first part, two cases are distinguished according to whether the variables appear under the function sign only, or also outside the function sign. One of the most interesting properties of FEs, namely that one equation may determine several unknown functions, is given one chapter by itself. Among the solution methods, reduction to differential and integral equations is treated; numerous applications to the theory of means and many other topics are given.

The main weakness of the book is the fact that only such equations are treated where—roughly speaking—the number of variables is higher than the number of variables in the unknown function. This excludes the first FE ever treated, and, the most important of all, the Abel - Schröder equation. The reason for this omission is that this type of equation is much more difficult to treat and requires different methods (for details on this problem, see the Introduction).


Let \( f(x + iy) = u(x, y) + iv(x, y) \)

Real and analytic part of an analytic function, \( u \) and \( v \) form a pair of conjugate harmonic functions; they are connected by the classical formula

\[ \nabla u(x, y) = \int \frac{u(x, y) - u(x', y)}{\sqrt{(x-x')^2 + (y-y')^2}} \, dx' \, dy \]

from the Cauchy Riemann formulae. The paper furnishes an alternative method for the determination of \( v \) from \( u \)

\[ \nabla v(x, y) = \int \lim_{(x', y') \to \infty} U(x + iy, 0) \]

where \( U \) is the complex extension of the (real) harmonic function \( u \).
provided \( f(0) \) is real on the real axis, apart from a possible imaginary constant; also, quite generally, the FE

\[
U(x+i\,y,0)+i\,V(x+i\,y,0)=U(0,y)+i\,V(0,y);
\]

\( U(x,y)+i\,V(x,y) \)

is necessary and sufficient for the Cauchy-Riemann equations to hold.

On the solutions of a FE (Sc).

The FE:

\[
f(x) f(x+1) / (x+1) \cdot (x+a) = 1
\]

is solved under more general conditions than before.

On certain iterated sequences (F).
Naučno Drustvo N, R. Bosne-Hercegovine dj. 4 odjeljenje priv.-

tehn. nauka 1 (1953) 1-33.

The FE:

\[
\sum_{j=0}^{2\,x} g(j) = f(x)
\]

is examined, where \( g \) is known and \( f \) unknown. Conditions are

stated under which a unique solution exists.

On certain iterated sequences II (F).
C. P. Peris 536 (1953) 263-269.

The FE:

\[
\psi(x+1) = f\left[ \frac{\psi(x)}{x} \right]
\]

(\( \psi \) unknown) is treated by relating it to the FE
dealt with in the preceding publication.

On certain solutions of two FEs (Sc).


On a FE.
Ser. II 12 (1957) 201-205.

It is shown that the FE

\[
\Omega(z) = \Omega \left\{ \Omega, \Omega \left[ \Omega(z) \right], \cdots \Omega \left[ \Omega \left( \Omega(z) \right) \right] \right\}
\]

has always a solution, under rather general conditions.
(Bajroktarevic, M.)

Monotone solution of a FE (F).

The FE mentioned in the title is
$$F(x) = E_0 f(x) + F \left[ \frac{q(x)}{q(x_0)} \right]$$

$E_0$ is constant, $f(x)$, and $q(x)$ are known functions. Existence and uniqueness of a strictly monotone solution is given under appropriate conditions.

C. mean value FE (F).

The existence of certain solutions of the above FE is shown; these solutions are explicitly given; the solutions are shown to be invariant under certain transformation; finally, a class of functions is given in which the FE is completely solved.

On a solution of the FE $q(x); q^2(x) = F(x)$

Results of Kirchoff are mentioned.

Baker, I. N.

Solutions of the FE $\frac{d}{dx} (x^2 f(x)) = -b(x)$ (E).

Solutions are given, under restrictions, for the FE in the title and for the FE $\frac{d}{dx} \left( \frac{f(x)}{f(2x)} \right) = b(x)$
which can be reduced to the former.

Bellman, R.

A Note on Linear Functions of Matrices (F).

Let $\varphi$ be a function of the $r$ variables $A_{ij}, i, j = 1, \ldots, k$,
arranged as a matrix $X$, and
$$\varphi(A) = (r, A)$$
for every pair $A, B$; then $\varphi(A)$ is a polynomial function of the coefficients of $A$ of the equation
$$\det (A - \lambda I) = 0$$
Berg van den, J.

On the FE $\varphi(x;x) - \beta \varphi(x) = f(x)$, I, II (G).


A treatment of the above FE which breaks the unfortunate habit of many authors of seeking for strictly increasing solutions only. First bounded solutions are investigated; later, some unbounded solutions are also considered.

Blum, J.R., Norris, M.J., and Wing, G.M.

Asymptotic behaviour of solutions of a FE (E). *


Boas, R.P.

Functions which are odd about several points (E).


The condition that $f(t)$ is odd about the point $t=x$ is expressed by the Jensen FE

$$f(x + t) + f(x - t) = 2f(x)$$

This FE is treated with respect to the nature of the set of $t$ and $x$ values on which it holds.

Boas, R.P.

Functions which are odd about several points. Addendum (E).

Nieuw Arch. Wisk. 5 (1957) 25.

The author points out the a lemma in his paper with the above title was already found by Basit in 1915. Some misprints are corrected.

Boswell, R.D.

Continuous Solutions of Two Functional Equations (E). *

Amer. math. Monthly 65 (1958) 476.

On Two FEs (E).


The only continuous solution of

$$f(x+y) = f(x) + f(y) + g(y)(1-A^x)$$

is

$$f(x) = 4x - g(1-A^x), \ A$$

being real and $A > 0$; the only continuous solution of

$$f(x+y) = A^x f(y) + A^y f(x)$$

is

$$f(x) = 4x A^x.$$
Carstoiu, I.

On some FEs and the symbolic calculus (F).

Five FEs (among them a difference equation) are solved using the Laplace transform. The method is restricted by the fact that existence of the first derivative had to be assumed.

Chaundy, T.W., and McLeod, J.B.

On a FE (E).

The FE \( f(x) = u(x, \phi(x)) + \int [\phi(u, \nu) x] \)

is investigated. \( x, \nu \) and \( \phi \) are variables, \( f, u \), and \( \nu \) are unknown functions, \( f \) is assumed to be continuous. The FE arises in a problem concerning statistical thermodynamics of mixtures.

Choczowski, B.

On continuous solutions of some FEs of the \( n \)-th order (E).

Continuous solutions of the following functional equations are investigated.

(1) \( \phi(x) = H \left[ x, \phi, \phi \left( \phi(x) \right) \right] \ldots \phi \left( \phi \left( \phi(x) \right) \right] \)

(2) \( \phi \left( \phi_{+}(x) \right) = \phi \left[ x, \phi(x), \phi \left( \phi_{+}(x) \right) \right] \ldots \phi \left( \phi_{+}(x) \right] \)

(3) \( \phi \left[ x, \phi(x), \phi \left( \phi(x) \right) \right] \ldots \phi \left( \phi(x) \right] = 0 \)

Climescu, A.C.

On the FE of associativity (F).

Introducing the generalized "multiplication" (group operation)

\( x * y = \phi \left( x, y \right) \)

the condition of associativity is expressed by the FE

\( \phi \left[ y, \phi \left( x, x \right) \right] = \phi \left[ \phi \left( x, \phi \left( x, x \right) \right), \phi \left( x, x \right) \right] \)

If \( u \) and \( \phi \) are defined and single valued on the range of \( x \) and \( y \)

\( u^{-1} \left[ \phi \left( u(x), u(y) \right) \right] \)

is also a solution. Several special cases are treated and applications given.

Doroczky, J.

Remarks on FEs (H).
Doróczy, Z. Necessary and sufficient conditions for the existence of non-constant solutions of functional equations (C).


The author starts from the following result of J. Bernstein: the FE \( f \left( \frac{x+y}{2} \right) = \frac{1}{2} \left( f(x) + f(y) \right) \) has constant solutions only if \( P + \frac{1}{2} = 1 \). As a generalization, the author investigates the FE

\[
\frac{1}{2} (x + y + c) = \frac{1}{2} \left( p f(x) + q f(y) + \lambda \right)
\]

and finds necessary and sufficient conditions on the coefficients for the the existence of non-constant solutions.

Dias Tavares, A. A Theorem on Real Functions of a Real Variable (Po). Revista científica 1 no. 1 (1964) 7-11.

Djakovic, D. (See Djakovic, D. i)

Djokovic, D. (See also Mijinovic, D.S.)

Djakovic, D. On some analytical functions \( g(x) \), which reduce to the equation of Grunau (E).* Publik. Math. Debrecen, Sect. A 16, 207-221.


The Grunau density function is shown to be the only density function satisfying

\[
\mathcal{F}(x) = \int (x^2 + y^2) \mathcal{F}(x) \mathcal{F}(y) 
\]

also \( \Sigma \mathcal{F}(x) = 1 \) is the only density satisfying

\[
\mathcal{F}(x) \mathcal{F}(y) = \mathcal{F}(x+y)
\]

Elyash, E.S., and Levine, N. A Note on the Function \( A_2 + b \) (E).


Let \( m(z, w) = \rho z + q w \) be with \( \rho > 0, q > 0, \quad \rho^2 + q = 1 \)

the weighted arithmetic mean of \( z \) and \( w \). The following theorem is proven: the only convex regular function satisfying the FE

\[
\mathcal{F} \left[ m(z, w) \right] = \mathcal{F} \left[ f(z), f(w) \right]
\]

is the linear function. A more general case includes the dependence of \( \rho \) and \( q \) on \( z \) and \( w \).
Erdös, J.

A Remark on the Paper "On some Functional Equations" by S. Kurepa (E).

It is shown that every continuous solution of the FE
\[ f(x+y, z) + f(x, y) = f(y, z) + f(x, y+z) \]
is of the form
\[ f(x, y) = g(x+y) - g(x) - g(y) \]
on the other hand, this is not the general solution of the FE.

Erdös, J., and Golomb, M.

Functions which are Symmetric about Several Points (E).

The "oddness" FE \[ f(x+\ell) + f(x-\ell) = 2f(x) \]
tricity earlier by H. P. Eure, is further investigated; the generalization
\[ \sum_{k=0}^{n} a_k f(x + C_k \ell) = f(x) \quad C_k \neq 0, \sum a_k = 1 \]
is treated.

Fenyö, I.

On a solution method for certain FE's (C).

The theory of distributions is used to transform certain FE's into distribution equations and to solve them.

Gödör, J.

A new definition of determinants (C).

A scalar function of a matrix is shown to be the determinant under some compatibility condition of some generality. Conditions all of which, except one, are very mild.

Gair, A.

On the analytical solutions of certain FE's (R).

Ghermanescu, M.

(G. see also Aczél, J.)

Ghermanescu, M.

Functional characterization of the trigonometric functions (F).

The addition theorems for the sine and the cosine are investigated assuming measurability of the solutions.
Measureable solutions of certain linear FE's in several variables. (F.).


Measureable solutions, especially polynomial solutions, are sought for a number of FE's, most of them difference equations.

Linear FE's (R). *


On the FE

\[ \sum_{i=0}^{\infty} A_i g(x + u_i) = 0 \]  \hspace{1cm} (R).


This FE is treated under various assumptions. A characteristic result:

\[ (2\pi i)^{-1} \int e^{ix} \psi(x) \, dx \]  \hspace{1cm} (R).

is a solution, where \( \psi(x) \) is a function of the form:

\[ \psi(x) = \sum_{i=0}^{\infty} A_i g(x + u_i) \]

and \( \psi(x) \) is arbitrary, but otherwise arbitrary.

On the FE

\[ \sum_{i=0}^{\infty} A_i g(x + u_i) = 0 \]  \hspace{1cm} (R).


Additional discussion is in the second part of the previous paper with the same title in the same volume of the journal.

On the FE

\[ \sum_{i=0}^{\infty} A_i g(x + u_i) = g(x) \]  \hspace{1cm} (R).


The FE is a generalization of the one discussed in the previous paper, the \( P_i \) being polynomials.

A system of FE's (R).


The FE

\[ \int f(x) \left( \sum_{i=0}^{\infty} A_i g(x + u_i) \right) \, dx = 0 \]

is treated; here the functions \( f(x) \) are known, so are the \( g_n(x) \) is the \( n \)-th harmonic function of a known \( x \) special attention is paid to the case where \( f(x) \) is a translation.
(Ghermanescu, M.) On FEs in two variables (R).

Sixteen results are given concerning a number of FEs, all in two variables. The results are quite general, since only measurability of the solutions is required.

FEs with n-periodic functional argument I (F).

Theorems on the existence of solutions for the FE
\[ \sum_{k=0}^{n-1} a_k f(J_k(x)) = g(x) \]
are given; here, f is unknown, \( J_k \) is known and \( J_k \) denotes its \( n \)-th iterate; \( g(x) \) is known and might also be identically zero. Finally, \( \forall \, J_k(x) \equiv x \) is assumed.

FEs with n-periodic functional argument II (F).

A continuation of the first part. Existence theorems are given.

A class of linear GFEs (F).

The FE
\[ f(P) + \sum_{k=1}^{n} a_k (P) f[J_k(P)] \]
is considered; P is a point in a multidimensional space, \( J_k(P) \) as usual the \( k \)-th iterate of \( J \); the coefficients satisfy
\[ a_k (P) \equiv q_k (P) \]
, i.e. they are invariants under the substitution \( J \) as generalized periodical functions in the sense of Rausenberger.

The characteristic equation
\[ \lambda(P) + \sum_{k=1}^{n} a_k (P) \lambda^k = 0 \]
is then defined; each solution
\[ \lambda_i (P) \equiv 1, \cdots, n \]
is also an invariant under the substitution \( J \). These solutions are used to construct the general solution of the FE.

Linear FEs with n-periodic functional argument (R).

The FE
\[ \sum_{k=0}^{n} a_k f[J_k(P)] = 0 \]
is studied; the \( a_k \) are constants n-periodicity of \( J \) is defined by
\[ J_n (P) \equiv J(P) \]
The inhomogeneous case
\[ \sum_{k=0}^{n} a_k f[J_k(P)] = g(P) \]
is also treated. P is a point in a multidimensional space.

Doubly automorphic functions (R)

These functions are defined by

\[ \sum \lambda_1(P) = \lambda_2(P) \]

where

\[ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \]

must satisfy various conditions.

On the FE of Cauchy (F)
1 (47) (1957) 33-46.

New methods are shown to solve the Cauchy equation

\[ f(x + iy) = f(x) + f(y) \]

assuming that the solutions are continuous, resp. measurable. Some related FE's are also treated, among them

\[ \Phi(x + iy) = \Phi(x) \]

\[ f(x + iy) - f(x) - f(y) = u(x, y) \]

etc.

A class of linear FE's (F)

The FE

\[ f(P) + \sum_{k=1}^{\infty} \alpha_k f(P) \]

is investigated. \( \alpha_k \) are known functions, the \( \lambda_k \) are the iterates of the unknown function \( \lambda \). P is a point in multidimensional space. The \( \alpha_k \) are assumed to be automorphic invariants with respect to \( \lambda \):

\[ \alpha_k(\lambda(P)) = \alpha_k(P) \]

On the functional definition of the trigonometric functions (F)

A simple method is given to solve the FE's

\[ f(x) + f(y) = \int_{0}^{1} (x - y) \]

\[ f(x) + f(y) = \int_{0}^{1} (x + y) \]

continuous monotone solutions are

\[ \alpha. \text{acos} \alpha \text{ resp. } \alpha. \text{asinh} \alpha. \]
On a class of linear FEs \((R)\),

Studii si cerc. mat. 9 (1953) 113-128.

(Functional Equations)
Equazioni funzionali \((R)\).

The book deals mainly with the results of Romanian authors. It
investigates various cases of the FE

\[(*) \quad E \left\{ P, f(P), f(S(P)), \ldots, f(S_n(P)) \right\} = 0 \]

where \(P\) is a point in multi-dimensional space, \(E\) and \(f\) known functions \((f\) may depend on a parameter) and \(S_n\) denotes the \(n\)-th
iterate of \(S\) may be linear, or non-linear, for the case
\(S(P) = P = P\), we obtain the difference equations, and two chapters of the book are accordingly dedicated
to the general theory of difference equations. The "impulsive"
case \(S(P) = P\) is treated, and so are many of the topics which come under the heading of the FE \((*)\).

Linear FEs with automorphic functional arguments \((F)\),

The FE
\[ \sum_{k=1}^{n} C_k(P) f(S_k(P)) = 0 \]

is investigated for the case that the coefficients are automorphic
(substitutional) invariants, with respect to some \(S\):
\[ C_k(S(P)) = C_k(P) \]

is a point in multi-dimensional space, \(S_k\) denotes the \(k\)-th iterate
of \(S\).

Gogab, J.

(See also Aczél, F.)

Gogab, S.

On the distributive law of real numbers \((G)\).

The equation
\[ g(f(x, y), z) = f(g(x, z), g(y, z)) \]
becomes \(x+y = xz + yz\) if \(f(x, y) = x + y, g(x, y) = xy \).

Simple conditions on the functions \(f\) and \(g\) are given under
which an automorphism of the operations \(x \oplus y = f(x, y)\)
and \(x \odot y = g(x, y)\) is established to
ordinary addition and multiplication on the field of real numbers.
(Golab, S.) On the FE \( f(\mathbf{x}) f(\mathbf{y}) = f(\mathbf{x} \cdot \mathbf{y}) \) (F).

Colloquium math. 10 (1957) 365.
(See also the next ref.)

On the equation: \( f(\mathbf{x}) f(\mathbf{y}) = f(\mathbf{x} \cdot \mathbf{y}) \) (F).

\( \mathbf{X} \) and \( \mathbf{Y} \) are \( 2 \times 2 \) matrices of complex numbers. If the above equations is satisfied for every \( \mathbf{X}, \mathbf{Y} \), then \( f = \mathbf{E}(\mathbf{O} \cdot \mathbf{X}) \).
Where \( \mathbf{E} \) is a complex-valued function with the property
\[ \mathbf{E}(\mathbf{X} \cdot \mathbf{Y}) = \mathbf{E}(\mathbf{X}) \mathbf{E}(\mathbf{Y}) \], i.e. under certain restrictions essentially a power of \( \mathbf{x} \).

Golomb, M. 
See Erdös, P. 1

Guinend, A.P. 
The FE of the form \( f(f(\mathbf{x})) = \mathbf{e} \mathbf{x} \mathbf{f}(\mathbf{x}) \) is solvable (F).

The equations:
\[ f[f(x, y, u, v)] = f[f(x, y, f(u, v))] \]
\[ f[f(x, y, u, v)] = f[f(y, u, x, v)] \]
\[ f[f(x, y, u, v)] = f[f(x, y, f(z, u, v))] \]
are solved.

Hajek, O. 
On the FE's of the trigonometric functions (Ku). *
Arch. math. J. S. (Czechoslovakia) 532-534.

Halperin, I. 
Non Measureable Sets and the Equation \( f(x + y) = f(x) + f(y) \) (F).

Some very refined set-theoretical investigations in connection with the above FE.


The classical results on the FE
$$f(x + y) = f(x) + f(y)$$
are true for the case where both sides of the FE are reduced mod 1; the same result of Hopf's holds as which the graph of $f$ is either
$$\sin x \in [0, 1]$$
$$\cos x \in [0, 1]$$
along the vertical
($y$ reduced mod 1). The FE of the exponential and of the trigonometric functions are treated in the same way.


Let $\eta \left( \frac{x}{2}, \frac{y}{2} \right)$ be a mean value function.

Conditions on $\eta$ are given so that the FE
$$\eta \left( \frac{x}{2}, \frac{y}{2} \right) = \eta \left[ \eta \left( x, y, \nu \right), \eta \left( x, y, \nu \right) \right]$$
has a continuous solution.

Hoszu, M. (See also Arzéti, J.)

Hoszu, M. On the FE of bionomy $g : H \to H$.


A Generalization of the FE of Bionomy $g : H$.

Studia math. 16 (1953) 100-104

The "generalized bionomy equation"
$$F \left[ \frac{\eta \left( x, y \right)}{\eta \left( x, y \right)}, H \left( \nu, \nu \right) \right] = \frac{\eta \left( \nu \left( \nu, \nu \right), \nu \left( \nu, \nu \right) \right]}{\nu}$$
is treated.

On the FE of Distributivity


The strictly monotone and twice differentiable solutions of the FE
$$F \left[ F \left( x, y \right), z \right] = F \left[ F \left( x, z \right), F \left( y, z \right) \right]$$
are determined.
On the FE of Transitivity \((E)\).


The transitivity condition
\[(x \circ y) \circ z = x \circ (y \circ z)\]
of operations between real numbers can be written, using the notation
\[x \circ y = F(x, y)\]
\[x \circ (y \circ z) = F(x, F(y, z))\]
This FE is solved under various conditions imposed.

On the FE of Autodistributivity \((E)\).


The monotone and once differentiable solutions kof
\[M [M(x, y, z) = M(M(x, y), M(z, y))]\]
\[M [M(x, y, z) = M(M(x, z), M(y, y))]\]
can be expressed in the form
\[M(x, y, z) = e^{-1} [a \cdot f(x) + b \cdot g(y)]\]
with
\[a + b = 1\]

Some FE's related with the associative law \((E)\)


"Associative type" relations and the corresponding FE's are investigated. A typical result: the most general strictly increasing solution of
\[x \circ a \circ y = 2 \cdot (a \cdot x)\]
is
\[x \circ y = e^{-1} [a \cdot f(x) + b \cdot g(y)]\]
Remark on a paper by H. Wunder. "On a FE in the Theory of heat conduction". It is shown (cf. Wunder, H.) that the only differentiable solution of the FE
\[f \left( \frac{x - y}{\sqrt{2} \cdot y} \right) = \frac{f(x) + f(y)}{2}\]
is constant.

Generalization of some FE's with several variables \((E)\).


This is a resumé of five papers by the author published between 1953 and 1956.
Unsymmetric means (H).

A continuous unsymmetric, quasi-linear interior mean is a continuous function of the form

\[ M_n(x,y) = \left( \left( \frac{1}{p} \int \phi(x) + \frac{q}{q} \phi(y) \right) \right) \]

where \( p > 0, q > 0 \) and \( p+q = 1 \).

A number of conditions in connection with theorems on such means is modified, with reference to previous work by other authors.

Unsymmetric means (Rus).
Colloquium math. 5 (1957-1958) 32-42.

A Russian version of the preceding paper.

A Generalization of the FE of Distributivity (E).

Introducing the "addition" \( x \overset{x}{\overset{y}{\overset{F}{H}}} y = F(x,y) \) and the multiplication \( x \overset{z}{\overset{y}{\overset{H}{I}}} y = H(x,y) \), the distributive condition \( (x + y) \overset{z}{\overset{y}{\overset{H}{I}}} z = x \overset{z}{\overset{y}{\overset{H}{I}}} z + y \overset{z}{\overset{y}{\overset{H}{I}}} z \) is expressed by the FE

\[ H(F(x,y), z) = F[H(x, z), H(y, z)] \]

This type of FE is discussed in detail with some applications.

Generalizations of the FE of distributivity (H).
Nemzeti Műszaki Egyetem közleményei 3 (1959) 151-166.

A Hungarian version of the preceding paper.

Nonsymmetric Means (E).

On the FE of translation (H).
In "2ème Congr. math. hongr. Budapest 1960".

On a FE (F).

The FE

\[
\begin{vmatrix}
\phi(x) & \phi(x+e) \\
\phi(x+e) & \phi(x+2e)
\end{vmatrix} = 0
\]

and its analogue for determinants of higher order is treated.
Jamm-Levy, J.

On the problem of general anamorphosis (Ru).

If the functional relation

\[ z = f(x, y) \]

can be written in the form

\[
\begin{bmatrix}
    g_1(x) & f_1(x) & 1 \\
    g_2(y) & f_2(y) & 1 \\
    g_3(z) & f_3(z) & 1
\end{bmatrix} = 0
\]

a nomogram can be constructed to "solve" the equation, i.e., find the value of one variable if the two others are given. The functions

\[ f_1, f_2, f_3, g_1, g_2, g_3 \]

determine the scales of the nomogram.

Janko, B.

On the method analogous to that of Tchebitcheff and to that of the tangent hyperbolas for the approximate solution of non-linear FEs (R). *

On a new generalization of the method of the tangent hyperbolas for the solution of non-linear functional equations defined in Banach spaces (R). *

Jewett, J.W.

(See Seebbeck, L.L.)

Kalmár, L.

(See Aczél, J.)

Kestelman, H.

On the FE \( f(x+y) = f(x) + f(y) \) (E).
Fundamenta math. 34 (1947) 144-147.

A simple proof is given of the classical result of Ostrowski that a real solution of this FE is linear, provided it is bounded on a set of positive measure.

Kleewetter, U.

(See also Aczél, J.)
Kiesewetter, H. Structure of linear FEs in connection with the Abel theorem ( G ).

Linear FEs of the form
\[ \sum_{i=1}^{p} a_i f(x_i) + \sum_{k=1}^{p} \Phi_k(x_1, \ldots, x_p) \cdot y_{sk} = Const. \]
can be investigated. A special case is
\[ \sum_{i=1}^{p} f(x_i) = \sum_{k=1}^{p} \Phi_k(x_1, \ldots, x_p) \]
which for \( p = 1 \) becomes the "addition theorem" (in a sense different from the generally used notion)
\[ f(x) + f(y) = \Phi([x, y]) \]
under some restrictions it was shown already by Abel that (3) has a non-trivial solution if and only if the operation
\[ x \cdot y = \Phi(x, y) \]
is associative. The existence of solutions of (3) is thus linked to the algebraic properties generated by \( \Phi(x, y) \).

For the more general case (2) the notion of group had to be generalized for commutative and associative algebraic structures where \( p \geq 2 \) simultaneous operations between \( p \) elements exist.

The paper is essentially devoted to the associative and the cyclical properties of the "argument function" \( \Phi \); the results are described in terms of geometrical models as spherical trigonometry. A bibliography of 35 relevant papers and books is given.
On some linear FEs (E).

Kordylewski, J. and Kuczma, M.

Continuous solutions of the FE

\[ \Phi[F(x)] = FL(x, \phi(x)) \]

with the function \( \phi(x) \) decreasing (E).

Sufficient conditions are given for the existence of a solution.

Kordylewski, J. and Kuczma, M.
Kuczma, M. (See also Aczél, J., Kordyłowski, J.)

Kuczma, M.

On convex solutions of the FE

\[ \mathcal{Q}(F(x)) - \mathcal{Q}(x) = \mathcal{Q}(x) \tag{E} \]


Conditions are found under which the above FE possesses at most one convex solution which takes a prescribed value at a given point \( c \). The conditions under which the theorem is proved are the following:

- \( \mathcal{Q}(x) \) is continuous, concave and

\[ \mathcal{Q}(x) \geq \mathcal{Q}(a) \quad \text{in } [a, \infty] \]

Moreover,

\[ \lim_{n \to \infty} \left[ \mathcal{Q}(\mathcal{Q}_n(a)) - \mathcal{Q}(\mathcal{Q}_{n-1}(a)) \right] = 0 \]

\( \mathcal{Q}_n \) denotes the \( n \)-th iterate of \( \mathcal{Q} \). Under these conditions at most one convex solution exists. In order to have one solution it is necessary that

\[ \lim_{x \to \infty} \frac{\mathcal{Q}(x)}{x} = L \]

The problem may be considered as a generalization of the FE

\[ \mathcal{Q}(x+1) = \mathcal{Q}(x) \quad \text{with} \quad \mathcal{Q}(0) = 0 \]

of which the function \( \mathcal{Q}(x) \) is the only convex solution.

On the FE

\[ \mathcal{Q}(x) + \mathcal{Q}(F(x)) = \mathcal{Q}(F(x)) \]

Ann. polon. math. 6 (1959) 231-237.

If \( \mathcal{Q}(x) \) and \( \mathcal{Q}(a) \) are continuous in the closed interval \([a,b]\), and

- \( \mathcal{Q}(x) \) is strictly increasing, there exists an infinite number of solutions continuous in the open interval \((a,b)\) while not more than one solution is continuous at \( a \).
Monotonic solutions of the FE
\[ g\left(x, f(x)\right) - g(x) = \varphi(x) \]
are sought which take a prescribed value at a given point. The result is related to the Beta function.

General solution of a FE (E).

This FE is
\[ g\left(x, f(x)\right) = \varphi\left(f(x)\right) \]

On continuous solutions of a FE (E).

The FE
\[ g\left(x, f(x)\right) = \varphi\left(f(x)\right) \]
is solved for \( \varphi(x) \) under special conditions.

On the form of solutions for some FE s (E). *

Remarks on some FE s (E). *

Kuczma, M. and Vopenka, P.

On the FE
\[ \lambda\left[f(x)\right] \lambda(x) + A(x) \lambda(x) + B(x) = 0 \quad (E) \]

Conditions are given under which a continuous solution exists.
A uniqueness theorem for a linear FE (E).


It is shown that under suitable restrictions for the FE
\[ f(x) = \int_{-\infty}^{x} q(y) \, dy \]
there exists a unique solution which - on the first derivatives of which -
takes prescribed values at a given point.

On the form of solutions of some FE's (E).

The solution
\[ \frac{1}{2} F(x) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left( \frac{1}{k} - \frac{1}{k+1} \right) \]
is given for the FE
\[ \phi(x, y) = \int_{-\infty}^{x} q(y) \, dy \]

The generalization
\[ \phi(x, y) = \int_{-\infty}^{x} q(y) \, dy \]
is considered.

General solution of the FE
\[ \phi(x, y) = \int_{-\infty}^{x} q(y) \, dy \]

On monotonic solutions of a FE I. (E)

The FE
\[ \phi_2(x) = \int_{-\infty}^{x} \phi_1(y) \, dy \]
is investigated, \( \phi_2(x) \) being the second iterate of \( \phi_0(x) \).

It is shown that under suitable conditions infinitely many strictly increasing and continuous solutions exist in an interval.
(Kuczma, M.)

On monogenic solutions of a FE II. 

On some FEs containing iterations of the unknown functions (E.).

Kurepa, S.

On some FEs (E.).

The solutions of the following FEs are given:

\[ f\left(x_1, x_2, x_3, x_4\right) + f\left(x_1, x_2, x_3, x_4\right) = f\left(x_1 + x_2, x_3, x_4\right) + f\left(x_1, x_2 + x_3, x_4\right) + f\left(x_1, x_2, x_3 + x_4\right) + f\left(x_1, x_2, x_3, x_4\right) \]

It turns out that the solutions are appropriate combinations of arbitrary functions; those are differentiable if the unknown function is assumed to be differentiable.

On some FEs in Banach space (E.).

On the FE

\[ f\left(x + y\right) = f\left(x\right)f\left(y\right) - g\left(x\right)g\left(y\right) \]  
(E).

The explicit solution of the above FE is given in terms of exponential functions.
On the FE

\[ \int f(x) \, dx = \mathcal{F} \left[ \mathcal{F} \left[ f(x) \right] \right] \quad (F). \]

Mémo. Sci. Univ. Kyoto, A. Math. 27 (1951) 139-144.

A special case of the problem of the previous paper is dealt with

\[ f(x+y, z+w) = R \left( f(x, z), f(x, w), f(y, z), f(y, w) \right) \]

where \( R \) is rational.

On the analytic function of four complex variables satisfying associativity (F).


The above FE is treated under the assumption that for some complex \( C \)

\[ f(C, C) = C \]

is valid and that \( f \) can be expanded into a power series.

On functions of two variables satisfying an algebraical addition theorem (F).


The "impossible addition theorem" in two variables

\[ P \left( f(x+y, z+w), f(x+z, y+w), f(x, y), f(z, w) \right) = 0 \]

is treated, \( P \) being a polynomial and \( f \) assumed to be analytic in both variables.

Lambek, J. (See Moser, L.)
Levine, N. (See Elyash, E. S.,)

McLeod, J. B. (See Chung, T. W.,)


Meynieux, R. On a theorem about the analyticity of the solution of a FE (F).

Now results on the piece-wise analyticity of the solutions of the FE

\[ \mathcal{F} \left[ \mathcal{L} \left( \frac{f(u)}{u}, \frac{g(y)}{y} \right) \right] = g(u + v) \]

are given.

On analyticity of the continuous solutions of a FE (F).
C.R. Acad. Sci. 254 (1962) 4412-4414
(See the preceding paper.)

Results of a similar type are given for the FE

\[ \mathcal{O} \left[ \mathcal{L} \left( \frac{f(u)}{u}, \frac{g(y)}{y} \right) \right] = g(u + v) \]

Mikusiński, J. G. (See also Aczél, J.,)


Mitrinović, D. S. On a process furnishing FEs the continuous and differentiable solutions of which can be determined (F).

The FE

\[ \left[ \mathcal{L} \left( f(x), A g(y) \right) \right] \left[ \mathcal{L} \left( g(y), B f(y) \right) \right] = f(x) \cdot g(y) \]

is solved besides this the paper deals with some differential-functional equations.

Mitrinović, D. S. and Doković, D. On certain FEs the general solutions of which can be determined (E).
Mitrinović, D.S., and Presić, S.B. On a cyclic, non-linear FE (F).

The FE
\[
F(x_1, \ldots, x_{2n}) = \left[ f(x_1, x_2) + \cdots + f(x_{2n-1}, x_{2n}) \right] x
\]
\[
\times \left[ f(x_{2n+1}, x_{2b+2}) + \cdots + f(x_{2n-1}, x_{2n}) \right]
\]

where \( F \) is known solution of a cyclical FE is given in terms of arbitrary functions.

Morgantini, E. On equations in six variables which can be represented by a point
topograph \((E)\).

Mycielski, J., and Paszkowski, S. On a problem of probability calculus \((F)\).
Studia math. 15 (1956) 188-200

The motion of a molecule on a straight line is considered by a
method involving FEs.

Milkman, J. Note on the FE
\[
f(x, y) = f(x) + f(y), \quad f(x^2) = f(f(x))(E).
\]

Solution of those FEs are given under assumptions on the set of which
the functions are defined.

The Logarithmic function is unique \((E)\).

The FE
\[
f(x) + g(y) = x\cdot y
\]

is treated by reducing it to \( F(x) + F(y) = F(xy) \).
Moser, L., and Lambek, J.


It is shown that

$$f(m, n) = f(n)f(n); \quad (m, n) = 1; f(n) \neq 0$$

and

$$f(m) \geq f(n) \quad \text{for} \quad m \geq n$$

implies

$$f(n) = n^{k/2}$$

$k$ being constant. The analogous case for continuous argument is well-known; the interest of the paper lies in the fact that $f$ is a number theoretical function, i.e. defined for positive integer arguments.

Méter, W.

On the cyclotomic structure of certain FE's $(G)$.


FE's with cyclotomic relations between the variables are studied.

Mitrinovic, D. S., and Djokovic, D.

On an extended class of FE's $(F)$.


An operation consisting of substitutions and summations is defined; the corresponding FE is solved in terms of the same operation.

Norris, M. J.

(See Bium, J.R.)

Pessidas, N.

On the FE's of Polya's type $(F)$. *

Composition Math. 10 (1932) 162-212.

Pazekowski, S.

(See Mycielski, J.)

Pietiläinen, T.

On continuous systems of functions with an algebraic addition theorem $(G)$.


Continuous functions \( f_1, f_2 \) \( on \) the real interval \( [0, 1] \)

are investigated under the assumption that
\[
\begin{align*}
\frac{f_1 (u + v)}{f_2 (u + v)} \quad \text{are algebraic functions of} \\
\frac{f_1 (u)}{f_2 (u)}, \frac{f_1 (v)}{f_2 (v)}, \frac{f_2 (u)}{f_2 (v)}
\end{align*}
\]

\[\left[0, T\right], \text{can be divided into a finite number of sub-intervals so that} \]
\[
\frac{f}{g} \text{ are analytic in each of them.}
\]

**Penzagl, J.**

Automatic foundations of a general theory of measurement (G).

This book is related to the theory of PEs through the theory of means, which plays a central part in it.

**Poporescu, N.**

On singular PEs (R).

FEs in the theory of Stochastic processes are investigated.

**Prestić, S. B.**

(See also Mirković, S. B.)

**Prestić, S.**

On the FE of translation (F).

On the FE

\[
\mathcal{F}(x) = \int \mathcal{G}(x) \]


The general solution of the above equation is given as well as examples.

**Radström, H.**

Some elementary PEs and Hilbert's Fifth problem (Sw).
Nordisk. mat. Tidskr. 3 (1955) 129-147.

In the context given in the title, the FE

\[
F(x + y) = F(x) + F(y)
\]
In dealing with linear functions, the general form is:

\[ y = f(x) \]

where \( y \) is the dependent variable, \( x \) is the independent variable, and \( f(x) \) is the function representing the relationship between \( x \) and \( y \).
Some new results are gained, other known results confirmed, dealing with problems in electromagnetic theory; the main interest lies in the method. Instead of starting from the Maxwell equations, the author derives his results from FE's, such as

\[ \mathcal{F}(x, y) \mathcal{F}^{-1}(x, y) = \mathcal{F}(x + y, y) \]

for a class of stationary transfer matrices in a multimode cavity.

On solutions of Riccati's equation \( \alpha \); functions of the initial values (E).

\( \text{J. reil. Math. Anal.} 3 (1951) 285-293. \)

Denote by \( \mathcal{F}(x) \) the solution of the Riccati equation which vanishes for \( x=1 \); for \( \mathcal{F}(x) \) generally, which equals \( \mathcal{F}(1) \) for \( x=1 \). Where \( \mathcal{F}(x) \) is given, introducing two auxiliary functions, the author shows that \( \mathcal{F}(x) \) satisfies a number of FE's.

\[ \mathcal{F}(x) = \mathcal{F}_1(x) + \mathcal{F}_2(x) \]

\[ \mathcal{F}_1(x) = \alpha x \quad \mathcal{F}_2(x) = \gamma x \]

and \( \mathcal{F}_1(x) = \alpha x + \beta x^2 \). With constant \( \alpha \) and constant and real \( \beta \), the only regular functions satisfying the FE

\[ \left| \mathcal{F}(x, y) \right| = \left| \mathcal{F}_1(x) + \mathcal{F}_2(y) \right| \]

\[ \mathcal{F}_1(x) = \mathcal{F}_2(x) \]

and \( \mathcal{F}_2(x) = \mathcal{F}_1(x) \). The authors derive, under some general and sophisticated conditions,
that \( k_1 x_1 \) and \( k_2 x_2 \sin k_2 x \) are the only solutions for complex \( x \), and \( c_1 x_1, c_2 \sin x_2, c_y \cosh c_y x \) are the only solutions for real \( x \).

Sakovich, G.N. Solution of a FE of several variables (\( q_m \)).

The
\[
\mathcal{L} \left[ \mathcal{F} \left( \frac{t}{2} \right) \right] = \mathcal{F} \left( \frac{t}{2} \right) C_x - \mathcal{F}
\]
is investigated; here \( \mathcal{F} \) is a real vector and \( C_x \) is a given sequence of non-singular matrices.

San Juan, R. An application of Diophantine representation to the FE
\[
\mathcal{F} \left( x_1, x_2 \right) = \mathcal{F}' \left( x_1 \right) + \mathcal{F}' \left( x_2 \right)
\]

It is shown that every finite solution of this equation is either linear or its graph is everywhere dense in the plane.

Segal, S.L. (See Rosenbaun, R.A.)

Slevet, E. (See Kalin, J.)

Schinzel, A. (See Golab, A.)

Seebach, L.L., and Jewett, J.W. A development of logarithms using the function concept. \( (E) \).

Sharkovski, A.N. On the solution of a class of FE (\( q_m \)).

The FE
\[
\mathcal{F} \left( \mathcal{L}, \mathcal{F}' \left( x \right), \mathcal{F} \left( \mathcal{F} \left( x \right) \right) \right) = 0
\]
is treated.
A class of mean formulas \((R)\).


A number of means value theorems is geometrically interpreted in terms of tangents to parametrically given curves.

On a property of the parabola and the solution of a FE \((R)\).


Remarks in connection with FEs \((R)\).


On the FE

\[
\ell_{\frac{f}{f}}(x + y) = \frac{f}{f}(x) + \frac{f}{f}(y) + \frac{f}{f}(x)\frac{f}{f}(y)
\] 

\((R)\).


The solution for the above addition theorem is given.

Contributions to the integration of a FE \((R)\).


On a class of FEs \((R)\).

Gaz. mat.-fiz. (A) 11 (1960) 587-598.

---

Straus, E. G.

(See Aczél, J.)

Szekeres, G.

Regular iteration of real and complex functions \((E)\).


The paper is devoted to the solution of the Schröder equation

\[
\ell\left[g(\ell(x))\right] = \lambda \ell(x)
\]
under more general conditions than those given by earlier authors.
The result is relevant to the question of non-integral iteration indices,
since it provides a solution of the translation equation

\[ F \left( \frac{1}{\lambda} - 1 \right) = F \left( \frac{1}{\lambda} + 1 \right) \]

since...

\[ F \left( \frac{1}{\lambda} - 1 \right) \]

is such a solution; thus

\[ g \left( \frac{1}{\lambda} - 1 \right) \]

can be considered as the \( n \)-th iterate of \( g(x) \), where \( n \) can be arbitrary real, or complex, under suitable conditions.

Targonski, Gp. I. (See also Targonski, Gp. II.)

Targonski, Gp. II. The representation of functions by means of chain series (C).

Public. Math. 2 (1951) 271-274.

The problem is solved under some restriction, to find the representation

\[ f(x) = x - g(1) + g(2) - g(3) + \cdots \]

where \( g(1), g(2), \ldots \) is the \( n \)-th iterate of the unknown function \( g \),
which is shown to be the solution of the FE

\[ f \left( \frac{g(x)}{x} \right) = f \left( \frac{f(x)}{x} \right) \]

Convergence and uniqueness is proved. Asymptotic expansions like

\[ \log(x) \approx \frac{1}{x} \left[ (x - 1) + \frac{1}{2} e^{-x/(x+1)} \right] \]

where \( \sin x \approx x(1 + (x^2 - x) \cdot \sin x \sqrt{1 - x^2}) \)

result for small \( x \).

Thielman, H. P. On generalized means (C).

On generalized Cauchy FE (E).

\[ F(x, y + ny) = \frac{G(x)}{y} e^{y} \]

is investigated under the condition
\[ x > \frac{1}{y} = \frac{1}{n}, \quad y > \frac{1}{n}. \]

The general solution turns out to be
\[ F(x) = a(x) \left( 1 + ny \right) \]
\[ g(x) = q (1 + nx)^{\frac{1}{y}} \]
\[ \frac{dF}{dx} = \frac{q}{y} \left( 1 + nx \right)^{\frac{1}{y}} \]
a, b, k, being constants.

On a pair of FE (E).

The FE
\[ F(x, y) = p(x, y) g(x) h(y) \]
\[ f(x, y) = q(x) f(y) \]

are solved by first reducing them to the pair
\[ \frac{f(x, y)}{g(x)} = \frac{q}{p(x)} \]
\[ f(x, y) = r(x) q(y) \]

and solving this latter pair.

A note on a FE (E).
Amer. J. Math. 73 (1951) 462-484.

Sufficient conditions are given under which an "operation between real numbers" \( x \circ y = \frac{F(x, y)}{G(x, y)} \) can be written in the form
\[ (x \circ y) = \left( \frac{1}{p} \right) \left[ f(x) + g(y) \right] \]
where \( f \) is continuous and strictly monotone. In particular, if \( x \to x_1 \) is a polynomial of degree higher than 1, then

\[
\int_{x_1}^{x_2} f(\theta) \, d\theta = \log(Ax + B) + \log(K) - \log(K_1) \tag{1}
\]

**Van den Berg, J.**

*Societé van den, J.*

**Vaughan, H. E.**

Characterization of the Sine and Cosine (G).


The well-known FE

\[
g_k(x, y) = g_k(x, y) + f(x, y)
\]

is solved.

**Vincze, E.**

On the characterization of associative functions of several variables (G).


The theorem that
\( x \cdot y = F(x, y) = \Phi^{-1}[\Phi(x) + \Phi(y)] \)

for any continuous, strictly monotone associative operation is generalized
to simultaneous operations on \( n \) variables; two different types of formulae arise according to whether \( n \) is odd or even.

(Vincze, E.)

A generalization of the FE of Abel-Poisson (H).


The generalization

\[ F(x+y) + G(x-y) = \Phi(x) \Phi(y) \]

of the d'Alambert-Poisson FE

\[ C(x+y) + C(x-y) = 2 C(x) C(y) \]

is solved in the most general form (complex solution).

Vopenka, P.

(See Kuczma, M.)

Wilner, J.A.

(See Wilner, I.A.)

Wing, G.M.

(See Blum, J.R.)

Wundt, H.

On a FE in the theory of heat conduction (G).


The FE:

\[ \int \left( \frac{x-y}{\log x - \log y} \right) = \frac{\Phi(x) + \Phi(y)}{2} \]

is treated in detail. The most general differentiable solution is given as

\[ f(x) = A \int \frac{\log t + t^{-1}}{t - \log t - 1} + B \]

Later, M. Haszhu showed that \( f(x) \) is a solution only if \( f(x) \) reduces
to a constant. (See under Haszhu, M.)
Young, G. S.

The linear FE \( (E) \)

This is a concise proof that every bounded solution of the FE

\[
\ell(x, \varphi) = \int_{\Omega} u(x) \varphi(x) \, dx
\]

is of the form:

\[
\varphi(x) = \frac{1}{\sqrt{\Omega}} \sum_{n=1}^{N} \hat{u}_n \varphi_n(x)
\]