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FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

N-63-4-3

ESD TDR 63-387

A STEREOGRAPHIC CO-ORDINATE SYSTEM FOR THE UTILIZATION OF DATA FROM SEVERAL RADARS

MARCH 1959

THE MITRE CORPORATION

SR-2

A STEREOGRAPHIC CO-ORDINATE SYSTEM

FOR THE UTILIZATION OF DATA

FROM SEVERAL RADARS

The research reported in this document was supported by the Department of the Air Force under Air Force Contract AF-33 (600) 37882.

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TABLE OF CONTENTS

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I. INTRODUCTION

The utilization of information from several radars tracking the same target requires that the data of slant range, azimuth, and height as reported by the various radars of the system be referenced to a common origin of co-ordinates. Experience has shown that the data are best processed in the form of rectangular co-ordinates. Therefore, it becomes necessary to determine a method of projecting the surface area of the earth covered by the radar system onto a plane containing a set of rectangular co-ordinates in order that the positions of, and data from, each radar may all be referenced to a common origin for the system.

Cartographers have devised many methods of projecting the earth onto a plane. Each was developed to satisfy various requirements of representation and navigation. For the problem at hand, the choice of the best method of projection must be governed by the simplicity with which slant-range, azimuth, and height data can be converted into rectangular co-ordinates relative to the radar and by the facility with which these co-ordinates can be referenced to the origin of the master grid of co-ordinates for the system. The accuracy of these processes will determine the accuracy in positioning a reported target relative to the origin as well as the degree of displacement between the positions of a single target as reported by several radars.

The problem has been separated into two aspects, conversion and transformation, not only for convenience in analysis but also because transformation is a separate problem unto itself, arising when it is necessary to communicate the position of a target in one system of radars to another system. The two aspects of the problem may be defined as follows:

(a) **Conversion** of the data of slant range, azimuth, and height to rectangular co-ordinates in a plane with the radar as the origin.

(b) **Transformation** of the rectangular co-ordinates of a target in one plane into the rectangular co-ordinates in another plane. The same method of projection must be applied to all radars within a system, both for conversion and transformation. An exchange of information between systems employing differing methods of projection is possible, but extremely complicated.

Reference 1 analyzes these problems for the gnomonic, stereographic, oblique Mercator, and Lambert conformal conic methods of projection and concludes that, for most practical radar networks, considerations of simplicity and accuracy make the stereographic projection the most suitable for accomplishing the required conversions and transformations of coordinates. This conclusion is based on the following considerations:

(a) The approximation

$$D = \sqrt{S^2 - H^2}$$

to the conversion equation,

where

D = the distance of the target from the radar measured in the plane of projection,

S = Slant range of target with respect to the radar, and

H = Target altitude,

yields the smallest possible maximum conversion error. This error is independent of the extent of the network and of radar range; this is not true of the other methods of projection.

(b) The value of D is independent of the target's azimuth with respect to the radar.

(c) If Taylor expansions are used to approximate the transformation equations, the sum of the maximum errors in conversion and transformation is less for the stereographic projection than for the other methods investigated under most practical conditions.

(d) The accuracy of projection is independent of the geographic latitudes of the radars in the network.

(e) Less and simpler equipment is required for exact solutions to the transformation equations by means of analog or wired program computers if the stereographic projection is used.

It is the purpose of this report to develop the equations of the stereographic projection more fully than is accomplished in reference 1. The equations will first be developed for a spherical earth and then modified to take the earth's ellipticity into account.

The stereographic projection is made by placing a plane tangent to the surface of the earth and projecting this surface onto this plane by lines drawn from the point diametrically opposite to the point of tangency through the points on the earth's surface to be projected (see Figure 1). It is sometimes advantageous to elevate the plane of projection; that is the projection is made onto a plane parallel to the tangent plane.

References 2, 3 and 4 discuss the applications of the stereographic projection to cartography.

II. THE TRANSFORMATION EQUATIONS

A. The Exact Transformation Equation

The transformation equation is derived in reference 1 and is given in complex notation by

$$w = \frac{w_o + z' e^{-i\beta}}{1 - \frac{\bar{w}_o z' e^{-i\beta}}{F^2}} \qquad \text{Eq. 1}$$

where

$w = u + iv$ = the rectangular coordinates of the target with respect to the origin of coordinates,

$w_o = u_o + iv_o = W_o e^{i\left(\frac{\pi}{2} - \gamma\right)}$ = the rectangular and polar coordinates, respectively, of the radar with respect to the origin of coordinates,

$\bar{w}_o = u_o - iv_o$,

$z' = x' + iy' = De^{i\left(\frac{\pi}{2} - \theta\right)}$ = the rectangular and polar coordinates, respectively, of the target with respect to the radar,

$F = 2E$ = the diameter of the spherical earth,

β = an angle which depends only on the coordinates of the radar with respect to the origin of coordinates. (γ and θ are measured clockwise from the positive y-axis.)

w_o and β are computed from the latitudes and longitudes of the radars and the center of coordinates as follows:

$$w_o = u_o + iv_o = 2E \frac{\sin \Delta L \cos L + i(\sin L \cos L_o - \cos L \sin L_o \cos \Delta\lambda)}{1 + \sin L \sin L_o + \cos L \cos L_o \cos \Delta\lambda} \qquad \text{Eq. 2}$$

$$\tan \beta = \frac{(\sin L_o + \sin L)\sin(\lambda - \lambda_o)}{\cos \Delta\lambda + \cos L_o \cos L + \sin L \sin L_o \cos \Delta\lambda} \qquad \text{Eq. 3}$$

where

L_o, λ_o = latitude and longitude, respectively, of the origin of coordinates

L, λ = latitude and longitude, respectively, of the radar

$\Delta\lambda = \lambda - \lambda_o$

β represents a rotation of the z'-plane with respect to the w-plane.. For longitudes west of Greenwich this rotation is clockwise when $\lambda - \lambda_o > o$ and counterclockwise when $\lambda - \lambda_o < o$. The direction of rotation is reversed for longitudes east of Greenwich. The effect of this rotation is to make the axes of the two planes more nearly parallel.

Since β is a constant for any radar with respect to a particular origin of coordinates, the $e^{-i\beta}$ terms in Eq. 1 can be eliminated either by changing the north orientation of the radar antenna or by adding a constant to each azimuth. Eq. 1 then becomes

$$w = \frac{w_o + z}{1 - \dfrac{\overline{w}_o z}{F^2}} \qquad \text{Eq. 4}$$

where $z = z' e^{-i\beta} = x + iy = De^{i\left(\frac{\pi}{2} - \theta - \beta\right)}$

Expanding Eq. 4 yields

$$w = (w_o + z)\left[1 + \sum_{n=1}^{\infty}\left(\frac{\overline{w}_o z}{F^2}\right)^n\right]$$

$$w = w_o + w_o \sum_{n=1}^{\infty}\left(\frac{\overline{w}_o z}{F^2}\right)^n + z + z\sum_{n=1}^{\infty}\left(\frac{\overline{w}_o z}{F^2}\right)^n$$

$$= w_o + W_o^2 \sum_{n=1}^{\infty} \frac{\overline{w}_o^{\,n-1} z^n}{F^{2n}} + z + \sum_{n=1}^{\infty} \frac{\overline{w}_o^{\,n} z^{n+1}}{F^{2n}}$$

$$= w_o + \left(\frac{W_o}{F}\right)^2 \sum_{n=o}^{\infty} \frac{\overline{w}_o^{\,n} z^{n+1}}{F^{2n}} + \sum_{n=o}^{\infty} \frac{\overline{w}_o^{\,n} z^{n+1}}{F^{2n}}$$

$$= w_o + \left[1 + \left(\frac{W_o}{F}\right)^2\right] \sum_{n=o}^{\infty} \frac{\overline{w}_o^{\,n} z^{n+1}}{F^{2n}}$$

$$= w_o + \left[1 + \left(\frac{W_o}{F}\right)^2\right] \sum_{n=o}^{\infty} \frac{W_o^{\,n} D^{n+1}}{F^{2n}} \exp.\left\{i\left[\frac{\pi}{2} - \theta - n(\theta - \gamma)\right]\right\} \qquad \text{Eq. 5a}$$

Equation 5a shows that for a particular radar, w is the vector sum of a number of terms whose amplitudes are rapidly decreasing but are independent of the azimuthal angle θ. Separating Eq. 5a into its real and imaginary parts, there results

$$u = u_o + \left[1 + \left(\frac{W_o}{F}\right)^2\right] \sum_{n=o}^{\infty} \frac{W_o^{\,n} D^{n+1}}{F^{2n}} \sin\left[\theta + n(\theta - \gamma)\right] \qquad \text{Eq. 5b}$$

$$v = v_o + \left[1 + \left(\frac{W_o}{F}\right)^2\right] \sum_{n=o}^{\infty} \frac{W_o^{\,n} D^{n+1}}{F^{2n}} \cos\left[\theta + n(\theta - \gamma)\right]. \qquad \text{Eq. 5c}$$

For some applications it is desirable to have exact expressions for u and v in closed form. This can be achieved by multiplying the numerator and denominator of Eq. 4 by the conjugate of the denominator.

The result is

$$u = \frac{u_o + x + \dfrac{(u_o^2 - v_o^2)\,x + 2u_o v_o y + u_o(x^2 + y^2)}{F^2}}{1 + 2\,\dfrac{u_o x + v_o y}{F^2} + \dfrac{(u_o^2 + v_o^2)(x^2 + y^2)}{F^4}} \qquad \text{Eq. 6a}$$

$$v = \frac{v_o + y + \dfrac{(v_o^2 - u_o^2)\,y + 2u_o v_o x + v_o(x^2 + y^2)}{F^2}}{1 + 2\,\dfrac{u_o x + v_o y}{F^2} + \dfrac{(u_o^2 + v_o^2)(x^2 + y^2)}{F^4}} \qquad \text{Eq. 6b}$$

These equations may also be expressed in the form

$$u = \frac{W_o \sin\gamma + D\sin\theta + W_o D\,\dfrac{W_o \sin(2\gamma - \theta) + D\sin\gamma}{F^2}}{1 + \dfrac{W_o D}{F^2}\left[\cos(\gamma - \theta) - \dfrac{W_o D}{F^2}\right]} \qquad \text{Eq. 7a}$$

$$v = \frac{W_o \cos\gamma + D\cos\theta + W_o D\,\dfrac{W_o \cos(2\gamma - \theta) + D\cos\gamma}{F^2}}{1 + \dfrac{W_o D}{F^2}\left[\cos(\gamma - \theta) - \dfrac{W_o D}{F^2}\right]} \qquad \text{Eq. 7b}$$

The first and second order approximations to Equations 5 are given below and the errors resulting from these approximations evaluated.

B. First Order Approximation

$$w = w_o + \left[1 + \left(\frac{W_o}{F}\right)^2\right] D \exp\left[i\left(\frac{\pi}{2} - \theta\right)\right] \qquad \text{Eq. 8a}$$

$$= w_o + D^I \exp\left[i\left(\frac{\pi}{2} - \theta\right)\right]$$

where $$D^I = \left[1 + \left(\frac{W_o}{F}\right)^2\right] D$$

$$u = u_o + D^I \sin\theta \qquad \text{Eq. 8b}$$

$$v = v_o + D^I \cos\theta. \qquad \text{Eq. 8c}$$

The transformation error ϵ_t is of the order of the next term of the series in Eq. 5

$$\left|\epsilon_t\right| \sim \frac{W_o D^2}{F^2}. \qquad \text{Eq. 8d}$$

C. Second Order Approximation

$$w = w_o + \left[1 + \left(\frac{W_o}{F}\right)^2\right] \left\{ D \exp\left[i\left(\frac{\pi}{2} - \theta\right)\right] + \frac{W_o D^2}{F^2} \exp\left[i\left(\frac{\pi}{2} - 2\theta + \gamma\right)\right] \right\}$$

$$= w_o + D' \exp\left[i\left(\frac{\pi}{2} - \theta\right)\right] + K(D')^2 \exp\left[i\left(\frac{\pi}{2} - 2\theta + \gamma\right)\right] \qquad \text{Eq. 9a}$$

where $K = \frac{W_o}{\left[1 + \left(\frac{W_o}{F}\right)^2\right] F^2}$ = constant for any particular radar

$$u = u_o + D' \sin\theta + K(D')^2 \sin(2\theta - \gamma) \qquad \text{Eq. 9b}$$

$$v = v_o + D' \cos\theta + K(D')^2 \cos(2\theta + \gamma) \qquad \text{Eq. 9c}$$

$$\left|\epsilon_t\right| \sim \frac{W_o^2 D^3}{F^4}. \qquad \text{Eq. 9d}$$

D. The Transformation Equation for Elevated Planes of Projection

Let h_r = the elevation of the plane of projection at the radar site

h_c = the elevation of the plane of projection at the center of coordinates

then

$$w = \frac{w_o + \dfrac{1 + \frac{h_c}{F}}{1 + \frac{h_r}{F}} z}{1 - \dfrac{\overline{W}_o z}{\left(1 + \frac{h_r}{F}\right)\left(1 + \frac{h_c}{F}\right) F^2}} \qquad \text{Eq. 10}$$

The $1 + \left(\frac{W_o}{F}\right)^2$ term in Eqs. 7, 8 and 9 now becomes

$$\frac{1 + \frac{h_c}{F}}{1 + \frac{h_r}{F}} + \frac{W_o^2}{\left(1 + \frac{h_c}{F}\right)\left(1 + \frac{h_r}{F}\right) F^2}.$$

III. THE CONVERSION EQUATIONS

The conversion problem is that of expressing the quantity D which occurs in the transformation equations as a function of the slant range S and the altitude H. It is useful to define a quantity R by

$$R = \sqrt{S^2 - H^2} = S\sqrt{1-\left(\frac{H}{S}\right)^2}. \qquad \text{Eq. 11}$$

If the plane of projection is elevated by a distance h_r above the earth's surface, the conversion equation for a spherical earth is given by

$$D = R \frac{1 + \frac{h_r}{2E}}{\sqrt{1 + \frac{H}{E} - \frac{R^2}{4E^2}}} \qquad \text{Eq. 12a}$$

$$\sim R\left[1 + \frac{h_r - H}{2E} + \frac{R^2}{8E^2}\right]. \qquad \text{Eq. 12b}$$

The fact that h_r and H in Eq. 12b are of opposite sign indicates that the error resulting from approximations to D may be minimized by a judicious choice of h_r. It is shown in reference 1 that if D is approximated by R, the maximum error is minimized when

$$h_r = \left\{\left[\frac{1 + \left(1 + \frac{H_M}{E}\right)^{1/3}}{2}\right]^{3/2} - 1\right\} 2E \sim \frac{H_M}{2} \qquad \text{Eq. 13}$$

where H_M is the maximum altitude at which targets can be detected.

The error resulting from approximating D by R in Eq. 12b after setting $h_r = \frac{1}{2} H_M$ is given by

$$\epsilon_c = \frac{H_M - 2H}{4E} R + \frac{R^3}{8E^2}. \qquad \text{Eq. 14a}$$

The maximum value of H is H_M, and its minimum value is the height corresponding to the radar's horizon. Setting $H = H_M$ in Eq. 14a yields

$$\epsilon_c = \frac{R^3}{8E^2} - \frac{H_M}{4E} R. \qquad \text{Eq. 14b}$$

It is shown in reference 1 that when the minimum value of H is used in Eq. 14a the result, over the range of interest, is very nearly equal and opposite to that obtained for $H = H_M$. Thus, ϵ_c varies between the limits

$$\frac{R^3}{8E^2} - \frac{H_M R}{4E} < \epsilon_c < \frac{H_M R}{4E} - \frac{R^3}{8E^2} . \qquad \text{Eq. 14c}$$

The maximum value of $|\epsilon_c|$ occurs at approximately

$$R = \sqrt{\frac{2}{3} H_M E} \qquad \text{Eq. 15a}$$

and is very nearly equal to

$$|\epsilon_c| \text{ (max.)} = \sqrt{\frac{2}{3}} \; \frac{1}{6} E \left(\frac{H_M}{E}\right)^{3/2} \qquad \text{Eq. 15b}$$

For $H_M = 10$, $|\epsilon_c|$ (max) is 0.073 n. mile at $R = 151$ n. miles.

It is shown in Appendix A that if the radar site is elevated by a distance H_R above the earth, the plane of projection should be elevated by

$$h_r = H_R + \frac{H_M}{2} . \qquad \text{Eq. 16}$$

This will increase the maximum conversion error by the factor

$$\sqrt{1 + \frac{H_R}{E}} .$$

IV. SUMMARY OF STEREOGRAPHIC PROJECTION ERRORS FOR A SPHERICAL EARTH

For a first order approximation to the transformation equations (Eqs. 8) the magnitude of the transformation error is approximately $W_o D^2/F^2$. If D is approximated by R, the maximum conversion error is given by Eq. 14c, that is

$$|\epsilon_c| = \frac{H_M R}{2F} - \frac{R^3}{2F^2} .$$

The total projection error is the sum of the transformation and conversion errors. Since $D \sim R \sim S$ this error can be expressed as a function of W_o, S, and H_M by

$$|\epsilon_p| \text{ (max)} = \frac{W_o S^2}{F^2} + \left| \frac{H_M S}{2F} - \frac{S^3}{2F^2} \right| \qquad \text{Eq. 17}$$

Figure 2 is a family of curves of $|\epsilon_p|$ (max.) versus S, with W_o as a parameter, $H_M = 10$ and $H_R = 0$. The curve for $W_o = 0$ represents the conversion error, and the difference between this curve and the curves for $W_o > 0$ is the error resulting from a first order approximation to the transformation equations. Figure 2 may be used to determine whether, for any particular combination of W_o and S, the error resulting from a first order approximation exceeds a prescribed limit so that a higher order approximation becomes necessary. For the range of values of W_o and S shown in Figure 2, the curve for $W_o = 0$ is also very nearly equal to $|\epsilon_p|$ max. for a second order approximation.

An inspection of Eq. 17 shows that the maximum projection error for a radar network varies linearly with the maximum value of W_o for that network. It is therefore desirable to choose the center of coordinates for a network so as to minimize the distance between this center and the radar from which it is furthest removed. The choice of h_c, the elevation of the plane of projection at the center of coordinates, allows a degree of freedom but no criterion for optimizing this choice seems to exist. It appears convenient to let $h_c = 0$.

V. THE EFFECT OF THE EARTH'S ELLIPTICITY ON THE STEREOGRAPHIC PROJECTION

A. Mathematical Figure of the Earth

In calculating the positions of points on the earth, it is necessary to assume some mathematical surface to represent the figure of the earth. The figure generally adopted is the oblate spheroid. Such a figure is generated by rotating an ellipse about its minor axis.

In Figure 3, the equation of the ellipse shown, with major and minor semiaxes a and b, referred to its own axes as coordinate axes, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ .$$

The eccentricity e is defined as the distance from the focus to the center divided by a and is defined by

$$e^2 = 1 - \frac{b^2}{a^2} \ .$$

The ellipticity (or flattening) is given by

$$f = 1 - \frac{b}{a} \ .$$

The normal to the surface of the spheroid is known as the geographic vertical, and the angle between this vertical and the equatorial plane is the geographic (or geodetic) latitude L.

The angle δ between the geographic vertical and the radius MO is called the deviation of the normal and is defined by

$$\tan \delta = \frac{e \left(1 - \frac{e}{2}\right) \sin 2L}{1 - (2e - e^2) \sin^2 L} \ .$$

The maximum value of δ is 11.59 minutes at L = 45.10 degrees. The distance MQ measured from M along the geographic vertical and terminating in the minor axis is called the normal N and is given by

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 L}} \ . \qquad \text{Eq. 18}$$

The dimensions of the earth to be used in the above expressions should be the same as those used by surveyors in preparing data for site locations. The International Ellipsoid (Reference 5) and the Clarke Spheroid of 1866 (Reference 6) are most commonly used for this purpose. These dimensions are as follows:

The International Ellipsoid

Semimajor axis – a:		6,378,388 meters
	=	3,441.734 727 U.S. nautical miles
	=	3,444.053 995 International nautical miles
Semiminor axis – b:		6,356,911.946 meters
	=	3,430.146 394 U.S. nautical miles
	=	3,432.457 854 International nautical miles
Ellipticity (flattening) – f:		1/297 = 0.006 722 670 022
Square of Eccentricity – e^2:		0.006 768 170 197

The Clarke Spheroid of 1866

Semimajor axis – a:		6,378,206.4 meters
	=	3,441.636 737 U.S. nautical miles
	=	3,443.995 939 International nautical miles
Semiminor axis – b:		6,356,583.8 meters
	=	3,429.969 329 U.S. nautical miles
	=	3,432.280 669 International nautical miles
Ellipticity (flattening) – f:		1/294.98
Square of Eccentricity – e^2:		0.006 768 657 997 291

1 U.S. nautical mile = 1,853.248 meters = 6,080.20 feet

1 International nautical mile = 1852 meters = 6,076.103 33 . . . feet

Effective July 1, 1954, the International nautical mile was adopted, in lieu of the U.S. nautical mile, for use in the Departments of Defense and Commerce (Reference 7).

It is emphasized that the units employed to represent the earth's dimensions, which in turn affect the constants and coefficients used in the conversion and transformation computations, must be consistent with the units in which the radars measure slant range.

B. The Effect of Ellipticity on the Transformation Equations

For conformal projections (the stereographic projection is conformal) the equations for projecting the spheroidal earth onto a plane are the same as those for the spherical earth, except that an angle, designated by ψ and known as the conformal latitude, must be used in place of the geographic latitude L. Thomas (reference 4, pp. 68 and 86) shows that a conformal projection of the spheroid is accomplished by using the conformal latitude ψ in place of the geographic latitude L and that this represents all conformal mapping of the spheroid on a plane. The conformal latitude is defined by

$$\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right) = \left[\frac{1 - e \sin L}{1 + e \sin L}\right]^{e/2} \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) . \qquad \text{Eq. 19a}$$

As an aid to computation Eq. 19a may be changed to the form

$$\sin \psi = \left[0.99327733 + 0.00666251 \sin^2 L + 0.00005959 \sin^4 L + 0.00000059 \sin^6 L + \ldots\right] \sin L \qquad \text{Eq. 19b}$$

where the constants have been calculated for the International Ellipsoid.

In addition to using the conformal latitudes ψ and ψ_o in place of the geodetic latitudes L and L_o to compute W_o and β in Eqs. 2 and 3 (ψ = L for a spherical earth), the registration error between observations of the same target by different radars will be minimized if the radius E of the spherical earth used in the equations of Section II is replaced by (see Reference 4, pp. 86 and 87)

$$E_o = N_o \frac{\cos L_o}{\cos \psi_o} . \qquad \text{Eq. 20}$$

N_o is the normal at the center of coordinates and is computed from Eq. 18.

C. The Effect of Ellipticity on the Conversion Equations.

Since the earth is an ellipsoid of revolution generated by revolving an ellipse about its minor axis, a cross-section of the earth is elliptical when obtained from a plane containing the polar axis, but a plane parallel to the equatorial plane will produce a circular cross-section. The effects of ellipticity will therefore be a maximum for targets on the same meridian as the radar, as shown in Figure 4. It is important to keep in mind that this figure is not drawn to scale. If it were, the distance of the target T from the position of the radar R would not exceed 3/8 inch, and the altitude H of the target above the earth's surface would have a maximum of 1/64 inch. The angle ϕ would then be smaller than 5 degrees, and the difference between the true altitude TP and the radial altitude TP$'$ is less than two parts per million. The angle P$'$TP is then very nearly equal to δ, the deviation of the normal, since the normal at P is essentially parallel to the normal P$'$. Also, the ellipticity of the earth, defined as the ratio of the difference between the equatorial and the polar radii to the equatorial radius, is 1/297. This is much smaller than is shown in the figure, so that the angle P$'$QR is very nearly equal to $\phi/2$, and the maximum value of δ is 11.6 minutes.

For a spherical earth, the projection is made onto the line RS$'$ which is perpendicular to the diameter QR. The point to be projected is P$'$ defined by the intersection of the surface with the extended radius OT. The projected distance is therefore

$$D = \overline{RS'} = 2E \tan\frac{\phi}{2} \qquad \text{Eq. 21}$$

For a spheroidal earth, the projection is made onto the line RS which is perpendicular to the normal at R. The point to be projection is P, defined by the normal to the surface from T. The distance $\overline{PP'}$ is Hδ, and the angle P$'$QP is approximated by Hδ/2E. The projected distance can then be found from the law of sines:

$$D_e = RS \sim \frac{\sin\left(\frac{\phi}{2} - \frac{H\delta}{2E}\right)}{\sin\left[180 - (90 - \delta) - \left(\frac{\phi}{2} - \frac{H\delta}{2E}\right)\right]} 2E = \frac{\sin\left(\frac{\phi}{2} - \frac{H\delta}{2E}\right)}{\cos\left[\frac{\phi}{2} - \left(1 + \frac{H}{2E}\right)\delta\right]} 2E \qquad \text{Eq. 22a}$$

Eq. 22a reduces to Eq. 21 when $\delta = 0$. By an exactly analogous procedure it can be shown that, when the target is at a latitude less than that of the radar, the projected distance is

$$D_e \sim \frac{\sin\left(\frac{\phi}{2} + \frac{H\delta}{2E}\right)}{\cos\left[\frac{\phi}{2} + \left(1 + \frac{H}{2E}\right)\delta\right]} 2E\ . \qquad \text{Eq. 22b}$$

If the angles ϕ and δ will be sufficiently small so that

$$\cos\left[\frac{\phi}{2} \pm \left(1 + \frac{H}{2E}\right)\delta\right] \sim 1$$

$$\sin\left(\frac{\phi}{2} \pm \frac{H\delta}{2E}\right) \sim \frac{\phi}{2} \pm \frac{H\delta}{2E}$$

$$\tan\frac{\phi}{2} \sim \frac{\phi}{2},$$

then from Eqs. 21 and 22

$$D_e \sim D \pm H\delta \qquad \text{Eq. 23a}$$

where the plus sign applies when the latitude of the target is smaller than that of the radar, and the minus sign indicates that the target's latitude is greater than the radar's. In general when target is not on the same meridian as the radar, but at an azimuth θ,

$$D_e \sim D - H\delta\cos\theta \; . \qquad \text{Eq. 23b}$$

When the plane of projection is elevated by a distance h_r above the earth's surface, both D_e and D are increased by the factor $1 - \frac{h_r}{2E}$. Eq. 23b then becomes

$$D_e \sim D - \left(1 + \frac{h_r}{2E}\right) H\delta\cos\theta \; . \qquad \text{Eq. 23c}$$

The conversion error due to ellipticity is therefore

$$e_c = D - D_e = \left(1 + \frac{h_r}{2E}\right) H\,\delta\cos\theta \qquad \text{Eq. 24}$$

The maximum value of e_c for $h_r = H_M/2$ and $H_M = 10$ is 0.034 nautical miles.

For the sake of completeness it should be mentioned that an error in both range and azimuth arises from the fact that the normals to the surface from two points not on the same meridian do not lie in the same plane. The magnitude of these errors may be evaluated by an extension of the analysis by Hosmer (Reference 8). The results show that these errors will be wholly negligible for most practical systems.

VI. REFERENCES

1. Lincoln Laboratory Technical Report TR 67, *"A Common Coordinate System for the Utilization of Data from Several Radars"* by D. Goldenberg and E. W. Wolf (1954).

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APPENDIX A

The Conversion Equations for Elevated Radar Sites

The derivations of the relations below, though somewhat more complex, are analogous to the corresponding relations in Reference 1. Only the results are shown.

Let H = altitude of target above sea level

H_R = altitude of radar above sea level

h_r = elevation of plane of projection above sea level

Define $R' = \sqrt{S^2 - (H - H_R)^2}$

$$\text{then } D = R' \frac{1 + \frac{h_r}{2E}}{\sqrt{1 + \frac{H + H_R}{E} + \frac{HH_R}{E^2} - \frac{(R')^2}{4E^2}}}$$

$$\sim R' \left[1 + \frac{h_r - (H + H_R)}{2E} - \frac{HH_R}{2E^2} + \frac{(R')^2}{8E^2}\right] \qquad \text{Eq. A1}$$

The maximum error resulting from approximating D by R' is minimized if

$$h_r = 2E \left\{\left[\frac{1}{2}\left(1 + \frac{H_R}{E}\right)^{1/3}\left[1 + \left(1 + \frac{H_M}{E}\right)^{1/3}\right]\right]^{3/2} - 1\right\}$$

$$\sim H_R + \frac{H_M}{2} \qquad \text{Eq. A2}$$

where H_M is the maximum altitude at which targets can be detected. For targets at altitude H_M, the maximum error is

$$\epsilon_M = 2E\sqrt{1 + \frac{H_R}{E}}\left[\frac{\left(1 + \frac{H_M}{E}\right)^{1/3} - 1}{2}\right]^{3/2} \sim \left(1 + \frac{1}{2}\frac{H_R}{E}\right) 2E\left(\frac{H_M}{6E}\right)^{3/2} . \qquad \text{Eq. A3}$$

The maximum error for targets at the radar's horizon is

$$|\epsilon_m| = 2E\sqrt{1 + \frac{H_R}{E}}\left[\frac{1 + \left(1 + \frac{H_M}{E}\right)^{1/3}}{2} - \left(1 + \frac{H_R}{E}\right)^{1/3}\right]^{3/2} . \qquad \text{Eq. A4}$$

Equations A3 and A4 are equal for $H_R = 0$. The effect of an elevated radar site is to increase the maximum error by the factor $\sqrt{1 + \frac{H_R}{E}}$.

APPENDIX B

Heading Angle Errors Caused by the Stereographic Projection

The stereographic projection does not project meridians as lines parallel to the v-axis in the w-plane. As a consequence the heading from true (geographic) north of a target with velocity components $\dot{u}$ and $\dot{v}$ is not given by $\tan^{-1}(\dot{u}/\dot{v})$. To obtain the correct heading the angle μ between the projection of the meridian passing through the point defined by the target's positional components u and v and the line parallel to the v-axis passing through that point must be added to $\tan^{-1}(\dot{u}/\dot{v})$. This angle is defined by

$$\tan\mu = \frac{(\sin\psi_o + \sin\psi)\sin(\lambda-\lambda_o)}{\cos(\lambda-\lambda_o) + \cos(\lambda-\lambda_o)\sin\psi_o\sin\psi + \cos\psi_o\cos\psi} \qquad \text{Eq. B1}$$

$$\sin\psi = \frac{2V\cos\psi_o + (1-U^2-V^2)\sin\psi_o}{1+U^2+V^2} \qquad \text{Eq. B2}$$

$$\tan(\lambda-\lambda_o) = \frac{2U}{(1-U^2-V^2)\cos\psi_o - 2V\sin\psi_o} \qquad \text{Eq. B3}$$

where

ψ, λ = longitude and conformal latitude of the target,

ψ_o, λ_o = longitude and conformal latitude of the center or coordinates,

$U = \frac{u}{F}$

$V = \frac{v}{F}$.

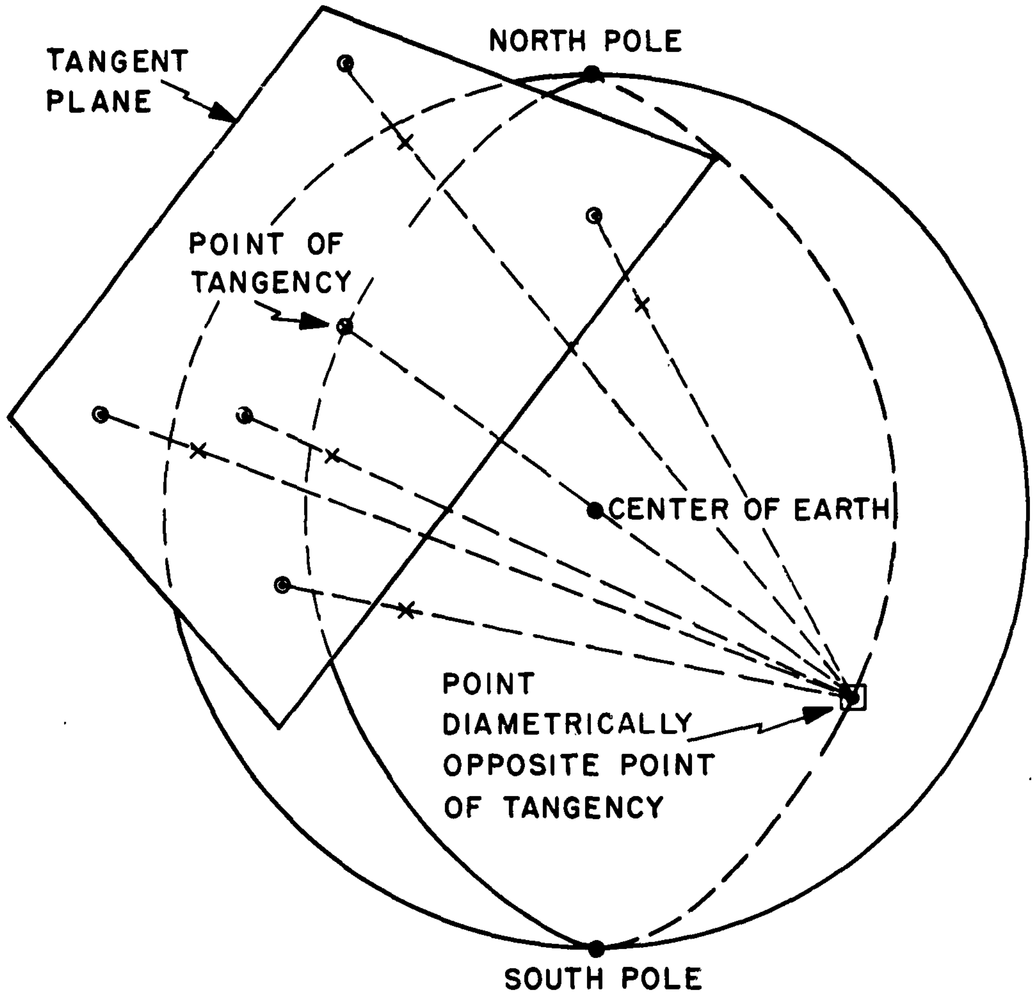
Note that the equation for μ is the same as that for β (Eq..3) and that Eqs. B2 and B3 represent the inverse solution of Eq. 2, permitting the computation of the latitude and longitude of a point from its u, v coordinates.

Equations B1 – B3 define μ as an explicit function of u and v. The first order term of the Taylor expansion of μ about the origin (u = v = o) is

$$\mu = \frac{\tan\psi_o}{E}\,u \ .$$

The heading of a target, measured clockwise from true north, is thus approximated without significant error by

$$\tan^{-1}\frac{\dot{u}}{\dot{v}} + \frac{\tan\psi_o}{E}\,u \ .$$



SYMBOLS :

× POINT ON EARTH

◉ PROJECTION ONTO TANGENT PLANE

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FIG. 1
STEREOGRAPHIC PROJECTION

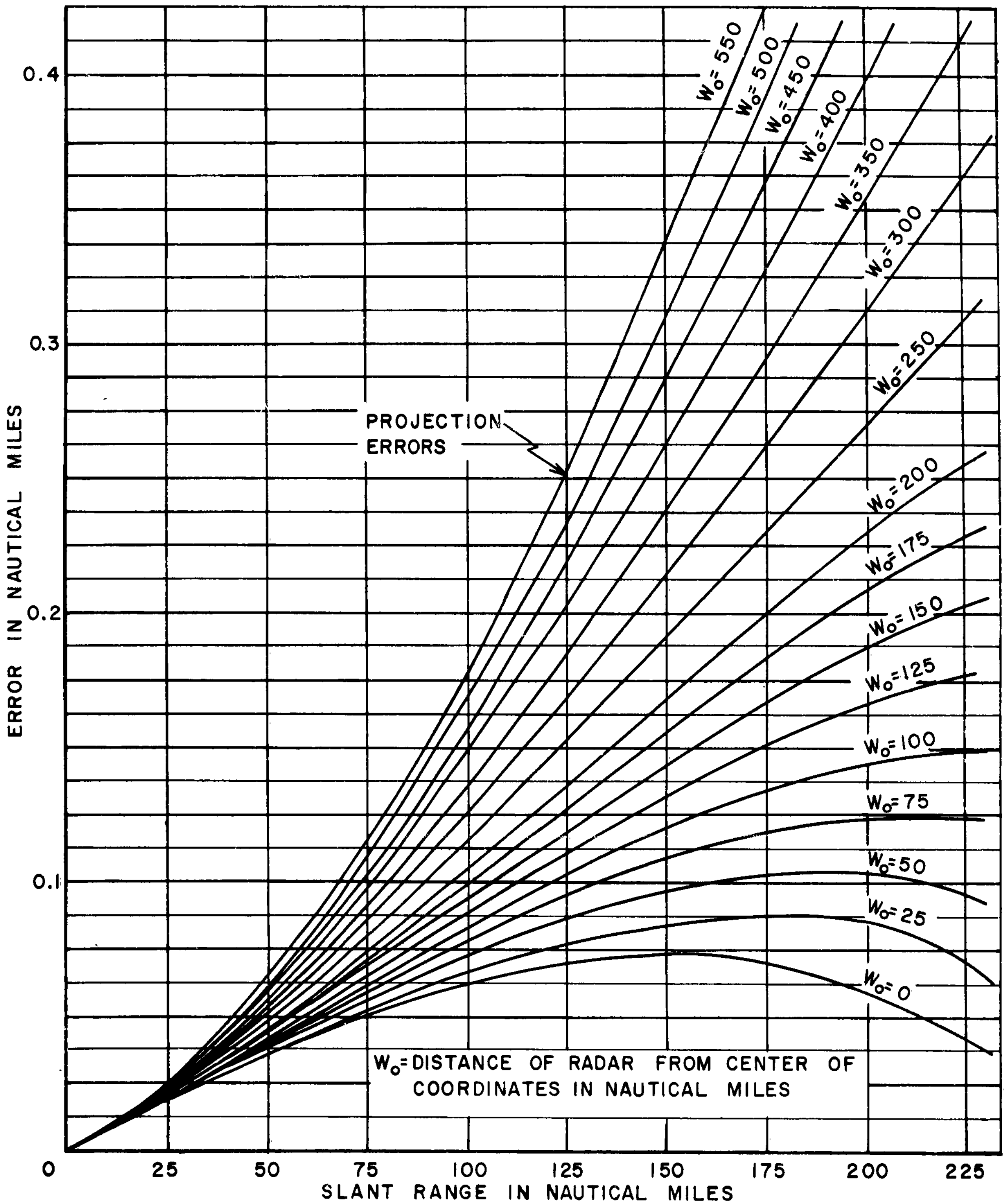

FIG. 2

MAXIMUM MAGNITUDE OF THE TOTAL PROJECTION ERRORS RESULTING FROM A FIRST ORDER APPROXIMATION TO THE TRANSFORMATION EQUATIONS vs. SLANT RANGE.

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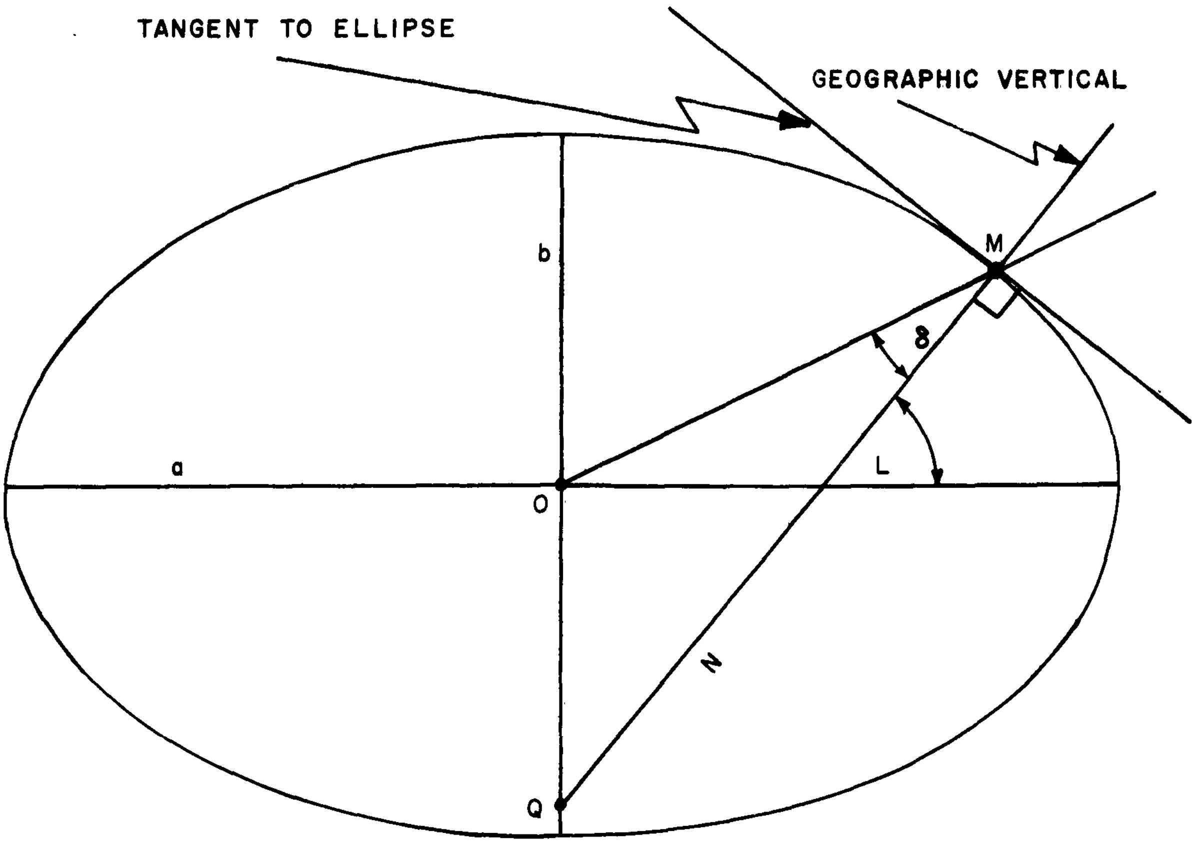

FIG. 3

MATHEMATICAL FIGURE OF THE EARTH

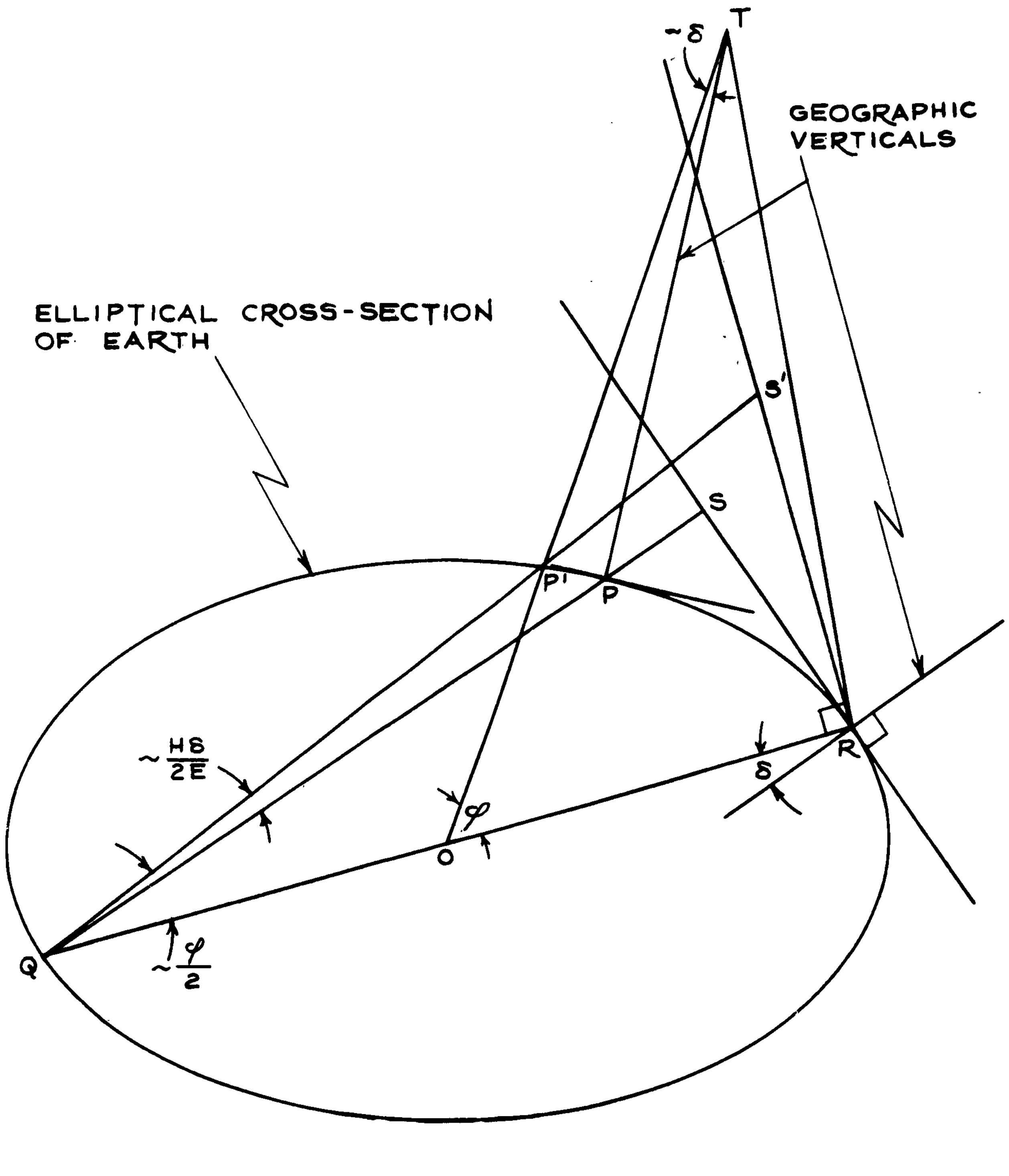

FIG. 4

STEREOGRAPHIC PROJECTION OF THE SPHEROID

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