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ADDENDUM

Proof of a General Relationship
Used in the Stability Test of Linear Discrete Systems

In the stability criterion developed for the linear discrete systems, the following general relationship is used to reduce the stability constraints.

\[
A_k^2 - B_k^2 = \frac{1}{2} \left[ (A_{k+1} + B_{k+1})(A_{k-1} - B_{k-1}) + (A_{k+1} - B_{k+1})(A_{k-1} + B_{k-1}) \right],
\]

\[k = 2, 3, \ldots, n - 1 \tag{1}\]

where \(A_k + B_k\) are given in the Appendix.

The proof of Eq. (1) is based on the combined use of the following three propositions. The reader should be familiar with Refs. 1 and 2 in order to follow the notations and the details of the proofs.

**Proposition 1:**

\[
\forall n, \forall k, \exists 2k - 1 < n, A_k + B_k \text{ are prime (not factorable) polynomials in the ring of polynomials of } n \text{ variables } a_0, a_1, a_2, \ldots, a_n.
\]

**Proof:**

We readily notice that \(A_1 + B_1\) is prime. Assume \(A_{k-1} + B_{k-1}\) is prime, we show \(A_k + B_k\) is prime too. The proof for \(A_k - B_k\) is analogous to \(A_k + B_k\).

Suppose, on the contrary, that \(A_k + B_k\) is not prime. Let \(P\) and \(Q\) be polynomials into which \(A_k + B_k\) factors,

\[
A_k + B_k = PQ \tag{2}
\]

Expanding the determinant (see Appendix) for \(A_k + B_k\) in cofactors of the last row we get:

\[
PQ = A_k + B_k = a_0G + a_nG^* \tag{3}
\]

where \(G\) and \(G^*\) are the cofactors of \(a_0\) and \(a_n\).
By inspection, it can be seen that except for a relabeling of some indices, \( G \) with \( a_n = 0 \) and \( G^* \) with \( a_0 = 0 \) are of exactly the same form as \( A_{k-1} + B_{k-1} \) (the restriction \( 2k - 2 < n \) is needed here and for later discussions we require \( 2k - 1 < n \)). Therefore \( G \mid a_n = 0 \) and \( G^* \mid a_0 = 0 \) are prime.

If we put \( a_n = 0 \) in (3) and divide by \( P \),

\[
\begin{align*}
Q \mid a_n = 0 &= \frac{a_0G}{a_n = 0} \\
\text{Also,} &
\end{align*}
\]

\[
\begin{align*}
Q \mid a_0 = 0 &= \frac{a_n G^*}{a_0 = 0}
\end{align*}
\]

From (4) and (5) and the fact that \( Q \) is a polynomial and \( G \mid a_n = 0 \), \( G^* \mid a_0 = 0 \) are prime, it follows that \( P \) must be of the form:

\[
P = (c_1a_n + c_2a_0 + c_3 + a_0a_nP')
\]

where \( P' \) is a polynomial and the \( c \)'s are constants.

Similarly for \( Q \) we can write

\[
Q = (c_4a_n + c_5a_0 + c_6 + a_0a_nQ')
\]

where \( Q' \) is a polynomial and the \( c \)'s are constants.

Because \( A_k + B_k \) is identically equal to \( PQ \), all of the constants in (6) and (7) are necessarily zero. This implies \( PQ = 0 \) if \( a_0 \) or \( a_n = 0 \) which is not true. Thus \( A_k + B_k \) does not factor and hence a prime. By similar reasoning \( A_k - B_k \) can be shown to be prime too.

Proposition 2:

\[
\delta_{i+2} \mid a_{n-(i+2)} \quad \text{for } i = 1, 2, 3, \ldots, n-1
\]

Proof: To prove Eq. (8), first we observe that the rules of calculating \( \delta_{i+2} \) (see Table 1 of Ref. 2) from the previous two rows in the table and forming the sums \( \delta_{i+2} \pm a_{n-(i+2)} \) are identical to those for calculating
\(\delta_2 + a_{n-2}^{(2)}\) from the first two rows. Hence, the relationship, as far as divisibility is concerned, between \(a_0 + a_n\) and \(\delta_2 + a_{n-2}^{(2)}\) is exactly the same as that between \(\delta_1 + a_{n-1}^{(1)}\) and \(\delta_{i+2} a_{n-(i+2)}\). Therefore, to prove (8) it is sufficient to show that \(a_0 + a_n\) divides \(\delta_2 + a_{n-2}^{(2)}\). This can be shown as follows:

From Table (1) of Ref. (2),

\[
\delta_2 = (a_0^2 - a_n^2) - (a_0 a_{n-1} - a_1 a_n)^2
\]

and

\[
a_{n-2}^{(2)} = (a_0^2 - a_n^2)(a_0 a_{n-2} - a_2 a_n) - (a_0 a_1 - a_{n-1} a_n)(a_0 a_{n-1} - a_1 a_n)
\]

In forming \(\delta_2 + a_{n-2}^{(2)}\), we obtain

\[
\delta_2 - a_{n-2}^{(2)} = (a_0 + a_n) \phi_1
\]

and

\[
\delta_2 + a_{n-2}^{(2)} = (a_0 - a_n) \phi_2
\]

where \(\phi_1\) and \(\phi_2\) are polynomials of the variables \(a_k\)'s.

Therefore \(a_0 + a_n\) divides \(\delta_2 + a_{n-2}^{(2)}\) and this proves proposition 2.

**Proposition 3:**

\(\forall n, \forall_k, \exists 2k - 1 < n\),

\[
\frac{\delta_k + a_{n-k}^{(k)}}{\delta_1 \delta_2 \delta_3 \delta_4 \cdots \delta_{k-2} (A_{k-1} + B_{k-1})} = A_{k+1} + B_{k+1}
\]

**Proof:** To prove the above relationship we introduce the following equation which can be obtained from Table 1 of Ref. 2:

\[
\delta_{k+1} = (\delta_k + a_{n-k}^{(k)})(\delta_k - a_{n-k}^{(k)}) \, k = 2, 3, \ldots, n-1
\]

Furthermore, Marden \(^3\) has proved the following identity in terms of the stability constants \(A_k\)'s and \(B_k\)'s. \(^1\)
We assume that (13) is valid for all \( k \leq m - 1 \); by induction we will show that (13) holds for \( k = m \).

Using (15) we get for \( k = m \)

\[
(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \frac{\delta_{m+1}}{\delta_1 \delta_2 \delta_3 \delta_2 \delta_3 \delta_2 \delta_3 \delta_{m-1}}
\]

(16)

Using (14) and (15) for \( \delta_{m-1} \), to obtain

\[
(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \frac{(\delta_m + a_{n-m})(\delta_m - a_{n-m})}{(\delta_1 \delta_2 \delta_3 \delta_2 \delta_3 \delta_2 \delta_3 \delta_{m-1})(\delta_m \delta_2 \delta_3 \delta_4 \delta_{m-2} \delta_{m-3} \delta_{m-4} \delta_{m-5})}
\]

(17)

The above equation can also be written as:

\[
(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \left\{ \frac{\delta_m + a_{n-m}}{\delta_1 \delta_2 \delta_3 \delta_2 \delta_3 \delta_2 \delta_3 \delta_{m-1}} \right\}
\]

\[
= \left\{ \frac{\delta_m - a_{n-m}}{\delta_1 \delta_2 \delta_3 \delta_2 \delta_3 \delta_2 \delta_3 \delta_{m-1}} \right\}
\]

\[
= F_1
\]

(18)

\[
F_2
\]

\[
\delta_2 = \Delta_2
\]

\[
\delta_3 = \delta_1 \delta_2 = \Delta_1 \Delta_2 \text{, since } \delta_1 = \Delta_1
\]

\[
\delta_4 = \delta_1^2 \delta_2 \delta_3 = \Delta_1^2 \Delta_2 \Delta_1 \Delta_2 = \Delta_1^3 \Delta_2^2
\]

..
\[
\Delta_{k+1} = (A_{k+1} + B_{k+1})(A_{k+1} - B_{k+1}) = \frac{\delta_{k+1}}{\delta_1 \delta_2 \delta_3 \delta_{k-2} \delta_{k-3} \delta_{k-4}}
\]

(15)\(^{+}\)

We assume that (13) is valid for all \(k < m - 1\); by induction we will show that (13) holds for \(k = m\).

Using (15) we get for \(k = m\)

\[
(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \frac{\delta_{m+1}}{\delta_1 \delta_2 \delta_3 \delta_{m-2} \delta_{m-3} \ldots \delta_{m-2} \delta_{m-1}}
\]

(16)

Using (14) and (15) for \(\delta_{m-1}\), to obtain

\[
(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \frac{(\delta_{m} + a^{(m)}_{n-m})(\delta_{m} - a^{(m)}_{n-m})}{(\delta_1 \delta_2 \delta_3 \delta_{m-2} \delta_{m-3} \ldots \delta_{m-2} \delta_{m-1})(\Delta_{m-1} \delta_{m-2} \delta_{m-3} \ldots \delta_{m-3})}
\]

(17)

The above equation can also be written as:

\[
(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \left\{ \begin{array}{c}
\frac{\delta_{m} + a^{(m)}_{n-m}}{\delta_1 \delta_2 \delta_3 \ldots \delta_{m-2} \delta_{m-3} \ldots \delta_{m-2} \delta_{m-1}} \\
\Delta_{m-1} \delta_{m-2} \delta_{m-3} \ldots \delta_{m-3} \ldots \delta_{m-2} (A_{m+1} - B_{m+1})
\end{array} \right\} F_1
\]

\[
\left\{ \begin{array}{c}
\frac{\delta_{m} - a^{(m)}_{n-m}}{\delta_1 \delta_2 \delta_3 \ldots \delta_{m-2} \delta_{m-3} \ldots \delta_{m-2} \delta_{m-1}} \\
\Delta_{m-1} \delta_{m-2} \delta_{m-3} \ldots \delta_{m-3} \ldots \delta_{m-2} (A_{m+1} + B_{m+1})
\end{array} \right\} F_2
\]

(18)

\(^{+}\)From this equation we can readily establish

\[
\delta_2 = \Delta_2
\]

\[
\delta_3 = \delta_1 \delta_2 = \Delta_1 \Delta_2 , \text{ since } \delta_1 = \Delta_1)
\]

\[
\delta_4 = \delta_1 \delta_2 \delta_3 = \Delta_1 \Delta_2 \Delta_1 \Delta_2 = \Delta_1 \Delta_2
\]

\[
\delta_5 = \delta_1 \delta_2 \delta_3 \delta_4 = \Delta_1 \Delta_2 \Delta_1 \Delta_2 \Delta_1 \Delta_2
\]

\[
\ldots
\]

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For simplicity we define \( F_1 \) and \( F_2 \) as indicated in the brackets of (18).

Next, we show that \( F_1 \) and \( F_2 \) are polynomials:

(a) \( A_{m-1} + B_{m-1} \) and \( [\delta_1^{m-2}\delta_2^{m-3} \ldots \delta_{m-2}] \) are coprime (i.e., have no common factors).

By proposition 1, \( A_{m-1} + B_{m-1} \) is prime. Using (15) for \( k = m-1 \),

\[
[\delta_1^{m-2}\delta_2^{m-3} \ldots \delta_{m-2}]
\]

can be written as a product of prime factors of the form \( A_k + B_k, k < m-1 \). Hence

\[
[A_{m-1} + B_{m-1}] \quad \text{and} \quad [\delta_1^{m-2}\delta_2^{m-3} \ldots \delta_{m-2}]
\]

are coprime.

(b) By assumption, (13) is valid for \( k \leq m-1 \) and in particular, \( k = m-2 \); therefore, by noting (13) we readily ascertain that \( A_{m-1} + B_{m-1} \) is a divisor of

\[
\delta_{m-2} + a_{(m-2)}\frac{m}{n-m}
\]

Now if we apply proposition 2, it is clear that \( A_{m-1} + B_{m-1} \) divides \( \delta_m + a_{(m)} \) from (18).

(c) Since \( \Delta_{m+1} \) is a polynomial, \( \delta_m^2 - (a_{n-m})^2 \) is divisible by

\[
[\delta_1^{m-2}\delta_2^{m-3} \ldots \delta_{m-2}]^2
\]

Furthermore,

\[
\Delta_m = \frac{\delta_m}{\delta_1^{m-2}\delta_2^{m-3} \ldots \delta_{m-2}}
\]

is a polynomial, it follows that \( a_{n-m}^{(m)} \) and therefore \( \delta_m + a_{n-m} \) are divisible by

\[
\delta_1^{m-2}\delta_2^{m-3} \ldots \delta_{m-2}
\]

Items (b) and (c) in combination with (a) account for all the terms of the denominators dividing into the numerators in \( F_1 \) and \( F_2 \). Hence, \( F_1 \) and \( F_2 \) are polynomials.
From Proposition 1, $A_{m+1} + B_{m+1}$ are both prime. Since $F_1 F_2$ in (18) is divisible by $A_{m+1} + B_{m+1}$, it follows that $A_{m+1} + B_{m+1}$ divides $F_1$ or $F_2$ and $A_{m+1} - B_{m+1}$ also divides $F_1$ or $F_2$. The degrees of $A_{m+1} + B_{m+1}$ and $F_1$ and $F_2$ are equal, so $A_{m+1} + B_{m+1}$ must be constant multiples of $F_1$ and $F_2$. Therefore, only these two possibilities may exist:

I. \[
\begin{align*}
A_{m+1} + B_{m+1} &= C_1 F_1 \\
A_{m+1} - B_{m+1} &= D_1 F_2
\end{align*}
\] (19)

II. \[
\begin{align*}
A_{m+1} - B_{m+1} &= C_2 F_1 \\
A_{m+1} + B_{m+1} &= D_2 F_2
\end{align*}
\] (20)

where $C_1$, $D_1$, $C_2$, and $D_2$ are constants.

From (18) we note

\[
F_1(A_{m-1} - B_{m-1}) + F_2(A_{m-1} + B_{m-1}) = \frac{2\delta_m}{\delta_1^{m-2}\delta_2^{m-3}\ldots\delta_{m-2}}
\] (21)

Using (15), we obtain

\[
2\Delta_m = F_1(A_{m-1} - B_{m-1}) + F_2(A_{m-1} + B_{m-1}) = 2(A_{m-1} - B_{m-1})(A_{m-1} + B_{m-1})
\] (22)

To show that for $F_1$ and $F_2$ only possibility (I) exists, we substitute in the above the second form, i.e., Eq. (20)

\[
2(A_{m-1} - B_{m-1})(A_{m-1} + B_{m-1}) = \frac{1}{C_2} (A_{m+1} - B_{m+1})(A_{m-1} + B_{m-1})
\]

\[
+ \frac{1}{D_2} (A_{m+1} + B_{m+1})(A_{m-1} + B_{m-1})
\] (23)

To show the above does not hold for all $m$'s, (1) we equate the coefficients of $a_0^{2m}$ on both sides to obtain $2 = 1/C_2 + 1/D_2$; (2) we also equate the coefficients of $a_n^{2m}$ on both sides. We notice that for $m$-even, the above (for finite $C_2$ and $D_2$) is not satisfied while possibility (I) is satisfied. Furthermore, by equating the coefficients of $a_0^{2m}$ on both sides of the above equation in addition to (1) we find that for $m$-odd the above is
also not satisfied while using (19), it is satisfied. Therefore (22) is satisfied only by using (19) for $F_1$ and $F_2$. Thus we finally obtain

$$A_m^2 - B_m^2 = \frac{1}{2} \left\{ \frac{1}{C_1} (M_{m+1} + B_{m+1}) (M_{m-1} + B_{m-1}) + \frac{1}{D_1} (M_{m+1} - B_{m+1}) (M_{m-1} - B_{m-1}) \right\}$$

(24)

We can show that $C_1 = D_1 = 1$ for possibility (1) by equating the coefficients of $a_m$ on both sides of Eq. (24).

$$0 = \pm \left( \frac{1}{C_1} a_n \Delta_{m-1} - \frac{1}{D_1} a_n \Delta_{m-1} \right)$$

or $1/C_1 = 1/D_1$ and from equating the coefficients of $a_0$ on both sides of Eq. (24), we get, $2 = 1/C_1 + 1/D_1$. Therefore, $C_1 = D_1 = 1$. This completes the proof of proposition 3.

If we substitute for $m$ the variable $k$ in (24),

$$\Delta_k = A_k^2 - B_k^2 = \frac{1}{2} \left\{ (A_{k+1} + B_{k+1}) (A_{k-1} - B_{k-1}) + (A_{k+1} - B_{k+1}) (A_{k-1} + B_{k-1}) \right\}$$

(25)

$$= A_{k-1} A_{k+1} - B_{k-1} B_{k+1}$$

which is (1) except for the restriction on $k$ to be such that $2k-1 < n$. This restriction, which was only used in proving proposition 1 in fact, is not necessary. To see that (25) holds for all values of $k < n$, consider for the moment $n$ to be fixed, say $n = n_0$. We have already shown (25) to be valid for all $k$ such that $2k-1 < n_0$, and it remains to show its validity for all other $2k-1 \geq n_0$, $k < n_0$. For these values of $k$, consider (25) for $n = 2n_0$. Since $k < n_0$, $2k-1 < n = 2n_0$ so (25) is valid for these values of $k$ and for $n = 2n_0$. Put $a_{2n_0} = a_n$, $a_{2n_0-1} = a_{n-1}$. The resulting determinants for $k < n_0$ are identical to those for $n = n_0$. Hence, (25) is valid for all $k < n_0$, which proves the validity (1).

+By noting Eqs. (18) and (19), one can establish that $C_1 D_1 = 1$. Using $C_1 D_1 = 1$ in combination with $2 = 1/C_1 + 1/D_1$, we can readily establish that $C_1 = D_1 = 1$. 

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