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(6) A DELIVERY-LAG INVENTORY MODEL WITH AN
EMERGENCY PROVISION

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A DELIVERY-LAG INVENTORY MODEL WITH AN EMERGENCY PROVISION

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I. THE SINGLE-PERIOD CASE

1. Introduction and general discussion. This report begins the study of an inventory model in which there is a fixed lag time of one period for delivery of orders, but in which there is also defined an emergency situation with respect to initial stock at any particular inventory point; when such an emergency situation obtains, an additional order of a specified fixed size is taken with immediate delivery. There is a certain additional cost for this immediate delivery (although the analysis may be specialized to the case of no additional cost). While the particular structure of emergency is laid down, there is a free parameter left to be chosen for each period to render the definition of emergency fully specific, and this parameter is determined as part of the optimization.

There is no attempt here? to achieve the fullest generality in every respect. The principal object is to investigate the indicated emergency character of the model. It is anticipated that the n-period cases, with n > 1, will yield their essential properties by the device of inductive argument, ~~as in [1: Chapt. 9]~~, and that from

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these, by suitable limit considerations, the stationary case may be studied. Therefore this first investigation is directed to the case of the single-period problem, so formulated as to present the features which may be expected to arise in the general n-period case.

Before we enter into the single-period problem, it will be helpful for orientation to discuss the general n-period model, and the attendant recursion relations.

Consider a time interval of n periods in length, with the inventory points labelled 1, 2,..., n + 1 from left to right, and the periods labelled I_i, i = 1, 2,..., n:

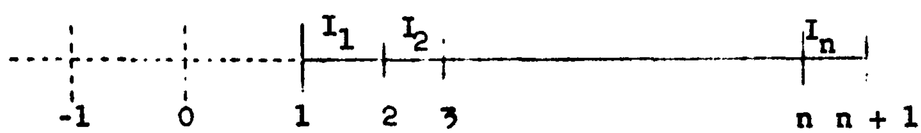

FIG. I

For i = 1, 2,..., n, let x_i denote what shall be called the intial stock in I_i, and is defined to be the (positive or negative) stock level at i inclusive of any orders that have been placed at inventory points t < i and which arrive at i, but not inclusive of an order which is placed at i and is immediately delivered. For purposes of deriving the recursion relation, it is necessary to consider the n-period interval in the context of a larger interval (as indicated in FIG. I, with inventory points 0, -1,...); thus, the initial stock x_1 takes account of orders at t = 1 which were placed at previous inventory points.

Now in fact, we consider that an order arrives at an inventory point only on either immediate delivery or delivery of one period lag. In more detail: at a point i, ordering is done according to the following scheme. There is a number γ_{n-i+1} such that (i) if $x_i > \gamma_{n-i+1}$, then an order for z_i units (the quantity z_i to be determined by optimization) is plased for delivery at $i + 1$, one period later, whereas (ii) if $x_i \leq \gamma_{n-i+1}$, then in addition to an order for z_i (= the optimal order size) units to be delivered at $i + 1$, an order for m units is placed and these m units will be immediately delivered. The quantity m is fixed, the same for all periods. The numbers γ_j are to be chosen in an optimal way.

The sense of this ordering scheme is clear. We are taking the simplest kind of characterization of an emergency situation; namely, when the initial stock is below a prescribed level (the number γ_j). And in this situation we are considering that the supplier will accommodate with an immediate delivery, though not of an arbitrarily large order, but, more realistically, of a fixed quantity m which he is able to produce with relative ease (but at perhaps somewhat higher cost--see below) for immediate delivery. The inventory manager is free to set the values of the γ_j, thus fully defining the states of stock-level emergency. We shall see that in certain cases there are simple optimal selections of the γ_j.

We shall employ the term starting stock at the inventory point i to designate (i) the initial stock itself if there is no ordering at i of the additional m units for immediate delivery (no emergency), or (ii) the initial stock plus m if there is ordering of the additional m units for immediate delivery (emergency).

Let the demands in the several inventory periods be independent and identically distributed, with distribution having a continuous density φ:

$$(1) \qquad \varphi(\xi) \begin{cases} = 0 \text{ for } < 0 \\ \geqq 0 \text{ and continuous for } \xi > 0. \end{cases}$$

We assume that $\varphi(0+)$ exists, and we shall denote this limit more simply by $\varphi(0)$.

We consider a holding cost function, h, and a penalty cost function, p, applicable in each period. For any particular period, if y is the starting stock level at the beginning of this period and ξ is the demand in this period, then the holding cost is $h(y - \xi)$ and the penalty cost is $p(\xi - y)$, these costs being charged at the end of the period. Of course, only one of these two costs may be positive in any particular case, as follows from the first assumptions on h and p:

$$(1.2) \qquad h(\eta) \begin{cases} = 0 \quad \text{for } \eta \leqq 0 \\ \geqq 0 \quad \text{for } \eta > 0 \\ \text{convex for all } \eta. \end{cases}$$

$$(1.3) \qquad p(\eta) \begin{cases} = 0 \quad \text{for } \eta \leqq 0 \\ \geqq 0 \quad \text{for } \eta > 0 \\ \text{convex for all } \eta. \end{cases}$$

We take a fixed cost c per unit of the item under consideration. However, this is the regular cost which is applicable only in the case of orders to be regularly delivered, that is, to be delivered one period later. In the case of an emergency, the m units that are delivered immediately will be considered to have a unit cost c_0, not less than c and in general greater than c. Thus, in an emergency, the m immediately delivered units will have the cost $c_0 m$, while the cost of the simultaneously placed, regular order, z, will be cz (the latter cost being incurred at the time of placement of the order).

We assume that there is no set-up cost. The discount factor will be denoted by α .

Now, for r = 1, 2,..., n, let

(1.4) $\bar{f}_r(x,z,\xi;\gamma_r)$ = conditional expected total cost, discounted to $t = n - r + 1$, for the r-period interval from $t = n - r + 1$ to $t = n + 1$, given that the initial stock at $n - r + 1$ is x, that a regular order for z units is placed at $n - r + 1$, and that the demand in the period $[n - r + 1, n - r + 2)$ is ξ; and given, also, that there is an optimal ordering rule in effect in $[n - r + 2, n + 1]$, and that γ_r specifies the emergency level at $n - r + 1$.

Let

$$(1.5) \quad f_r^o(x,z;\gamma_r) = \int_{\xi=0}^{\infty} \bar{f}_r(x,z,\xi;\gamma_r)\ \varphi(\xi)\ d\xi.$$

And, finally, let

(1.6) $f_r(x)$ = optimal value of $f_r^o(x,z;\gamma_r)$ among all choices of

(i) the constant γ_r independent of x, and

(ii) the quantity z dependent upon x and the choice of γ_r.

We have purposely not used the word "minimal" in (1.6) where the word "optimal" appears, because it is not in general true that there is a unique minimum for $f_r^o(x,z;\gamma_r)$ under the condition (i) in (1.6) which demands that γ_r be independent of x. To formulat this more precisely, let

$$(1.7) \quad f_r^{(1)}(x;\gamma_r) = \min_{z\geq 0} f_r^o(x,z;\gamma_r)$$

and

$$(1.8) \quad f_r^{(2)}(x) = \min_{\gamma_r} f_r^{(1)}(x;\gamma_r).$$

Then, it makes sense to substitute the word "minimal" in (1.6) only if there is some value of γ_r, say γ_r', such that

$$(1.9) \quad f_r^{(2)}(x) = f_r^{(1)}(x;\gamma_r'), \text{ all } x;$$

and in this case the value γ_r' will be taken to define the emergency level, and the optimal cost function f_r of (1.6) will be $f_r^{(2)}$. If there is no such value γ_r', then some new convention must be made as to what "optimal" shall mean.

For each r, we consider that such a new convention is at hand, if necessary, to provide the functions f_r of (1.6).

Now, let us examine this question of minimality more closely by looking at f_r^o as given by the recursion relation. For $r > 1$ we have

$$(1.10)\quad \bar{f}_r(x,z,\xi;\gamma_r) = \begin{cases} cz + 1\,(x - \xi) + \alpha\; f_{r-1}(x + z - \xi), & x > \gamma_r, \\ cz + c_o m + \;1\,(x + m - \xi) & \\ \quad + \alpha\; f_{r-1}(x + m + z - \xi), & x \leqq \gamma_r, \end{cases}$$

where 1 is the function defined by

$$(1.11)\qquad 1\,(\eta) = \begin{cases} h(\eta), & \eta > 0, \\ p(-\eta), & \eta \leqq 0. \end{cases}$$

If we set

$$(1.12)\quad L(\eta) = \begin{cases} \int_{\xi=0}^{\eta} h(\eta - \xi)\varphi(\xi)\; d\xi + \int_{\xi=\eta}^{\infty} p(\xi - \eta)\varphi(\xi)\; d\xi, & \eta > 0, \\ \int_{\xi=0}^{\infty} p(\xi - \eta)\varphi(\xi)\; d\xi, & \eta \leqq 0, \end{cases}$$

then we have, from (1.10),

$$(1.13)\quad f_r^o(x,z;\gamma_r) = \begin{cases} cz + L(x) + \alpha\int_{\xi=0}^{\infty} f_{r-1}(x + z - \xi)\varphi(\xi)d\xi, & \\ \qquad\qquad\qquad x > \gamma_r & \\ cz + c_o m + L(x + m) & \\ \quad + \alpha\int_{\xi=0}^{\infty} f_{r-1}(x + m + z - \xi)\varphi(\xi)d\xi, & \\ \qquad\qquad\qquad x \leqq \gamma_r. & \end{cases}$$

Let us set, for $r \geq 1$:

(1.14) $$G_r(z;\eta) = cz + \alpha \int_{\xi=0}^{\infty} f_r(\eta + z - \xi)\varphi(\xi)d\xi.$$

Then (1.13) becomes

(1.15) $$f_r^o(x,z;\gamma_r) = \begin{cases} L(x) + G_{r-1}(z;x), & x > \gamma_r, \\ c_o m + L(x+m) + G_{r-1}(z;x+m), & x \leq \gamma_r. \end{cases}$$

Let us put

(1.16) $$H_r(\eta) = \min_{z \geq 0} G_{r-1}(z;\eta),$$

and

(1.17) $$J_r(\eta) = L(\eta) + H_r(\eta).$$

Then we have, according to (1.7),

(1.18) $$f_r^{(1)}(x;\gamma_r) = \begin{cases} J_r(x), & x > \gamma_r, \\ c_o m + J_r(x+m), & x \leq \gamma_r. \end{cases}$$

We see thus that $f_r^{(1)}$ has a rather special form, and we can inquire after conditions under which (1.9) holds.

From (1.18) we find that

(1.19) $$f_r^{(2)}(x) = \min\left[J_r(x),\ c_o m + J_r(x+m)\right].$$

Hence, if there is a γ_r' such that (1.9) holds, we have

(1.20) $$\begin{cases} J_r(x) \leq c_o m + J_r(x+m), & x > \gamma_r', \\ J_r(x) \geq c_o m + J_r(x+m), & x \leq \gamma_r'. \end{cases}$$

Conversely, if there is a γ_r' satisfying (1.20), then (1.9) holds. Thus, the existence of a γ_r' such that (1.20) holds is, in our case, the characteristic condition on the function J_r in order that optimality in (1.6) be definable as minimality.

Notice that the function $c_o m + J_r(x + m)$ is, geometrically, the translate of the function $J_r(x)$ first to the left by m and then up by the amount $c_o m$. If the function J_r is U-shaped, the picture may be as follows:

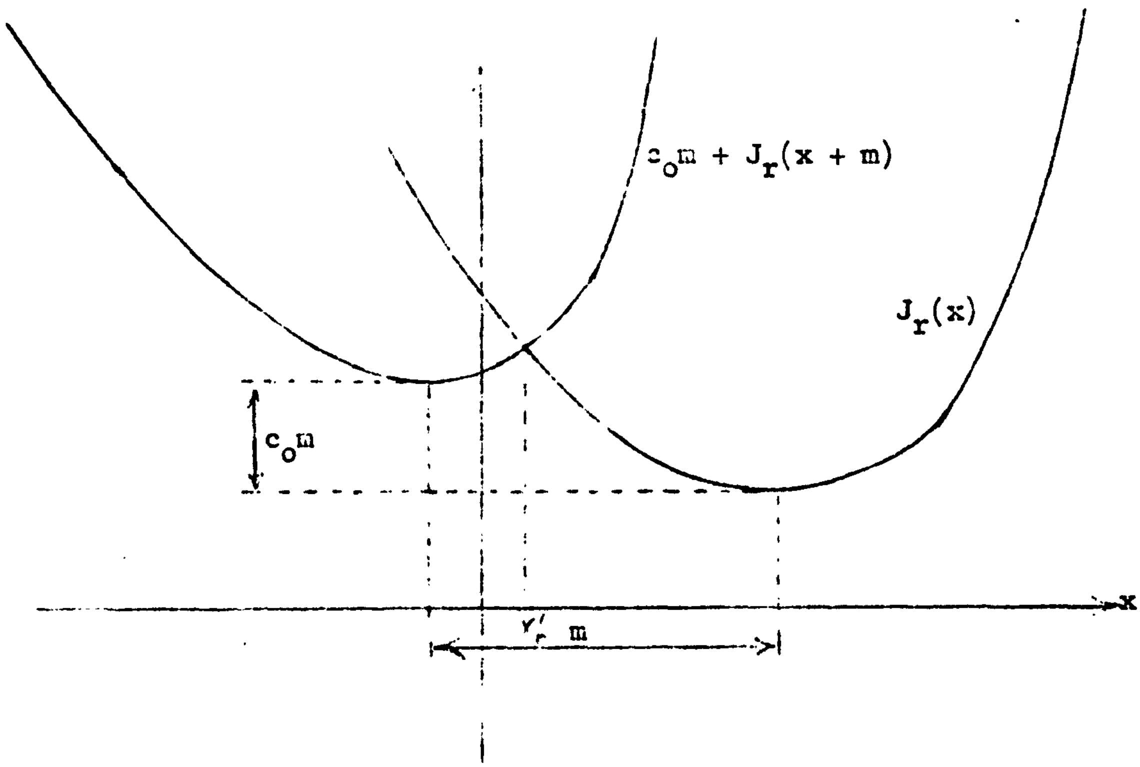

FIG. II

In this case--that is, a situation of the type depicted in FIG. II--the γ_r' indicated in the figure is the unique number verifying (1.20).

But observe that--the nature of the function J_r permitting--c_o might be so large that, instead of the situation in FIG. II, we have the following picture:

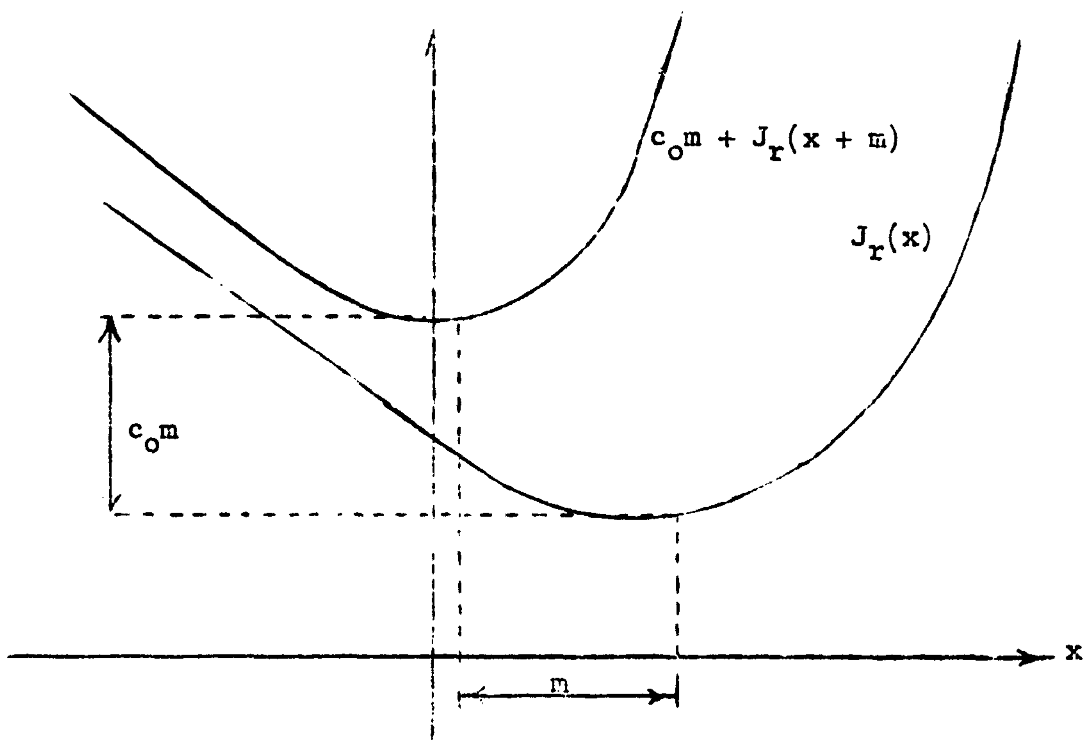

FIG. III

This is the situation that

$$(1.21) \qquad J_r(x) \leqq c_o m + J_r(x + m), \quad \text{all } x.$$

In this case we may say that (1.20) holds with $\gamma_r' = -\infty$. And operationally this means that we shall not admit any emergency situation:--a natural consequence of too large an emergency cost as compared with penalties.

In subsequent investigations we shall study the question of convexity of the successive J_r's, and the existence of finite values γ_r', $r = 1, 2, \ldots$ as depicted in FIG. II. For the present we direct the necessary first attention to the single-period case, and find that under natural assumptions we do have the indicated convexity of J_1 and consequent analysis according to FIG. II or FIG. III.

2. *Formulation of the single-period case.* As we have already said, our intention in fully formulating and analyzing the one-period case is to accomplish the prototype of argumentation that can be expected to be called for in the general n-th stage discussed above. We shall therefore consider that there is an initial stock level of x at the beginning of our single-period, that there is an emergency level γ, as previously described, to be determined (in particular, it might turn out that we would want to take $\gamma = -\infty$ (disallowance of any emergency situation), and that a regular order for z units is placed at the beginning of the period for delivery at the end of the period. To complete the description, we consider that the z units arriving at the end of the period may be used to fill any demands still outstanding. But, because such filling of orders is late, we assume that there is a gain

function v which does not in general cancel the holding cost that is calculated at the end of the period before arrival of the order of z units. Finally, we assume that if, after late filling of demands, there is still some stock left over, it is sold for salvage, the salvage gain function being w.

The functions v and w will be supposed concave for positive arguments; we assume, in fact, the following properties of v and w and their first and second derivatives:

$$(2.1)\qquad \begin{cases} v(\eta) = 0, \ \eta \leqq 0, \\ v'(\eta) > 0, \ \eta \geqq 0 \quad (v'(0) \text{ denotes } v'(0+)), \\ v''(\eta) \leqq 0, \ \eta \geqq 0 \quad (v''(0) \text{ denotes } v''(0+)). \end{cases}$$

$$(2.2)\qquad \begin{cases} w(\eta) = 0, \quad \eta \leqq 0, \\ w'(\eta) > 0, \quad \eta \geqq 0 \quad (w'(0) \text{ denotes } w'(0+)), \\ w''(\eta) \leqq 0, \quad \eta \geqq 0 \quad (w''(0) \text{ denotes } w''(0+)). \end{cases}$$

Correspondingly, we make the following more detailed assumptions concerning the holding and penalty functions h and p:

$$(2.3)\qquad \begin{cases} h(\eta) = 0, \quad \eta \leqq 0 \\ h'(\eta) > 0, \quad \eta \geqq 0 \quad (h'(0) \text{ denotes } h'(0+)), \\ h''(\eta) \geqq 0, \quad \eta \geqq 0 \quad (h''(0) \text{ denotes } h''(0+)). \end{cases}$$

$$(2.4)\qquad \begin{cases} p(\eta) = 0, \quad \eta \leqq 0 \\ p'(\eta) > 0, \quad \eta \geqq 0 \quad (p'(0) \text{ denotes } p'(0+)), \\ p''(\eta) \geqq 0, \quad \eta \geqq 0 \quad (p''(0) \text{ denotes } p''(0+)). \end{cases}$$

Our analysis will be carried out under several assumptions regarding the interrelations between the functions v, w, h, and p, and the cost constant c. These are the following:

(2.5) $\quad v'_\infty > w'(0), \qquad (v'_\infty \equiv \lim_{\eta \to \infty} v'(\eta)),$

(2.6) $\quad v'_\infty > c > w'_\infty, \qquad (w'_\infty \equiv \lim_{\eta \to \infty} w'(\eta)),$

(2.7) $\quad v'(0) < p'(0),$

and

(2.8) $\quad h'_\infty > w'_\infty, \qquad (h'_\infty \equiv \lim_{\eta \to \infty} h'(\eta)).$

The significance of condition (2.5) is that there is always a greater gain in filling demands late, no matter how large the amount of such demands, than in leaving some of these demands unfulfilled and taking the salvage value of the left-over stock instead. This interpretation makes use, of course, of the fact that the second derivatives v'' and w'' are nonpositive for all nonnegative arguments, so that we have, in fact,

(2.9) $$v'(\eta_1) \geqq v'_\infty > w'(0) \geqq w'(\eta_2) \quad \underline{\text{for all}}\ \eta_1 \geqq 0\ \underline{\text{and}}\ \eta_2 \geqq 0.$$

The left-hand inequality in (2.6) means--again taking account of the fact that $v'(\eta) \geqq v'_\infty$ for all $\eta \geqq 0$--that every demand filled late, no matter how large the amount of such demands, represents a gain in excess of the (regular) purchase cost of the quantity of item needed to fill that demand. In other words, even late filling of demands, however many, has sufficient worth that the purchase cost of the item is no deterrent.

Out assumptions on the function w imply, of course, that w' is monotonely nonincreasing from 0 to ∞. In view of this, the right-hand inequality in (2.6) means that there is a level beyond which salvage returns do not make up purchase cost. Indeed, if the reverse were true there would be gain to be had in the purely subsidiary operation of purchasing for resale to salvage.

The relation (2.7)--again together with the consequences of the second derivative conditions on v and p--asserts the realistic situation that the gain achieved by filling demands late does not make up all of the penalty that is paid for this lateness.

And finally, taking account of the second derivative condition on w and h, conditions (2.8) means that there is no level of stock sufficiently large that beyond this level salvage returns make up holding costs.

Let γ denote the (to be optimally determined) single-period emergency level, and let $\bar{f}$ stand for $\bar{f}_1$. Then we have

$$(2.10)\quad \bar{f}(x,z,\xi;\gamma) = \begin{cases} cz + l\,(x - \xi) - v(\min[z,\xi - x]) \\ \qquad - w(x + z - \xi),\ x > \gamma, \\ cz + c_o m + l\,(x + m - \xi) \\ \qquad - v(\min\,[z,\xi - (x + m)]) \\ \qquad - w(x + m + z - \xi),\quad x \leqq \gamma. \end{cases}$$

Letting f^o stand for f_1^o, we thus have

$$(2.11)\quad f^o(x,z;\gamma) = \begin{cases} cz + L(x) - V(x,z) - W(x + z), & x > \gamma, \\ cz + c_o m + L(x + m) - V(x + m,\ z) \\ \qquad - W(x + m + z), & x \leqq \gamma, \end{cases}$$

where

$$(2.12)\quad V(x,z) = \begin{cases} \int_0^{x+z} v(\xi - x)\varphi(\xi)d\xi + v(z)\int_{x+z}^{\infty} \varphi(\xi)d\xi, \\ \qquad\qquad x + z > 0, \\ v(z), \qquad x + z \leqq 0, \end{cases}$$

and

$$(2.13)\qquad W(x) = \int_0^{\infty} w(x - \xi)\varphi(\xi)d\xi.$$

Define

$$(2.14)\qquad G(z;x) = cz - V(x,z) - W(x + z);$$

then (2.11) becomes

$$(2.15)\qquad f^{o}(x,z;\gamma) = \begin{cases} L(x) + G(z;x), & x > \gamma, \\ c_o m + L(x+m) + G(z;x+m), & x \leqq \gamma. \end{cases}$$

We put

$$(2.16)\qquad H(x) = \min_{z \geqq 0} G(z;x)$$

and

$$(2.17)\qquad J(x) = L(x) + H(x).$$

Then we have, letting $f^{(1)}$ stand for $f_1^{(1)}$,

$$(2.18)\qquad f^{(1)}(x;\gamma) = \begin{cases} J(x), & x > \gamma, \\ c_o m + J(x+m), & x \leqq \gamma. \end{cases}$$

We see that $f^{(1)}$ is of the same form as $f_r^{(1)}$ for $r > 1$, as given by (1.18). Hence, the deliberations in Section 1 subsequent upon formula (1.18) are applicable here in the single-period case (our present G, H, and J being G_o, H_1, and J_1, resp., in the notation of Section 1), and the problem facing us is the study of the function J with the goal of determining whether or not there exists a γ' such that

$$(2.19)\qquad \begin{cases} J(x) \leqq c_o m + J(x+m), & x > \gamma', \\ J(x) \geqq c_o m + J(x+m), & x \leqq \gamma'. \end{cases}$$

3. Study of the minimization of $G(z;x)$. Our first major task toward understanding the behavior of the function J is to get at the function H; that is, to study the minimization of $G(z;x)$ with respect to $z \geqq 0$.

We notice, to begin with, that for each x, $V(x,z)$ and $W(x+z)$ are continuous functions of $z \geqq 0$. Hence, $G(z;x)$ is, for each x, a continuous function of z. In fact, G is differentiable with respect to z, and to calculate this partial derivative and other dominatives further along in our analysis, we set down the following more explicit (than (2.12) and (2.13)) formulas:

$$(3.1)\quad V(x,z) = \begin{cases} \int_0^{x+z} v(\xi - x)\varphi(\xi)d\xi + v(z)\int_{x+z}^{\infty}\varphi(\xi)d\xi, \\ \qquad\qquad x+z>0,\quad x \leqq 0, \\ \int_x^{x+z} v(\xi - x)\varphi(\xi)d\xi + v(z)\int_{x+z}^{\infty}\varphi(\xi)d\xi, \\ \qquad\qquad x+z>0,\quad x>0, \\ v(z), \quad x+z \leqq 0, \end{cases}$$

$$(3.2)\quad W(x+z) = \begin{cases} \int_0^{x+z} w(x+z-\xi)\varphi(\xi)d\xi, & x+z>0, \\ 0, & x+z \leqq 0. \end{cases}$$

From (3.1)--or, in fact, also from (2.12)--we have

$$(3.3)\quad \frac{\partial V}{\partial z} = \begin{cases} v'(z)\int_{x+z}^{\infty}\varphi(\xi)d\xi, & x+z>0, \\ v'(z), & x+z \leqq 0; \end{cases}$$

and from (3.2) we get (on taking account of the fact that $w(0) = 0$)

$$(3.4)\quad \frac{\partial W(x+z)}{\partial z} = \begin{cases} \int_0^{x+z} w'(x+z-\xi)\varphi(\xi)d\xi, & x+z>0, \\ 0, & x+z \leqq 0. \end{cases}$$

Hence, we have

$$(3.5)\quad \frac{\partial G}{\partial z} = \begin{cases} c - v'(z)\int_{x+z}^{\infty}\varphi(\xi)d\xi - \int_0^{x+z} w'(x+z-\xi)\varphi(\xi)d\xi, \\ \qquad\qquad x+z>0, \\ c - v'(z), \quad x+z \leqq 0. \end{cases}$$

We see readily that, for each x, $\partial G/\partial z$ is continuous in $z > 0$. In fact, $\partial G/\partial z$ is differentiable with respect to z; but before we go ahead to find $\partial^2 G/\partial z^2$, we first examine the behavior of $\partial G/\partial z$.

From (3.5) we find

$$(3.6)\quad \left(\frac{\partial G}{\partial z}\right)_{z=0+} = \begin{cases} c - v'(0)\int_x^{\infty}\varphi(\xi)d\xi - \int_0^{x} w'(x-\xi)\varphi(\xi)d\xi, \\ \qquad\qquad x>0, \\ c - v'(0), \quad x \leqq 0. \end{cases}$$

By (2.1) and (2.6) we have

$$(3.7)\qquad v'(0) \geqq v'_{\infty} > c.$$

Applying this to (3.6) we see that

$$(3.8)\qquad \left(\frac{\partial G}{\partial z}\right)_{z=0+} < 0 \quad \underline{\text{for each}}\ x \leqq 0.$$

The function $(\partial G/\partial z)_{z=0+}$ is a continuous function of x, constant and negative for $x \leqq 0$. Let us examine its derivative for $x > 0$:

$$(3.9)\quad \frac{d}{dx}\left[\left(\frac{\partial G}{\partial z}\right)_{z=0+}\right] = v'(0)\varphi(x) - w'(0)\varphi(x) - \int_0^x w''(x-\xi)\varphi(\xi)d\xi, \quad x > 0.$$

By (2.1) and (2.5),

$$(3.10)\qquad v'(0) \geqq v'_\infty > w'(0)$$

Combining this with the fact that $\varphi(x) \geqq 0$, and remembering that $w'' \leqq 0$, we see that

$$(3.11)\qquad \frac{d}{dx}\left[\left(\frac{\partial G}{\partial z}\right)_{z=0+}\right] \geqq 0, \qquad x > 0.$$

Hence, $(\partial G/\partial z)_{z=0+}$ is monotone nondecreasing in x. We find, furthermore, from (3.6),

$$(3.12)\qquad \lim_{x\to\infty}\left(\frac{\partial G}{\partial z}\right)_{z=0+} = c - w'_\infty > 0$$

as a consequence of (2.6). It follows that $(\partial G/\partial z)_{z=0+}$ vanishes for some unique positive value of x or for some closed interval of positive values of x. Let x_0 be this unique value of x, if it exists, or the lower endpoint of this interval of values of x, if an interval exists. Then, x_0 is either the unique or the minimum number verifying

$$(3.13) \qquad c - v'(0)\int_{x_0}^{\infty} \varphi(\xi)d\xi - \int_0^{x_0} w'(x_0 - \xi)\varphi(\xi)d\xi = 0.$$

To sum up what we have found thus far:

(3.14) <u>There is a positive number x_0 such that the continuous, monotone nondecreasing function (3.6) satisfies</u>

$$\left(\frac{\partial G}{\partial z}\right)_{z=0+} \begin{cases} < 0, & x < x_0, \\ \geqq 0, & x \geqq x_0. \end{cases}$$

<u>The number x_0 is the smallest solution of (3.13).</u>

Let us observe that if φ is strictly positive for positive arguments then the inequality sign holds in (3.11). And as a consequence of this the function (3.6) cannot vanish on an interval. Hence:

(3.15) <u>If $\varphi(\eta) > 0$ for $\eta > 0$, then the number x_0 of (3.14) is the unique solution of (3.13), and is consequently the only point at which the function (3.6) vanishes</u>.

Let us now study the second derivative (obtained from (3.5)):

$$(3.16) \qquad \frac{\partial^2 G}{\partial z^2} = \begin{cases} - v''(z)\displaystyle\int_{x+z}^{\infty} \varphi(\xi)d\xi - \int_0^{x+z} w''(x + z - \xi)\varphi(\xi)d\xi \\ \qquad + \left[v'(z) - w'(0)\right]\varphi(x + z), \\ \qquad\qquad x + z > 0, \\ - v''(z), \qquad x + z < 0. \end{cases}$$

Since v'' and w'' are nonpositive, and φ is nonnegative and $v'(z) > w'(0)$ (see (2.9)), we see by (3.16) that $\partial^2G/\partial z^2$ is, for each x, nonnegative for all $z \geq 0$ for which it is defined, namely, (i) for all nonnegative z in case $x \geq 0$, and (ii) for all nonnegative z except possibly $z = -x$ in case $x < 0$. In either case the derivative $\partial G/\partial z$ is therefore monotone nondecreasing in z. For $x < 0$ we have

$$\lim_{z\uparrow -x} \frac{\partial^2 G}{\partial z^2} = -v''(-x), \qquad x < 0, \tag{3.17}$$

and

$$\lim_{z\downarrow -x} \frac{\partial^2 G}{\partial z^2} = -v''(-x) + [v'(-x) - w'(0)]\varphi(0), \qquad x < 0. \tag{3.18}$$

Thus, by (2.9),

$$\lim_{z\downarrow -x} \frac{\partial^2 G}{\partial z^2} \geq \lim_{z\uparrow -x} \frac{\partial^2 G}{\partial z^2}, \qquad x < 0. \tag{3.19}$$

Strict inequality holds here if $\varphi(0) > 0$, as we see by (2.9) and (3.18). We have, as noted, the result that $\partial G/\partial z$ is monotone nondecreasing in z, for each x. For $x \geq x_0$ (see (3.14)) this means immediately that $G(z,x)$ has a minimum with respect to z and it takes on this minimum value at $z = 0$. In particular, for an $x > x_0$ such that $(\partial G/\partial z)_{z=0+} > 0$, the minimum of G is taken on only at $z = 0$. Hence, under the condition of (3.15), $z = 0$ is the unique minimum point of G for each $x > x_0$. For $x = x_0$ we see by (3.16) that

$$(3.20)\qquad \left[\left(\frac{\partial^2 G}{\partial z^2}\right)_{z=0+}\right]_{x=x_o} = -\,v''(0)\int_{x_o}^{\infty}\varphi(\xi)d\xi$$

$$-\int_0^{x_o} w''(x_o-\xi)\varphi(\xi)d\xi + [v'(0) - w'(0)]\varphi(x_o),$$

and this is strictly positive under the condition that $\varphi(\eta) > 0$ for $\eta > 0$. Hence, under the condition of (3.15) it is true also for $x = x_o$ that the minimum of G is taken on only at $z = 0$.

We must now examine the minimization of G, with respect to z, for values of $x < x_o$. For such a value of x, $G(z;x)$ starts out at $z = 0$ with a negative slope (see (3.14)). This slope is monotone nondecreasing, by virtue of the nonnegativity of $\partial^2G/\partial z^2$. We ask now if this slope eventually becomes positive. By (3.5) we have

$$(3.21)\qquad \lim_{z\to\infty}\frac{\partial G}{\partial z} = c - w'_\infty > 0.$$

Thus, the answer to our question is in the affirmative; in fact, $\partial G/\partial z$ tends to the same positive limit in $z \to \infty$ for every x. Therefore, in the case of concern to us, namely, $x < x_o$, the derivative $\partial G/\partial z$ vanishes at some unique positive value of z or for all values of z in some positive interval, and at this unique point, or at all the points of this interval, as the case may be, the function G takes on its minimum value in z. Now, by (2.6) and the monotone nonincreasing character of v', we see in (3.5) that $\partial G/\partial z < 0$ for every z such that $x + z \leqq 0$. It follows that the minimizing value or values of z must be such that $x + z > 0$. Then, by (3.16) we see that under the condition of (3.15), namely, $\varphi(\eta) > 0$ for $\eta > 0$, the second derivative $\partial^2G/\partial z^2$ is strictly positive at a minimizing z. Consequently, there is a unique minimizing z under this condition. And this minimizing z is the value that causes the vanishing of the first expression in (3.5).

Let us now gather together our results on the minimization of G:

Proposition I. For each $x \geq x_0$, $G(z;x)$ is monotone, nondecreasing, convex in $z \geq 0$. Its minimum is therefore taken on either at $z = 0$ only or at all points of a closed interval with left end-point $z = 0$. If $\varphi(\eta) > 0$ for $\eta > 0$, then $z = 0$ is the unique minimum point.

For each $x < x_0$ $G(z;x)$ is first decreasing and then increasing as z runs from 0 to ∞, and it is convex. Its minimum is taken on either at a unique positive point or at every point of a closed positive interval. Such a minimum point z satisfies $x + z > 0$ and is a root of the equation

$$(3.22)\quad c - v'(z)\int_{x+z}^{\infty}\varphi(\xi)d\xi - \int_0^{x+z} w'(x + z - \xi)\varphi(\xi)d\xi = 0.$$

If $\varphi(\eta) > 0$ for $\eta > 0$, then there is a unique minimum point for $G(z;x)$, and this point is the unique solution of (3.22).

From this stage on we shall work under the following assumption, which has been shown above to be important for uniqueness of the policy function:

(3.23) ASSUMPTION: $\varphi(\eta) > 0$ <u>for</u> $\eta > 0$.

With this assumption we may now define

(3.24) $z^*(x) \overset{\text{def}}{=}$ <u>the unique value of z at which G(z,x) takes on its minimum value</u>.

From proposition I we have:

$$(3.25) \qquad z^*(x) = \begin{cases} 0, & \underline{\text{for}} \quad x \geq x_0, \\ \underline{\text{the unique solution of}}\ (3.22), & \underline{\text{for}}\ x < x_0, \end{cases}$$

and the additional fact that

$$(3.26) \qquad x + z^*(x) > 0 \ \underline{\text{for all}}\ x.$$

4. Study of the function $H(x)$. We have

$$(4.1) \qquad H(x) = \min_{z \geq 0} G(z;x) = G(z^*(x);\ x),$$

and we shall examine the first and second derivatives of H.

In general,

$$(4.2) \qquad H'(x) = \left(\frac{\partial G}{\partial x}\right)_{z=z^*(x)} + \left(\frac{\partial G}{\partial z}\right)_{z=z^*(x)} \frac{dz^*}{dx}$$

<u>for</u> $x \neq x_0$ (<u>possibly</u>).

But we have

$$(4.3) \qquad \begin{cases} \left(\dfrac{\partial G}{\partial z}\right)_{z=z^*(x)} = 0 & \underline{\text{for}} \quad x < x_0, \\ \dfrac{dz^*}{dx} = 0 & \underline{\text{for}} \quad x > x_0. \end{cases}$$

Hence, (4.2) reduces to

$$(4.4) \qquad H'(x) = \left(\frac{\partial G}{\partial x}\right)_{z=z^*(x)}, \quad x \neq x_o.$$

From (2.14),

$$(4.5) \qquad \frac{\partial G}{\partial x} = -\frac{\partial V}{\partial x} - \frac{\partial W(x+z)}{\partial x} .$$

By virtue of (3.26) we see that the first two expressions in (3.1) and the first expression in (3.2) are the pertinent ones for evalutation of the terms on the right-hand side of (4.5). We find

$$(4.6) \quad \frac{\partial G}{\partial x} = \begin{cases} \int_0^{x+z} v'(\xi - x)\varphi(\xi)d\xi - \int_0^{x+z} w'(x+z-\xi)\varphi(\xi)d\xi, & x \leqq 0, \\ \int_x^{x+z} v'(\xi - x)\varphi(\xi)d\xi - \int_0^{x+z} w'(x+z-\xi)\varphi(\xi)d\xi, & x > 0. \end{cases}$$
$$\text{for } x+z>0$$

From (3.13) and (3.22) it follows that z^* is a continuous function, and this together with the continuity properties of $\partial G/\partial x$ in (4.6) implies that H' is a continuous function for all x. Thus, the exception noted in (4.4), namely, $x \neq x_o$, may be discarded.

We may obtain "explicit" expressions for $H'(x)$ by substituting $z^*(x)$ for z in (4.6). If we do this and avail ourselves of (3.22), we obtain:

$$(4.7)\quad H'(x) = \begin{cases} \displaystyle\int_0^{x+z^*(x)} v'(\xi - x)\varphi(\xi)d\xi + v'(z^*(x))\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi - c, & x \leqq 0, \\ \displaystyle\int_x^{x+z^*(x)} v'(\xi - x)\varphi(\xi)d\xi + v'(z^*(x))\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi - c, & 0 < x < x_o, \\ \displaystyle -\int_0^x w'(x-\xi)\varphi(\xi)d\xi, \quad x \geqq x_o. \end{cases}$$

From the third expression here we get immediately

$$(4.8)\qquad \lim_{x\to\infty} H'(x) = -w'_c .$$

To gain information about the behavior of H' as $x \to -\infty$, we must study z^* a little more closely. From (3.22) we find:

$$(4.9)\quad \frac{dz^*}{dx} = -\frac{[v'(z^*(x))-w'(0)]\varphi(x+z^*(x)) - \int_0^{x+z^*(x)} w''(x+z^*(x)-\xi)\varphi(\xi)d\xi}{\left\{\begin{array}{l}[v'(z^*(x))-w'(0)]\varphi(x+z^*(x)) \\ -\int_0^{x+z^*(x)} w''(x+z^*(x)-\xi)\varphi(\xi)d\xi - v''(z^*(x))\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi\end{array}\right\}}$$

<u>for</u> $x < x_o$.

We see that, for $x < x_o$, dz^*/dx is negative and bounded by 1 in absolute value:

$$(4.10)\qquad -1 \leqq \frac{dz^*}{dx} < 0, \qquad x < x_o.$$

Hence, z^* is monotone decreasing in x. From (4.10) we get

$$\frac{d}{dx}\left[x + z^*(x)\right] \geqq 0, \tag{4.11}$$

and this is valid for all x. From (4.9) we see also that if $v''(\eta) < 0$ for $\eta > 0$, then the strict inequality signs hold in (4.10) and (4.11). The result (4.11) tells us that $x + z^*(x)$ is monotone nondecreasing. Let us define

$$\rho = \lim_{x \to -\infty} \left[x + z^*(x)\right]. \tag{4.12}$$

By (3.26) we have that $\rho \geqq 0$. The determining equation for ρ is obtained from (3.22) by letting $x \to -\infty$, and noting that in this case $z^*(x) \to +\infty$, by virtue of (3.26). We get:

$$c - v'_\infty \int_\rho^\infty \varphi(\xi)d\xi - \int_0^\rho w'(\rho - \xi)\varphi(\xi)d\xi = 0. \tag{4.13}$$

Since $c > 0$ we see that, in fact, $\rho > 0$.

If we now consider $x \to -\infty$ in (4.7) we get

$$\lim_{x \to -\infty} H'(x) = v'_\infty - c. \tag{4.14}$$

From (2.6) and the fact that $w'_\infty \geqq 0$ we have then:

$$\lim_{x \to -\infty} H'(x) > 0 \geqq \lim_{x \to \infty} H'(x). \tag{4.15}$$

$H'(x)$ actually changes sign as x runs from $-\infty$ to $+\infty$. In fact there is a sign change between $-\infty$ and x_o, as we see from the fact that

$$H'(x_o) = -\int_0^{x_o} w'(x_o - \xi)\varphi(\xi)d\xi < 0. \tag{4.16}$$

We now calculate the second derivative of H; from (4.7) we get:

$$(4.17)\quad H''(x) = \Big\{[v'(z^*(x)) - w'(0)]\varphi(x+z^*(x)) - \int_0^{x+z^*(x)} w''(x+z^*(x)-\xi)\varphi(\xi)d\xi\Big\}\left(1 + \frac{dz^*}{dx}\right) - \int_0^{x+z^*(x)} v''(\xi - x)\varphi(\xi)d\xi$$

<u>for</u> $x < 0$,

and

$$(4.18)\quad H''(x) = \Big\{[v'(z^*(x)) - w'(0)]\varphi(x+z^*(x)) \cdot - \int_0^{x+z^*(x)} w''(x+z^*(x)-\xi)\varphi(\xi)d\xi\Big\}\left(1 + \frac{dz^*}{dx}\right) - v'(0)\varphi(x) - \int_x^{x+z^*(x)} v''(\xi - x)\varphi(\xi)d\xi$$

<u>for</u> $0 < x < x_o$,

and

$$(4.19)\quad H''(x) = -\,w'(0)\varphi(x) - \int_0^x w''(x-\xi)\varphi(\xi)d\xi$$

<u>for</u> $x > x_o$.

The information gathered above concerning the function H will suffice to investigate the function J.

5. Study of the function $J(x)$. We have

$$J(x) = L(x) + H(x) \tag{5.1}$$

and

$$L(x) = \begin{cases} \int_0^x h(x-\xi)\varphi(\xi)d\xi + \int_x^\infty p(\xi - x)\varphi(\xi)d\xi, & x > 0, \\ \int_0^\infty p(\xi - x)\varphi(\xi)d\xi, & x \leqq 0. \end{cases} \tag{5.2}$$

H and L are continuous functions, and therefore J is continuous. From (5.2) we calculate:

$$L'(x) = \begin{cases} \int_0^x h(x-\xi)\varphi(\xi)d\xi - \int_x^\infty p'(\xi - x)\varphi(\xi)d\xi, & x > 0, \\ -\int_0^\infty p'(\xi - x)\varphi(\xi)d\xi, & x < 0. \end{cases} \tag{5.3}$$

From this we see that also L' is continuous. We have already seen that H' is continuous. Hence, J' is continuous.

Differentiating once again, we get

$$L''(x) = \begin{cases} [h'(0)+p'(0)]\varphi(x) + \int_0^x h''(x-\xi)\varphi(\xi)d\xi \\ \qquad + \int_x^\infty p''(\xi-x)\varphi(\xi)d\xi, & x > 0, \\ \int_0^\infty p''(\xi - x)\varphi(\xi)d\xi, & x < 0. \end{cases} \tag{5.4}$$

Combining this with (4.17)--(4.19) we get

$$(5.5) \quad J''(x) = \left\{ [v'(z^*(x)) - w'(0)]\varphi(x+z^*(x)) - \int_0^{x+z^*(x)} w''(x+z^*(x)-\xi)\varphi(\xi)d\xi \right\} \left(1 + \frac{dz^*}{dx}\right) - \int_0^{x+z^*(x)} v''(\xi-x)\varphi(\xi)d\xi + \int_0^{\infty} p''(\xi-x)\varphi(\xi)d\xi$$

<u>for</u> $x < 0$

and

$$(5.6) \quad J''(x) = \left\{ [v'(z^*(x)) - w'(0)]\varphi(x+z^*(x)) - \int_0^{x+z^*(x)} w''(x+z^*(x)-\xi)\varphi(\xi)d\xi \right\} \left(1 + \frac{dz^*}{dx}\right)$$

$$- \int_x^{x+z^*(x)} v''(\xi-x)\varphi(\xi)d\xi + [h'(0)+p'(0)-v'(0)]\varphi(x)$$

$$+ \int_0^x h''(x-\xi)\varphi(\xi)d\xi + \int_x^{\infty} p''(\xi-x)\varphi(\xi)d\xi$$

<u>for</u> $0 < x < x_0$

and

$$(5.7) \quad J''(x) = -\int_0^x w''(x-\xi)\varphi(\xi)d\xi + [h'(0) + p'(0) - w'(0)]\varphi(x)$$

$$+ \int_0^x h''(x-\xi)\varphi(\xi)d\xi + \int_x^{\infty} p''(\xi-x)\varphi(\xi)d\xi$$

<u>for</u> $x > x_0$.

We see immediately, by (2.1), (2.2), (2.4), (2.9) and (4.11), that J'' is nonnegative for $x < 0$. Taking account as well of (2.3) and (2.7), we see that J'' is strictly positive for $0 < x < x_o$. And by virtue of these also, noting that

(5.8) $$p'(0) > v'(0) > w'(0),$$

we see that, finally, J'' is strictly positive for $x > x_o$. Hence, we have the result that J is a convex function.

Let

(5.9) $$p'_\infty \overset{\text{def}}{=} \lim_{\eta \to \infty} p'(\eta).$$

(p'_∞ may be infinite.) From (5.3) we see that

(5.10) $$\lim_{x \to -\infty} L'(x) = -p'_\infty \ .$$

Combining this with (4.14) we have

(5.11) $$\lim_{x \to -\infty} J'(x) = v'_\infty - p'_\infty - c.$$

From the properties of v and p it follows by (2.7) that

(5.12) $$v'_\infty < p'_\infty.$$

Hence,

(5.13) $$\lim_{x \to -\infty} J'(x) < 0.$$

Again from (5.3),

(5.14) $$\lim_{x \to \infty} L'(x) = h'_\infty.$$

Combining this with (4.8), we have

(5.15) $$\lim_{x \to \infty} J'(x) = h'_\infty - w'_\infty.$$

Thus, by (2.8),

$$\lim_{x \to \infty} J'(x) > 0. \tag{5.16}$$

Hence, the function J is U-shaped.

Now we are at the point of investigating J with regard to the question centering around (2.19). For this purpose we state state--without proof--the following

Lemma I. Let $g(x)$ be defined for all $x \in (-\infty, \infty)$, and let it have the properties

$$\begin{cases} \lim_{x \to -\infty} g'(x) < 0 < \lim_{x \to +\infty} g'(x) \\ g''(x) \geqq 0 \end{cases} \tag{5.17}$$

Furthermore, let α and β be positive numbers, and let

$$\psi(x) = g(x + \alpha) + \beta. \tag{5.18}$$

Then, the graph of the function ψ is either entirely above the graph of g, the two graphs having no point in common, or the graph of ψ cuts that of g in just one point.

If the graph of ψ fails to cut the graph of g for some positive β, then

$$\lim_{x \to -\infty} g'(x) > -\infty. \tag{5.19}$$

Conversely, if (5.19) holds and $g(x)$ has an asymptote as $x \to -\infty$, then for all sufficiently large β the graph of ψ fails to cut the graph of g.

We apply this lemma to our function J. From (5.11) we have that sine v'_∞ is finite, the question of whether or not (5.19) of the lemma obtains in our case depends on p'_∞. It follows immediately from the lemma, then, that, whatever be m and c_0, there is a γ' satisfying (2.19) if $p'_\infty = \infty$.

On the other hand, if $p'_\infty < \infty$ and $J(x)$ has an asymptote as $x \to -\infty$, then the graphs of $J(x)$ and $J(x + m) + c_0 m$ will intersect for all sufficiently small positive values of $c_0 m$, and will fail to intersect for all values of $c_0 m$ greater than or equal to the pertinent critical value. This critical value depends only on the slope of the asymptote of J, which is given by (5.11). However, for completeness, we shall determine the asymptote of J fully explicitly.

To investigate this asymptote we shall make the convenient (and reasonable) assumption:

$$(5.20) \qquad \text{ASSUMPTION:} \quad \mu = \int_0^\infty \xi\varphi(\xi)d\xi < \infty;$$

that is, that the demand variable has finite expectation.

If g is a function which has an asymptote as $x \to -\infty$, then the vertical intercept of the asymptote is given by

$$(5.21) \qquad \lim_{x \to -\infty} [g(x) - xg'(x)].$$

The same formula applies for $x \to +\infty$ if there is an asymptote in this direction. In our present case, we shall see that H has an asymptote as $x \to -\infty$ provided the function v has one as $x \to +\infty$, and that L has one as $x \to -\infty$ provided p does as $x \to +\infty$. J is the sum of L and H, and the formula (5.21) is linear. Hence the vertical intercept of the asymptote of J is the sum of those for L and H. Let λ_L and λ_H denote the latter two vertical intercepts, respectively. We first calculate λ_L.

We suppose that the function p has an asymptote as $x \to \infty$. Let λ_p be the vertical intercept of this asymptote. Applying (5.21) to (5.2) and (5.3) we have:

$$\lambda_L = \lim_{x \to -\infty}\left[\int_0^\infty p(\xi-x)\varphi(\xi)d\xi + x\int_0^\infty p'(\xi-x)\varphi(\xi)d\xi\right] \tag{5.22}$$

$$= \lim_{x \to -\infty}\int_0^\infty [p(\xi-x) + xp'(\xi-x)] \quad \varphi(\xi)d\xi$$

$$= \lim_{x \to -\infty}\int_0^\infty \{[p(\xi-x) - (\xi-x)p'(\xi-x)]$$

$$+ \xi p'(\xi-x)\} \ \varphi(\xi)d\xi$$

$$= \int_0^\infty \{\lambda_p + p'_\infty\xi\} \ \varphi(\xi)d\xi,$$

and so

$$\lambda_L = \lambda_p + p'_\infty \mu. \tag{5.23}$$

Next we calculate λ_H. Utilizing (2.14), (3.1), (3.2) and (4.7), we have, for $x < 0$,

$$(5.24)\quad H(x) - xH'(x) = cz^*(x) -$$

$$- \int_0^{x+z^*(x)} v(\xi-x)\varphi(\xi)d\xi - v(z^*(x))\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi$$

$$- \int_0^{x+z^*(x)} w(x + z^*(x) - \xi)\varphi(\xi)d\xi$$

$$- x\int_0^{x+z^*(x)} v'(\xi-x)\varphi(\xi)d\xi - xv'(z^*(x))\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi$$

$$+ cx$$

$$= c(x+z^*(x)) - \int_0^{x+z^*(x)} [v(\xi-x)+xv'(\xi-x)]\varphi(\xi)d\xi$$

$$- [v(z^*(x)) + xv'(z^*(x))]\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi$$

$$- \int_0^{x+z^*(x)} w(x + z^*(x) - \xi)\varphi(\xi)d\xi$$

$$= c\cdot(x+z^*(x)) - \int_0^{x+z^*(x)} [v(\xi-x)-(\xi-x)v'(\xi-x)]\varphi(\xi)d\xi$$

$$- \int_0^{x+z^*(x)} \xi v'(\xi-x)\varphi(\xi)d\xi$$

$$- [v(z^*(x)) - z^*(x)v'(z^*(x))]\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi$$

$$-(x + z^*(x))\int_{x+z^*(x)}^{\infty}\varphi(\xi)d\xi$$

$$- \int_0^{x+z^*(x)} w(x + z^*(x) - \xi)\varphi(\xi)d\xi.$$

In this last form we see that every term has a limit, provided the function $v(\eta)$ has an asymptote as $\eta \longrightarrow \infty$. We assume that this is so, and we denote the vertical intercept of this asymptote by λ_v. Then, taking limits, we get:

$$(5.25) \quad \lambda_H = c\rho - \lambda_v \int_0^\rho \varphi(\xi)d\xi - v'_\infty \int_0^\rho \xi\varphi(\xi)d\xi$$

$$- \lambda_v \int_\rho^\infty \varphi(\xi)d\xi - \rho \int_\rho^\infty \varphi(\xi)d\xi - \int_0^\rho w(\rho-\xi)\varphi(\xi)d\xi,$$

or, simplifying,

$$(5.26) \quad \lambda_H = c\rho - \lambda_v - \rho \int_\rho^\infty \varphi(\xi)d\xi - v'_\infty \int_0^\rho \xi\varphi(\xi)d\xi - \int_0^\rho w(\rho-\xi)\varphi(\xi)d\xi.$$

And now combining (5.23) and (5.26) we have

$$(5.27) \quad \lambda_J = (\lambda_p - \lambda_v) + p'_\infty \mu - v'_\infty \int_0^\rho \xi\varphi(\xi)d\xi$$

$$+ c\rho - \rho \int_\rho^\infty \varphi(\xi)d\xi - \int_0^\rho w(\rho - \xi)\varphi(\xi)d\xi$$

Thus, finally, we have, by (5.11) and with λ_J as given by (5.27), the following equation for the asymptote of J as $x \longrightarrow -\infty$:

$$(5.28) \qquad y = (v'_\infty - p'_\infty - c)x + \lambda_J.$$

Now, from (5.28) we are able to calculate that the asymptote of $J(x + m)$ is

$$(5.29) \qquad y = (v'_\infty - p'_\infty - c)\, x + \left[m(v'_\infty - p'_\infty - c) + \lambda_J\right].$$

And we see that the line (5.29) is at a vertical distance

(5.30) $$m(c + p'_\infty - v'_\infty)$$

below the line (5.28). (Notice that this quantity (5.30) does not depend on λ_J, and could have been reasoned to without explicit knowledge of λ_J.)

The quantity (5.30) is the critical value to which we referred at the beginning of our discussion of asymptote of J. Thus, if $c_o m$ is less than (5.30), then the graphs of $J(x)$ and $J(x + m) + c_o m$ will intersect; on the other hand, if $c_o m$ is greater than or equal to (5.30), then these two graphs will not intersect. In the case of intersection, the abscissa of the point of intersection is the number γ' fulfilling (2.19). Thus, we may state:

Proposition II. *If (in the case of existence of asymptotes)*

(3.31) $$c_o < c + p'_\infty - v'_\infty,$$

then there is a number γ' *such that (2.19) holds. This* γ' *is the abscissa of the unique point of intersection of the graphs of* $J(x)$ *and* $J(x + m) + c_o m$.

If

(5.32) $$c_o \geqq c + p'_\infty - v'_\infty,$$

then the graph of $J(x + m) + c_o m$ *lies everywhere above the graph of* $J(x)$, *no emergency level is to be defined*.

We state also formally the additional result noted above:

Proposition III. If $p'_\infty = \infty$, then there is a γ' such that (2.19) holds. Again, this number is the abscissa of the unique point of intersection of the graphs of $J(x)$ and $J(x + m) + c_o m$.

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