CORNELL UNIVERSITY

GRADUATE SCHOOL OF AEROSPACE ENGINEERING

A THEORY FOR UNSTEADY MOTIONS OF JET-FLAPPED THIN AIRFOILS

By

John Clifford Erickson, Jr.

SEPTEMBER, 1962

Ithaca, New York

Best
Available
Copy

A THEORY FOR UNSTEADY MOTIONS OF JET-FLAPPED THIN AIRFOILS

J. C. Erickson, Jr.

Submitted to the Office of Scientific
Research of the Air Force Air
Research and Development
Command, in partial fulfillment
of contract #AF 49(638)-544.

W. R. Sears

FOREWORD

This report summarizes research carried out under contract AF 49(638)-544 at the Graduate School of Aerospace Engineering, Cornell University, Ithaca, New York. This contract has been monitored by the Mechanics Division of the Office of Scientific Research, USAF.

The work reported here was begun before the extended visit of Professor D. A. Spence in 1959-60, was pursued under his direction during that academic year and more recently has been continued in his absence. As the reader will discover, the conclusions reached are in some degree critical of the solutions provided by Prof. Spence for the integro-differential equations set up by the author, John C. Erickson, Jr. Since Prof. Spence's return to England, where he is employed at the Royal Aircraft Establishment, he has continued his interest in the investigation and sincere thanks must go to him for his many comments.

This is the last Technical Note to be prepared under the above-mentioned contract, which has now been replaced by a grant.

ABSTRACT

A linearized model for the incompressible, inviscid, irrotational, and unsteady flow about a thin airfoil with jet-flap is formulated. The unsteady problems considered are the transient and oscillatory deflection of the jet, plunging and pitching of the airfoil, deflection of a "blown-flap," and also the penetration of a sharp-edged gust. Justification is given for representation of the jet, in the limit of high speed, small thickness, and constant momentum-flux strength, by a vortex sheet, across which there is a pressure difference proportional to the momentum-flux strength and inversely proportional to the local radius of curvature of the jet. The dynamic and kinematic interaction of the main stream with the vortex sheets representing the airfoil and jet are shown to be described by a coupled set of equations consisting of a third-order partial differential equation and a singular integral equation, along with appropriate boundary conditions. The properties of these equations and their relationship to classical unsteady thin-airfoil theory and steady jet-flap theory are discussed.

For small momentum-flux strength, and for either small-time after initiation of transient motion or high frequency of oscillation, a transformation is made which leads to a simplified form of the governing equations. All types of airfoil motion are reduced in this approximation to a single set of equations, whose solution is found in terms of the solution for jet deflection.

Spence (1961B) attempted, by making a further approximation, a solution for small-time after instantaneous jet deflection. This solution is found to be incorrect, and it is shown that no solution can be found in the sense of this further approximation. Spence's related solution for high-frequency oscillatory jet deflection is also found to be incorrect, and a tentative, corrected solution is proposed and discussed.

TABLE OF CONTENTS

INTRODUCTION

The jet-flap principle has been extensively studied, both experimentally and theoretically, in recent years since the pioneering work of Davidson (1956), Stratford (1956), Malavard, Poisson-Quinton and Jousserandot (1956), and Helmbold (1955). The principle is briefly this: a thin, high-speed jet of air is ejected at or near the trailing edge of an airfoil. Besides the direct-reaction lift, which acts upon the internal jet ducting, additional lift is obtained due to the effect of the curved jet on the airfoil external pressure distribution. Furthermore, this modified external pressure distribution accounts for recovery of very nearly the total thrust of the jet, independent of jet deflection angle, the so-called "thrust hypothesis," cf. Yen (1960). In the jet-flap, then, has been found a promising means of integrating the lift and propulsion of an airfoil. A thorough review of the jet-flap literature has been given recently by Korbacher and Sridhar (1960).

This integration has generated interest for application not only to airplane wings for STOL performance, but also to airfoil applications, e.g., helicopter-rotor blades, cf. Richards and Jones (1956) and Dorand (1959), jet-engine compressor blading, cf. Clark and Ordway (1959), Brocher (1961) and Paulon (1959), and very recently to hydrofoils, cf., Ho (1961).

In view of the proposed applications of the jet-flap, it would seem desirable to extend the analysis to unsteady problems.

For if the jet-flap is used as a controlling device, e.g., for cyclic control of helicopter-rotor blades, or for aircraft in slow flight, the lift response to time-dependent jet deflections is of great importance. In flutter stability calculations, the lift and pitching moments of an oscillating jet-flapped airfoil are required. The motivation for the study of unsteady jet-flapped airfoil theory therefore arises from the same considerations which motivated the classical theory of unsteady airfoil motion.

To study the steady-state lifting properties of jet-flapped airfoils, a model has been formulated independently by Malavard (1957), Helmbold (1955) and Spence (1956) - which will be referred to subsequently as I. In this model, the non-homogeneous flow of the jet embedded in the main-stream is treated by representing the jet by a vortex sheet, across which there is a pressure difference proportional to the jet momentum flux and inversely proportional to the jet radius of curvature. This model has been shown by Spence in I to be a good approximation when the jet velocity is very much greater than the free-stream velocity, and when the jet is sufficiently thin. With this as the model for the jet, the problem can be linearized in typical thin-airfoil fashion. The linearized, two-dimensional problem has been solved in a rheoelectric analogy by Malavard (1957), numerically by Spence in I and Spence (1958), and finally analytically by Spence (1961A) - to be referred to as II. The corresponding problem for supercavitating hydrofoils has been solved numerically, following I, by Ho (1961). Three-dimensional theories have also been put forth, but are beyond the scope of this present research.

In the present research, the unsteady two-dimensional lifting problem of a jet-flapped thin airfoil in incompressible flow is formulated as an extension of the Malavard-Helmbold-Spence model. The problems considered are those where (i) the airfoil is performing some time-dependent motion normal to a mean, steady position, or ii) the jet-deflection angle is time-dependent. The jet momentum flux at its exit from the airfoil is assumed constant, independent of time. Furthermore, the jet is assumed to be fully developed in length prior to the onset of the unsteady motion, i.e., the time-dependent motion is superimposed upon a fully developed steady-state configuration.

In Chapter 1 the non-homogeneous flow problem of a jet and mainstream of different total pressures is treated to find a model for the dynamical interaction of the jet and mainstream. The limit of a very thin, very high speed, constant-momentum-flux jet is taken, and the consequent representation of the jet by a vortex sheet is justified. This vortex sheet is characterized, exactly as in the steady problem, by its support of a pressure difference proportional to the jet momentum flux and inversely proportional to the instantaneous local radius of curvature of the jet.

The resultant flow problem is linearized by the assumptions of classical thin-airfoil theory in Chapter 2, the airfoil also being represented by a vortex sheet with the appropriate downwash boundary condition. Consideration of the pressure difference across the vortex sheet representing the jet leads to a third-order partial differential equation relating the jet vortex strength and the jet ordinate, or alternately, the same-order equation relating the jet vortex strength and the downwash on the jet. The integral equation for the downwash on the airfoil and jet, calculated by the Biot-Savart Law, or alternately considered, the mixed boundary-value problem in the main-stream velocity perturbations, is solved, resulting in a singular integral equation relating the jet vortex strength and the downwash on the jet. Boundary conditions on the jet ordinate and slope at the trailing edge are specified. A proof due to Spence (1961B) - to be referred to as III - is given that the potential difference across the jet, at a given instant of time after initiation of the unsteady motion, vanishes if a point sufficiently far downstream is considered. This condition is shown to be stated alternately in the form of the Wagner integral condition of the classical theory of unsteady thin airfoils without jets. With this condition the formulation is completed and some properties of the equations of the problem are discussed.

Expressions for the lift and pitching-moment coefficients are derived in Chapter 3 by relating the pressure distribution on the airfoil to the vortex distribution representing the jet. These expressions are compared to their counterparts in the classical unsteady thin-airfoil theory.

Detailed equations for the application of this model to five fundamental problems of flat-plate airfoils are given in Chapter 4, and are applicable for time-dependence of both transient and steady-state oscillatory nature. These problems are:

i) An airfoil aligned with the free-stream direction and having a time-dependent jet deflection angle at the trailing edge, the "jet-deflection" or "singular-blowing" problem.

ii) An airfoil performing a purely plunging motion about a mean position aligned with the free stream, the jet always remaining tangential to the trailing edge.

iii) An airfoil performing a pitching motion about some axis in its plane, the jet being always tangential to the trailing edge.

iv) An airfoil having time-dependent deflection of a mechanical flap, over which a jet is blown from the hinge point and leaves tangentially at the trailing edge, the so-called "blown flap" or "jet-augmented flap" problem.

v) Entrance of the airfoil, previously aligned with the free stream, into a sharp-edged gust of constant upwash amplitude. The jet remains tangential at the trailing edge, and the relative speed between the airfoil and gust is arbitrary, but constant. These problems, being linear, may be superimposed in any desired fashion.

The reduction of the equations of Chapters 3 and 4 to the case of airfoils without jets is discussed in Chapter 5. The classical unsteady theory of thin-airfoils is then outlined, bringing out certain features of the flow pattern, e.g., the downwash distribution behind the airfoil, which have particular importance in understanding the extension of this theory to the jet-flap case. The steady limit of the equations is also found, and Spence's steady-state solutions of I, (1958), and II discussed.

In Chapter 6, the "boundary-layer" nature of the equations for small values of the jet momentum and small times after initiation of transient motion or high-frequency steady-state oscillations are investigated and the equations are given in terms of new "boundary-layer" coordinates. All the airfoil motion problems - ii) to v) above - are shown to reduce to the same equationsfor a first approximation in these coordinates. The relation of the first approximation solution of this airfoil motion problem to that of the jet-deflection problem is shown.

The further small-time, or high-frequency, approximation of neglecting x- derivatives with respect to t- derivatives is then examined closely in Chapter 7. Errors made in III in the solution of the jet-deflection problem in terms of the jet ordinate and jet vortex distribution for small times are pointed out. Attempts to correct these errors by considering the downwash on the jet and the jet vortex distribution as unknowns and a similar approach to the problem of airfoil motion demonstrate the failure of this approximation to give valid solutions. However, for high-frequency steady-state oscillations, tentative solutions are proposed for both the jet-deflection and airfoil-motion problems to replace the erroneous ones given in III.

Chapter 8 briefly gives some conclusions of this research and points out some areas of, and approaches to, the problem of the unsteady motion of jet-flapped airfoils where further work is necessary. These are felt worthy of further research.

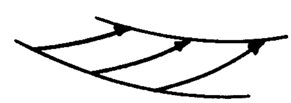
CHAPTER 1 - DYNAMIC INTERACTION OF THE JET AND MAIN STREAM

1.1 Assumptions

For the purposes of this section, no assumptions about the airfoil need be made, except that it has a jet emerging at the trailing edge, and that it has some means of causing unsteady motion of the jet. This means might be its own motion or the motion of the jet ducting within the airfoil.

The jet is assumed to be fully developed in length, i.e., it extends infinitely far downstream at any instant its motion is being considered. The momentum flux of the jet at the airfoil trailing edge is taken constant in time. The flow in the jet is assumed inviscid, incompressible and irrotational.

It is also assumed that the local velocity, v, of the jet is very much greater than the local velocity, u, of the main stream in the vicinity of the jet. This velocity, u, is composed of the undisturbed free-stream velocity at infinity upstream, U_0, plus perturbations due to the interaction between the two flows, including the velocity of downward translation of the jet boundaries. For the practical jet-flap applications which have been proposed, $v \gg U_0$, so the perturbations, in particular those normal to U_0, i.e., due to the downward translation of the jet boundaries, must be very small compared to v. For $v \gg u$, it follows that the instantaneous streamlines of the flow in the jet at the boundaries between the jet and main stream are substantially parallel to the instantaneous shape of these boundaries. That is, with reference to Figure 1, the jet velocities, v_1 and v_2, at the boundaries are assumed to be parallel to the boundaries. Furthermore, if the jet is assumed thin, significant mean properties of the flow variables at any position along the jet may be defined, and the streamlines of the jet flow, in addition to being parallel to the boundaries at the boundaries, will be all parallel to each other, hence concentric at any position along the jet. This property leads to a great simplification in the equations governing the problem, as will be seen in the next section. To clarify this point, consider the excluded case, $u = O(v)$, due, say, to large downward velocities of the jet boundaries. The instantaneous streamline pattern, even for a thin jet, would intersect the jet boundaries at an appreciable angle, as shown below. Such a geometry must be excluded in order to formulate a tractable model.



The restriction, $v \gg u$, will be assumed to be met in all types of airfoil and jet motions to be considered in Section 2.2. In order to actually attempt solutions of these problems in Chapter 7, discontinuous motions of the airfoil and jet, as represented by the unit-step-function (equations 2-17), will be considered. These motions will always be treated in the sense that they are mathematical idealizations of continuous motions with $v \gg u$.

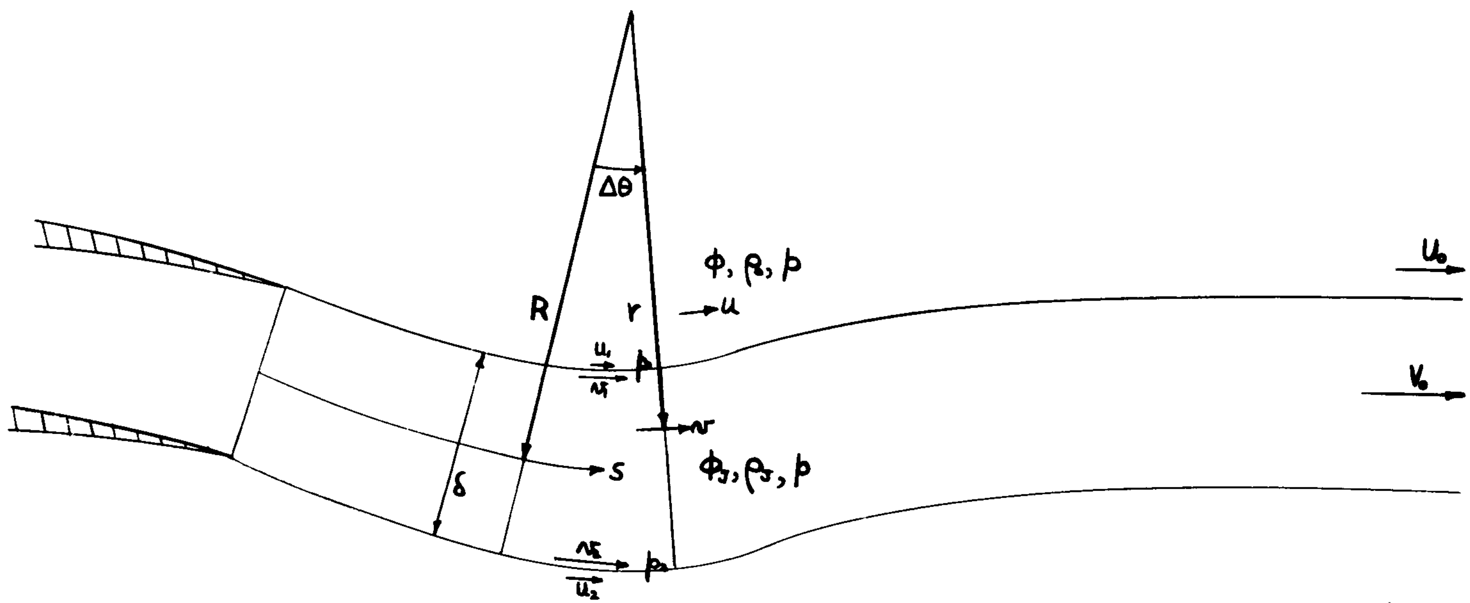


Figure 1 - Flow in an Element of the Jet (Schematic)

The main-stream flow is also assumed inviscid, incompressible, and irrotational. The total pressure in the jet is, for $v >> u$, greater than that in the main stream. This and the continuity of static pressure across each boundary between the flows require, in the absence of viscous mixing (since both flows are assumed inviscid), a vortex sheet at each boundary to satisfy the velocity discontinuity there.

In the main stream, the integrated form of Euler's equation is the so-called "unsteady Bernoulli equation," which is, here,

$$p + \frac{\rho_0 u^2}{2} + \rho_0 \frac{\partial \phi}{\partial t} = p_0 + \frac{\rho_0 U_0^2}{2}, \tag{1-1}$$

where p, ϕ, and u are the local static pressure, velocity potential and velocity, respectively; ρ_0 is the (constant) density. p_0 and U_0 are the static pressure and velocity, evaluated at the undisturbed conditions at infinity, hence independent of time.

In the jet, similarly,

$$p + \frac{\rho_J v^2}{2} + \rho_J \frac{\partial \phi_J}{\partial t} = p_0 + \frac{\rho_J V_0^2}{2}, \tag{1-2}$$

where p, ϕ_J, and v are as above; ρ_J is the (constant) density, in general different from ρ_0. p_0 and V_0 are the static pressure and velocity evaluated at the undisturbed conditions at infinity downstream,* also independent of time. The total

---

* See Section 2.6 for remarks about this assumption of undisturbed conditions in the jet at infinity downstream.

pressures in the two flows, i.e., the right-hand sides of (1-1) and (1-2), are in general different.

The condition of irrotationality in the jet can be written

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{1-3}$$

where r is the radius of curvature of the streamlines. This immediately integrates to

$$r v = \text{constant}.$$

1.2 Pressure Difference Across the Jet

Consider an incremental element of jet as in Figure 1, described by its position, s, downstream from the trailing edge at an instant of time, t. The subscripts 1 and 2 refer to the upper and lower boundaries, respectively, of the jet. Having assumed concentric streamlines within the jet, as discussed in the preceding section, the element may be described further by a radius of curvature, $R(s,t)$, to the centerline of the jet, a thickness, $\delta(s,t)$, such that $R(s,t) \mp \frac{\delta(s,t)}{2}$ are the upper and lower boundaries, respectively, and an incremental angle, $\Delta\theta(s,t)$, (positive counter-clockwise) subtended by the jet element. The pressure difference across the jet is, from (1-2),

$$\begin{aligned} p_1(s,t) - p_2(s,t) &= -\rho_J \frac{v_1^2(s,t) - v_2^2(s,t)}{2} - \rho_J \left\{ \frac{\partial \phi_{J_1}(s,t)}{\partial t} - \frac{\partial \phi_{J_2}(s,t)}{\partial t} \right\} \\ &= -\rho_J \frac{v_1(s,t) + v_2(s,t)}{2} \left[v_1(s,t) - v_2(s,t) \right] \\ &\quad - \rho_J \frac{\partial}{\partial t} \left[\phi_{J_1}(s,t) - \phi_{J_2}(s,t) \right]. \end{aligned} \tag{1-5}$$

By the arguments of Section 1.1, the radii of curvature are normal to the streamlines inside the jet, so they are equipotentials of the jet flow. Therefore,

$$\phi_{J_1}(s,t) - \phi_{J_2}(s,t) = 0\,. \tag{1-6}$$

The mean velocity of the jet may be defined as

$$V(s,t) = \frac{v_1(s,t) + v_2(s,t)}{2}\,. \tag{1-7}$$

The irrotational condition within the jet, (1-4), becomes

$$v_1(s,t)\left[R(s,t) - \frac{\delta(s,t)}{2}\right] = v_2(s,t)\left[R(s,t) + \frac{\delta(s,t)}{2}\right], \tag{1-8}$$

and with (1-7), solving (1-8) for the velocity difference, $v_1(s,t) - v_2(s,t)$, gives

$$v_1(s,t) - v_2(s,t) = \frac{V(s,t)\,\delta(s,t)}{R(s,t)}\,. \tag{1-9}$$

Substituting (1-6), (1-7), and (1-9) into (1-5) gives the pressure difference across the jet as

$$\begin{aligned} p_1(s,t) - p_2(s,t) &= -\frac{\rho_J V^2(s,t)\,\delta(s,t)}{R(s,t)} \\ &= -\frac{J(s,t)}{R(s,t)}\,, \end{aligned} \tag{1-10}$$

where the jet momentum flux, $J(s,t)$, is defined as

$$J(s,t) = \rho_J V^2(s,t)\,\delta(s,t)\,. \tag{1-11}$$

Therefore it has been found that the pressure difference across the jet is, as in Spence's steady formulation of I, proportional to the jet momentum flux and inversely proportional to the radius of curvature of the jet. Here the pressure difference is a function of time as well as position along the jet. The essential simplification of the vanishing of the velocity potential difference across the jet has resulted from the assumption of Section 1.1 that $v \gg u$.

1.3 Velocity Difference Across the Jet in the Main Stream.

By treating the flow in the main stream and relating it to that in the jet by the condition of constant static pressure across the boundaries, the main-stream velocity difference across the jet can be found. This velocity difference is required to determine the strength of the vortex distribution necessary to represent the effects of the jet on the main stream.

Considering again the jet element of Figure 1, equation (1-1) evaluated in the main stream across the jet is

$$\begin{aligned} p_1(s,t) - p_2(s,t) &= -\rho_0 \frac{u_1^2(s,t) - u_2^2(s,t)}{2} - \rho_0 \left\{ \frac{\partial \phi_1(s,t)}{\partial t} - \frac{\partial \phi_2(s,t)}{\partial t} \right\} \\ &= -\rho_0 \frac{u_1(s,t) + u_2(s,t)}{2}\left[u_1(s,t) - u_2(s,t)\right] \\ &\quad - \rho_0 \frac{\partial}{\partial t}\left[\phi_1(s,t) - \phi_2(s,t)\right]. \end{aligned} \tag{1-12}$$

The mean velocity of the main stream across the jet may be defined as

$$U(s,t) = \frac{u_1(s,t) + u_2(s,t)}{2} \,.$$

(1-13)

Substituting (1-13) into (1-12) and solving for the velocity difference, $u_1(s,t) - u_2(s,t)$, gives

$$u_1(s,t) - u_2(s,t) = -\frac{p_1(s,t) - p_2(s,t)}{\rho_0 U(s,t)} - \frac{1}{U(s,t)} \frac{\partial}{\partial t}\left[\phi_1(s,t) - \phi_2(s,t)\right].$$

(1-14)

Since the static pressure is continuous across the boundaries between the flows, the pressure difference found considering the jet alone, (1-10), may be substituted into (1-14), giving

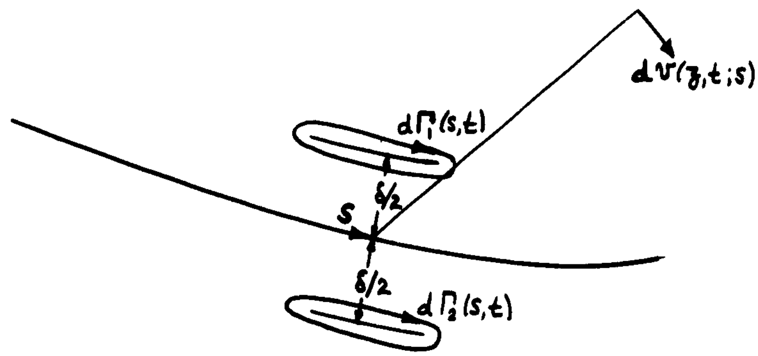
$$u_1(s,t) - u_2(s,t) = \frac{J(s,t)}{\rho_0 U(s,t) R(s,t)} - \frac{1}{U(s,t)} \frac{\partial}{\partial t}\left[\phi_1(s,t) - \phi_2(s,t)\right].$$

(1-15)

This result differs significantly from the steady case of I by the time derivative of the velocity-potential difference.

1.4 Velocity Induced in the Main Stream by the Jet

By considering the elemental vortices comprising the jet boundaries, the velocity induced in the main stream by the jet can be calculated using the Biot-Savart Law. The strength of these vortices is related to the main-stream velocity at the jet boundaries. For convenience, the complex velocity at a field point, z, in the complex plane, due to the element of jet at ζ will be treated. The coordinates are as follows:



The velocity induced by the pair of elemental vortices, $d\Gamma_1(s,t)$ and $d\Gamma_2(s,t)$, is

$$dv(\zeta,t;s) = \frac{i}{2\pi}\left\{\frac{d\Gamma_1(s,t)}{(\zeta-s)-i\frac{\delta(s,t)}{2}} + \frac{d\Gamma_2(s,t)}{(\zeta-s)+i\frac{\delta(s,t)}{2}}\right\} .$$

(1-16)

Assuming $\zeta >> \delta$, since the jet is thin,

$$\begin{aligned} dv(\zeta,t;s) &= \frac{i}{2\pi}\left\{\frac{d\Gamma_1(s,t)}{\zeta-s}\left[1+i\frac{\delta(s,t)}{2(\zeta-s)}+O\left(\frac{\delta^2}{(\zeta-s)^2}\right)\right]\right. \\ &\quad \left. + \frac{d\Gamma_2(s,t)}{\zeta-s}\left[1-i\frac{\delta(s,t)}{2(\zeta-s)}+O\left(\frac{\delta^2}{(\zeta-s)^2}\right)\right]\right\} \\ &= \frac{i\left[d\Gamma_1(s,t)+d\Gamma_2(s,t)\right]}{2\pi(\zeta-s)} - \frac{\left[d\Gamma_1(s,t)-d\Gamma_2(s,t)\right]\delta(s,t)}{4\pi(\zeta-s)^2} + O\left[\frac{\delta^2}{(\zeta-s)^3}\right] . \end{aligned}$$

(1-17)

This may be recognized as the increment of velocity induced by a vortex of strength

$$d\Gamma(s,t) = d\Gamma_1(s,t) + d\Gamma_2(s,t) ,$$

(1-18)

and a doublet, directed downstream, of strength

$$d\mu(s,t) = -\frac{[d\Gamma_1(s,t) - d\Gamma_2(s,t)]}{2}\,\delta(s,t)\,,$$

(1-19)

both of which lie on the jet center line, plus higher-order singularities whose strengths are higher-order in $\delta(s,t)$.

Define a vortex strength per unit length along the center line, $\gamma_J(s,t)$, and a doublet strength per unit length along the center line, $\mu_J(s,t)$, by

$$\left.\begin{aligned} d\Gamma(s,t) &= \gamma_J(s,t)\,R(s,t)\,\Delta\theta(s,t) \\ d\mu(s,t) &= \mu_J(s,t)\,R(s,t)\,\Delta\theta(s,t) \end{aligned}\right\}.$$

(1-20)

The strengths of the elemental vortices $d\Gamma_1(s,t)$ and $d\Gamma_2(s,t)$ may be written in terms of the velocity they induce in the main stream at the boundaries, namely,

$$\left.\begin{aligned} d\Gamma_1(s,t) &= u_1(s,t)\left[R(s,t) - \frac{\delta(s,t)}{2}\right]\Delta\theta(s,t) \\ d\Gamma_2(s,t) &= -u_2(s,t)\left[R(s,t) + \frac{\delta(s,t)}{2}\right]\Delta\theta(s,t) \end{aligned}\right\}.$$

(1-21)

Rewriting (1-18) and (1-19) using (1-20) and (1-21) gives

$$\gamma_J(s,t)\,R(s,t)\Delta\theta(s,t) = u_1(s,t)\left[R(s,t) - \frac{\delta(s,t)}{2}\right]\Delta\theta(s,t) - u_2(s,t)\left[R(s,t) + \frac{\delta(s,t)}{2}\right]\Delta\theta(s,t)$$

and

$$\mu_J(s,t)\,R(s,t)\,\Delta\theta(s,t) = -\left\{u_1(s,t)\left[R(s,t) - \frac{\delta(s,t)}{2}\right]\Delta\theta(s,t) - u_2(s,t)\left[R(s,t) + \frac{\delta(s,t)}{2}\right]\Delta\theta(s,t)\right\}\frac{\delta(s,t)}{2},$$

or

$$\left.\begin{aligned} \gamma_J(s,t) &= \left[u_1(s,t)-u_2(s,t)\right] - \left[\frac{u_1(s,t)+u_2(s,t)}{2}\right]\frac{\delta(s,t)}{R(s,t)} \\ \text{and} \quad \mu_J(s,t) &= -\left[\frac{u_1(s,t)+u_2(s,t)}{2}\right]\delta(s,t) + \left[u_1(s,t)-u_2(s,t)\right]\frac{\delta^2(s,t)}{4R(s,t)} \end{aligned}\right\}. \tag{1-22}$$

Substituting (1-13) and (1-15) into (1-22) gives

$$\left.\begin{aligned} \gamma_J(s,t) &= \frac{J(s,t)}{\rho_0 U(s,t)R(s,t)} - \frac{1}{U(s,t)}\frac{\partial}{\partial t}\left[\phi_1(s,t)-\phi_2(s,t)\right] - \frac{U(s,t)\delta(s,t)}{R(s,t)} \\ \text{and} & \\ \mu_J(s,t) &= -U(s,t)\delta(s,t) + \frac{J(s,t)\delta^2(s,t)}{4\rho_0 U(s,t)R^2(s,t)} - \frac{\delta^2(s,t)}{4U(s,t)R(s,t)}\frac{\partial}{\partial t}\left[\phi_1(s,t)-\phi_2(s,t)\right] \end{aligned}\right\}. \tag{1-23}$$

The influence of the jet on the main stream can thus be calculated by replacing the jet by vortex and doublet distributions along the jet center line, their strengths being given by (1-23). In order to make precise the neglect of the higher-order singularities in (1-17), and to simplify the strengths of the singularities in (1-23), the limiting case of a thin, high-speed jet will be treated in the next section.

1.5 Limiting Case of Thin, High-Speed Jet

By order of magnitude considerations, the limiting case of a thin, high-speed jet of constant momentum flux, J_0 , will be deduced.

The (constant) mass flux of the incompressible jet is defined as

$$m = \rho_J V(s,t)\delta(s,t), \tag{1-24}$$

so the jet momentum flux may be written using (1-11) and (1-24) as

$$J(s,t) = m V(s,t) .$$

(1-25)

Writing the subscript $(\ )_o$ to refer to conditions at infinity downstream where all the quantities are assumed constant in time, (1-25) may be written between some jet position, s, and infinity as

$$\frac{J(s,t)}{J_o} = 1 + \frac{V(s,t) - V_o}{V_o}$$

$$= 1 + \frac{\bar{\Delta} V(s,t)}{V_o} ,$$

(1-26)

where $\bar{\Delta} V(s,t) = V(s,t) - V_o$, and similarly $\bar{\Delta} U(s,t) = U(s,t) - U_o$ are the perturbations of $U(s,t)$ and $V(s,t)$ from the conditions at infinity. It is important to note here that these perturbations are not necessarily small for the following analysis. They are written this way for convenience.

Equations (1-1) and (1-2) may be written, between an arbitrary field point in or near the jet and the quiescent conditions at infinity, as

$$\left.\begin{aligned} p(s,t) - p_o + \frac{\rho_J}{2}\left[V^2(s,t) - V_o^2\right] + \rho_J \frac{\partial \phi_J}{\partial t} = 0 \\ p(s,t) - p_o + \frac{\rho_o}{2}\left[U^2(s,t) - U_o^2\right] + \rho_o \frac{\partial \phi}{\partial t} = 0 \end{aligned}\right\}$$

(1-27)

For an order-of-magnitude analysis it is sufficient to consider the pressure, $p(s,t)$, in each of these equations as the same, say an average pressure in the neighborhood of the jet. Eliminating $p(s,t) - p_0$ from this pair of equations and solving for $\bar{\Delta} V(s,t)$, gives

$$\bar{\Delta} V(s,t) = \frac{\rho_0}{\rho_J} \frac{[2U_0 + \bar{\Delta} U(s,t)]\,\bar{\Delta} U(s,t)}{2V_0 + \bar{\Delta} V(s,t)} + 2\frac{\rho_0}{\rho_J} \frac{\frac{\partial \phi(s,t)}{\partial t}}{2V_0 + \bar{\Delta} V(s,t)} - 2\,\frac{\frac{\partial \phi_J(s,t)}{\partial t}}{2V_0 + \bar{\Delta} V(s,t)}\,.$$

(1-28)

To estimate the order of the time derivatives of the velocity potentials, write

$$\frac{\partial \phi(s,t)}{\partial t} \simeq \frac{\bar{\Delta} \phi(s,t)}{\bar{\Delta} t}\,,$$

and

$$\frac{\partial \phi_J(s,t)}{\partial t} \simeq \frac{\bar{\Delta} \phi_J(s,t)}{\bar{\Delta} t}\,.$$

In the main stream, a characteristic time-dependent $\bar{\Delta}\phi(s,t)$ would be the product of the characteristic perturbation speed $\bar{\Delta} U(s,t)$ and a characteristic length, say the chord, c. Likewise a characteristic time, $\bar{\Delta} t$, would be the chord, c, divided by the free-stream speed, U_0. Thus,

$$\frac{\partial \phi(s,t)}{\partial t} \simeq \frac{c\,\bar{\Delta} U(s,t)}{c/U_0} = U_0\,\bar{\Delta} U(s,t)\,.$$

(1-29)

In the jet, a similar analysis would use the jet speeds, $\bar{\Delta} V(s,t)$ and $V(s,t)$, and the characteristic jet dimension, $\delta(s,t)$. Thus,

$$\frac{\partial \phi_J(s,t)}{\partial t} \simeq \frac{\delta(s,t)\,\bar{\Delta} V(s,t)}{\delta(s,t)/V_0} = V_0\,\bar{\Delta} V(s,t)$$

(1-30)

Substituting (1-29) and (1-30) into (1-28) gives

$$\bar{\Delta}V(s,t) \simeq -\frac{\rho_0}{\rho_J}\,\frac{U_0\,\bar{\Delta}U(s,t)}{V_0}\left[1 + O\left(\frac{\bar{\Delta}U}{U_0}\right) + O\left(\frac{\bar{\Delta}V}{V_0}\right)\right]$$

$$+\frac{\rho_0}{\rho_J}\,\frac{U_0\bar{\Delta}U(s,t)}{V_0}\left[1+O\left(\frac{\bar{\Delta}V}{V_0}\right)\right] - \frac{V_0\bar{\Delta}V(s,t)}{V_0}\left[1+O\left(\frac{\bar{\Delta}V}{V_0}\right)\right].$$

(1-31)

Assuming that ρ_0/ρ_J is $O(1)$, and that the terms in brackets are of most $O(1)$, (1-31) becomes

$$\bar{\Delta}V(s,t) \sim \frac{U_0}{V_0}\,\bar{\Delta}U(s,t) ,$$

or

$$\frac{\bar{\Delta}V(s,t)}{V_0} \sim \left(\frac{U_0}{V_0}\right)^2 \frac{\bar{\Delta}U(s,t)}{U_0} .$$

Since in the mainstream $\bar{\Delta}U(s,t)$ is of $O(U_0)$,*

$$\frac{\bar{\Delta}V(s,t)}{V_0} = O\left(\frac{U_0}{V_0}\right)^2 . \tag{1-32}$$

---

* In the case of a jet or airfoil deflection having a unit-step-function time dependence, $\frac{\partial\Delta\phi(s,t)}{\partial t}$, and consequently $\bar{\Delta}U(s,t)$, would have in the first instant the infinity of a Dirac delta function, in clear violation of the assumption $\bar{\Delta}U(s,t) = O(U_0)$. However, as discussed in Section 1-1, unit-step functions are considered because of their mathematical convenience, and may be considered as the generalization of a physically realistic deflection of finite rate, where $\bar{\Delta}U(s,t) = O(U_0)$. This is the same consideration made in classical linearized unsteady airfoil theory, where the infinities introduced by the derivatives of the step function clearly violate the small-perturbation assumption, unless understood in the above sense.

Substituting (1-32) into (1-26) gives, finally,

$$\frac{J(s,t)}{J_0} = 1 + O\left(\frac{U_0}{V_0}\right)^2 . \tag{1.33}$$

In most practical jet-flap applications, the jet speed, V_0 , is very much greater than the free-stream speed, U_0 or

$$\frac{U_0}{V_0} = o(1) .$$

Consider now the limit as $\delta(s,t)$ vanishes, such that the jet momentum flux remains finite. In this limit $V(s,t)$ must become infinite although the flow is still considered incompressible. In detail, from (1-33),

$$J(s,t) = J_0 = constant = J , \tag{1.34}$$

and as $\delta(s,t)$ vanishes,

and

$$\left.\begin{aligned} V(s,t) &\sim [\delta(s,t)]^{-1/2} \\ m &\sim [\delta(s,t)]^{1/2} \end{aligned}\right] . \tag{1.35}$$

The important relations from the earlier analysis may be written in terms of this approximation, treating $\delta(s,t)$ as the small parameter tending to zero. Equations (1-10), (1-15), and (1-23) become

$$p_1(s,t) - p_2(s,t) = -\frac{J}{R(s,t)} , \tag{1-36}$$

$$u_1(s,t) - u_2(s,t) = \frac{J}{\rho_0 U(s,t) R(s,t)} - \frac{1}{U(s,t)} \frac{\partial}{\partial t}\left[\phi_1(s,t) - \phi_2(s,t)\right] , \tag{1-37}$$

$$\gamma_J(s,t) = \frac{J}{\rho_0 U(s,t) R(s,t)} - \frac{1}{U(s,t)} \frac{\partial}{\partial t}\left[\phi_1(s,t) - \phi_2(s,t)\right] + O(\delta), \tag{1-38}$$

$$\mu_J(s,t) = O(\delta). \tag{1-39}$$

The jet vortex strength is zeroeth order in the jet thickness, $\delta(s,t)$, and becomes, to this order, equal to the main-stream velocity difference across the jet. The doublet strength is of first order in jet thickness and will thus be neglected in the lifting problem. Therefore the lifting problem is independent of thickness in this approximation. Furthermore, the neglect of the higher-order singularities in (1-17) is justified by this limiting analysis, as they are higher-order in the jet thickness.

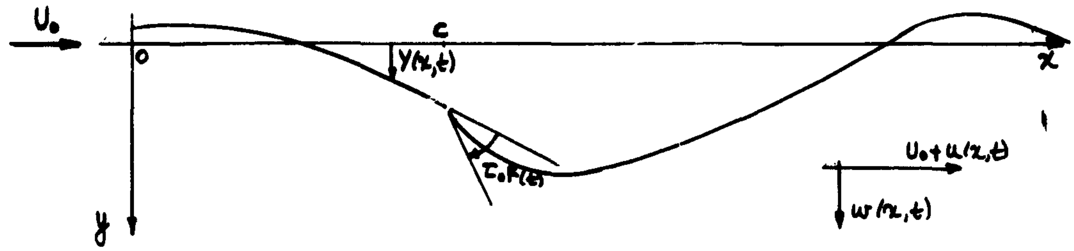
Therefore, the non-homogeneous flow problem of a thin, high-speed jet embedded in the main stream has been approximated, representing the jet by a vortex distribution along the jet center line. This vortex distribution, whose strength is given by (1-38), interacts dynamically with the main stream in the same manner as the jet, in the limit of a vanishingly thin, constant-momentum-flux jet.

CHAPTER 2 - DERIVATION OF THE BASIC EQUATIONS OF THE LINEARIZED UNSTEADY PROBLEM

2.1 Linearization of the Problem

No treatment of the airfoil was required in Chapter 1, except indication that it was the source of the jet and the cause of its unsteady motion. In principle, then, a theory with no restrictions on the airfoil or its motion could be developed using the model of Chapter 1 as a representation of a thin, high-speed jet-flap, so long as none of the assumptions of Section 1.1 were violated. Practically, it is desirable to make the thin-airfoil assumptions and linearize the problem.

Assume that the airfoil is thin and symmetrical about a mean camber line, and has chord, c . Choose x- and y-coordinates, so that x is the free-stream direction and y is positive downwards, for convenience. If $Y(x,t)$ is the ordinate of the mean camber line for $0 < x < c$, and is the ordinate of the jet center line for $c < x < \infty$, and if $\delta(x,t)$, $0 < x < c$, is the thickness of the airfoil (the jet having been assumed of zero thickness in Chapter 1), the problem may be linearized by assuming $\frac{\partial Y(x,t)}{\partial x}$, $\frac{\partial \delta(x,t)}{\partial x}$, $\frac{\partial Y(x,t)}{\partial t}$, and $\frac{\partial \delta(x,t)}{\partial t}$ to be small quantities with respect to unity. The velocities induced due to these quantities are then small compared to the free-stream velocity at infinity, U_0 ,* and the squares of these quantities may be neglected. Consistent with this, the airfoil



and jet are considered to lie along the x- axis, regardless of their actual shape, and the unsteady kinematic boundary condition that they be instantaneously streamlines is to be satisfied on the x- axis, too.

As is well known for linearized flow, cf. Robinson and Laurmann (1956), pps. 129 and 170, the thickness and lifting problems decouple and may be considered separately. For the lifting problem, the airfoil may be represented by a vortex distribution, as the jet has been. The above kinematic boundary condition that the airfoil and jet surfaces be streamlines is satisfied if the downwash, $w(x,t)$, is given by the linearized convective derivative of the airfoil or jet ordinate,

$$w(x,t) = \frac{\partial Y(x,t)}{\partial t} + U_0 \frac{\partial Y(x,t)}{\partial x} \equiv \frac{DY(x,t)}{Dt} \quad , 0 < x < \infty, \tag{2-1}$$

as given in Robinson and Laurmann (1956), p. 3.

---

* Here, as in Section 1.3, infinities implied by the step-function time variation must be considered as the limit of finite-rate processes which do not violate the small-perturbation approximation.

Consistent with the linearization, the local mean main-stream speed across the jet, $U(s,t)$, given by (1-13), may be taken to be U_0 everywhere. The x- and y- components of the local stream velocity may be written as $U_0 + u(x,t)$ and $v(x,t)$ respectively, where u, v are to denote the velocity perturbations for the remainder of this report. No confusion with their previous usage should occur.

Define Δ of a quantity to mean the difference in its value just below the x- axis from its value just above, i.e.,

$$\Delta(\quad) \equiv (\quad)_1 - (\quad)_2 \equiv (\quad)_{y=0+} - (\quad)_{y=0-} . \tag{2-2}$$

The coordinate s, denoting distance along the jet from the trailing edge linearizes to $x-c$.

The curvature of the jet is given geometrically by

$$\frac{1}{R(s,t)} = -\left[1 + \left(\frac{\partial Y(x,t)}{\partial x}\right)^2\right] \frac{\partial^2 Y(x,t)}{\partial x^2} , \quad c < x < \infty ,$$

which is, linearized,

$$\frac{1}{R(s,t)} = -\frac{\partial^2 Y(x,t)}{\partial x^2} , \quad c < x < \infty . \tag{2-3}$$

If the jet-momentum-flux coefficient is defined as

$$C_J \equiv \frac{J}{\frac{1}{2}\rho_0 U_0^2 c} , \tag{2-4}$$

the pressure difference across the jet, (1-36), and the jet vortex strength, (1-38), may be written in linearized form as

$$\Delta p(x,t) = \frac{1}{2}\rho_0 U_0^2 c C_J \frac{\partial^2 Y(x,t)}{\partial x^2}, \quad c < x < \infty, \tag{2-5}$$

and

$$\gamma(x,t) = -\frac{1}{2} U_0 c C_J \frac{\partial^2 Y(x,t)}{\partial x^2} - \frac{1}{U_0}\frac{\partial \Delta\phi(x,t)}{\partial t}, \quad c < x < \infty, \tag{2-6}$$

where the subscript, $(\ )_J$ on the vortex strength has been omitted for simplicity.

From the definition of the velocity potential, and the strength of the vortex distribution in terms of the free-stream velocity perturbation,

$$2u(x,0+,t) = \gamma(x,t) = \frac{\partial \Delta\phi(x,t)}{\partial x}, \quad 0 < x < \infty. \tag{2-7}$$

The jet vortex strength, (2-6), written using (2-7), gives an equation in the potential difference across the jet, namely

$$\frac{D\Delta\phi(x,t)}{Dt} = \frac{\partial \Delta\phi(x,t)}{\partial t} + U_0 \frac{\partial \Delta\phi(x,t)}{\partial x} = -\frac{1}{2} U_0^2 c C_J \frac{\partial^2 Y(x,t)}{\partial x^2}, \quad c < x < \infty. \tag{2-8}$$

Taking the linearized convective derivative of this and using (2-1) gives

$$\frac{D^2\Delta\phi(x,t)}{Dt^2} = \frac{\partial^2 \Delta\phi(x,t)}{\partial t^2} + 2U_0 \frac{\partial^2 \Delta\phi(x,t)}{\partial t\,\partial x} + U_0^2 \frac{\partial^2 \Delta\phi(x,t)}{\partial x^2} = -\frac{1}{2} U_0^2 c C_J \frac{\partial^2 w(x,t)}{\partial x^2},$$
$$c < x < \infty. \tag{2-9}$$

Differentiating (2-8) with respect to x and using (2-7) gives

$$\frac{D\gamma(x,t)}{Dt} = \frac{\partial\gamma(x,t)}{\partial t} + U_0\frac{\partial\gamma(x,t)}{\partial x} = -\frac{1}{2}U_0^2 c C_J \frac{\partial^3 Y(x,t)}{\partial x^3}, \quad c<x<\infty. \tag{2-10}$$

From (2-9) or (2-10), it also follows that

$$\frac{D^2\gamma(x,t)}{Dt^2} = \frac{\partial^2\gamma(x,t)}{\partial t^2} + 2U_0\frac{\partial^2\gamma(x,t)}{\partial t\partial x} + U_0^2\frac{\partial^2\gamma(x,t)}{\partial x^2} = -\frac{1}{2}U_0^2 c C_J \frac{\partial^3 w(x,t)}{\partial x^3}, \quad c<x<\infty. \tag{2-11}$$

The unsteady Bernoulli equation, (1-1), evaluated across the x- axis is, with (2-7),

$$\frac{\partial\Delta\phi(x,t)}{\partial t} + U_0\frac{\partial\Delta\phi(x,t)}{\partial x} = -\frac{\Delta p(x,t)}{\rho_0}, \quad 0<x<\infty. \tag{2-12}$$

If $\Delta p(x,t)$ given by the analysis of the jet, (2-5), is substituted directly into (2-12), (2-8) seems to follow immediately, just from the linearization of the problem. The analysis of Sections 1.3 to 1.5 is necessary, however, to justify the representation of the jet by a vortex sheet as being a valid approximation to the jet flow, i.e., that the jet thickness is really a higher-order effect for thin, high-speed jets.

2.2 Nature of the Motion and its Time Dependence

The unsteady motions to be treated are of the following types:

i) The airfoil is fixed and the jet deflection angle, $\tau(t)$, relative to the slope of the airfoil at the trailing edge varies time-dependently. This motion is given by

$$\tau(t) = \tau_0 F(t), \tag{2-13}$$

where τ_0 is the small amplitude of the deflection. Each point on the jet boundary moves normally to the x- axis only.

ii) The airfoil is performing some small-amplitude time-dependent motion about a mean position, e.g., plunging, pitching, or rotation of a mechanical flap with the jet deflection angle at the trailing edge fixed relative to the airfoil. Such motion can be characterized by

$$Y(x,t) = \hat{Y}(x) F(t), \quad 0 < x < c. \tag{2-14}$$

Each point on the airfoil and jet boundary moves normally to the x- axis.

iii) The airfoil and jet deflection angle at the trailing edge remain fixed and a sharp-edged gust passes over the airfoil and jet from the leading edge. Such a gust can be represented by the downwash distribution

$$W(x,t) = W_g(x) F\left(t - \frac{\lambda x}{U_0}\right), \tag{2-15}$$

where $W_g(x)$ is the small amplitude and U_0/λ is the speed relative to the airfoil. Again the jet boundary moves normally to the x- axis only. Despite the vortex sheet required at the edge of the gust, this problem has been treated in classical unsteady-airfoil theory as if it were a potential flow with an imposed, $W(x,t)$, cf. von Kármán and Sears (1938) and Miles (1956).

The $\hat{Y}(x)$, $W_g(x)$, and $F\left(t-\frac{\lambda x}{V_0}\right)$ functions are assumed to be of a form such that in any case the downwash on the airfoil may be represented by a power series in x . Such a power series is equivalent to the familiar Glauert Series for the downwash on the airfoil, Glauert (1943), hence there is little loss of generality in the choice of functions which can be considered.

Two types of time dependence, $F(t)$, are of particular interest. First are the transient problems where

$$F(t) = f(t)\, 1(t). \tag{2-16}$$

The function $f(t)$ is any non-dimensional function of time, and $1(t)$ is the unit-step function, defined by

$$1(t) = \begin{cases} 0 , & t \leq 0 \\ 1 , & t > 0 \end{cases} \tag{2-17}$$

The fundamental transient problem is that of response to the unit-step function, i.e., $f(t) \equiv 1$, since the response to all other $f(t)$ may be found from it by Duhamel superposition. Second are the cases of steady-state oscillations, where

$$F(t) = e^{i\omega t} , \; \mathscr{I}(\omega) \leq 0 . \tag{2-18}$$

The motion is assumed to have begun at $t = -\infty$, with $\mathscr{R}(\omega)$ being the circular frequency of the oscillations, and $\mathscr{I}(\omega)$ being the damping factor. For $\mathscr{I}(\omega) < 0$, the oscillations diverge exponentially with time, but began with zero amplitude at $t = -\infty$, and are thus meaningful for $t < \infty$. The $\mathscr{I}(\omega) > 0$, i.e., exponentially damped, oscillations must be excluded since they arise from the physically unrealistic initial condition of infinite amplitude at $t = -\infty$. Exponentially damped oscillations beginning at finite amplitude at $t = 0$ could be treated as a transient problem, with $f(t) = e^{i\omega t}$, $\mathscr{I}(\omega) > 0$. For a consideration of these damping considerations in classical unsteady-airfoil theory, see the paper by Luke and Dengler (1951) and the subsequent notes by Van de Vooren (1952), Laitone (1952), W. P. Jones (1952) and Dengler, Goland and Luke (1952).

2.3 Downwash Conditions on the Airfoil and Jet

The downwash due to the vortex distributions representing both the airfoil and jet may be found by using the Biot-Savart Law for the velocity induced by an incremental vortex element, and integrating over the distributions, giving

$$w(x,t) = \frac{DY(x,t)}{Dt} = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi,t)\,d\xi}{\xi - x} - \frac{1}{2\pi}\int_c^\infty \frac{\gamma(\xi,t)\,d\xi}{\xi - x}, \quad 0 < x < \infty. \tag{2-19}$$

This is, in fact, two equations, one for $0 < x < c$ where the first integral exists in the sense of Cauchy's Principal Value, the second for $c < x < \infty$ where the second integral exists in the same sense. Whereas (2-8) to (2-11) express the dynamic interaction of the jet and main stream, (2-19) expresses a kinematic interaction.

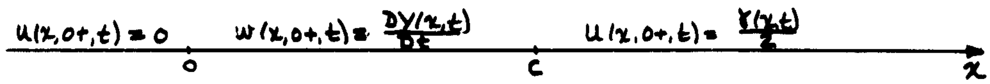
The two equations (2-19) and equation (2-11) are three equations in the four functions $w(x,t)$, $0 < x < c$; $w(x,t)$, $c < x < \infty$; $\gamma(x,t)$, $0 < x < c$; $\gamma(x,t)$, $c < x < \infty$. Alternately, combinations of $Y(x,t)$ and $\gamma(x,t)$, $w(x,t)$ and $\Delta\phi(x,t)$, or $Y(x,t)$ and $\Delta\phi(x,t)$ might be considered, also giving three equations in four unknown functions. In the cases of practical interest, the airfoil shape, $Y(x,t), 0<x<c$ and from (2-1) then, the downwash, $w(x,t), 0<x<c$, is prescribed, and along with the boundary conditions to be discussed in Sections 2.5 and 2.6, the equations (2-19) and (2-11) are then three equations in the remaining three unknown functions.

The downwash equations (2-19) may be inverted, using the prescribed airfoil shape, $Y(x,t), 0<x<c$, in a way to mathematically decouple the airfoil vortex distribution, $\gamma(x,t)$, $0<x<c$, from the functions describing the jet. That is, a form of the equations may be found where the jet ordinate or downwash and vortex distribution or potential difference may be solved for independently of the airfoil vortex distribution. Upon solving for the former, the latter may then be evaluated.

There are several alternate methods of inverting (2-19), all of which have a similar mathematical basis, although the individual techniques are different. Spence used a "null-transform" technique (cf. Section D12 of Heaslet and Lomax (1954)) in I. In II he used a generalization of a result first given by Carleman (1922) for the inversion of a certain singular integral equation. For the present report, the technique put forth by Cheng and Rott (1954) is used. This is based on the solution of the equivalent mixed boundary-value problem in the velocity perturbations.

The downwash equations (2-19) will not be considered directly, but the corresponding mixed boundary-value problem in the complex velocity function, $v(\zeta,t) = u(\zeta,t) + iw(\zeta,t)$, is treated. In this form, boundary values are given in terms of $u(x,0+,t)$ and $w(x,0+,t)$ on the alternate segments of the

x- axis corresponding to the airfoil, jet and the region upstream of the leading edge. From the strength of the vortex distribution representing the jet, (2-7), and from the downwash required on the airfoil, the boundary-value problem here is as follows:



$$\left.\begin{aligned} u(x,0+,t) &= 0 \quad , -\infty < x < 0 \\ w(x,0+,t) &= \frac{DY(x,t)}{Dt} \, , \; 0 < x < c \\ u(x,0+,t) &= \frac{1}{2}\gamma(x,t), \; c < x < \infty \end{aligned}\right\} ,$$

(2-20)

where $w(x,t)=\frac{DY(x,t)}{Dt}, 0<x<c$ is known, but $\gamma(x,t)$, $c<x<\infty$ is still unknown and leads to another integral equation after inversion.

To solve this problem, the corresponding homogeneous solution in the complex velocity is found, i.e., for the problem where

$$\left.\begin{aligned} u_H(x,0+,t) &= 0 \;, -\infty < x < 0 \\ w_H(x,0+,t) &= 0 \;, \quad 0 < x < c \\ u_H(x,0+,t) &= 0 \;, \quad c < x < \infty \end{aligned}\right\} .$$

(2-21)

This homogeneous solution, $H_0(\zeta,t)$, say, is constructed by considering the behavior required of the inhomogeneous solution at the edge points, i.e., the leading and trailing edges, $x=0$ and $x=c$. Here the usual $x^{-1/2}$

singularity is permitted at the leading edge $x = 0$, while at $x = c$ an $(x-c)^{-1/2}$ singularity is not permitted since the jet emerges here, preventing flow around the trailing edge.

In the classical non-jet-flap case, flow around the trailing edge is excluded by the Kutta condition. With the jet-flap a stronger condition is imposed, since the angle of deflection of the jet relative to the slope of the airfoil at the trailing edge, $\tau(t)$, may also be prescribed. Anticipating later results, it is necessary to remark that exclusion of the 1/square-root singularity does not prevent a logarithmic singularity at the trailing edge. Such a singularity is weaker and represents not flow completely around the trailing edge, but only deflection of the flow by some angle. This corresponds to a discontinuity in the downwash at the trailing edge. In a similar fashion, logarithmic singularities on the airfoil surface are permitted and will, in fact, result from prescribed discontinuities in the downwash. The classical example of such a singularity is that at the kink of a bent flat plate, first pointed out by Glauert (1927) - cf. also Spence (1958) - in his representation of an airfoil with flap.

To make the condition at the trailing edge more precise, write

$$\lim_{x\to c} |x-c|^{1/2} u(x,0+,t) = \frac{1}{2}\lim_{x\to c} |x-c|^{1/2}\gamma(x,t) = \frac{1}{2}\lim_{x\to c} |x-c|^{1/2}\frac{\partial\Delta\phi(x,t)}{\partial x} = 0 \,. \tag{2-22}$$

Since no singularities stronger than 1/square root are permitted in the velocity on the airfoil, the circulation around the airfoil, i.e., $\lim_{x\to c}\Delta\phi(x,t)$, is regular, and since its time derivative must be also,

$$\lim_{x \to c} \frac{\partial \Delta\phi(x,t)}{\partial t} = \text{Finite}. \tag{2-23}$$

From (2-22) and (2-23), therefore,

$$\lim_{x \to c} |x-c|^{1/2} \left\{ \frac{\partial \Delta\phi(x,t)}{\partial t} + U_0 \frac{\partial \Delta\phi(x,t)}{\partial x} \right\} = 0, \tag{2-24}$$

and using (2-8), therefore

$$\lim_{x \to c} |x-c|^{1/2} \frac{\partial^2 Y(x,t)}{\partial x^2} = 0. \tag{2-25}$$

The homogeneous solution is, then, from Cheng and Rott (1954),

$$H_0(z,t) = i\left(\frac{z-c}{z}\right)^{1/2}. \tag{2-26}$$

The complex velocity function, $F(z,t)$, say, which is the quotient of the inhomogeneous and homogeneous solutions,

$$F(z,t) = \frac{\mathcal{N}(z,t)}{H_0(z,t)} \tag{2-27}$$

has the property that its imaginary part, i.e., its downwash, is known everywhere along the x- axis. Thus the problem of finding $F(z,t)$ is the so-called direct, or thickness problem of thin-airfoil theory, whose solution can be written down immediately by considering the velocity induced by a source distribution whose strength is twice the downwash. This technique will become clearer upon working through the problem.

For reference, write out the complex velocity components of the three functions,

$$\left.\begin{aligned} F(z,t) &= p(x,y,t) + i\,g(x,y,t) \\ v(z,t) &= u(x,y,t) + i\,w(x,y,t) \\ H_0(z,t) &= u_H(x,y,t) + i\,w_H(x,y,t) \end{aligned}\right\}$$

(2-28)

In order to remain on the same branch of the function $H_0(z,t)$, it is necessary to restrict $0 \leq \arg z \leq \pi$.

For $-\infty < x < 0$, $y = 0+$, and $\arg|z-c| = \pi$, $\arg|z| = \pi$, from (2-26),

$$H_0(x,0+,t) = i\left|\frac{c-x}{x}\right|^{1/2}, \quad -\infty < x < 0.$$

(2-29)

From (2-20), (2-27), (2-28), and (2-29)

$$F(x,0+,t) = p(x,0+,t) + i\,g(x,0+,t) = \frac{i\,w(x,0+,t)}{i\left|\frac{c-x}{x}\right|^{1/2}}, \quad -\infty < x < 0,$$

(2-30)

and so the imaginary part of $F(x,0+,t)$ is

$$g(x,0+,t) = 0, \quad -\infty < x < 0.$$

(2-31)

In a similar fashion, for $0 < x < c$, $y = 0+$, and $\arg|z-c| = \pi$, $\arg|z| = 0$, from (2-19),

$$H_0(x,0+,t) = -\left(\frac{c-x}{x}\right)^{1/2}, \quad 0 < x < c.$$

(2-32)

From (2-20), (2-27), (2-28), and (2-32),

$$F(x,0+,t) = p(x,0+,t) + i\,g(x,0+,t) = \frac{u(x,0+,t) + i\,\frac{DY(x,t)}{Dt}}{-\left(\frac{c-x}{x}\right)^{1/2}}, \quad 0<x<c,$$

(2-33)

and so the imaginary part of $F(x,0+,t)$ is

$$g(x,0+,t) = -\left(\frac{x}{c-x}\right)^{1/2} \frac{DY(x,t)}{Dt}, \quad 0<x<c.$$

(2-34)

Finally, for $c<x<\infty$, $y=0+$, and $\arg(z-c)=0$, $\arg(z)=0$, from (2-19),

$$H_0(x,0+,t) = i\left(\frac{x-c}{x}\right)^{1/2}, \quad c<x<\infty,$$

(2-35)

From (2-20), (2-27), (2-28), and (2-35),

$$F(x,0+,t) = p(x,0+,t) + i\,g(x,0+,t) = \frac{\frac{\gamma(x,t)}{2} + i\,w(x,0+,t)}{i\left(\frac{x-c}{x}\right)^{1/2}}, \quad c<x<\infty,$$

(2-36)

and so the imaginary part of $F(x,0+,t)$ is

$$g(x,0+,t) = -\frac{1}{2}\left(\frac{x}{x-c}\right)^{1/2} \gamma(x,t), \quad c<x<\infty.$$

(2-37)

From the three expressions (2-31), (2-34), and (2-37), it is seen that the imaginary part of $F(x,0+,t)$, i.e., $g(x,0+,t)$ for $-\infty<x<\infty$ is known. By equation (1) of Cheng and Rott (1954), which is just the velocity induced by a source distribution of strength $2g(x,0+,t)$,

$$F(z,t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi,0+,t)\,d\xi}{\xi - z},$$

or using the definition of $F(z,t)$, (2-27),

$$v(z,t) = \frac{1}{\pi} H_0(z,t) \int_{-\infty}^{\infty} \frac{g(\xi,0+,t)\,d\xi}{\xi - z} . \tag{2-38}$$

With (2-1), (2-19), (2-26), (2-24), (2-27), (2-31), and (2-37),

$$v(z,t) = -\frac{i}{\pi}\left(\frac{z-c}{z}\right)^{1/2}\left\{\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi - z} + \frac{1}{2}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi - z}\right\} \tag{2-39}$$

This gives the complex velocity, $v(z,t)$, everywhere in the field.

To get the desired inversion results, in addition to recovering the boundary values, the limit $y \to 0+$ of equation (2-32) must be taken on each segment of the x- axis. The results are

$$v(x,0+,t) = u(x,0+,t) + i w(x,0+,t) = \begin{cases} -\dfrac{i}{\pi}\left|\dfrac{c-x}{x}\right|^{1/2}\left\{\displaystyle\int_0^c\left(\frac{\xi}{c-\xi}\right)^{1/2}\frac{w(\xi,t)\,d\xi}{\xi - x} + \frac{1}{2}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi - x}\right\}, \quad -\infty < x < 0, & (2\text{-}40) \\[2ex] \dfrac{1}{\pi}\left(\dfrac{c-x}{x}\right)^{1/2}\left\{\displaystyle\oint_0^c\left(\frac{\xi}{c-\xi}\right)^{1/2}\frac{w(\xi,t)\,d\xi}{\xi - x} + \frac{1}{2}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi - x}\right\} + i\,\dfrac{DY(x,t)}{Dt}, \quad 0 < x < c, & (2\text{-}41) \\[2ex] \dfrac{\gamma(x,t)}{2} - \dfrac{i}{\pi}\left(\dfrac{x-c}{x}\right)^{1/2}\left\{\displaystyle\int_0^c\left(\frac{\xi}{c-\xi}\right)^{1/2}\frac{w(\xi,t)\,d\xi}{\xi - x} + \frac{1}{2}\oint_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi - x}\right\}, \quad c < x < \infty. & (2\text{-}42) \end{cases}$$

The boundary values, (2-20) are apparent in (2-40)-(2-42), and by using (2-7) and (2-1), the desired inversion results are

$$\gamma(x,t) = \frac{2}{\pi}\left(\frac{c-x}{x}\right)^{1/2}\left\{ \oint_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi-x} + \frac{1}{2}\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} \right\}, \quad 0<x<c, \tag{2-43}$$

$$w(x,t) = \frac{DY(x,t)}{Dt} = -\frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2}\left\{ \int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi-x} + \frac{1}{2}\oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} \right\}, \quad c<x<\infty. \tag{2-44}$$

The integral equations, (2-19), for the downwash have been inverted and the equations (2-43) and (2-44) are the results. The airfoil vortex distribution $\gamma(x,t)$, $0<x<c$, does not appear in what are now the two governing equations of the problem. The problem, now, is to solve (2-11) and (2-44) for the downwash on the jet, $w(x,t)$, and the jet vortex distribution, $\gamma(x,t)$, both $c<x<\infty$,using the boundary conditions to be discussed in Sections 2.5 and 2.6. Alternately $Y(x,t)$ and $\gamma(x,t)$, $w(x,t)$ and $\Delta\phi(x,t)$, or $Y(x,t)$ and $\Delta\phi(x,t)$ may be considered to be the unknown functions. The airfoil vortex distribution, $\gamma(x,t)$, $0<x<c$, may then be evaluated from (2-43), as could the donwash upstream of the leading edge from (2-40).

For future reference, equation (2-43) may be integrated with respect to x , giving, by (2-7), the potential difference across the airfoil. Since there are no u- velocity perturbations on the x- axis upstream of the airfoil leading edge, (2-20),

$$\Delta\phi(0,t) = 0 , \tag{2-45}$$

so

$$\Delta\phi(x,t) = \int_0^x \gamma(\xi,t)\,d\xi \;, \quad 0<x<c$$

$$= \frac{2}{\pi}\int_0^x \left(\frac{c-s}{s}\right)^{1/2} ⨍_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi-s}\,ds \;+\; \frac{1}{\pi}\int_0^x \left(\frac{c-s}{s}\right)^{1/2} \int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-s}\,ds,$$

$$0<x<c. \tag{2-46}$$

To simplify this expression, interchange the order of integration. This is permissible according to Section D12 of Heaslet and Lomax (1954), since the behavior of $\gamma(x,t)$ at the trailing edge, (2-22), and the assumption that $w(x,t)$ on the airfoil can be represented by a power series in x eliminate any residual singularities of the integration. This gives

$$\Delta\phi(x,t) = -2\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,\frac{1}{\pi}⨍_0^x \left(\frac{c-s}{s}\right)^{1/2} \frac{ds}{s-\xi}\,d\xi \;-\; \int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,\frac{1}{\pi}\int_0^x \left(\frac{c-s}{s}\right)^{1/2} \frac{ds}{s-\xi}\,d\xi,$$

$$0<x<c. \tag{2-47}$$

Using the results of Appendix A, (A-48) and (A-49), the potential difference across the airfoil is

$$\Delta\phi(x,t) = \frac{4}{\pi}\sin^{-1}\left[\left(\frac{x}{c}\right)^{1/2}\right]\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi \;+\; \frac{2}{\pi}\int_0^c w(\xi,t)\,\ln\left|\frac{\left(\frac{c-x}{x}\right)^{1/2}+\left(\frac{c-\xi}{\xi}\right)^{1/2}}{\left(\frac{c-x}{x}\right)^{1/2}-\left(\frac{c-\xi}{\xi}\right)^{1/2}}\right|d\xi$$

$$+\frac{2}{\pi}\sin^{-1}\left[\left(\frac{x}{c}\right)^{1/2}\right]\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi \;-\; \frac{2}{\pi}\int_c^\infty \gamma(\xi,t)\cot^{-1}\left[\left(\frac{c-x}{x}\right)^{1/2}\left(\frac{\xi}{\xi-c}\right)^{1/2}\right]d\xi,$$

$$0<x<c. \tag{2-48}$$

In particular, if $x=c-$, (2-48) becomes the circulation around the airfoil,

$$\Delta\phi(c-,t) = \Gamma(t) = 2\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi \;+\; \int_c^\infty \left\{\left(\frac{\xi}{\xi-c}\right)^{1/2} - 1\right\}\gamma(\xi,t)\,d\xi. \tag{2-49}$$

2.4 Identification of the Airfoil-Quasi-Steady Terms

The first term on the right-hand sides of (2-43), (2-44), (2-49), and the first two terms on the right-hand side of (2-48) will now be identified. A quasi-steady quantity is for the present purposes defined as one dependent only on the instantaneous unsteady motion, independent of the previous time history of the flow. Since the terms in question are functions only of the airfoil motion, and are independent of the jet, they will be referred to here as the airfoil-quasi-steady terms. They are calculated such that the instantaneous time-dependent boundary condition that the airfoil surface be a streamline is satisfied, but neglecting any effects due to shed vorticity or the jet. These terms are results of the classical steady-state solution for a thin airfoil with time-dependent downwash on the airfoil, i.e., the mixed boundary-value problem treated in Section 2.3, with

$u(x,0+,t)=0 \quad , \quad c<x<\infty \quad$, or

$$\begin{aligned} u(x,0+,t)&=0 \quad , \quad -\infty<x<0 \\ w(x,0+,t)&=\frac{DY(x,t)}{Dt}, \quad 0<x<c \\ u(x,0+,t)&=0 \quad , \quad c<x<\infty \,. \end{aligned} \tag{2-50}$$

Since the same requirements are made on this solution at the leading and trailing edges (although the classical Kutta condition is invoked at the trailing edge instead of the argument of page 33), solution follows the procedure of Section 2.3 directly. Equations (2-43) and(2-44) with $\gamma(x,t)=0 \quad , \quad c<x<\infty$, are then the airfoil-quasi-steady values.

The airfoil-quasi-steady vortex distribution is, from (2-43),

$$\gamma_0(x,t) = \frac{2}{\pi}\left(\frac{c-x}{x}\right)^{1/2} \oint_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi - x}\,, \quad 0<x<c. \tag{2-51}$$

From (2-48), the airfoil-quasi-steady potential difference across the airfoil is

$$\Delta\phi_0(x,t) = \frac{4}{\pi}\sin^{-1}\left[\left(\frac{x}{c}\right)^{1/2}\right]\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi + \frac{2}{\pi}\int_0^c w(\xi,t)\ln\left|\frac{\left(\frac{c-x}{x}\right)^{1/2}+\left(\frac{c-\xi}{\xi}\right)^{1/2}}{\left(\frac{c-x}{x}\right)^{1/2}-\left(\frac{c-\xi}{\xi}\right)^{1/2}}\right|, \quad 0<x<c, \tag{2-52}$$

and the airfoil-quasi-steady circulation is then

$$\Gamma_0(t) = \Delta\phi_0(c^-,t) = 2\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi\;. \tag{2-53}$$

The airfoil-quasi-steady downwash behind the airfoil is, from (2-44),

$$w_0(x,t) = -\frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2}\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi - x}\,, \quad c<x<\infty\,. \tag{2-54}$$

Since the airfoil and gust shapes were assumed in Section 2.2 to give power-series representations of the downwash on the airfoil, it can be shown that this downwash is continuous at the trailing edge with the airfoil-quasi-steady downwash behind the airfoil. The power series may be written

$$w(x,t) = \sum_{n=0}^{\infty} G_n(t)\,x^n\,, \quad 0<x<c, \tag{2-55}$$

where $G_n(t)$ is the appropriate function of time. Evaluating (2-54) with this, using (A-33), gives

$$w_0(x,t) = \sum_{n=0}^{\infty} G_n(t)\left\{x^n - \sum_{k=0}^{n}\binom{2k}{k} 2^{-2k} c^k x^{n-k}\left(\frac{x-c}{x}\right)^{1/2}\right\}, \quad c<x<\infty. \tag{2-56}$$

Therefore

$$w_0(c+,t) = \sum_{n=0}^{\infty} G_n(t) c^n = w(c-,t). \tag{2-57}$$

Since problems with the airfoil shape, $Y(x,t)$, $0<x<c$, prescribed are to be considered, these airfoil-quasi-steady quantities may be calculated immediately in a given case.

2.5 Boundary Conditions at the Trailing Edge

There are two boundary conditions on the jet ordinate at the trailing edge of the airfoil. First, the jet must be continuous with the airfoil there; i.e.,

$$Y(c+,t) = Y(c-,t). \tag{2-58}$$

Secondly, as mentioned in Sections 2.1 and 2.3, the angle of deflection of the jet relative to the slope of the airfoil at the trailing edge may be prescribed, giving, using (2-13),

$$\frac{\partial Y(c+,t)}{\partial x} = \frac{\partial Y(c-,t)}{\partial x} + \tau_0 F(t). \tag{2-59}$$

In terms of the downwash, (2-58) and (2-59) may be combined, using (2-1) to give the single condition

$$w(c+,t) = w(c-,t) + U_0 \tau_0 F(t),$$

and from (2-57),

$$w(c+,t) = w_0(c+,t) + U_0 \tau_0 F(t) \,. \tag{2-60}$$

Finally, these conditions can be expressed in terms of an integral over the jet vortex distribution or potential difference. From (2-44), (2-54) and (2-60),

$$\lim_{x \to c+} \frac{1}{\pi} \left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi - c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x} = -2U_0 \tau_0 F(t) \,, \tag{2-61}$$

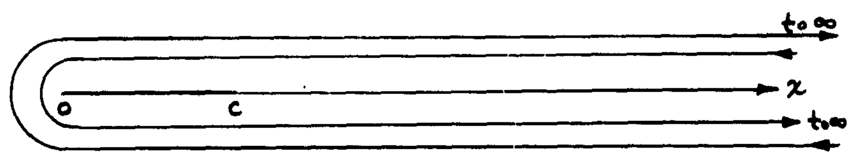
or, using (2-7),

$$\lim_{x \to c+} \frac{1}{\pi} \left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi - c}\right)^{1/2} \frac{\partial \Delta\phi(\xi,t)}{\partial \xi} \frac{d\xi}{\xi - x} = -2U_0 \tau_0 F(t) \,. \tag{2-62}$$

In the steady jet-flap analysis of I and II, the boundary condition at the trailing edge can be expressed in exactly the form of (2-61) and (2-62), less the time dependence. Spence has shown in I that this condition can be satisfied if there is a logarithmic singularity in the vortex distribution at the trailing edge, and his analytic solution of II confirms this. These singularities have been discussed in Section 2.3. If there is no jet deflection, $\tau_0 = 0$, (2-61) implies that the vortex distribution is regular at the trailing edge.

2.6 Proof of Constancy of Circulation

Since the problem is linear, solutions for various boundary conditions may be superimposed. Therefore a transient problem, e.g., one described by (2-13), (2-14), or (2-15) and (2-16), may be considered independently of any other steady or unsteady configuration the system of airfoil plus jet might have, provided the assumption of a jet fully established in length before initiation of the transient motion is met. The airfoil and jet may then be assumed to consist of a flat plate with an infinitely long jet tangent at the trailing edge, both aligned with the free stream before initiation of the motion. Then the circulation around the following simply connected contour is initially zero:



In the classical unsteady theory of airfoils without jets, the flow about the airfoil is doubly connected. Therefore the circulation about a contour drawn around the airfoil is zero before initiation of the transient motion and must remain zero for all times after the motion has begun. This is a direct conclusion from Kelvin's Theorem, cf. Sears (1954). In the jet-flap case, the simple connectivity of the flow eliminates this argument. Instead, a proof of the constancy of circulation in the jet-flap case given by Spence in III will be reproduced.

This proof is based on the physical property that sufficiently far downstream at a given instant of time after initiation of the motion, the jet must return to its initial undisturbed position. Moreover, its slope and curvature will also vanish; i.e.,

$$\lim_{\substack{x \to \infty \\ t \text{ fixed}}} Y(x,t) = \lim_{\substack{x \to \infty \\ t \text{ fixed}}} \frac{\partial Y(x,t)}{\partial x} = \lim_{\substack{x \to \infty \\ t \text{ fixed}}} \frac{\partial^2 Y(x,t)}{\partial x^2} = 0 \ . \tag{2-63}$$

This property may be visualized in the following way. If the velocity, V_0, of the flow in the jet is much greater than the velocity of the jet boundaries, as expressed by the downwash velocity, $w(x,t)$, the jet will be continuous from the trailing edge to infinity downstream. If the reverse inequality held, the jet might cease to be a continuous flow and break in some fashion. For the continuous jet, the assumption of inviscid flow implies that the jet is impermeable; hence a particle of air above (or below) the jet initially must always remain above (or below). If, then, the jet were displaced from its initial position at infinity downstream, an infinite amount of work would have been done in a finite time to move the infinite amount of air above and below the jet. The condition that $V_0 >> w(x,t)$ is no restriction, since in Section 1.5 the assumption of $V_0 >> U_0$ has been made, and in Section 2.1 the small-perturbation assumption of $U_0 >> w(x,t)$ has been made. Again, as in Sections 1.5 and 2.1, it must be mentioned that although unit-step functions and their derivatives may be treated, they are considered as the limiting cases of flows in which $V_0 >> w(x,t)$.

To find the circulation in the system at any time, i.e., the potential difference $\Delta\phi(x,t)$ as $x \to \infty$, the unsteady Bernoulli equation, (2-12), is solved for $\Delta\phi(x,t)$. This solution is

$$\Delta\phi(x,t) = \begin{cases} -\frac{1}{\rho_0 U_0}\int_0^x \Delta p\left(\xi, t+\frac{\xi-x}{U_0}\right)d\xi + F(x-U_0t), & x<U_0t \\ -\frac{1}{\rho_0 U_0}\int_{x-U_0t}^x \Delta p\left(\xi, t+\frac{\xi-x}{U_0}\right)d\xi + F(x-U_0t), & x>U_0t, \end{cases}$$

(2-64)

where the motion is assumed to have been initiated at $t=0$, and where there is no pressure difference across the x- axis upstream of the leading edge. $F(x-U_0t)$ is an arbitrary function of the integration, but since (2-64) must be valid for all positive and negative x and t, the vanishing of $\Delta\phi(x,t)$ for $x<0$ and $t<0$ require $F(x-U_0t)$ to be zero, identically.

An alternate integration of (2-12), which is related to (2-64) by the transformation $\eta = t + \frac{\xi-x}{U_0}$ is, since $F(x-U_0t) \equiv 0$,

$$\Delta\phi(x,t) = \begin{cases} -\frac{1}{\rho_0}\int_{t-x/U_0}^{t} \Delta p(x+U_0\eta-U_0t, \eta)\,d\eta, & x<U_0t \\ -\frac{1}{\rho_0}\int_0^t \Delta p(x+U_0\eta-U_0t, \eta)\,d\eta, & x>U_0t. \end{cases}$$

(2-65)

For $x-U_0t>c$, with the pressure difference across the jet, (2-5), (2-64) becomes

$$\Delta\phi(x,t) = -\frac{1}{2}U_0 c C_J \int_{x-U_0t}^{x} \frac{\partial^2 Y\left(\xi, t+\frac{\xi-x}{U_0}\right)}{\partial \xi^2}\,d\xi, \quad x>c+U_0t.$$

Now, if

$$K(x,t) = \max \left| \frac{\partial^2 Y(\xi, t + \frac{\xi - x}{U_0})}{\partial \xi^2} \right| \quad \text{for} \quad x - U_0 t \le \xi \le x$$

then

$$\left| \Delta\phi(x,t) \right| \le \frac{1}{2} U_0 c G (U_0 t) \, K(x,t) ,$$

and for a fixed t , by (2-63),

$$\lim_{\substack{x \to \infty \\ t \text{ fixed}}} \left| \Delta\phi(x,t) \right| = 0 . \tag{2-66}$$

Therefore the circulation in the system is zero. Note that the limit is non-uniform with respect to t , since for longer times it is necessary to go farther downstream to take the limit.

For a jet-flapped airfoil in unsteady motion, then, it is no longer just a case of shedding vorticity into the wake at the trailing edge in equal and opposite amounts to the changes in circulation around the airfoil, as in the classical unsteady-airfoil theory. Rather, equal and opposite amounts of vorticity must be shed from all points of the airfoil and jet, in such a way that the total circulation in the system vanishes. Birnbaum's concept of bound and shed vortices, cf. Cicala (1941), is a good physical way of considering the problem. The airfoil and jet are thought of as being represented by bound vortex distributions whose strengths at a point are equal to $\frac{\Delta p(x,t)}{\rho_0 U_0}$. Each of these bound vortices sheds, as its strength varies, the amount of vorticity necessary to satisfy (2-12). This shed vorticity is convected downstream at the free-stream speed, U_0 .

An alternate equation expressing the constancy of circulation may be found by considering (2-49) along with (2-7) and (2-53), giving

$$\Delta\phi(c-,t) = \Gamma_0(t) + \int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi - \Delta\phi(\infty,t) + \Delta\phi(c+,t). \tag{2-67}$$

With (2-66), then, since the potential difference must be continuous at the trailing edge,

$$\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi = -\Gamma_0(t) \tag{2-68}$$

expresses the constancy of circulation. This is recognized as the Wagner (1925) integral condition of classical unsteady-airfoil theory. The form is identical, but $\gamma(x,t)$ here contains the effects of the jet, in addition to the vorticity shed from the airfoil due to its motion.

The above proof and results are readily extended to cover divergent steady-state oscillations, $\mathcal{J}(\omega) < 0$. At $t = -\infty$ these started with zero amplitude, so the jet must return to its initial position at infinity downstream. For pure oscillations, $\mathcal{J}(\omega) = 0$, care must be exercised, since the oscillations began at finite amplitude at $t = -\infty$. The same difficulty arises in the classical unsteady-airfoil theory and will be discussed in Section 5.3.

2.7 Complete Equations for the System: Some Properties of Them

The complete set of equations required to state the problem have now been derived and will be collected here for convenience. Since the equations can be written in a variety of forms, depending on which functions are taken as the primary unknowns, several alternatives will be given.

The dynamic equation coupling the jet to the main stream is, from (2-8) to (2-11), either

$$\frac{D\,\Delta\phi(x,t)}{Dt} = -\frac{1}{2}U_0^2 c C_J \frac{\partial^2 Y(x,t)}{\partial x^2} \quad , \quad c<x<\infty \, ,$$

(2-69)

or

$$\frac{D^2\,\Delta\phi(x,t)}{Dt^2} = -\frac{1}{2}U_0^2 c C_J \frac{\partial^2 w(x,t)}{\partial x^2} \quad , \quad c<x<\infty \, ,$$

(2-70)

or

$$\frac{D\gamma(x,t)}{Dt} = -\frac{1}{2}U_0^2 c C_J \frac{\partial^3 Y(x,t)}{\partial x^3} \quad , \quad c<x<\infty \, ,$$

(2-71)

or

$$\frac{D^2\gamma(x,t)}{Dt^2} = -\frac{1}{2}U_0^2 c C_J \frac{\partial^3 w(x,t)}{\partial x^3} \quad , \quad c<x<\infty \, .$$

(2-72)

The kinematic equation giving the downwash required to make the jet be a streamline at each instant of time is, using (2-44), (2-54) and (2-7), either

$$w(x,t) = \frac{DY(x,t)}{Dt} = w_0(x,t) - \frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x} \quad , \quad c<x<\infty \, ,$$

(2-73)

or

or

$$w(x,t) = \frac{DY(x,t)}{Dt} = w_0(x,t) - \frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\partial\Delta\phi(\xi,t)}{\partial\xi} \frac{d\xi}{\xi-x}, \quad c<x<\infty, \tag{2-74}$$

where

$$w_0(x,t) = -\frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} \frac{w(\xi,t)\,d\xi}{\xi-x}, \quad c<x<\infty. \tag{2-75}$$

The boundary conditions at the trailing edge of the airfoil are from (2-58) to (2-62), either

$$Y(c+,t) = Y(c-,t) \tag{2-76}$$

and

$$\frac{\partial Y(c+,t)}{\partial x} = \frac{\partial Y(c-,t)}{\partial x} + \tau_0 F(t), \tag{2-77}$$

or

$$w(c+,t) = w_0(c+,t) + U_0\tau_0 F(t), \tag{2-78}$$

or

$$\lim_{x\to c+} \frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} = -2U_0\tau_0 F(t), \tag{2-79}$$

or

$$\lim_{x \to c+} \frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\partial \Delta\phi(\xi,t)}{\partial \xi} \frac{d\xi}{\xi-x} = -2U_0\tau_0 F(t) .$$

(2-80)

The conservation of circulation in the system is expressed, using (2-66) to (2-68), by either

$$\lim_{\substack{x \to \infty \\ t \text{ fixed}}} \Delta\phi(x,t) = 0 ,$$

(2-81)

insuring that $\Delta\phi(x,t)$ is continuous at the trailing edge, or

$$\int_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\, d\xi = -\Gamma_0(t) ,$$

(2-82)

or

$$\int_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\partial \Delta\phi(\xi,t)}{\partial \xi} d\xi = -\Gamma_0(t) .$$

(2-83)

By explicit use of the Wagner integral condition (2-82) or (2-83), the downwash equations (2-73) or (2-74) can be cast in a form useful in exhibiting certain properties of the solution of these equations. Using the identity

$$\frac{1}{\xi - x} = \frac{(\xi - c)}{(x-c)(\xi - x)} - \frac{1}{x-c} ,$$

(2-84)

equations (2-73) and (2-75) may be written

$$w(x,t) = \frac{1}{\pi x^{1/2}(x-c)^{1/2}} \int_0^c \frac{\xi^{1/2}(c-\xi)^{1/2} w(\xi,t)\,d\xi}{\xi - x} + \frac{1}{\pi x^{1/2}(x-c)^{1/2}} \int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi$$

$$+ \frac{1}{2\pi x^{1/2}(x-c)^{1/2}} \int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi - \frac{1}{2\pi x^{1/2}(x-c)^{1/2}} \oint_c^\infty \frac{\xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi}{\xi - x},$$

$$c < x < \infty .$$

From (2-53) and the Wagner integral condition, (2-82), the second and thrid integrals cancel, leaving either

$$w(x,t) = \frac{DY(x,t)}{Dt} = w_1(x,t) - \frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x}, \quad c < x < \infty, \tag{2-85}$$

or

$$w(x,t) = \frac{DY(x,t)}{Dt} = w_1(x,t) - \frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\partial\Delta\phi(\xi,t)}{\partial\xi}\frac{d\xi}{\xi - x}, \quad c < x < \infty, \tag{2-86}$$

where

$$w_1(x,t) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^c \left[\frac{\xi(c-\xi)}{x(x-c)}\right]^{1/2} \frac{w(\xi,t)\,d\xi}{\xi - x} \\ \\ w_0(x,t) - \dfrac{\Gamma_0(t)}{2\pi x^{1/2}(x-c)^{1/2}}, \end{cases} \qquad c < x < \infty . \tag{2-87}$$

Finally, although decoupled by the inversion of Section 2.3 from the equations now required to solve the problem, the vortex distribution on the airfoil is, from (2-43) and (2-51),

$$\gamma(x,t) = \gamma_0(x,t) + \frac{1}{\pi}\left(\frac{c-x}{x}\right)^{1/2}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x}, \quad 0<x<c, \tag{2-88}$$

and can be evaluated once the jet vortex distribution is found.

It is of interest to notice that the effect of the jet appears explicitly, i.e., by appearance of C_J in the equations, only in (2-69) to (2-72). These are dynamic equations involving the pressure difference across the jet, which is proportional to C_J. The fundamental downwash equations (2-19), and the results obtained therefrom by inversion do not explicitly contain C_J, although the effect of C_J is implicit. Furthermore, the boundary conditions at the trailing edge are kinematic, and although a dynamic argument was used to prove the constancy of circulation, it may be expressed by (2-81) to (2-83), which are found from the kinematical equations.

In the case of the functions $\gamma(x,t)$ and $\Delta\phi(x,t)$, single equations may be found by eliminating $w(x,t)$ from the fundamental pair of equations. For the vortex distribution, differentiate (2-73) three times with respect to x and eliminate $\frac{\partial^3 w(x,t)}{\partial x^3}$ between this and (2-72), giving

$$\frac{D^2\gamma(x,t)}{Dt^2} = \frac{U_0^2 c C_J}{4\pi}\frac{\partial^3}{\partial x^3}\left\{\left(\frac{x-c}{x}\right)^{1/2}\oint_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x}\right\} - \frac{1}{2}U_0^2 c C_J\frac{\partial^3 w_0(x,t)}{\partial x^3}, \quad c<x<\infty. \tag{2-89}$$

Using the transformation $\xi = c+(x-c)\eta$ in the integral, the three derivatives may be taken, giving the complete result

$$\frac{D^2\gamma(x,t)}{Dt^2} = \frac{U_0^2 c C_J}{x^{1/2}4\pi(x-c)^{3/2}}\left\{\frac{3c}{8x}\int_c^\infty \frac{(\xi-c)^{3/2}\gamma(\xi,t)\,d\xi}{\xi^{5/2}} + \frac{3c(x-c)}{4x^2}\int_c^\infty \frac{(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi}{\xi^{3/2}}\right.$$

$$+\frac{15c(x-c)^2}{8x^3}\int_c^\infty \frac{\gamma(\xi,t)\,d\xi}{\xi^{1/2}(\xi-c)^{1/2}} - \frac{3c}{4x}\int_c^\infty \frac{(\xi-c)^{3/2}}{\xi^{3/2}}\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi - \frac{9(x-c)}{4x^2}\int_c^\infty \left(\frac{\xi-c}{\xi}\right)^{1/2}\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi$$

$$\left.+\frac{3c}{2x}\int_c^\infty \frac{(\xi-c)^{3/2}}{\xi^{1/2}}\frac{\partial^2\gamma(\xi,t)}{\partial\xi^2}\,d\xi + \oint_c^\infty \frac{\xi^{1/2}(\xi-c)^{5/2}}{\xi-x}\frac{\partial^3\gamma(\xi,t)}{\partial\xi^3}\,d\xi\right\}$$

$$-\frac{U_0^2 c C_J}{2}\frac{\partial^3 w_0(x,t)}{\partial x^3}, \qquad c<x<\infty.$$

(2-90)

The alternate form of this equation, explicitly incorporating the Wagner integral condition, is, from (2-72) and (2-85),

$$\frac{D^2\gamma(x,t)}{Dt^2} = \frac{U_0^2 c C_J}{4\pi x^{1/2}(x-c)^{7/2}}\left\{\frac{3c}{8x}\int_c^\infty \left(\frac{\xi-c}{\xi}\right)^{5/2}\gamma(\xi,t)\,d\xi + \frac{3c(x-c)}{4x^2}\int_c^\infty \left(\frac{\xi-c}{\xi}\right)^{3/2}\gamma(\xi,t)\,d\xi\right.$$

$$+\frac{15c(x-c)^2}{8x^3}\int_c^\infty \left(\frac{\xi-c}{\xi}\right)^{1/2}\gamma(\xi,t)\,d\xi - \frac{3c}{4x}\int_c^\infty \frac{(\xi-c)^{5/2}}{\xi^{3/2}}\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi - \frac{9c(x-c)}{4x^2}\int_c^\infty \frac{(\xi-c)^{3/2}}{\xi^{1/2}}\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi$$

$$\left.+\frac{3c}{2x}\int_c^\infty \frac{(\xi-c)^{5/2}}{\xi^{1/2}}\frac{\partial^2\gamma(\xi,t)}{\partial\xi^2}\,d\xi + \oint_c^\infty \frac{\xi^{1/2}(\xi-c)^{7/2}}{\xi-x}\frac{\partial^3\gamma(\xi,t)}{\partial\xi^3}\,d\xi\right\}$$

$$-\frac{U_0^2 c C_J}{2}\frac{\partial^3 w_1(x,t)}{\partial x^3}, \qquad c<x<\infty.$$

(2-91)

Either of these equations can then be solved, using the conditions (2-79) and (2-82).

Following the same procedure, (2-70) and (2-74) yield for the potential difference across the jet

$$\frac{D^2\Delta\phi(x,t)}{Dt^2} = \frac{U_0^2 c C_J}{4\pi x^{1/2}(x-c)^{3/2}}\left\{-\frac{c}{4x}\int_c^\infty \frac{(\xi-c)^{1/2}}{\xi^{3/2}}\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}d\xi - \frac{3c(x-c)}{4x^2}\int_c^\infty \frac{1}{\xi^{1/2}(\xi-c)^{1/2}}\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}d\xi\right.$$

$$\left. + \frac{c}{x}\int_c^\infty\left(\frac{\xi-c}{\xi}\right)^{1/2}\frac{\partial^2\Delta\phi(\xi,t)}{\partial\xi^2}d\xi + \oint_c^\infty \frac{\xi^{1/2}(\xi-c)^{3/2}}{\xi-x}\frac{\partial^3\Delta\phi(\xi,t)}{\partial\xi^3}d\xi\right\}$$

$$- \frac{U_0^2 c C_J}{2}\frac{\partial^2 w_0(x,t)}{\partial x^2}, \quad c<x<\infty, \tag{2-92}$$

or, in the form explicitly incorporating the Wagner integral condition,

$$\frac{D^2\Delta\phi(x,t)}{Dt^2} = \frac{U_0^2 c C_J}{4\pi x^{1/2}(x-c)^{3/2}}\left\{-\frac{c}{4x}\int_c^\infty\left(\frac{\xi-c}{\xi}\right)^{3/2}\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}d\xi - \frac{3c(x-c)}{4x^2}\int_c^\infty\left(\frac{\xi-c}{\xi}\right)^{1/2}\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}d\xi\right.$$

$$\left. + \frac{c}{x}\int_c^\infty \frac{(\xi-c)^{3/2}}{\xi^{1/2}}\frac{\partial^2\Delta\phi(\xi,t)}{\partial\xi^2}d\xi + \oint_c^\infty \frac{\xi^{1/2}(\xi-c)^{5/2}}{\xi-x}\frac{\partial^3\Delta\phi(\xi,t)}{\partial\xi^3}d\xi\right\}$$

$$- \frac{U_0^2 c C_J}{2}\frac{\partial^2 w_1(x,t)}{\partial x^2}, \quad c<x<\infty. \tag{2-93}$$

Either of these may be solved, using the conditions (2-80) and (2-81) or (2-83).

These equations in the jet vortex distribution and potential difference are unwieldy. However, upon approximation for small values of C_J in Chapter 6, they simplify greatly and are useful forms. Single equations in the jet ordinate or downwash on the jet cannot be found, since the jet vortex distribution appearing in the integral term cannot be eliminated completely.

By considering some special limits of the equations of the problem, important asymptotic properties of the solution near $x = \infty$ may be found.

First, if the downwash equation, (2-73), is multiplied by x, and the limit as $x \to \infty$ is taken for finite fixed t, the result is

$$\lim_{x\to\infty} x\, w(x,t) = \frac{1}{\pi}\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi + \frac{1}{2\pi}\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi \,. \tag{2-94}$$

Using (2-53), the Wagner integral condition (2-82) makes the right-hand side of (2-94) vanish. Therefore

$$\lim_{x\to\infty} x\, w(x,t) = 0 \,, \tag{2-95}$$

or, asymptotically in x,

$$w(x,t) \sim x^{-n} \tag{2-96}$$

where $n > 1$, but as yet undetermined. Equation (2-72) for the jet then implies that

$$\frac{D^2\gamma(x,t)}{Dt^2} \sim x^{-n-3} \tag{2-97}$$

or

$$\gamma(x,t) \sim x^{-n-3} .$$

(2-98)

These latter conditions may also be shown to follow from (2-90) by multiplying it by x^4, taking the limit as $x \to \infty$, and integrating by parts the integrals containing $\gamma(\xi,t)$, thus reducing them to a form which vanishes by the Wagner integral condition.

Next treat the downwash equation in its form, (2-85), which incorporates the Wagner integral condition. Multiplying it by x^2 and taking the limit as $x \to \infty$ gives

$$\lim_{x\to\infty} x^2 w(x,t) = -\frac{1}{2\pi}\left\{2\int_0^c \xi^{1/2}(c-\xi)^{1/2} w(\xi,t)\,d\xi - \int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi\right\},$$

(2-99)

the integral over $\gamma(\xi,t)$ existing because of (2-98). The right-hand side of this equation is in general not zero, and will be shown in Chapter 3 to be related to the lift on the airfoil. The downwash, then, behaves asymptotically in x like

$$w(x,t) \sim x^{-2},$$

(2-100)

i.e., like the downwash of a doublet, which follows since the circulation in the system is conserved. Since $n = 2$ in (2-96), (2-97) and (2-98) give

$$\frac{D^2\gamma(\chi,t)}{Dt^2} \sim \chi^{-5},$$

(2-101)

and

$$\gamma(\chi,t) \sim \chi^{-5} .$$

(2-102)

These latter results may also be shown directly from (2-91), where multiplication by χ^5, taking the limit of $\chi \to \infty$, and several integrations by parts yields

$$\lim_{\chi \to \infty} \chi^5 \frac{D^2\gamma(\chi,t)}{Dt^2} = -\frac{6U_0^2 c C_J}{\pi}\left\{2\int_0^c \xi^{1/2}(c-\xi)^{1/2} w(\xi,t)\,d\xi - \int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi\right\}.$$

(2-103)

Similar considerations of the potential difference equations, (2-92) and (2-93), show that

$$\lim_{\chi \to \infty} \chi^4 \frac{D^2\Delta\phi(\chi,t)}{Dt^2} = \frac{3U_0^2 c C_J}{2\pi}\left\{2\int_0^c \xi^{1/2}(c-\xi)^{1/2} w(\xi,t)\,d\xi - \int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi\right\},$$

(2-104)

indicating that asymptotically in χ,

$$\Delta\phi(\chi,t) \sim \chi^{-4} .$$

(2-105)

CHAPTER 3 - CALCULATION OF THE LIFT AND PITCHING-MOMENT COEFFICIENTS

3.1 Calculation of the Lift Coefficient

The total lift on the airfoil is the integral of the pressure difference across the airfoil plus the vertical component of the jet momentum flux. This jet-momentum-flux reaction acts upon the internal ducting of the airfoil and is not an external pressure force. The lift, positive upwards, is written in the linearized approximation, using (2-1), as

$$L(t) = -\int_0^c \Delta p(x,t)\,dx + J\,\frac{\partial Y(c+,t)}{\partial x}. \tag{3-1}$$

Defining the usual non-dimensional lift coefficient, using the unsteady Bernoulli equation (2-12), and using the definition of the jet-momentum-flux coefficient (2-4), (3-1) may be rewritten

$$C_L(t) \equiv \frac{L(t)}{\frac{\rho}{2}U_0^2 c} = \frac{2}{U_0^2 c}\int_0^c \left[\frac{\partial \Delta\phi(x,t)}{\partial t} + U_0\,\frac{\partial \Delta\phi(x,t)}{\partial x}\right] dx + C_J\,\frac{\partial Y(c+,t)}{\partial x}. \tag{3-2}$$

von Kármán and Sears (1938) argue that since the total circulation in the system of an airfoil without jet vanishes, the vortices occur in equal and opposite pairs. The lift may then be found by taking the negative of the time derivative of the impulse of the flow. Despite the fact that part of the lift in the jet-flap case is given by the internal jet-reaction force, the total lift is again given by this result. To show this, consider the following integral, using (2-8),

$$\frac{2}{U_0^2 c}\int_c^\infty\left[\frac{\partial\Delta\phi(x,t)}{\partial t}+U_0\frac{\partial\Delta\phi(x,t)}{\partial x}\right]dx=-C_J\int_c^\infty\frac{\partial^2 Y(x,t)}{\partial x^2}dx=-C_J\left.\frac{\partial Y(x,t)}{\partial x}\right]_c^\infty$$

$$=C_J\frac{\partial Y(c+,t)}{\partial x},$$

(3-3)

where the upper limit vanishes by (2-63). Therefore, although the jet-momentum-flux reaction acts upon the internal ducting, this force is represented by the integral of the pressure difference across the jet. Substituting (3-3) into (3-2) gives

$$C_L(t)=\frac{2}{U_0^2 c}\int_0^\infty\left[\frac{\partial\Delta\phi(x,t)}{\partial t}+U_0\frac{\partial\Delta\phi(x,t)}{\partial x}\right]dx.$$

(3-4)

The total lift of the system, then, is given by the integral of the pressure difference across both the airfoil and the jet. Integrating the $\partial\Delta\phi(x,t)/\partial x$ term immediately, and taking the time derivative outside the integral,

$$C_L(t)=\frac{2}{U_0^2 c}\frac{d}{dt}\int_0^\infty\Delta\phi(x,t)\,dx+\frac{2}{U_0 c}\left.\Delta\phi(x,t)\right]_0^\infty$$

$$=\frac{2}{U_0^2 c}\frac{d}{dt}\int_0^\infty\Delta\phi(x,t)dx,$$

(3-4)

the second term vanishing at both limits by (2-45) and (2-105). Integrating the remaining integral by parts and using (2-7) yields

$$C_L(t)=\frac{2}{U_0^2 c}\frac{d}{dt}\left\{\left.x\Delta\phi(x,t)\right]_0^\infty-\int_0^\infty x\gamma(x,t)dx\right\},$$

where the first term vanishes, again by (2-45) and (2-105). Therefore,

$$C_L(t) = -\frac{2}{U_0^2 c}\frac{d}{dt}\int_0^\infty x\gamma(x,t)\,dx. \tag{3-6}$$

The lift is also given in this case by the negative of the time derivative of the impulse of the flow.

To get the lift coefficient in forms useful for calculation, it is more convenient to treat (3-2). Integrating the $\partial\Delta\phi(x,t)/\partial t$ term by parts,

$$\begin{aligned}\frac{2}{U_0^2 c}\int_0^c \frac{\partial\Delta\phi(x,t)}{\partial t}\,dx &= -\frac{2}{U_0^2 c}(c-x)\frac{\partial\Delta\phi(x,t)}{\partial t}\Big]_0^c + \frac{2}{U_0^2 c}\int_0^c (c-x)\frac{\partial\gamma(x,t)}{\partial t}\,dx \\ &= \frac{2}{U_0^2 c}\int_0^c (c-x)\frac{\partial\gamma(x,t)}{\partial t}\,dx,\end{aligned} \tag{3-7}$$

the first term vanishing at both limits of integration by (2-22) and (2-45). Substituting (3-7) and (2-7) into (3-2) and taking the time derivative outside the integral gives

$$C_L(t) = \frac{2}{U_0^2 c}\frac{d}{dt}\int_0^c (c-x)\gamma(x,t)\,dx + \frac{2}{U_0 c}\int_0^c \gamma(x,t)\,dx + C_J\frac{\partial Y(c+,t)}{\partial x}, \tag{3-8}$$

a general expression for the lift coefficient in terms of the airfoil vortex distribution alone.

To get the lift coefficient in terms of the vortex distribution representing the jet, substitute (2-43) and (2-51) into (3-8), giving

$$C_L(t) = \frac{2}{U_0^2 c}\frac{d}{dt}\int_0^c (c-x)\gamma_0(x,t)\,dx + \frac{2}{\pi U_0^2 c}\frac{d}{dt}\int_0^c \left(\frac{c-x}{x}\right)^{1/2}(c-x)\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x}\,dx$$

$$+\frac{2}{U_0 c}\int_0^c \gamma_0(x,t)\,dx + \frac{2}{\pi U_0 c}\int_0^c\left(\frac{c-x}{x}\right)^{1/2}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x}\,dx + C_J\,\frac{\partial Y(c+,t)}{\partial x}\,.$$

(3-9)

The order of integration may be interchanged in (3-9), using the arguments following (2-46), and the resulting integrals over x evaluated by (A-51) and (A-53). Using these results and the classical lift terms discussed below, the lift coefficient becomes

$$C_L(t) = C_{L_0}(t) + C_{L_1}(t) + \frac{c}{2U_0}\frac{dC_{L_0}(t)}{dt} - \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^\infty\left[\left(\xi-\frac{3c}{2}\right)\left(\frac{\xi}{\xi-c}\right)^{1/2}-(\xi-c)\right]\gamma(\xi,t)\,d\xi$$

$$+\frac{2}{U_0 c}\int_c^\infty\left[\left(\frac{\xi}{\xi-c}\right)^{1/2}-1\right]\gamma(\xi,t)\,d\xi + C_J\,\frac{\partial Y(c+,t)}{\partial x}\,.$$

(3-10)

This is a completely general result for the unsteady lift coefficient. The constancy of circulation in the system is not incorporated in this result.

The integrals over the airfoil-quasi-steady vortex distribution, $\gamma_0(x,t)$, have been identified as the same terms which arise in the classical unsteady-airfoil theory. These have been investigated by von Kármán and Sears (1938), whose notation ana definitions are to be used here. The airfoil-quasi-steady lift is that circulatory force due to the airfoil shape and motion alone, neglecting any wake or jet contributions as discussed in Section 2.4. Its coefficient is denoted by $C_{L_0}(t)$ and is, using (2-53),

$$C_{L_0}(t) \equiv \frac{2}{U_0 c}\int_0^c \gamma_0(x,t)\,dx = \frac{2}{U_0 c}\Gamma_0(t) = \frac{4}{U_0 c}\int_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2} w(\xi,t)\,d\xi \,. \tag{3-11}$$

The airfoil apparent-mass lift is the force of non-circulatory origin and depends on the airfoil shape and motion only. Its coefficient is denoted by $C_{L_1}(t)$ and is defined by

$$C_{L_1}(t) \equiv -\frac{2}{U_0^2 c}\frac{d}{dt}\int_0^c \left(x-\frac{c}{2}\right)\gamma_0(x,t)\,dx \,. \tag{3-12}$$

To get $C_{L_1}(t)$ in terms of the downwash on the airfoil, substitute (2-51) into (3-12), giving

$$C_{L_1}(t) = -\frac{2}{U_0^2 c}\frac{d}{dt}\int_0^c \left(x-\frac{c}{2}\right)\frac{2}{\pi}\left(\frac{c-x}{x}\right)^{1/2} \oint_0^c \left(\frac{\xi}{c-\xi}\right)^{1/2}\frac{w(\xi,t)\,d\xi}{\xi-x}\,dx \,.$$

Inversion of the order of integration using the arguments following (2-46), and evaluation of the resulting integrals over x by (A-50) and (A-52) gives

$$C_{L_1}(t) = \frac{4}{U_0^2 c}\frac{d}{dt}\int_0^c \xi^{1/2}(c-\xi)^{1/2}\,w(\xi,t)\,d\xi \,. \tag{3-13}$$

Since the potential difference must vanish at infinity downstream as discussed in Section 2.6, (2-66), this result, (3-10), can be simplified. Identification of certain of the integral terms in (3-10) by the Wagner integral condition, (2-82), and by the definition of the airfoil-quasi-steady lift coefficient, (3-11), substitution of $\partial\Delta\phi(x,t)/\partial x$ for $\gamma(x,t)$, and substitution of (3-3) for $C_J \frac{\partial Y(c+,t)}{\partial x}$ give, upon cancellation of terms,

$$C_L(t) = C_{L_1}(t) - \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty} \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi + \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty}(\xi-c)\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}\,d\xi$$

$$+ \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty}\Delta\phi(\xi,t)\,d\xi. \qquad (3\text{-}14)$$

Integrating the second integral of (3-14) by parts yields

$$\frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty}(\xi-c)\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}\,d\xi = \frac{2}{U_0^2 c}\frac{d}{dt}\Big[(\xi-c)\Delta\phi(\xi,t)\Big]_c^{\infty} - \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty}\Delta\phi(\xi,t)\,d\xi$$

$$= -\frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty}\Delta\phi(\xi,t)\,d\xi,$$

the contributions at the limits of integration vanishing by (2-22) and (2-105). The integral remaining cancels the last integral in (3-14), giving the compact form for the lift coefficient

$$C_L(t) = C_{L_1}(t) - \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^{\infty}\xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi. \qquad (3\text{-}15)$$

In this expression, all the lift of a circulatory nature is included in the integral.

The lift coefficient can be found in terms of the limiting behavior of the downwash, vortex distribution and potential difference at infinity downstream, as promised in Section 2.7. Examination of (3-13) and (3-15), in conjunction with (2-99), (2-103) and (2-104) gives

$$C_L(t) = -\frac{4\pi}{U_0^2 c}\frac{d}{dt}\lim_{x\to\infty} x^2 w(x,t),$$

(3-16)

$$C_L(t) = -\frac{\pi}{3U_0^2 c^2 C_J}\frac{d}{dt}\lim_{x\to\infty} x^5 \frac{D^2\gamma(x,t)}{Dt^2},$$

(3-17)

and

$$C_L(t) = \frac{4\pi}{3U_0^3 c^2 C_J}\frac{d}{dt}\lim_{x\to\infty} x^4 \frac{D^2\Delta\phi(x,t)}{Dt^2}.$$

(3-18)

For the sake of completeness, an expression for the lift coefficient can be found which is of the form of the von Kármán and Sears (1938) result. To show this, add and subtract $C_{L_0}(t)$ from (3-15) using the Wagner integral condition, (2-82), take the time derivative inside the integral in (3-15), and substitute (2-10) for $\partial\gamma(x,t)/\partial t$ inside the integral, giving

$$C_L(t) = C_{L_0}(t) + C_{L_1}(t) + \frac{2}{U_0 c}\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi$$
$$+ \frac{2}{U_0^2 c}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\left[U_0\frac{\partial\gamma(\xi,t)}{\partial\xi} + \frac{1}{2}U_0^2 c C_J \frac{\partial^3\gamma(\xi,t)}{\partial\xi^3}\right]d\xi.$$

(3-19)

Integrating the integral over $\partial\gamma(\xi,t)/\partial\xi$ by parts gives

$$\frac{2}{U_0 c}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\frac{\partial\gamma(\xi,t)}{\partial\xi}d\xi = \frac{2}{U_0 c}\left[\xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\right]_c^\infty - \frac{2}{U_0 c}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi + \frac{1}{U_0}\int_c^\infty \frac{\gamma(\xi,t)\,d\xi}{\xi^{1/2}(\xi-c)^{1/2}}$$
$$= -\frac{2}{U_0 c}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi + \frac{1}{U_0}\int_c^\infty \frac{\gamma(\xi,t)\,d\xi}{\xi^{1/2}(\xi-c)^{1/2}},$$

since the contribution at the limits of integration vanish by (2-22) and (2-102). Substituting this into (3-19) simplifies it to

$$C_L(t) = C_{L_0}(t) + C_{L_1}(t) + C_{L_2}(t) + C_J \int_c^{\infty} \xi^{1/2}(\xi - c)^{1/2} \frac{\partial^3 Y(\xi,t)}{\partial \xi^3}\, d\xi, \tag{3-20}$$

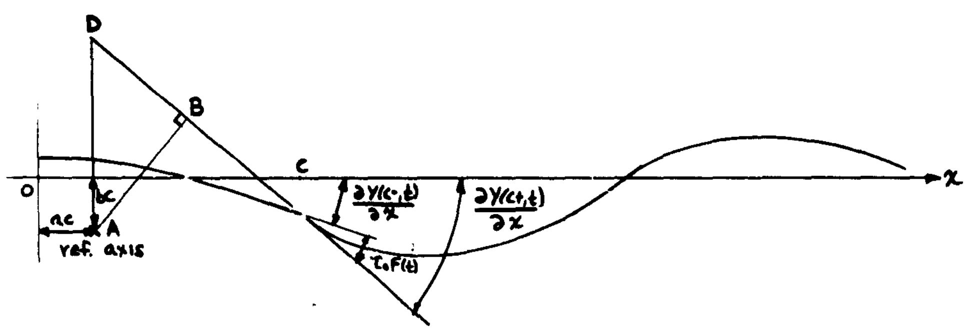
where

$$C_{L_2}(t) \equiv \frac{1}{U_0} \int_c^{\infty} \frac{\gamma(\xi,t)\, d\xi}{\xi^{1/2}(\xi - c)^{1/2}} . \tag{3-21}$$

In the classical case, $C_{L_2}(t)$ is the wake-effect term but here includes the jet effects as does the last term. This term vanishes with C_J, leaving the von Kármán and Sears (1938) result.

3.2 Calculation of the Pitching-Moment Coefficient

The total pitching moment, positive nose downward about some reference axis, is the integral of the moments of the individual lifting elements of the airfoil about this axis, plus the moment of the jet-momentum-flux reaction about this axis. To determine this direct-jet-reaction moment, consider the following arbitrary airfoil at some instant of its motion.



By the linearized approximation, $\overline{AB} \simeq \overline{AD}$ since $\cos\left[\frac{\partial y(c+,t)}{\partial x}\right] \simeq 1 + O\left(\frac{\partial y}{\partial x}\right)^2$. Geometrically,

$$\overline{AD} = \frac{\partial y(c+,t)}{\partial x}(1-a)c - y(c+,t) + bc.$$

The moment due to this force is

$$M(t) = J\left[(1-a)c\,\frac{\partial y(c+,t)}{\partial x} - y(c+,t) + bc\right].$$

The total moment is, then,

$$M(t) = -\int_0^c (x-ac)\,\Delta p(x,t)\,dx + J\left[(1-a)c\,\frac{\partial y(c+,t)}{\partial x} - y(c+,t) + bc\right]. \tag{3-22}$$

Defining the non-dimensional pitching-moment coefficient, and using the unsteady Bernoulli equation, (2-12), and the definition of the momentum-flux coefficient, (2-4), (3-22) becomes

$$C_m(t) \equiv \frac{M(t)}{\frac{\rho}{2}U_0^2c^2} = \frac{2}{U_0^2c^2}\int_0^c (x-ac)\left[\frac{\partial\Delta\phi(x,t)}{\partial t} + U_0\,\frac{\partial\Delta\phi(x,t)}{\partial x}\right]dx$$

$$+ C_J\left[(1-a)\,\frac{\partial y(c+,t)}{\partial x} - \frac{y(c+,t)}{c} + b\right]. \tag{3-23}$$

As for the lift, this equation may be shown to reduce to a form analogous to that which von Kármán and Sears (1938) derived considering the time derivative of the moment of impulse. To show how this differs in this case due to the moment of the internal jet-reaction force, consider the following integral, using (2-8),

$$\frac{2}{U_0^2c^2}\int_c^\infty (x-ac)\left[\frac{\partial\Delta\phi(x,t)}{\partial t} + U_0\,\frac{\partial\Delta\phi(x,t)}{\partial x}\right]dx = -\frac{C_J}{c}\int_c^\infty (x-ac)\,\frac{\partial^2 y(x,t)}{\partial x^2}\,dx,$$

which upon integration by parts

$$= -\frac{C_J}{c}\left\{(x-ac)\frac{\partial y(x,t)}{\partial x}\Big]_c^\infty - \int_c^\infty \frac{\partial y(x,t)}{\partial x}\,dx\right\}.$$

A further integration and use of (2-63) in evaluating the limits of the integrals gives, therefore,

$$\frac{2}{U_0^2c^2}\int_c^\infty (x-ac)\left[\frac{\partial\Delta\phi(x,t)}{\partial t}+U_0\frac{\partial\Delta\phi(x,t)}{\partial x}\right]dx = C_J\left\{(1-a)\frac{\partial y(c,t)}{\partial x} - \frac{y(c,t)}{c}\right\}. \tag{3-24}$$

Substituting (3-24) into (3-23) yields

$$C_M(t) = \frac{2}{U_0^2c^2}\int_0^\infty (x-ac)\left[\frac{\partial\Delta\phi(x,t)}{\partial t}+U_0\frac{\partial\Delta\phi(x,t)}{\partial x}\right]dx + C_J b. \tag{3-25}$$

Integrating (3-25) by parts, and using (2-7),

$$C_M(t) = \frac{2}{U_0^2c^2}\left\{U_0(x-ac)\Delta\phi(x,t)\Big]_0^\infty - U_0\int_0^\infty \Delta\phi(x,t)\,dx + \frac{(x-ac)^2}{2}\frac{\partial\Delta\phi(x,t)}{\partial t}\Big]_0^\infty - \frac{1}{2}\int_0^\infty (x-ac)^2\frac{\partial\gamma(x,t)}{\partial t}\,dx\right\} + C_J b.$$

The limits of the first and third terms vanish by (2-45) and (2-105), so the moment expression becomes

$$C_M(t) = -\frac{1}{U_0^2c^2}\frac{d}{dt}\int_0^\infty (x-ac)^2\gamma(x,t)\,dx - \frac{2}{U_0c^2}\int_0^\infty \Delta\phi(x,t)\,dx + C_J b. \tag{3-26}$$

This corresponds to the von Kármán and Sears (1938) expression except for the term $C_J b$. This term is the moment of the momentum flux at infinity downstream times the distance of the reference axis below the undisturbed position of the jet at infinity downstream. It is not surprising that this additional term appears, since the moment due directly to the jet reaction acts on the internal ducting of the airfoil, not as a pressure force on the airfoil surface.

To get the moment coefficient in forms useful for calculation, similar treatment to that made in the lift-coefficient case will be made. Integrating the $\partial\Delta\phi(x,t)/\partial t$ term in (3-23) by parts gives

$$\frac{2}{U_0^2c^2}\int_0^c (x-ac)\frac{\partial\Delta\phi(x,t)}{\partial t}dx = -\frac{1}{U_0^2c^2}\left[(c-x)(x+c-2ac)\frac{\partial\Delta\phi(x,t)}{\partial t}\right]_0^c + \frac{1}{U_0^2c^2}\int_0^c\left[c^2(1-2a)+2acx-x^2\right]\frac{\partial\gamma(x,t)}{\partial t}dx$$

$$= \frac{1}{U_0^2c^2}\int_0^c\left[c^2(1-2a)+2acx-x^2\right]\frac{\partial\gamma(x,t)}{\partial t}dx \,,$$

the integral vanishing at both limits of integration by (2-22) and (2-45). Substituting this result and (2-7) into (3-23) and taking the time derivative outside the integral gives a general expression for the pitching-moment coefficient in terms of the airfoil vortex distribution alone,

$$C_M(t) = \frac{1}{U_0^2c^2}\frac{d}{dt}\int_0^c\left[c^2(1-2a)+2acx-x^2\right]\gamma(x,t)dx + \frac{2}{U_0c^2}\int_0^c(x-ac)\gamma(x,t)dx$$

$$+ C_J\left[(1-a)\frac{\partial Y(c+,t)}{\partial x} - \frac{Y(c+,t)}{c} + b\right].$$

(3-27)

The pitching-moment coefficient also can be expressed in terms of the vortex distribution representing the jet. Substitute (2-43) and (2-51) into (3-27), giving

$$C_M(t) = \frac{1}{U_0^2 c^2}\frac{d}{dt}\int_0^c \left[c^2(1-2a)+2ac\chi-\chi^2\right]\gamma_0(\chi,t)\,d\chi + \frac{2}{U_0 c^2}\int_0^c (\chi-ac)\,\gamma_0(\chi,t)\,d\chi$$

$$+\frac{1}{\pi U_0^2 c^2}\frac{d}{dt}\int_0^c\left(\frac{c-\chi}{\chi}\right)^{1/2}\left[c^2(1-2a)+2ac\chi-\chi^2\right]\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-\chi}\,d\chi$$

$$+\frac{2}{\pi U_0 c^2}\int_0^c\left(\frac{c-\chi}{\chi}\right)^{1/2}(\chi-ac)\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-\chi}\,d\chi + C_J\left[(1-a)\frac{\partial Y(c+,t)}{\partial \chi}-\frac{Y(c+,t)}{c}+b\right].$$

(3-28)

The order of integration again may be interchanged by the argument following (2-46) and the resulting integrals over χ evaluated by (A-51), (A-53) and (A-55). Using this and the classical terms discussed below, the pitching-moment coefficient reduces to

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \tfrac{1}{2}(1-2a)C_{L_0}(t) + \tfrac{1}{2}(1-2a)C_{L_1}(t) + \frac{5c}{16U_0}\left(1-\frac{8a}{5}\right)\frac{dC_{L_0}(t)}{dt}$$

$$-\frac{1}{U_0^2c^2}\frac{d}{dt}\int_c^\infty\left\{\left[\xi^2-\frac{c}{2}(1+4a)\xi-\frac{5c^2}{8}\left(1-\frac{8a}{5}\right)\right]\left(\frac{\xi}{\xi-c}\right)^{1/2}-\left[\xi^2-2ac\xi-c^2(1-2a)\right]\right\}\gamma(\xi,t)\,d\xi$$

$$+\frac{2}{U_0c^2}\int_c^\infty\left\{\left[\xi-\frac{c}{2}(1+2a)\right]\left(\frac{\xi}{\xi-c}\right)^{1/2}-(\xi-ac)\right\}\gamma(\xi,t)\,d\xi + C_J\left[(1-a)\frac{\partial Y(c+,t)}{\partial\chi}-\frac{Y(c+,t)}{c}+b\right].$$

(3-29)

This is a general result, the constancy of circulation not having been used.

The integrals over the airfoil-quasi-steady vortex distribution in (3-29) have been identified in terms of the airfoil-quasi-steady and apparent-mass lift coefficients defined in (3-11) to

(3-13), and the analogous pitching-moment coefficients discussed by von Kármán and Sears (1938). The airfoil-quasi-steady pitching moment is that moment of circulatory origin due to the airfoil shape and motion alone, neglecting wake and jet contributions. Denoted by $C_{M_0}(t)$ its coefficient is

$$C_{M_0}(t) \equiv \frac{2}{U_0^2 c}\int_0^c \left(x-\frac{c}{2}\right)\gamma_0(x,t)\,dx . \tag{3-30}$$

By the same argument leading from (3-12) to (3-13), this can be written, using (A-50) and (A-52), as

$$C_{M_0}(t) = -\frac{4}{U_0 c^2}\int_0^c \xi^{1/2}(c-\xi)^{1/2}\, w(\xi,t)\,d\xi . \tag{3-31}$$

By convention, this is a moment about midchord $\left(a = \frac{1}{2}\right)$. The airfoil apparent-mass pitching moment, also about midchord, is of non-circulatory origin and depends on the airfoil shape and motion only. Denoted by $C_{M_1}(t)$, its coefficient is defined by

$$C_{M_1}(t) \equiv -\frac{1}{U_0^2 c^2}\frac{d}{dt}\int_0^c \left(x^2 - cx + \frac{c^2}{8}\right)\gamma_0(x,t)\,dx. \tag{3-32}$$

By the argument leading from (3-12) to (3-13), this also can be written, using (A-50), (A-52), and (A-54), as

$$C_{M_1}(t) = -\frac{2}{U_0^2 c^2}\frac{d}{dt}\int_0^c \xi^{1/2}(c-\xi)^{1/2}\left(\frac{c}{2}-\xi\right) w(\xi,t)\,d\xi. \tag{3-23}$$

By use of the condition that the potential difference at infinity downstream vanishes, (2-66), this result, (3-29), can be simplified. Identification of certain of the integral terms in (3-29) using the Wagner integral condition, (2-82), and the definition of the airfoil-quasi-steady lift coefficient, (3-11), substitution of $\partial\Delta\phi(x,t)/\partial x$ for $\gamma(x,t)$ by (2-7) where needed, and substitution of (3-24) for $C_J\left[(1-a)\frac{\partial\gamma(c+,t)}{\partial x} - \frac{\gamma(c+,t)}{c}\right]$ give, upon cancellation of terms,

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \tfrac{1}{2}(1-2a)C_{L_1}(t)$$

$$-\frac{1}{U_0^2c^2}\frac{d}{dt}\int_c^\infty\left[\xi^2-\frac{c}{2}(1+4a)\xi-\frac{c^2}{2}(1-4a)\right]\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi+\frac{1}{U_0^2c^2}\frac{d}{dt}\int_c^\infty\left[\xi^2-2ac\xi-c^2(1-2a)\right]\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}\,d\xi$$

$$+\frac{2}{U_0c^2}\int_c^\infty\xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi+\frac{2}{U_0^2c^2}\frac{d}{dt}\int_c^\infty(\xi-ac)\Delta\phi(\xi,t)\,d\xi+C_J b. \tag{3-34}$$

Integrating the second integral in this expression by parts yields

$$\frac{1}{U_0^2c^2}\frac{d}{dt}\int_c^\infty\left[\xi^2-2ac\xi-c^2(1-2a)\right]\frac{\partial\Delta\phi(\xi,t)}{\partial\xi}\,d\xi=\frac{1}{U_0^2c^2}\frac{d}{dt}\Big[(\xi-c)(\xi+c-2ac)\Delta\phi(\xi,t)\Big]_c^\infty$$

$$-\frac{2}{U_0^2c^2}\frac{d}{dt}\int_c^\infty(\xi-ac)\Delta\phi(\xi,t)\,d\xi$$

$$=-\frac{2}{U_0^2c^2}\frac{d}{dt}\int_c^\infty(\xi-ac)\Delta\phi(\xi,t)\,d\xi,$$

the contributions at the limits of integration vanishing by (2-22) and (2-105). The remaining integral cancels the last integral in (3-34), which, upon factoring of the first integral becomes

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \tfrac{1}{2}(1-2a)C_{L_1}(t) - \frac{3(1-\frac{4a}{3})}{2U_0^2 c}\frac{d}{dt}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi$$

$$-\frac{1}{U_0^2c^2}\frac{d}{dt}\int_c^\infty \xi^{1/2}(\xi-c)^{3/2}\gamma(\xi,t)\,d\xi + \frac{2}{U_0c^2}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi + C_J b.$$

(3-35)

To calculate the pitching-moment coefficient by (3-35), only one more integral over the jet vortex distribution need be calculated besides the integral required for the lift coefficient in (3-15).

An expression for the pitching-moment coefficient of the form of the von Kármán and Sears (1938) expression is found from (3-35) by adding and subtracting $\frac{1}{2}(1-2a)\,C_{L_0}(t)$ from (3-35) using the Wagner integral condition, (2-82), taking the time derivative inside the integral and substituting (2-10) for $\partial\gamma(x,t)/\partial t$. This gives

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \tfrac{1}{2}(1-2a)C_{L_1}(t) + \tfrac{1}{2}(1-2a)\,C_{L_0}(t)$$

$$+\frac{(1-2a)}{U_0c}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi + \frac{3(1-\frac{4a}{3})}{2U_0c}\int_c^\infty \xi^{1/2}(\xi-c)^{3/2}\,\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi$$

$$+\frac{C_J}{2c}\int_c^\infty \xi^{1/2}(\xi-c)^{3/2}\,\frac{\partial^2\gamma(\xi,t)}{\partial\xi^2}\,d\xi + \frac{2}{U_0c^2}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi + C_J b.$$

(3-36)

Integrating the $\partial\gamma(\xi,t)/\partial\xi$ terms by parts gives

$$\frac{3(1-\frac{4a}{3})}{2U_0 c}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi + \frac{1}{U_0c^2}\int_c^\infty \xi^{1/2}(\xi-c)^{3/2}\frac{\partial\gamma(\xi,t)}{\partial\xi}\,d\xi =$$

$$\frac{3(1-\frac{4a}{3})}{2U_0c}\left[\xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\right]_c^\infty + \frac{3(1-\frac{4a}{3})}{4U_0}\int_c^\infty\frac{\gamma(\xi,t)\,d\xi}{\xi^{1/2}(\xi-c)^{1/2}} - \frac{3(1-\frac{4a}{3})}{2U_0c}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi$$

$$+\frac{1}{U_0c^2}\left[\xi^{1/2}(\xi-c)^{3/2}\gamma(\xi,t)\right]_c^\infty - \frac{2}{U_0c^2}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi + \frac{1}{2U_0c}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi$$

$$-\frac{1}{2U_0}\int_c^\infty\frac{\gamma(\xi,t)\,d\xi}{\xi^{1/2}(\xi-c)^{1/2}} = \frac{(1-4a)}{4U_0}\int_c^\infty \xi^{-1/2}(\xi-c)^{-1/2}\gamma(\xi,t)\,d\xi$$

$$-\frac{(1-2a)}{U_0c}\int_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi - \frac{2}{U_0c^2}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi\,,$$

where the contributions at the limits of integration vanish by (2-22) and (2-102). Substituting this into (3-36) and cancelling terms gives

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \tfrac{1}{2}(1-2a)\left[C_{L_0}(t) + C_{L_1}(t)\right] + \frac{1-4a}{4}C_{L_2}(t)$$

$$+\frac{3(1-\frac{4a}{3})}{4}C_J\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\frac{\partial^3\gamma(\xi,t)}{\partial\xi^3}\,d\xi + \frac{1}{2c}C_J\int_c^\infty \xi^{1/2}(\xi-c)^{3/2}\frac{\partial^3\gamma(\xi,t)}{\partial\xi^3}\,d\xi + C_J b.$$

(3-37)

This form corresponds to the result of von Kármán and Sears (1938). In addition to their terms, there are two additional integrals, proportional to C_J , and also the $C_J b$ term discussed above. If $C_J \equiv 0$, the classical results are recovered, and the moment contains only the same terms which were evaluated to find the lift coefficient, (3-20).

There are no expressions for the pitching-moment coefficient in terms of the behavior of the jet at infinity downstream as were found for the lift coefficient.

The pair of equations (3-15) and (3-35) are the simplest to use for the calculation of the lift and pitching-moment coefficients, since only two integrals over the jet vortex distribution need be calculated. For convenience they are written here,

$$C_L(t) = C_{L_1}(t) - \frac{2}{U_0^2 c}\frac{d}{dt}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi \,, \tag{3-38}$$

and

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \frac{1}{2}(1-2a)\,C_{L_1}(t) - \frac{3\left(1-\frac{4a}{3}\right)}{2U_0^2 c}\frac{d}{dt}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi$$

$$- \frac{1}{U_0^2 c^2}\frac{d}{dt}\int_c^\infty \xi^{1/2}(\xi-c)^{3/2}\gamma(\xi,t)\,d\xi + \frac{2}{U_0 c^2}\int_c^\infty \xi^{1/2}(\xi-c)^{1/2}\gamma(\xi,t)\,d\xi + C_J b. \tag{3-39}$$

CHAPTER 4 - EQUATIONS FOR PARTICULAR PROBLEMS

4.1 Jet-Deflection Problem

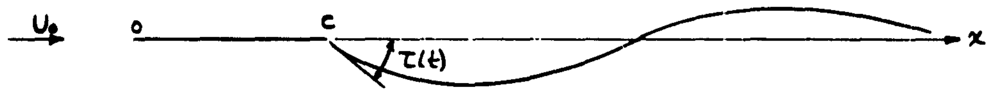
The fundamental jet-flap problem, both in the steady and unsteady cases is that of "jet deflection," i.e., the emergence of the jet from the trailing edge at a prescribed angle with respect to the slope of the airfoil ordinate there. This problem has been called by Malavard (1957) the "singular-blowing" problem in recognition of the logarithmic singularity in the vortex distribution at the trailing edge required to satisfy (2-61). Practically, jet deflection might be achieved by a very small flap at the trailing edge, or by internal ducting, and is assumed to be adequately represented by the present model. The importance of the problem resides in its being unique to the jet-flap, for if $C_J \equiv 0$, specification of the jet deflection angle can no longer be made and the problem is trivial.

As a model for the jet-deflection case, a flat-plate airfoil aligned with the free-stream direction is chosen, the jet having the time-dependent angle of deflection, (2-13),

$$\tau(t) = \tau_0 F(t). \tag{4-1}$$

For transient inputs this problem is of considerable practical importance, since the jet-flap has been proposed as a control mechanism and the response of the airfoil to such inputs is basic to the full understanding of such usage. Oscillating inputs are also important, since use of the jet flap for cyclic pitch control of helicopter rotor blades has been suggested by Dorand (1959).

Since the airfoil is aligned with the free stream,



$$\gamma(x,t) = 0, \quad 0<x<c, \tag{4-2}$$

and hence from (2-1),

$$w(x,t) = 0, \quad 0<x<c. \tag{4-3}$$

All the airfoil-quasi-steady and airfoil apparent-mass terms are identically zero.

The basic equations are, in terms of $\gamma(x,t)$ and $w(x,t)$ - the others are omitted here and in the remainder of this chapter for simplicity - from (2-69) to (2-87),

$$\frac{D^2\gamma(x,t)}{Dt^2} = -\frac{U_0^2 c}{2} C_J \frac{\partial^3 w(x,t)}{\partial x^3}, \quad c<x<\infty, \tag{4-4}$$

and either

$$w(x,t) = -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x}, \quad c<x<\infty, \tag{4-5}$$

or

$$w(x,t) = -\frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x}\,, \quad c<x<\infty, \tag{4-6}$$

with either

$$w(c+,t) = U_0 \tau_0 F(t), \tag{4-7}$$

or

$$\lim_{x\to c+} \frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} = -2U_0\tau_0 F(t), \tag{4-8}$$

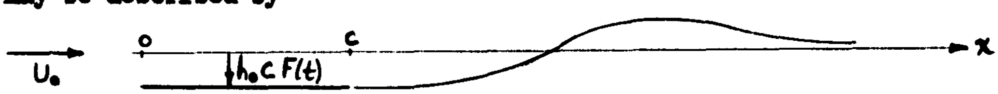
and

$$\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi = 0\,. \tag{4-9}$$

Once this set has been solved, the other properties of interest, e.g., the lift and pitching-moment coefficients, may be found from the appropriate results of earlier sections.

4.2 Problem of Airfoil in Plunging Motion

The problem of a flat-plate airfoil moving in a purely plunging motion with a jet emerging tangentially from the trailing edge may be described by



$$Y(x,t) = h_0 c\, F(t)\,, \quad 0<x<c\,, \tag{4-10}$$

where h_0 is the non-dimensional amplitude of the plunging motion, and is a small parameter consistent with linearization of the problem. This problem does not have much application in transient motion, but in oscillating motion its results are important for determining the coupled binary bending-torsion and tertiary bending-torsion, control-surface-rotation flutter stability criteria.

From (2-1) and (4-10), the downwash on the airfoil is

$$w(x,t) = h_0 c\, F'(t)\,, \quad 0<x<c. \tag{4-11}$$

With (4-11), the important airfoil-quasi-steady terms in the basic equations are from (2-51), (2-53), (2-54) and (2-87), using (A-9), (A-38), and (A-39)

$$\gamma_0(x,t) = 2 h_0 c \left(\frac{c-x}{x}\right)^{1/2} F'(t)\,, \quad 0<x<c, \tag{4-12}$$

$$w_0(x,t) = h_0 c\left[1-\left(\frac{x-c}{x}\right)^{1/2}\right] F'(t)\,, \quad c<x<\infty, \tag{4-13}$$

$$\Gamma_0(t) = \pi h_0 c^2\, F'(t)\,, \tag{4-14}$$

and

$$w_1(x,t) = h_0 c\left[1-\left(\frac{x-c}{x}\right)^{1/2} - \frac{c}{2x^{1/2}(x-c)^{1/2}}\right] F'(t)\,, \quad c<x<\infty\,. \tag{4-15}$$

The basic equations are, then, from (2-69) to (2-87),

$$\frac{D^2\gamma(x,t)}{Dt^2} = -\frac{U_0^2 c}{2} C_J \frac{\partial^3 w(x,t)}{\partial x^3}, \quad c<x<\infty, \tag{4-16}$$

and either

$$w(x,t) = -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} + h_0 c\left[1-\left(\frac{x-c}{x}\right)^{1/2}\right] F'(t), \quad c<x<\infty, \tag{4-17}$$

or

$$w(x,t) = -\frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} + h_0 c\left[1-\left(\frac{x-c}{x}\right)^{1/2} - \frac{c}{2x^{1/2}(x-c)^{1/2}}\right] F'(t), \quad c<x<\infty, \tag{4-18}$$

with either

$$w(c+,t) = h_0 c F'(t), \tag{4-19}$$

or

$$\lim_{x\to c+} \frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} = 0, \tag{4-20}$$

and

$$\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi = -\pi h_0 c^2 F'(t). \tag{4-21}$$

Upon solution, the other properties may be found. The airfoil-quasi-steady and airfoil apparent-mass coefficients are, given by (3-11), (3-13), (3-31) and (3-33), using (A-18) and (A-21),

$$C_{L_0}(t) = 2\pi h_0 \frac{cF'(t)}{U_0} , \tag{4-22}$$

$$C_{L_1}(t) = \frac{\pi h_0}{2} \frac{c^2 F''(t)}{U_0^2} , \tag{4-23}$$

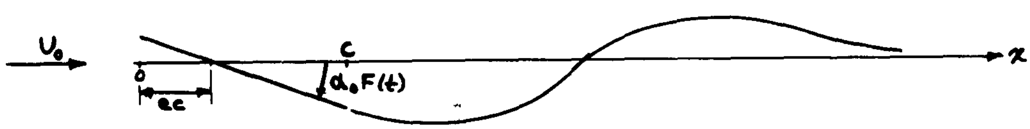
$$C_{M_0}(t) = -\frac{\pi h_0}{2} \frac{cF'(t)}{U_0} , \tag{4-24}$$

and

$$C_{M_1}(t) = 0 . \tag{4-25}$$

4.3 Problem of Airfoil in Pitching Motion

A flat-plate airfoil moving in a pitching motion about an axis a distance ec behind the leading edge may be described by



$$y(x,t) = \alpha_0 (x - ec) F(t) , \quad 0 < x < c , \tag{4-26}$$

where α_0 is the amplitude of the incidence angle and is small consistent with the linearization. The jet emerges from the airfoil tangentially at the trailing edge. Like the plunging case, the results of this problem for oscillatory motions are important in the binary and tertiary flutter analyses.

The downwash on the airfoil is, by (2-1) and (4-26),

$$w(x,t) = U_0 \alpha_0 F(t) + \alpha_0 (x - ec) F'(t), \quad 0 < x < c. \tag{4-27}$$

The important airfoil-quasi-steady quantities in the basic equations are, from (2-51), (2-53), (2-54) and (2-87), using (A-9), (A-11), (A-38), (A-39), (A-42) and (A-43),

$$\gamma_0(x,t) = 2U_0\alpha_0\left(\frac{c-x}{x}\right)^{1/2}\left\{F(t) + \left[x + \left(\tfrac{1}{2}-e\right)c\right]\frac{F'(t)}{U_0}\right\}, \quad 0 < x < c, \tag{4-28}$$

$$w_0(x,t) = U_0\alpha_0 F(t)\left[1 - \left(\frac{x-c}{x}\right)^{1/2}\right] + \alpha_0 F'(t)\left\{(x-ec) - \left[x + \left(\tfrac{1}{2}-e\right)c\right]\left(\frac{x-c}{x}\right)^{1/2}\right\}, \quad c < x < \infty, \tag{4-29}$$

$$\Gamma_0(t) = \pi U_0 \alpha_0 c F(t) + \pi \alpha_0 c^2 \left(\tfrac{3}{4} - e\right) F'(t), \tag{4-30}$$

and

$$w_1(x,t) = U_0\alpha_0 F(t)\left[1 - \left(\frac{x-c}{x}\right)^{1/2} - \frac{c}{2x^{1/2}(x-c)^{1/2}}\right]$$

$$+ \alpha_0 F'(t)\left\{(x-ec) - \left[x + \left(\tfrac{1}{2}-e\right)c\right]\left(\frac{x-c}{x}\right)^{1/2} - \frac{c^2\left(\tfrac{3}{4}-e\right)}{2x^{1/2}(x-c)^{1/2}}\right\}, \quad c < x < \infty. \tag{4-31}$$

The basic equations are, again from (2-69) to (2-87),

$$\frac{D^2\gamma(x,t)}{Dt^2} = -\frac{U_0^2 c}{2}\, C_J\, \frac{\partial^3 w(x,t)}{\partial x^3}, \quad c<x<\infty, \tag{4-32}$$

and either

$$w(x,t) = -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2}\oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x} + U_0\alpha_0\left[1-\left(\frac{x-c}{x}\right)^{1/2}\right]F(t)$$
$$+\alpha_0\left\{(x-ec)-\left[x+\left(\tfrac{1}{2}-e\right)c\right]\left(\frac{x-c}{x}\right)^{1/2}\right\}F'(t), \quad c<x<\infty, \tag{4-33}$$

or

$$w(x,t) = -\frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x} + U_0\alpha_0\left[1-\left(\frac{x-c}{x}\right)^{1/2}-\frac{c}{2x^{1/2}(x-c)^{1/2}}\right]F(t)$$
$$+\alpha_0\left\{(x-ec)-\left[x+\left(\tfrac{1}{2}-e\right)c\right]\left(\frac{x-c}{x}\right)^{1/2}-\frac{c^2\left(\frac{3}{4}-e\right)}{2x^{1/2}(x-c)^{1/2}}\right\}F'(t),$$
$$c<x<\infty, \tag{4-34}$$

with either

$$w(c+,t) = U_0\alpha_0 F(t) + \alpha_0(1-e)cF'(t), \tag{4-35}$$

or

$$\lim_{x\to c+}\frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2}\oint_c^\infty\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi,t)\,d\xi}{\xi-x} = 0, \tag{4-36}$$

and

$$\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi = -\pi U_0 \alpha_0 c F(t) - \pi \alpha_0 c^2 \left(\tfrac{3}{4}-e\right) F'(t). \tag{4-37}$$

The airfoil-quasi-steady and airfoil apparent-mass lift and pitching-moment coefficients are, by (3-11), (3-13), (3-31) and (3-33), using (A-18), (A-21) and (A-23),

$$C_{L_0}(t) = 2\pi\alpha_0 F(t) + 2\pi\alpha_0 \left(\tfrac{3}{4}-e\right) \frac{cF'(t)}{U_0}, \tag{4-38}$$

$$C_{L_1}(t) = \frac{\pi\alpha_0}{2} \frac{cF'(t)}{U_0} + \frac{\pi\alpha_0}{2}\left(\tfrac{1}{2}-e\right) \frac{c^2 F''(t)}{U_0^2}, \tag{4-39}$$

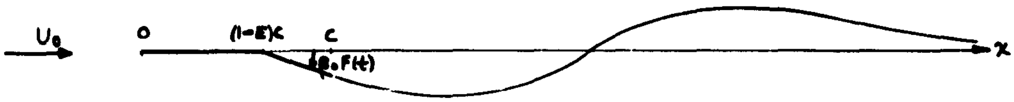
$$C_{M_0}(t) = -\frac{\pi\alpha_0}{2} F(t) - \frac{\pi\alpha_0}{2}\left(\tfrac{1}{2}-e\right) \frac{cF'(t)}{U_0} \tag{4-40}$$

and

$$C_{M_1}(t) = \frac{\pi\alpha_0}{64} \frac{c^2 F''(t)}{U_0^2}. \tag{4-41}$$

4.4 Problem of Blown Flap in Unsteady Motion

Another practical jet-flap system which has been considered is the so-called "blown flap," or "jet-augmented flap." In this configuration the airfoil has an ordinary trailing-edge flap, the jet emerging at the hinge point, following the flap, and leaving tangentially at the trailing edge. This has been represented in the thin-airfoil model as a bent flat plate with a jet emerging at the trailing edge. Spence (1958) has discussed the physical considerations of such a model. Here the problem concerns the time-dependent motion of the flap about its hinge point, which is located a distance εc ahead of the trailing edge, i.e.,



$$y(x,t) = \begin{cases} 0 \; , & 0 < x < (1-\varepsilon)c \\ \beta_0 [x-(1-\varepsilon)c] F(t) \; , & (1-\varepsilon)c < x < c . \end{cases}$$

(4-42)

Both the transient and oscillatory responses are of importance for this problem, the fomer because of the control use of the flap, the latter because of its importance in the tertiary bending-torsion, control-surface-rotation flutter characteristics.

Defining an angle χ by

$$\varepsilon = \sin^2 \frac{\chi}{2} \, ,$$

(4-43)

the downwash on the airfoil is, from (2-1) and (4-42),

$$w(x,t) = \begin{cases} 0 \; , & 0 < x < c\cos^2 \frac{\chi}{2} \\ U_0 \beta_0 F(t) + \beta_0 [x - c\cos^2 \frac{\chi}{2}] F'(t) \; , & c\cos^2 \frac{\chi}{2} < x < c . \end{cases}$$

(4-44)

The airfoil-quasi-steady quantities in the basic equations are, from (2-51), (2-53), (2-54) and (2-87), using (A-36), (A-37), (A-40), (A-41), (A-8), and (A-10),

$$\gamma_0(x,t) = \frac{2U_0\beta_0}{\pi}\left\{\chi\left(\frac{c-x}{x}\right)^{1/2} + \ln\left|\frac{\tan\frac{\chi}{2} + \left(\frac{c-x}{x}\right)^{1/2}}{\tan\frac{\chi}{2} - \left(\frac{c-x}{x}\right)^{1/2}}\right|\right\}F(t)$$

$$-\frac{\beta_0}{\pi}\left\{\left[c\chi\cos\chi - c\sin\chi - 2\chi x\right]\left(\frac{c-x}{x}\right)^{1/2} + \left[c\cos^2\frac{\chi}{2} - x\right]\ln\left|\frac{\tan\frac{\chi}{2} + \left(\frac{c-x}{x}\right)^{1/2}}{\tan\frac{\chi}{2} - \left(\frac{c-x}{x}\right)^{1/2}}\right|\right\}F'(t),$$

$$0 < x < c,$$

(4-45)

$$w_0(x,t) = \frac{U_0\beta_0}{\pi}\left\{2\tan^{-1}\left[\left(\frac{x}{x-c}\right)^{1/2}\tan\frac{\chi}{2}\right] - \chi\left(\frac{x-c}{x}\right)^{1/2}\right\}F(t)$$

$$+\frac{\beta_0}{\pi}\left\{2\left(x - c\cos^2\frac{\chi}{2}\right)\tan^{-1}\left[\left(\frac{x}{x-c}\right)^{1/2}\tan\frac{\chi}{2}\right] + \left[\frac{c}{2}\chi\cos\chi - \frac{c}{2}\sin\chi - \chi x\right]\left(\frac{x-c}{x}\right)^{1/2}\right\}F'(t),$$

$$c < x < \infty,$$

(4-46)

$$\Gamma_0(t) = U_0\beta_0 c(\chi + \sin\chi)F(t) + \frac{\beta_0 c^2}{4}\left[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\right]F'(t),$$

(4-47)

and

$$w_1(x,t) = \frac{U_0\beta_0}{\pi}\left\{2\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right] - \chi\left(\frac{x-c}{x}\right)^{1/2} - \frac{c(\chi+\sin\chi)}{2x^{1/2}(x-c)^{1/2}}\right\}F(t)$$

$$+\frac{\beta_0}{\pi}\left\{2\left[x - c\cos^2\frac{\chi}{2}\right]\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right] - \left[\chi x - \frac{c}{2}(\chi\cos\chi - \sin\chi)\right]\left(\frac{x-c}{x}\right)^{1/2}\right.$$

$$\left.-\frac{c^2}{8}\frac{\left[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\right]}{x^{1/2}(x-c)^{1/2}}\right\}F'(t), \quad c < x < \infty.$$

(4-48)

The basic equations are, from (2-69) to (2-87),

$$\frac{D^2\gamma(x,t)}{Dt^2} = -\frac{U_0^2 c}{2} c_3 \frac{\partial^3 w(x,t)}{\partial x^3}, \quad c < x < \infty, \tag{4-49}$$

and either

$$w(x,t) = -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x} + \frac{U_0\beta_0}{\pi}\left\{2\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right] - \chi\left(\frac{x-c}{x}\right)^{1/2}\right\} F(t)$$

$$+\frac{\beta_0}{\pi}\left\{2\left(x - c\cos^2\frac{\chi}{2}\right)\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right] - \left[\chi x - \frac{c}{2}(\chi\cos\chi - \sin\chi)\right]\left(\frac{x-c}{x}\right)^{1/2}\right\} F'(t),$$

$$c < x < \infty, \tag{4-50}$$

or

$$w(x,t) = -\frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x} + \frac{U_0\beta_0}{\pi}\left\{2\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right] - \chi\left(\frac{x-c}{x}\right)^{1/2}\right.$$

$$\left. - \frac{c}{2}\frac{(\chi + \sin\chi)}{x^{1/2}(x-c)^{1/2}}\right\} F(t) + \frac{\beta_0}{\pi}\left\{2\left[x - c\cos^3\frac{\chi}{2}\right]\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right]\right.$$

$$\left. - \left[\chi x - \frac{c}{2}(\chi\cos\chi - \sin\chi)\right]\left(\frac{x-c}{x}\right)^{1/2} - \frac{c^2}{8}\frac{[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)]}{x^{1/2}(x-c)^{1/2}}\right\} F'(t),$$

$$c < x < \infty, \tag{4-51}$$

with either

$$w(c+,t) = U_0 \beta_0 F(t) + \beta_0 c \sin^2 \frac{\chi}{2} F'(t), \tag{4-52}$$

or

$$\lim_{x \to c+} \frac{1}{\pi} \left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x} = 0, \tag{4-53}$$

and

$$\int_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi = -U_0 \beta_0 c(\chi + \sin\chi) F(t) - \frac{\beta_0 c^2}{4}\Big[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\Big] F'(t) \,. \tag{4-54}$$

Airfoil-quasi-steady and airfoil apparent-mass lift and pitching-moment coefficients are, from (3-11), (3-13), (3-31), and (3-33), using (A-17), (A-20) and (A-22),

$$C_{L_0}(t) = 2\beta_0(\chi + \sin\chi) F(t) + \frac{\beta_0}{2}\Big[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\Big] \frac{cF'(t)}{U_0}, \tag{4-55}$$

$$C_{L_1}(t) = \frac{\beta_0}{2}(\chi - \sin\chi\cos\chi) \frac{cF'(t)}{U_0} + \frac{\beta_0}{2}\left[-\frac{\chi\cos\chi}{2} + \frac{\sin\chi}{3} + \frac{\sin\chi\cos^2\chi}{6}\right] \frac{c^2 F''(t)}{U_0^2}, \tag{4-56}$$

$$C_{M_0}(t) = -\frac{\beta_0}{2}(\chi - \sin\chi\cos\chi) F(t) - \frac{\beta_0}{2}\left[-\frac{\chi\cos\chi}{2} + \frac{\sin\chi}{3} + \frac{\sin\chi\cos^2\chi}{6}\right] \frac{cF'(t)}{U_0}, \tag{4-57}$$

and

$$C_{M_1}(t) = \frac{\beta_0}{12} \sin^3\chi \frac{cF'(t)}{U_0} + \frac{\beta_0}{64}\left[\chi - \frac{\sin\chi}{6}(10\cos\chi - 4\cos^3\chi)\right] \frac{c^2 F''(t)}{U_0^2}. \tag{4-58}$$

If the limit $E \to 0$, or $\chi \to 0$, is taken in this problem, the equations reduce to the pure jet-deflection equations of Section 4.1, provided care is taken at the trailing edge to preserve the jet-deflection angle in the limit of vanishing flap chord. Also, in the limit $E \to 1$, or $\chi \to \pi$, the equations reduce to those of an airfoil in pitching motion about its leading edge, i.e., the limit of $e \to 0$ in the equations of Section 4.3.

4.5 Problem of Airfoil Entering Sharp-Edged Gust

As seen from equation (2-15), there are many possibilities of gust configurations which might be considered. The fundamental transient gust problem which has been considered in the literature is that of a sharp-edged gust of constant upwash amplitude moving over a flat plate airfoil with relative speed U_0/λ, i.e.,

$$W(x,t) = -\bar{W}\,1\left(t - \frac{\lambda x}{U_0}\right). \tag{4-59}$$

For $\lambda = 1$, this is the well-known Küssner (1936) problem, cf. von Kármán and Sears (1938). Miles (1956) generalized it for all positive and negative values of λ, positive values corresponding to gusts proceeding over the airfoil from the leading edge to the trailing edge, and negative values to gusts overtaking the airfoil from behind. The corresponding oscillatory problem is defined by

$$W(x,t) = -\bar{W}\,e^{i\omega\left(t - \frac{\lambda x}{U_0}\right)}. \tag{4-60}$$

This is the Sears (1941) problem of a sinusoidal gust of constant amplitude if $\lambda = 1$, and was extended by Kemp (1952) for a general, complex λ.

For the jet-flapped airfoil, the presence of the jet extending to infinity downstream, along with the assumption, implicit in Section 2.3, that the disturbances of the jet must die out at infinity downstream indicate that the transient, sharp-edged gust moving over a flat plate airfoil with tangential jet from the leading edge towards infinity downstream, $\lambda > 0$, is a reasonable problem to treat.

The downwash distribution of the gust, (4-59), implies that the basic downwash equations for the system, (2-19), rewritten to express the additional downwash which must be induced by the vortex distribution to cancel (4-59) must be

$$w(x,t) + \bar{W}\, 1(t - \frac{\lambda x}{U_0}) = -\frac{1}{2\pi} \oint_0^\infty \frac{\gamma(\xi,t)\, d\xi}{\xi - x}, \quad 0 < x < \infty, \tag{4-61}$$

where $w(x,t)$ is still given by (2-1). The inversion of Section 2.3 again holds for this equation, so the airfoil-quasi-steady expressions may again be calculated in the usual fashion, replacing $w(x,t)$ by $\bar{W}\, 1(t - \frac{\lambda x}{U_0})$ in (2-51), (2-53), (2-54) and (2-87) and using (A-7), (A-34) and (A-35), giving

$$\gamma_0(x,t) = \begin{cases} -\frac{2\bar{W}}{\pi} \ln \left| \dfrac{\left(\frac{\lambda c - U_0 t}{U_0 t}\right)^{1/2} + \left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{\lambda c - U_0 t}{U_0 t}\right)^{1/2} - \left(\frac{c-x}{x}\right)^{1/2}} \right| + \frac{4\bar{W}}{\pi} \left(\frac{c-x}{x}\right)^{1/2} \sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right], & 0 < t < \frac{\lambda c}{U_0} \\ & 0 < x < c, \\ 2\bar{W} \left(\frac{c-x}{x}\right)^{1/2} 1(t - \frac{\lambda c}{U_0}), & \frac{\lambda c}{U_0} < t < \infty \end{cases} \tag{4-62}$$

$$w_0(x,t) = \begin{cases} \dfrac{2\bar{w}}{\pi} \tan^{-1}\left[\left(\dfrac{U_0 t}{\lambda c - U_0 t}\right)^{1/2}\left(\dfrac{x-c}{x}\right)^{1/2}\right] - \dfrac{2\bar{w}}{\pi}\left(\dfrac{x-c}{x}\right)^{1/2}\sin^{-1}\left[\left(\dfrac{U_0 t}{\lambda c}\right)^{1/2}\right], & 0 < t < \dfrac{\lambda c}{U_0}, \\ & c < x < \infty, \\ \bar{w}\left[1 - \left(\dfrac{x-c}{x}\right)^{1/2}\right] \mathbf{1}\left(t - \dfrac{\lambda c}{U_0}\right), & \dfrac{\lambda c}{U_0} < t < \infty. \end{cases}$$

(4-63)

$$\Gamma_0(t) = \begin{cases} 2\bar{w}c\left\{\sin^{-1}\left[\left(\dfrac{U_0 t}{\lambda c}\right)^{1/2}\right] - \dfrac{\left[U_0 t(\lambda c - U_0 t)\right]^{1/2}}{\lambda c}\right\}, & 0 < t < \dfrac{\lambda c}{U_0}, \\ \pi \bar{w} c \, \mathbf{1}\left(t - \dfrac{\lambda c}{U_0}\right), & \dfrac{\lambda c}{U_0} < t < \infty, \end{cases}$$

(4-64)

$$w_1(x,t) = \begin{cases} \dfrac{2\bar{w}}{\pi} \tan^{-1}\left[\left(\dfrac{U_0 t}{\lambda c - U_0 t}\right)^{1/2}\left(\dfrac{x-c}{x}\right)^{1/2}\right] - \dfrac{2\bar{w}}{\pi}\left(\dfrac{x-c}{x}\right)^{1/2}\left[1 + \dfrac{c}{2(x-c)}\right]\sin^{-1}\left[\left(\dfrac{U_0 t}{\lambda c}\right)^{1/2}\right] & \\ \qquad + \dfrac{\bar{w}}{\pi}\left[\dfrac{U_0 t(\lambda c - U_0 t)}{\lambda^2 x(x-c)}\right]^{1/2}, & 0 < t < \dfrac{\lambda c}{U_0}, \\ & c < x < \infty, \\ \bar{w}\left[1 - \left(\dfrac{x-c}{x}\right)^{1/2} - \dfrac{c}{2x^{1/2}(x-c)^{1/2}}\right] \mathbf{1}\left(t - \dfrac{\lambda c}{U_0}\right), & \dfrac{\lambda c}{U_0} < t < \infty, \end{cases}$$

(4-65)

The fundamental equations are, then, using the inversion results of Section 2.3 with (4-59), and (2-69) to (2-87),

$$\frac{D^2\gamma(x,t)}{Dt^2} = -\frac{U_0^2 c}{2} C_J \frac{\partial^3 w(x,t)}{\partial x^3}, \quad c<x<\infty, \tag{4-66}$$

and either

$$w(x,t) = \begin{cases} -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} + \frac{2\bar{w}}{\pi}\tan^{-1}\left[\left(\frac{U_0 t}{\lambda x-U_0 t}\right)^{1/2}\left(\frac{x-c}{x}\right)^{1/2}\right] \\ \qquad -\frac{2\bar{w}}{\pi}\left(\frac{x-c}{x}\right)^{1/2}\sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right], \quad 0<t<\frac{\lambda c}{U_0} \\ \qquad\qquad c<x<\infty, \\ -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} + \bar{w}\left[1-\left(\frac{x-c}{x}\right)^{1/2}\right]1\left(t-\frac{\lambda c}{U_0}\right) \\ \qquad -\bar{w}\,1\left(t-\frac{\lambda x}{U_0}\right), \quad \frac{\lambda c}{U_0}<t<\infty \end{cases} \tag{4-67}$$

or

$$w(x,t) = \begin{cases} -\frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} + \frac{2\bar{w}}{\pi}\tan^{-1}\left[\left(\frac{U_0 t}{\lambda x-U_0 t}\right)^{1/2}\left(\frac{x-c}{x}\right)^{1/2}\right] \\ \qquad -\frac{2\bar{w}}{\pi}\left(\frac{x-c}{x}\right)^{1/2}\left[1+\frac{c}{2(x-c)}\right]\sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right] + \frac{\bar{w}}{\pi}\left[\frac{U_0 t(\lambda c-U_0 t)}{\lambda^2 x(x-c)}\right]^{1/2}, \quad 0<t<\frac{\lambda c}{U_0} \\ \qquad\qquad c<x<\infty, \\ -\frac{1}{2\pi}\oint_c^\infty \left[\frac{\xi(\xi-c)}{x(x-c)}\right]^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi-x} + \bar{w}\left[1-\left(\frac{x-c}{x}\right)^{1/2} - \frac{c}{2x^{1/2}(x-c)^{1/2}}\right]1\left(t-\frac{\lambda c}{U_0}\right) \\ \qquad -\bar{w}\,1\left(t-\frac{\lambda x}{U_0}\right), \quad \frac{\lambda c}{U_0}<t<\infty \end{cases} \tag{4-68}$$

with either

$$w(c+,t) = 0 \, , \tag{4-69}$$

or

$$\lim_{x \to c+} \frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{\gamma(\xi,t)\,d\xi}{\xi - x} = 0 \, , \tag{4-70}$$

and

$$\int_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma(\xi,t)\,d\xi = \begin{cases} -2\bar{w}c\left\{\sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right] - \frac{(U_0 t)^{1/2}(\lambda c - U_0 t)^{1/2}}{\lambda c}\right\}, & 0 < t < \frac{\lambda c}{U_0} \\ -\pi \bar{w} c \, 1\!\left(t - \frac{\lambda c}{U_0}\right) \, , & \frac{\lambda c}{U_0} < t < \infty \, . \end{cases} \tag{4-71}$$

The airfoil-quasi-steady and airfoil apparent-mass lift and pitching-moment coefficients are, calculated in the same fashion as (4-62) to (4-65), from (3-11), (3-13), (3-31) and (3-33), using (A-16) and (A-19)

$$C_{L_0}(t) = \begin{cases} \frac{4\bar{W}}{U_0}\left\{\sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right] - \frac{\left[U_0 t(\lambda c - U_0 t)\right]^{1/2}}{\lambda c}\right\}, & 0 < t < \frac{\lambda c}{U_0} \\ 2\pi \frac{\bar{W}}{U_0} \, , & \frac{\lambda c}{U_0} < t < \infty \, , \end{cases} \tag{4-72}$$

$$C_{L_1}(t) = \begin{cases} \frac{4\bar{W}}{U_0} \frac{\left[U_0 t(\lambda c - U_0 t)\right]^{1/2}}{\lambda^2 c} \, , & 0 < t < \frac{\lambda c}{U_0} \\ 0 \, , & \frac{\lambda c}{U_0} < t < \infty \, , \end{cases} \tag{4-73}$$

$$C_{M_0}(t) = \begin{cases} -\dfrac{\bar{W}}{U_0}\left\{\sin^{-1}\left[\left(\dfrac{U_0 t}{\lambda c}\right)^{1/2}\right] + \dfrac{(2U_0 t-\lambda c)\left[U_0 t(\lambda c-U_0 t)\right]^{1/2}}{\lambda^2 c^2}\right\}, & 0<t<\dfrac{\lambda c}{U_0}, \\ -\dfrac{\pi \bar{W}}{2U_0}, & \dfrac{\lambda c}{U_0}<t<\infty, \end{cases}$$

(4-74)

$$C_{M_1}(t) = \begin{cases} \dfrac{\bar{W}(2U_0 t-\lambda c)\left[U_0 t(\lambda c-U_0 t)\right]^{1/2}}{U_0 \lambda^2 C^2}, & 0<t<\dfrac{\lambda c}{U_0}, \\ 0, & \dfrac{\lambda c}{U_0}<t<\infty. \end{cases}$$

(4-75)

CHAPTER 5 - LIMITING THEORIES OF THE UNSTEADY JET-FLAP THEORY

5.1 Reduction of the Equations to the Classical Unsteady-Airfoil Theory

The classical theory of the unsteady motion of thin airfoils without jets is contained in the above formulation. To recover the fundamental equations of that theory it is sufficient to set $C_J \equiv 0$ formally, provided certain remarks about the equations are made. The vortex distribution behind the airfoil, $\gamma(x,t)$, $c < x < \infty$, and the downwash behind the airfoil, $w(x,t)$, $c < x < \infty$, are convenient functions to treat.

As discussed in Section 2.7, the jet effects appear explicitly only in the dynamic jet-interaction equations (2-69) to (2-72). All the equations derived from the inversion of the kinematic downwash equations, (2-19), do not change if $C_J \equiv 0$, provided the 1/square-root singularity at the trailing edge is understood to be excluded by the Kutta condition. Therefore (2-73) and (2-88) remain the same. In the absence of the jet, it is no longer possible to prescribe the slope of the jet deflection at the trailing edge. Hence, at the trailing edge the downwash is always continuous, (2-57). The circulation in the system is constant by the argument of Section 2.6 using Kelvin's Theorem, so the Wagner integral condition, (2-68), which follows from a kinematic equation, remains the same. Therefore the only equation which formally changes is the jet-interaction equation, (2-71), say. The pressure difference across the x- axis behind the airfoil is proportional to C_J , so in the absence of the jet,

$$\frac{D\gamma(x,t)}{Dt} = 0, \quad c < x < \infty. \tag{5-1}$$

This equation is a statement of Helmholtz' Theorem for the conservation of vorticity of a fluid particle convecting with the stream, cf. Sears (1954). Equation (2-69) becomes, when $C_J \equiv 0$, using (2-7), (2-45) and (2-49),

$$\gamma(x,t) = -\frac{1}{U_0}\frac{d\Gamma(t)}{dt} - \frac{1}{U_0}\frac{\partial}{\partial t}\int_c^x \gamma(\xi,t)\,d\xi, \quad c < x < \infty, \tag{5-2}$$

and, in particular for $x = c+$,

$$\gamma(c+,t) = -\frac{1}{U_0}\frac{d\Gamma(t)}{dt}. \tag{5-3}$$

A clear picture of the physical nature of the problem can be seen from (5-1) and (5-3). As the circulation around the airfoil changes with time, vorticity is shed off the trailing edge in equal and opposite amounts according to (5-3). This vorticity then convects with the free stream, its strength remaining constant by (5-1).

Solution of the problem has become considerably simplified by the disappearance of the downwash from (5-1), because it can be solved immediately for the vortex distribution, without simultaneously solving for the downwash. According to Chapter II of Webster (1955), the solution of the first-order, homogeneous, partial differential equation, (5-1), is that the vortex distribution must be a function of the characteristic variable $t - \frac{x-c}{U_0}$, namely,

$$\gamma(x,t) = \gamma\left(t - \frac{x-c}{U_0}\right), \quad c < x < \infty. \tag{5-4}$$

The functional form of $\gamma\left(t - \frac{x-c}{U_0}\right)$ is then determined by considering the Wagner integral condition, (2-82), as an integral equation for γ,

$$\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} \gamma\left(t - \frac{\xi-c}{U_0}\right) d\xi = -\Gamma_0(t). \tag{5-5}$$

This is the problem considered by von Kármán and Sears (1938), and many others, although the equations were derived there in a more physically direct manner.

Once the vortex distribution in the wake behind the airfoil is known, the downwash behind the airfoil, (2-73), the vortex distribution on the airfoil, (2-88), and the other properties of the airfoil and wake may be found. The lift and pitching moment coefficients may be calculated from (3-20) and (3-37) as discussed in Chapter 3.

A sharp contrast may be drawn now between the classical problem and the jet-flap problem. Classically the vortex strength in the wake behind the airfoil depends, cf. (5-1) and (5-3), on the history of the circulation around the airfoil, or as expressed through the Wagner integral equation (5-5), on the airfoil-quasi-steady circulation. Thus $\Gamma_0(t)$ is the important input to the problem, the actual distribution of downwash on the airfoil having been integrated. Many different chordwise downwash distributions would give the same vortex distribution in the wake if they led to $\Gamma_0(t)$ of the same time dependence. In the jet-flap problem the vortex distribution representing the jet depends not only on the history of the circulation around the airfoil, but also on the details of the downwash distribution at every point of the jet, through the dynamic coupling expressed by (2-69) to (2-72).

5.2 Properties of the Classical Transient Solutions

The two types of transient problems are those in which the motion of the airfoil is initiated at $t=0$, in accordance with (2-14) and (2-16), and those in which the leading edge of the airfoil enters a gust at $t=0$, in accordance with (2-15) and (2-16). In these linearized problems the shed vortex wake behind the airfoil grows in length with rate U_0 after initiation of motion, so, at any time, t, it extends downstream to $x=c+U_0t$. The fundamental equation to be solved, then, is the Wagner integral equation, (5-5), written as

$$\int_c^{c+U_0t}\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma\left(t-\frac{\xi-c}{U_0}\right)d\xi=-\Gamma_0(t)\,. \tag{5-6}$$

For airfoil motion, the fundamental problem is the so-called Wagner (1925) problem, where the airfoil-quasi-steady circulation has step function time dependence,

$$\Gamma_0(t)=U_0c\,\mathbb{1}(t)\,. \tag{5-7}$$

Once the solution of this problem has been found, solutions for other transient $\Gamma_0(t)$ may be found by Duhamel superposition. The application of the Wagner problem to actual airfoil motions of the types considered in Sections 4.2 to 4.4 has been misunderstood at times in the literature. Reference to the particular problems of the above Sections will clarify this assertion. For an airfoil in plunging motion with $F(t) = \frac{U_0 t}{c} \mathbb{1}(t)$, i.e., plunging at a constant speed, (4-14) gives

$$\Gamma_0(t) = \pi h_0 U_0 c \, \mathbb{1}(t), \tag{5-8}$$

a direct application of the Wagner problem. On the other hand, for an airfoil in pitching motion with $F(t) = \mathbb{1}(t)$, i.e., the airfoil being snapped up instantaneously to an angle of incidence, (4-30) gives

$$\Gamma_0(t) = \pi \alpha_0 U_0 c \, \mathbb{1}(t) + \pi \alpha_0 c^2 \left(\tfrac{3}{4} - e\right) \delta(t), \tag{5-9}$$

where the Dirac delta function, $\delta(t)$, is

$$\delta(t) = \frac{d\mathbb{1}(t)}{dt},$$

and has the properties that

$$\left.\begin{aligned} \delta(t) &= 0, \quad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\,dt &= 1 . \end{aligned}\right\} \tag{5-10}$$

but such that (second line above)

It is important to recognize that the $\delta(t)$ term must not be neglected, as it is, for instance, by Robinson and Laurmann (1956) on p. 505. Physically this term represents the limit of the motion, which, although occurring in a vanishingly small time interval, occurs at such a rapid rate that the result is a finite rotation through the angle α_0. Considering this term as such a limit, it is clear that it will have an important effect, particularly for small times. Therefore, only for an airfoil pivoted about the three-quarter chord point, $e = 3/4$, is the Wagner problem directly applicable to the airfoil snapped to angle of incidence. Likewise, the trailing-edge flap snapped to an angle of deflection with $F(t) = 1(t)$ gives an airfoil-quasi-steady circulation, (4-47), which contains both $1(t)$ and $\delta(t)$,

$$\Gamma_0(t) = U_0 \beta_0 c (\chi + \sin\chi) 1(t) + \frac{\beta_0 c^2}{4}\left[\chi(1 - 2\cos\chi) + \sin\chi(2 - \cos\chi)\right] \delta(t). \tag{5-11}$$

The fundamental gust problem, that of a sharp-edged gust of constant amplitude $\bar{w}$ moving with a speed U_0/λ relative to the airfoil, $\lambda > 0$, is formulated in Section 4.5. The airfoil-quasi-steady circulation is, from (4-64),

$$\Gamma_0(t) = \begin{cases} 2\bar{w}c\left\{\sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right] - \frac{\left[U_0 t(\lambda c - U_0 t)\right]^{1/2}}{\lambda c}\right\}, & 0 < \frac{U_0 t}{\lambda} < c \\ \pi \bar{w} c\, 1\left(t - \frac{\lambda c}{U_0}\right), & c < \frac{U_0 t}{\lambda} < \infty . \end{cases} \tag{5-12}$$

Sharp-edged gusts overtaking the airfoil from behind, $\lambda < 0$, can also be treated, cf. Miles (1956), but cannot be extended readily to the jet-flap case and will not be treated here.

A method of solution, valid for all transient $\Gamma_0(t)$, will now be outlined and certain results noted. Upon the transformation

$$t - \frac{\xi - c}{U_0} = \tau ,$$

(5-13)

(5-6) becomes

$$\int_0^t \left(\frac{t-\tau+\frac{c}{U_0}}{t-\tau} \right)^{1/2} \gamma(\tau)\, d\tau = -\frac{1}{U_0} \Gamma_0(t).$$

(5-14)

In a similar fashion the lift coefficient due to the wake, $C_{L_2}(t)$, is, from (3-21),

$$C_{L_2}(t) = \frac{1}{U_0} \int_0^t \frac{\gamma(\tau)\, d\tau}{\left[(t-\tau)(t-\tau+\frac{c}{U_0})\right]^{1/2}} ,$$

(5-15)

and finally, from (2-67), (2-68), (2-7) and (5-13),

$$\int_0^t \gamma(\tau)\, d\tau = -\frac{1}{U_0} \Gamma(t) .$$

(5-16)

Sears (1940) observed that the integrals in (5-14) to (5-16) are of the convolution type, and used Laplace transforms as a convenient technique for treating them. Defining the Laplace transform of a function by

$$\bar{F}(p) \equiv \int_0^\infty e^{-pt} F(t)\, dt \quad , \ \mathcal{R}(p) > 0 ,$$

(5-17)

A method of solution, valid for all transient $\Gamma_0(t)$, will now be outlined and certain results noted. Upon the transformation

$$t - \frac{\xi - c}{U_0} = \tau ,$$

(5-13)

(5-6) becomes

$$\int_0^t \left(\frac{t - \tau + \frac{c}{U_0}}{t - \tau} \right)^{1/2} \gamma(\tau) d\tau = - \frac{1}{U_0} \Gamma_0(t) .$$

(5-14)

In a similar fashion the lift coefficient due to the wake, $C_{L_2}(t)$, is, from (3-21),

$$C_{L_2}(t) = \frac{1}{U_0} \int_0^t \frac{\gamma(\tau) d\tau}{\left[(t-\tau)(t - \tau + \frac{c}{U_0}) \right]^{1/2}} ,$$

(5-15)

and finally, from (2-67), (2-68), (2-7) and (5-13),

$$\int_0^t \gamma(\tau) d\tau = - \frac{1}{U_0} \Gamma(t) .$$

(5-16)

Sears (1940) observed that the integrals in (5-14) to (5-16) are of the convolution type, and used Laplace transforms as a convenient technique for treating them. Defining the Laplace transform of a function by

$$\bar{F}(p) \equiv \int_0^\infty e^{-pt} F(t) dt \quad , \quad \mathcal{R}(p) > 0 ,$$

(5-17)

Sears (1940) showed that

$$\bar{\gamma}(p) = -\frac{2e^{-\frac{cp}{2U_0}}\bar{\Gamma}_0(p)}{c\left[K_0\left(\frac{cp}{2U_0}\right)+K_1\left(\frac{cp}{2U_0}\right)\right]}\,, \tag{5-18}$$

$$\bar{\Gamma}(p) = \frac{2U_0}{cp}\frac{e^{-\frac{cp}{2U_0}}\bar{\Gamma}_0(p)}{\left[K_0\left(\frac{cp}{2U_0}\right)+K_1\left(\frac{cp}{2U_0}\right)\right]}\,, \tag{5-19}$$

and

$$\bar{C}_{L_2}(p) = -\frac{2}{U_0 c}\frac{K_0\left(\frac{cp}{2U_0}\right)\bar{\Gamma}_0(p)}{\left[K_0\left(\frac{cp}{2U_0}\right)+K_1\left(\frac{cp}{2U_0}\right)\right]}\,, \tag{5-20}$$

where $K_0\left(\frac{cp}{2U_0}\right)$ and $K_1\left(\frac{cp}{2U_0}\right)$ are the Modified Bessel Functions of the Second Kind. From the definition of the airfoil-quasi-steady lift coefficient, (3-11), and using the airfoil apparent-mass lift coefficient appropriate to the particular motion considered, (3-13), the lift coefficient, (3-20), is upon transformation,

$$\bar{C}_L(p) = \bar{C}_{L_1}(p) + \frac{2}{U_0 c}\frac{K_1\left(\frac{cp}{2U_0}\right)\bar{\Gamma}_0(p)}{\left[K_0\left(\frac{cp}{2U_0}\right)+K_1\left(\frac{cp}{2U_0}\right)\right]}\,. \tag{5-21}$$

For the Wagner problem, (5-7),

$$\bar{\Gamma}_0(p) = \frac{U_0 c}{p}\,, \tag{5-22}$$

and the well-known results emerge. Upon inversion of the Laplace transforms, for instance, the small- and large-time lift limits are

$$\lim_{t \to 0+} \frac{C_L(t)}{C_L(\infty)} = \lim_{t \to 0+} \frac{C_{L_1}(t)}{C_L(\infty)} + \frac{1}{2} 1(t) + O(t), \tag{5-23}$$

and

$$\lim_{t \to \infty} \frac{C_L(t)}{C_L(\infty)} = \lim_{t \to \infty} \frac{C_{L_1}(t)}{C_L(\infty)} + 1 - \frac{c}{2U_0 t} + o\left(\frac{1}{t}\right). \tag{5-24}$$

It is of interest to note in the application of (5-23) to the plunging case that the lift coefficient as $t \to 0+$ is, using (4-23),

$$\lim_{t \to 0+} \frac{C_L(t)}{C_L(\infty)} = \frac{1}{4} \delta(t) + \frac{1}{2} 1(t) + O(t). \tag{5-25}$$

Therefore, for small time, airfoil apparent-mass lift is dominant. This follows for all types of airfoil motion, since $\Gamma_0(t)$, and hence $C_{L_0}(t)$ and $C_{L_2}(t)$ have, for small time, the same time dependence as the airfoil downwash, cf. (5-20) and (3-11), while $C_{L_1}(t)$ has time dependence like the time derivative of the airfoil downwash, cf. (3-13). That is, $C_{L_0}(t)$ and $C_{L_2}(t)$ are, for small time, proportional to the velocity of the airfoil, while $C_{L_1}(t)$ is proportional to its acceleration.

In view of the nature of the jet-flap equations, some of the local properties of the flow are of interest. From (5-18) and (5-22), it can be shown that, upon taking the inverse Laplace transforms,

$$\gamma\left(t-\frac{x-c}{U_0}\right) = -\frac{4}{\pi}\left[\frac{c}{U_0\left(t-\frac{x-c}{U_0}\right)}\right]^{1/2} I\left(t-\frac{x-c}{U_0}\right) + o\left(t-\frac{x-c}{U_0}\right)^{1/2}, \quad t-\frac{x-c}{U_0} \ll 1. \tag{5-26}$$

This 1/square-root singularity in the vortex distribution at the trailing edge of the wake vortex sheet is the mathematical representation of the "starting vortex" observed by Prandtl in his striking pictures of these flows, cf. Prandtl and Tietjens (1934). The downwash behind the airfoil can be calculated from (2-73) or (2-85), in principle, once $\gamma\left(t-\frac{x-c}{U_0}\right)$ has been found. Hobbs (1957) made such calculations in order to estimate the loading on the tail of an aircraft whose wing is flying through a gust field. Without details of the calculations, Hobbs gave curves for the Wagner problem which indicate that the downwash is finite as $x \to c+U_0t-$ on the wake vortex sheet, has a square-root singularity as $x \to c+U_0t+$ off the sheet, and then dies off at infinity like $1/x^2$. In analogy with the behavior of the downwash induced by the singularity of the vortex distributions near the leading edge of a flat-plate airfoil, this behavior at $x = c+U_0t$ is consistent with (5-26).

As $t \to 0$, the end of the shed vortex sheet moves in towards the trailing edge. Also, inversion of (5-19) with (5-22) shows that $\Gamma(t) \sim t^{1/2}$ as $t \to 0$. In the absence of circulation around the airfoil, there must be flow around the trailing edge in the first instant, $t = 0+$, after initiation of the motion, in order to satisfy the downwash boundary condition on the airfoil. Presence of the jet for $C_J > 0$, with the requirement that it always remain tangential to the trailing edge of the airfoil, (2-57), will therefore greatly modify the flow pattern near $x = c + U_0 t$ by preventing flow around the trailing edge in the first instant, and thus for all later time.

If the response to $\Gamma_0(t) = \frac{c}{U_0}\delta(t)$ is desired in order to treat the problems of an airfoil snapped up to incidence and the impulsive deflection of a trailing-edge flap, (5-9) and (5-11), $\bar{\Gamma}_0(p) = \frac{c}{U_0}$ may be substituted formally into (5-18) to (5-21). However, consideration of $\bar{\gamma}(p)$ for this input indicates that, for $p \to \infty$, $\bar{\gamma}(p) \sim p^{1/2}$. Inversion of this transform cannot be carried out because it is too singular, corresponding, as seen by differentiation of (5-26), to $\gamma(\tau) \sim \tau^{-3/2}$ for small τ. This implies that although $\bar{\Gamma}_0(p) = \frac{c}{U_0}$ can be substituted formally into (5-18) to (5-21), the integrals (5-14) to (5-16) and these transforms fail to exist. Therefore the cases involving $\Gamma_0(t) \sim \delta(t)$ are too singular to permit a solution. However, the step-function inputs are themselves idealizations of physically realizable inputs, so these physical inputs can be treated by direct application of the results (5-18) to (5-21) with their $\bar{\Gamma}_0(p)$.

For the sharp-edged gust problem, as defined by (5-12), the transform of $\Gamma_0(t)$ may be found to be, cf. Sears (1940), or Miles (1956),

$$\bar{\Gamma}_0(p) = \frac{\pi \bar{w} c}{p} e^{-\frac{\lambda c p}{2U_0}} \left\{ I_0\left(\frac{\lambda c p}{2U_0}\right) - I_1\left(\frac{\lambda c p}{2U_0}\right) \right\}, \tag{5-27}$$

where $I_0\left(\frac{\lambda c p}{2U_0}\right)$ and $I_1\left(\frac{\lambda c p}{2U_0}\right)$ are the Modified

Bessel Functions of the First Kind. Substitution of (5-27) into (5-18) to (5-21) gives among its results the fascinating one noticed by Sears (1940) that, for the Küssner problem, $\lambda = 1$,

$$\bar{C}_L(p) = \frac{4\pi \bar{W}}{cp^2} e^{-\frac{cp}{2U_0}} \frac{1}{\left[K_0\left(\frac{cp}{2U_0}\right) + K_1\left(\frac{cp}{2U_0}\right)\right]} , \tag{5-28}$$

which has the same dependence as $\bar{\Gamma}(p)$ for the Wagner problem, using (5-19) and (5-22). The lift coefficient for an airfoil flying through a stationary gust depends on t in the same fashion as the circulation in the Wagner problem.

The vortex distribution at the trailing edge of the shed vortex sheet is, upon inversion of (5-18) with (5-27),

$$\gamma\left(t-\frac{x-c}{U_0}\right) = -\frac{\bar{W}U_0}{\lambda^{1/2}c}\left(t-\frac{x-c}{U_0}\right)\mathbf{1}\left(t-\frac{x-c}{U_0}\right) + o\left(t-\frac{x-c}{U_0}\right), \quad t-\frac{x-c}{U_0} \ll 1, \tag{5-29}$$

vanishing as $x \to c + U_0 t -$ on the sheet, hence continuous at $x = c + U_0 t$. Hobbs (1957) also gave curves of downwash for the gust case with several $\lambda > 0$. He found that, consistent with the continuity in the vortex strength at the end of the shed vortex sheet, the downwash is continuous there. However a discontinuity in downwash appears at $x = c + U_0\left(t - \frac{\lambda c}{U_0}\right)$, which is the point in the wake corresponding to the arrival of the gust at the trailing edge of the airfoil. By the arguments of p.33 , this discontinuity in downwash implies a logarithmic singularity in the vortex distribution at that point. That the discontinuity arises at this point rather than at $x = c + U_0 t$, which corresponds to the arrival of the gust at the leading edge of the airfoil, can be seen by considering $\Gamma_b(t)$, (5-12) . This function of time is continuous and has a continuous slope at $t = 0$, but is continuous with a discontinuous slope at $t = \frac{\lambda c}{U_0}$, the discontinuity in slope leading to the downwash discontinuity.

The presence of the jet for $C_J > 0$ in this problem will modify the flow pattern near $x = c + U_0 (t - \frac{\lambda c}{U_0})$, since the jet is required to remain tangential to the airfoil at the trailing edge, (4-69), preventing a velocity discontinuity when the gust arrives there.

5.3 Properties of the Classical Solutions for Steady-State Oscillations

The appropriate functions of time for steady-state oscillations of airfoils have been discussed in Section 2.2 Airfoils in infinite sinusoidal gusts have been treated by Sears (1941) and Kemp (1952), but will be omitted here. For oscillations beginning at $t = -\infty$, the shed vortex wake behind the airfoil will be infinite in length at the finite times under consideration.

The fundamental problem for airfoil motion is the Theodorsen (1934) problem, in which the airfoil-quasi-steady circulation is exponential in time; i.e.,

$$\Gamma_0(t) = U_0 c \, e^{i\omega t}, \; \mathcal{I}(\omega) \leq 0, \tag{5-30}$$

so the governing Wagner integral equation is

$$\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\gamma(\xi,t)\,d\xi = -U_0 c\, e^{i\omega t}. \tag{5-31}$$

This problem was shown by Sears (1940) to be related to the Wagner problem of Section 5.2 in the fashion typical of linear systems; i.e., the steady-state response to an exponential input like (5-30) is found from the Laplace transform of the unit-step-function response by replacing the transform variable p by $(i\omega)$, and multiplying the response by $(i\omega)e^{i\omega t}$. It is of certain interest to go through the solution in the manner of von Kármán and Sears (1938). The application of the Theodoresen problem to the airfoil motions considered in Chapter 4 presents no difficulties of the type encountered in the Wagner problem. Considering the airfoil in plunging motion, for $F(t) = e^{i\omega t}$, $\Gamma_0(t)$ is

$$\Gamma_0(t) = i\pi h_0 c^2 \omega e^{i\omega t}, \tag{5-32}$$

and is a direct application of the Theodorsen problem. The airfoil oscillating in pitch and the airfoil with oscillating trailing-edge flap, both with $F(t) = e^{i\omega t}$ are also direct applications of the Theodorsen problem, since, from (4-30) and (4-47),

$$\Gamma_0(t) = \pi U_0 \alpha_0 c\, e^{i\omega t} + i\pi\alpha_0 c^2\omega\left(\tfrac{3}{4}-e\right) e^{i\omega t}, \tag{5-33}$$

and

$$\Gamma_0(t) = U_0\beta_0 c\,(\chi+\sin\chi)\, e^{i\omega t} + i\,\frac{\beta_0\omega c^2}{4}\left[\chi(1-2\cos\chi)+\sin\chi(2-\cos\chi)\right]e^{i\omega t}, \tag{5-34}$$

respectively.

For steady-state oscillations, all the functions have time dependence $\sim e^{i\omega t}$, therefore

$$\gamma\left(t-\frac{x-c}{U_0}\right) = \hat{\gamma}(\omega)\, e^{i\omega\left(t-\frac{x-c}{U_0}\right)}, \tag{5-35}$$

and the Wagner integral equation, (5-30), becomes

$$\hat{\gamma}(\omega)\int_c^\infty \left(\frac{\xi}{\xi-c}\right)^{1/2} e^{-\frac{i\omega(\xi-c)}{U_0}}\, d\xi = U_0 c\ , \quad \mathcal{J}(\omega) \le 0. \tag{5-36}$$

The integral in (5-36) exists for $\mathcal{J}(\omega) < 0$, but for pure oscillations, $\mathcal{J}(\omega) = 0$, the integral is finitely oscillatory. A common approach to this, cf., Robinson and Laurmann (1956), has been to solve the problem for $\mathcal{J}(\omega) < 0$ and then argue by analytic continuation that the result is valid for $\mathcal{J}(\omega) = 0$, too. von Kármán and Sears (1938) treated this in another way. They considered as the basic equation not the Wagner integral equation, (5-5), but the equation (5-3), from which (5-5) can be derived if $\Delta\phi(\infty, t)$ vanishes. Substituting $\frac{\partial}{\partial t}$ of (2-67) into (5-3), and using (5-30) and (5-35) gives

$$\hat{\gamma}(\omega)\left\{1 + \frac{i\omega}{U_0}\int_c^\infty \left[\left(\frac{\xi}{\xi-c}\right)^{1/2} - 1\right] e^{-\frac{i\omega(\xi-c)}{U_0}}\, d\xi\right\} = -\,i\omega c. \tag{5-37}$$

The integral on the lefthand side of this exists for $\mathcal{J}(\omega) \le 0$. To evaluate that integral, von Kármán and Sears (1938) used the integral representation of the Modified Bessel Function of the Second Kind, cf. Jahnke and Emde (1945), with identically equal integrals added and subtracted, i.e.,

$$K_0\left(\frac{i\omega}{2U_0}\right) = \int_0^\infty \frac{e^{-\frac{i\omega}{U_0}\left(\xi+\frac{c}{\xi}\right)}}{\xi^{1/2}(\xi-c)^{1/2}}\,d\xi - \int_0^\infty \frac{e^{-\frac{i\omega}{U_0}\left(\xi+\frac{c}{\xi}\right)}}{\xi+\frac{c}{\xi}}\,d\xi + \int_{\omega c/2U_0}^\infty \frac{e^{-it}}{t}\,dt, \quad \mathcal{I}(\omega) \le 0. \tag{5-38}$$

Differentiating this with respect to $i\omega c/2U_0$, and using the identity

$$K_0'(z) = -K_1(z), \tag{5-39}$$

the integral in (5-37) can be evaluated to give

$$\gamma(\omega) = -i\omega c\, e^{-\frac{i\omega c}{2U_0}} \mathcal{S}\left(\frac{\omega c}{2U_0}\right), \quad \mathcal{I}(\omega) \le 0, \tag{5-40}$$

where

$$\mathcal{S}\left(\frac{\omega c}{2U_0}\right) \equiv \frac{2U_0}{i\omega c\left[K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)\right]}, \quad \mathcal{I}(\omega) \le 0, \tag{5-41}$$

is the Sears Function, and was tabulated by Kemp (1952). The circulation, also from (2-67), is

$$\Gamma(t) = U_0 c\, e^{i\omega\left(t-\frac{c}{2U_0}\right)} \mathcal{S}\left(\frac{\omega c}{2U_0}\right), \quad \mathcal{I}(\omega) \le 0. \tag{5-42}$$

The circulatory lift coefficient may be calculated from (3-20), using (5-35), there being no difficulties with convergence, giving

$$C_{L_0}(t) + C_{L_2}(t) = 2\,\mathcal{C}\left(\frac{\omega c}{2U_0}\right) e^{i\omega t}, \quad \mathcal{I}(\omega) \le 0, \tag{5-43}$$

where

$$C\left(\frac{\omega c}{2U_0}\right) \equiv \frac{K_1\left(\frac{i\omega c}{2U_0}\right)}{K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)} \quad , \quad \mathcal{J}(\omega) \leq 0, \tag{5-44}$$

is the Theodorsen Function, tabulated in detail by Luke and Dengler (1951).

Of particular interest for the jet-flap analysis is the shed vortex distribution in the wake, given by (5-35) and (5-40), i.e.,

$$\gamma(x,t) = -i\omega c\, C\left(\frac{\omega c}{2U_0}\right) e^{i\omega\left(t - \frac{x-c}{U_0}\right)} \quad , \mathcal{J}(\omega) \leq 0, \tag{5-45}$$

which has the above-mentioned oscillatory properties at infinity if $\mathcal{J}(\omega) = 0$. The downwash behind an airfoil performing steady-state, purely oscillatory, $\mathcal{J}(\omega) = 0$, plunging motion was calculated by Lapin, Crookshanks and Hunter (1952). Their results were

$$\begin{aligned} w(x,t) = i\omega c h_0 e^{i\omega t} \Bigg\{ & 1 - \left(\frac{x-c}{x}\right)^{1/2} + \frac{K_0\left(\frac{i\omega c}{2U_0}\right)}{\left[K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)\right]} \left(\frac{x-c}{x}\right)^{1/2} \\ & - \frac{e^{-\frac{i\omega}{U_0}\left(x-\frac{c}{2}\right)} \ln\left|\frac{2}{c}\left(x-\frac{c}{2}\right) + \frac{2}{c}x^{1/2}(x-c)^{1/2}\right|}{\left[K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)\right]} \\ & - \frac{\pi}{2} \frac{e^{-\frac{i\omega}{U_0}\left(x-\frac{c}{2}\right)}}{\left[K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)\right]} \frac{x^{1/2}(x-c)^{1/2}}{x-\frac{c}{2}} He^{(2)}\left[\frac{c}{2\left(x-\frac{c}{2}\right)}, \frac{\omega\left(x-\frac{c}{2}\right)}{U_0}\right] \Bigg\}, \quad c < x < \infty \end{aligned} \tag{5-46}$$

where

$$He^{(2)}\left(\alpha, \frac{\beta}{\alpha}\right) \equiv \frac{2i}{\pi\alpha}\int_0^{\beta} K_0(i\rho)\, e^{i\frac{\rho}{\alpha}}\, d\rho \ , \tag{5-47}$$

is a function defined, discussed and tabulated by Schwarz (1943). A few limiting cases of (5-46), for which simple expressions may be found, were given in their paper. For $\omega \to 0$, (5-46) reduces to the airfoil-quasi-steady downwash distribution, $w_0(x,t)$ of (4-13). In the limit of $\omega \to \infty$,

$$w(x,t) \sim \begin{cases} \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\omega c}{2U_0}\right)^{1/2} e^{-\frac{i\omega x}{U_0} - i\frac{\pi}{4}} + o(\omega), & x > c + \frac{U_0}{\omega}, \\ 1 - \left[i\,\frac{2}{\sqrt{c}}\left(\frac{\omega c}{2U_0}\right) e^{-\frac{i\omega c}{U_0}} + o(\omega)\right](x-c)^{1/2} + o(x-c)^{1/2}, & x \ll c + \frac{U_0}{\omega}. \end{cases}$$

(5-48)

The latter expression of (5-48), although not given by them, may be found using equation (30*) of Schwarz (1943). Away from the trailing edge, the downwash increases like $\omega^{1/2}$ as $\omega \to \infty$. Finally, for arbitrary ω, but $x \to \infty$, Lapin, Crookshanks and Hunter (1952) gave - they have neglected a minus sign -

$$w(x,t) \sim \frac{\pi h_0 \omega c\, e^{i\omega\left(t - \frac{x-c}{U_0}\right)}}{K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)}, \quad x \to \infty.$$

(5-49)

This result may be shown to be the downwash induced at a point due to a doubly infinite distribution of vortices of strength

$$\gamma(x,t) = -\frac{2\pi i h_0 \omega c\, e^{i\omega\left(t - \frac{x-c/2}{U_0}\right)}}{K_0\left(\frac{i\omega c}{2U_0}\right) + K_1\left(\frac{i\omega c}{2U_0}\right)}, \quad -\infty < x < \infty,$$

(5-50)

This is the vortex distribution shed by an airfoil in plunging oscillations as seen by application of (5-45) to this case, using (5-32). The downwash and vortex distributions in the wake oscillate both in space and time far downstream. For $\mathfrak{I}(\omega) < 0$, these results damp to zero as $x \to \infty$, and it is also clear to see from these results why $\mathfrak{I}(\omega) > 0$ must be excluded from consideration.

5.4 Reduction of the Equations to the Steady Jet-Flap Theory

The steady jet-flap theory is also included in the above formulation. It will be the limiting solution of the equations for $t \to \infty$ in the transient case if, $\lim_{t \to \infty} F(t) = 1$. For steady oscillations, it is the limiting solution as $\omega \to 0$. Formally, however, the steady equations can be recovered simply by setting

$$\left.\begin{aligned} &\frac{\partial}{\partial t} \equiv 0, \\ &F(t) \equiv 1, \\ &F'(t) \equiv 0, \end{aligned}\right\} \tag{5-51}$$

and

Using (2-1) and (2-7), these reduce (2-69), (2-73), (2-78) and (2-79), for example, to the set

$$\gamma(x) = -\frac{c C_J}{2} w'(x) , \quad c < x < \infty , \tag{5-52}$$

and

$$w(x) = -\frac{1}{2\pi}\left(\frac{x-c}{x}\right)^{1/2}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi)\,d\xi}{\xi-x} + w_0(x)\,,\quad c < x < \infty, \tag{5-53}$$

with either

$$w(c+) = w_0(c) + U_0\tau_0 \tag{5-54}$$

or

$$\lim_{x\to c+}\frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2}\frac{\gamma(\xi)\,d\xi}{\xi-x} = -2U_0\tau_0\,. \tag{5-55}$$

The assumptions (5-51) imply that $t\to\infty$ much faster than $x\to\infty$, therefore the circulation in the system is no longer zero, and equations (2-81) to (2-83) and (2-85) to (2-88) are not valid. Likewise the lift and pitching-moment coefficients are not given by (3-38) and (3-39). The lift coefficient, using (5-5) in (3-10), (3-11) and (3-13), and using (5-52) in the integral over $\gamma(\xi)$, becomes

$$C_L = C_{L_0} - \frac{C_J}{U_0}\int_c^{\infty}\left(\frac{\xi}{\xi-c}\right)^{1/2} w'(\xi)\,d\xi\,. \tag{5-56}$$

The pitching-moment coefficient in similar fashion reduces from (3-29) to

$$C_M = C_{M_0} + C_{L_0} - (\tfrac{1}{2}+a)\,C_L$$

$$- \frac{C_J}{U_0 c}\int_c^{\infty} \xi\left[\left(\frac{\xi}{\xi-c}\right)^{1/2}-1\right] w'(\xi)\,d\xi \;+\; C_J\left[\frac{w(c+)}{U_0} - \frac{Y(c+)}{c} + b\right].$$

(5-57)

Eliminating $\gamma(x)$ between (5-52) and (5-53), the single equation treated by Spence in I and II

$$w(x) = w_0(x) + \frac{cC_J}{4\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{w'(\xi)\,d\xi}{\xi-x}\,, \quad c < x < \infty,$$

(5-58)

with either

$$w(c+) = w_0(c) + U_0\tau_0\,,$$

(5-59)

or

$$\lim_{x\to c+} \frac{1}{\pi}\left(\frac{x-c}{x}\right)^{1/2} \oint_c^{\infty} \left(\frac{\xi}{\xi-c}\right)^{1/2} \frac{w'(\xi)\,d\xi}{\xi-x} = \frac{4U_0\tau_0}{cC_J}\,,$$

(5-60)

is found.

In I, Spence assumed in the jet-deflection problem (Section 4.1 above) that

$$w(x) = \frac{4U_0\tau_0}{\pi C_J}\left\{2\left[1-\left(\frac{c}{x}\right)^{1/2}\right]\ln\left[\left(\frac{x}{c}\right)^{1/2}-1\right]-2\left[1+\left(\frac{c}{x}\right)^{1/2}\right]\ln\left[\left(\frac{x}{c}\right)^{1/2}+1\right]\right\}$$

$$-\frac{4U_0\tau_0}{C_J}\sum_{n=0}^{\infty}\frac{A_n}{4n^2-1}\left[2\cos\frac{\phi}{2}\cos n\phi+4n\sin\frac{\phi}{2}\sin n\phi\right],$$

(5-61)

where

$$x = \frac{c}{\cos^2\frac{\phi}{2}}.$$

(5-62)

The first term was introduced to satisfy (5-60), the series being regular at $x = c$. Having substituted this expression into (5-58), the resulting equation was approximately solved by a collocation scheme, i.e., the first N coefficients, A_n, were found by satisfying the equation exactly at N points. This approximation converged rapidly, and detailed results for $N = 9$ were found. Spence also solved, in I, the incidence problem (Section 4.3 above with assumptions (5-51)) in the same fashion, having assumed

$$w(x) = -\frac{4U_0\alpha_0}{C_J}\sum_{n=0}^{\infty}\frac{B_n}{4n^2-1}\left[2\cos\frac{\phi}{2}\cos n\phi+4n\sin\frac{\phi}{2}\sin n\phi\right].$$

(5-63)

Finally, Spence (1958) solved the jet-augmented-flap problem (Section 4.4 above with assumptions (5-51)) by collocation, where

$$w(x) = -\frac{4U_0\beta_0}{C_J}\sum_{n=0}^{\infty}\frac{D_n}{4n^2-1}\left[2\cos\frac{\phi}{2}\cos n\phi+4n\sin\frac{\phi}{2}\sin n\phi\right].$$

(5-64)

Later, in II, Spence solved (5-58) to (5-60) analytically, in the jet-deflection and incidence problems, for $\mu < 1$, where

$$\mu \equiv \frac{C_J}{4} . \tag{5-65}$$

This was done by transforming the downwash distribution by

$$w(x) = U_0 \tau_0 \left(\frac{c}{x}\right)^{1/2} f_\tau\left(\frac{x-c}{\mu c}\right) + U_0 \alpha_0 \left[1 + 2\left(\frac{\mu c}{x}\right)^{1/2} f_\alpha\left(\frac{x-c}{\mu c}\right)\right] . \tag{5-66}$$

A detailed discussion of this transformation and its implications will be made in Section 6.1 for the full equations of the unsteady problem. It suffices here to say that substitution of (5-66) into (5-58), with $w_0(x)$ given by (4-29) (under the assumptions (5-51)), led to an iterative expansion of $f_\tau(x')$ and $f_\alpha(x')$ of the forms

$$\begin{aligned} f_\tau(x') = {} & f_0(x') + \mu f_1(x') + \frac{1}{\pi^{3/2}}(\mu^2 \ln \mu) f_2(x') \\ & + \mu^2\left[f_3(x') - \frac{1 + \ln(4e^{\gamma})}{\pi^{3/2}} f_2(x')\right] + o(\mu^2), \end{aligned} \tag{5-67}$$

and

$$f_\alpha(x') = \left[1 + \frac{\mu}{2\pi}\left(\ln\frac{\mu}{4e^{\gamma}} + 2\right)\right] F_2(x') - \frac{\pi^{1/2}}{4}\mu \int_0^{x'} f_1(\xi)\, d\xi + o(\mu), \tag{5-68}$$

where γ is Euler's constant. Equations for the functions $f_0(x')$, $f_1(x')$, $f_2(x')$ and $f_3(x')$ were found to be

$$\left.\begin{aligned} L\,f_0(x') &= 0 \ , \quad 0<x'<\infty \\ f_0(0) &= 1, \end{aligned}\right\} \tag{5-69}$$

$$\left.\begin{aligned} L\,f_1(x') &= -\frac{1}{2\pi}\,\mathrm{P}\!\!\int_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{f_0(\xi')\,d\xi'}{\xi'-x'} \ , \quad 0<x'<\infty \\ f_1(0) &= 0 , \end{aligned}\right\} \tag{5-70}$$

$$\left.\begin{aligned} L\,f_2(x') &= -\tfrac{1}{2}\,x'^{1/2} \ , \quad 0<x'<\infty \\ f_2(0) &= 0, \end{aligned}\right\} \tag{5-71}$$

$$\left.\begin{aligned} L\,f_3(x') &= -\frac{1}{2\pi}\,\mathrm{P}\!\!\int_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{f_1(\xi')-x'f_0(\xi')}{\xi'-x'}\,d\xi' \ , \quad 0<x'<\infty \\ f_3(0) &= 0, \end{aligned}\right\} \tag{5-72}$$

where

$$L\,f(x') \equiv f(x') - \frac{1}{\pi}\,\mathrm{P}\!\!\int_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{f'(\xi')\,d\xi'}{\xi'-x'} \ . \tag{5-73}$$

These equations were then solved by a Mellin-Transform procedure, a technique to be discussed in Section 6.2. The lift coefficients found from this solution agreed very well with the numerical solution of I, even up to $\mu = 1$.

CHAPTER 6 - "BOUNDARY-LAYER" NATURE OF THE PROBLEM: TRANSFORMATION OF THE EQUATIONS TO "BOUNDARY-LAYER" COORDINATES

There are many problems in applied mechanics which may be typified as "boundary-layer" problems. The name arises from the mathematical similarity of these problems to Prandtl's (1904) "grenzschicht," or, as it has been translated, "boundary-layer," in the flow of viscous fluids around bodies. Properties and examples of "boundary-layer" problems have been discussed by Carrier (1953), Friedrichs (1955), and others.

Carrier (1953) states three requisite criteria for consideration of a problem as of "boundary-layer" type. First, the coefficient of the highest-order derivative appearing in the equations of the problem should be very small compared to $O(1)$. Second, the other important terms in the equations should have coefficients of $O(1)$. Finally, the domain of the problem in the coordinate system chosen must be characterized by lengths of $O(1)$.

Spence first realized in II that the equations for a jet-flapped airofil in steady motion were of "boundary-layer" type, provided the jet momentum coefficient, C_J, was sufficiently small. In most practical applications proposed for the jet-flap the required C_J are within the limits of the restriction. Spence then made the transformation to "boundary-layer" coordinates - equation (5-66) of Section 5.4 - and solved the steady problem within that framework. For the unsteady problem, Spence made a similar approach in III. It is important to discuss the unsteady problem in detail with respect to Carrier's criteria in order to determine the validity of this approach.

First, the highest-order derivative appearing in the problem is always on the right-hand side of the particular dynamic interaction equation, (2-69)-(2-72), used. This derivative is multiplied by μ, (5-65), so if $\mu = o(1)$ the highest-order derivative is $o(1)$. As mentioned in Section 2.7 this is the only explicit appearance of μ in the equations of the problem, hence Carrier's second criterion is also met. Considering the problem in the form of the single equation in $\gamma(x,t)$ or $\Delta\phi(x,t)$, (2-90)-(2-93), these criteria are again seen to be met. Therefore these two criteria apply in the unsteady problem exactly as in the steady one.

The characteristic-length considerations of the domain in the unsteady problem are somewhat different from those of the steady one. In the steady problem, Spence's numerical solution in I clearly indicated that the downwash and vortex distributions die off rapidly within a few chord lengths behind the airfoil. Thus the chord length, c, is a significant characteristic length. In the discussion of the classical unsteady transient solutions in Section 5.2, however, discontinuities in the downwash distributions and the associated singularities in the shed vortex distributions were noted. In particular, $U_0 t$ and $U_0(t - \frac{\lambda c}{U_0})$ are significant lengths for airfoil motion and gust penetration, respectively, in addition to the airfoil chord, c.
Similarly, equation (5-45) indicates that for steady oscillations

the wave length, $\frac{2\pi U_0}{\omega c}$, of the shed vortex distribution is an important length. Therefore, for times, $\frac{U_0 t}{c}$, of $O(1)$ or smaller in the transient case, or for frequencies, $\frac{\omega c}{2\pi U_0}$, of $O(1)$ or larger in the oscillating case, the airfoil chord, c, is again a sufficient characteristic length since $U_0 t$, $U_0(t-\frac{\lambda c}{U_0})$ and $\frac{2\pi U_0}{\omega c}$ are of the same order as c. Furthermore, (2-100) and (2-102) show that the downwash and vortex distributions die off at infinity even more rapidly than for the steady case, for $x > U_0 t$. For times so large, or frequencies so small, that $U_0 t$, $U_0(t-\frac{\lambda c}{U_0})$, and $\frac{2\pi U_0}{\omega c}$ are much greater than c, a "boundary-layer" based on c alone is inadequate, since important effects would be well outside the layer. Spence, in III, treated the region near $x = c + U_0 t$ as of primary importance for large times in the transient case, along with the region near the trailing edge.

Realizing the time or frequency limitations just discussed, the appropriate transformation of the coordinates for small μ is, following III,

$$x = c(1+\mu x'), \tag{6-1}$$

$$t = \frac{c}{U_0}\mu t', \tag{6-2}$$

and

$$\omega = \frac{U_0}{c}\frac{\gamma}{\mu}. \tag{6-3}$$

The leading edge, $x=0$, transforms to $x' = -\frac{c}{\mu}$, so for vanishing μ , the airfoil becomes semi-infinite in length. In conjunction with the coordinate transformation, it is convenient to follow Spence's procedure of II and III and transform the dependent variables representing the jet by

$$\Delta\phi(x,t) = \mu\, 2U_0 c A \left(\frac{c}{x}\right)^{1/2} f(x',t') , \tag{6-4}$$

$$\gamma(x,t) = 2U_0 A \left(\frac{c}{x}\right)^{1/2} g(x',t') , \tag{6-5}$$

$$Y(x,t) = B(x,t) + \mu c A \left(\frac{c}{x}\right)^{1/2} h(x',t') , \tag{6-6}$$

and

$$w(x,t) = \frac{DB(x,t)}{Dt} + U_0 A \left(\frac{c}{x}\right)^{1/2} k(x',t') . \tag{6-7}$$

The time dependence transforms by

$$F(t) = \mathcal{F}(t') . \tag{6-8}$$

Determination of the non-dimensional constant A , and the function $B(x,t)$ will be discussed after transformation of the equations.

The equations relating downwash and jet ordinate, (2-1), and potential difference and vortex distribution, (2-7), transform to

$$k(x',t') = \frac{\partial h(x',t')}{\partial t'} + \frac{\partial h(x',t')}{\partial x'} - \mu \frac{h(x',t')}{2(1+\mu x')} \,, \tag{6-9}$$

and

$$g(x',t') = \frac{\partial f(x',t')}{\partial x'} - \mu \frac{f(x',t')}{2(1+\mu x')} \,. \tag{6-10}$$

Trnasforming the various foms of the equations of the jet, main-stream dynamic interaction, (2-69) to (2-72), gives

$$\frac{\partial f(x',t')}{\partial t'} + \frac{\partial f(x',t')}{\partial x'} - \mu \frac{f(x',t')}{2(1+\mu x')} = -\frac{\partial^2 h(x',t')}{\partial x'^2} + \mu \frac{1}{1+\mu x'} \frac{\partial h(x',t')}{\partial x'} - \mu^2 \frac{3h(x',t')}{4(1+\mu x')^2}, \quad 0 < x' < \infty, \tag{6-11}$$

$$\frac{\partial^2 f(x',t')}{\partial t'^2} + 2\frac{\partial^2 f(x',t')}{\partial t' \partial x'} + \frac{\partial^2 f(x',t')}{\partial x'^2} - \mu \frac{1}{1+\mu x'}\left[\frac{\partial f(x',t')}{\partial t'} + \frac{\partial f(x',t')}{\partial x'}\right] + \mu^2 \frac{3f(x',t')}{4(1+\mu x')^2} = -\frac{\partial^2 k(x',t')}{\partial x'^2} + \mu \frac{1}{1+\mu x'} \frac{\partial k(x',t')}{\partial x'} - \mu^2 \frac{3k(x',t')}{4(1+\mu x')^2}, \quad 0 < x' < \infty, \tag{6-12}$$

$$\frac{\partial g(x',t')}{\partial t'} + \frac{\partial g(x',t')}{\partial x'} - \mu \frac{g(x',t')}{2(1+\mu x')} = -\frac{\partial^3 h(x',t')}{\partial x'^3} + \mu \frac{3}{2(1+\mu x')} \frac{\partial^2 h(x',t')}{\partial x'^2} - \mu^2 \frac{9}{4(1+\mu x')^2} \frac{\partial h(x',t')}{\partial x'} + \mu^3 \frac{15 h(x',t')}{8(1+\mu x')^3}, \quad 0 < x' < \infty, \tag{6-13}$$

and

$$\frac{\partial^2 g(x',t')}{\partial t'^2} + 2\frac{\partial^2 g(x',t')}{\partial t'\partial x'} + \frac{\partial^2 g(x',t')}{\partial x'^2} - \mu\frac{1}{1+\mu x'}\left[\frac{\partial g(x',t')}{\partial t'} + \frac{\partial g(x',t')}{\partial x'}\right] + \mu^2\frac{3g(x',t')}{4(1+\mu x')^2} =$$

$$-\frac{\partial^3 h(x',t')}{\partial x'^3} + \mu\frac{3}{2(1+\mu x')}\frac{\partial^2 h(x',t')}{\partial x'^2} - \mu^2\frac{9}{4(1+\mu x')}\frac{\partial h(x',t')}{\partial x'} + \mu^3\frac{15h(x',t')}{8(1+\mu x')^3},$$

$$0 < x' < \infty, \qquad (6\text{-}14)$$

provided

$$\frac{\partial^2 B(x,t)}{\partial x^2} \equiv 0 \, . \qquad (6\text{-}15)$$

The two forms of the downwash equation, (2-73) and (2-85), transform to

$$h(x',t') = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{g(\xi',t')\,d\xi'}{\xi'-x'} + \frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right], \quad 0 < x' < \infty, \qquad (6\text{-}16)$$

and

$$h(x',t') = -\frac{1}{\pi}\oint_0^\infty \left(\frac{\xi'}{x'}\right)^{1/2} \frac{g(\xi',t')\,d\xi'}{\xi'-x'} + \frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right], \quad 0 < x' < \infty. \qquad (6\text{-}17)$$

The various forms of the trailing-edge boundary conditions, (2-76) to (2-80), become

$$h(0+,t') = \frac{\gamma(c-,t) - B(c+,t)}{\mu c A} \, , \qquad (6\text{-}18)$$

$$\frac{\partial h(0+,t')}{\partial x'} = \frac{1}{A}\left[\frac{\partial \gamma(c-,t)}{\partial x} - \frac{\partial B(c+,t)}{\partial x}\right] + \frac{\gamma(c-,t) - B(c+,t)}{2c}$$

$$+ \frac{\tau_0\,\xi(t')}{A} \, , \qquad (6\text{-}19)$$

$$k(0+,t') = \frac{1}{U_0 A}\left[w_0(c+,t) - \frac{DB(c+,t)}{Dt}\right] + \frac{\tau_0 \mathcal{F}(t')}{A}\,, \tag{6-20}$$

and

$$\lim_{x'\to 0+} \frac{1}{\pi}\oint_0^{\infty}\left(\frac{x'}{s'}\right)^{1/2}\frac{g(s',t')\,ds'}{s'-x'} = -\frac{\tau_0 \mathcal{F}(t')}{A}\,, \tag{6-21}$$

while the Wagner integral condition, (2-82), becomes

$$\int_0^{\infty}\frac{g(s',t')\,ds'}{s'^{1/2}} = -\frac{\Gamma_0'(t')}{2U_0 C A \mu^{1/2}}\,, \tag{6-22}$$

where

$$\Gamma_0(t) \cdot \Gamma_0'(t')\,. \tag{6-23}$$

The lift coefficient, (3-15), and the pitching-moment coefficient, (3-35), become, under this transformation,

$$C_L(t') = C_{L_1}(t') - 4\mu^{1/2} A \frac{d}{dt'}\int_0^{\infty} s'^{1/2} g(s',t')\,ds'\,, \tag{6-24}$$

and

$$C_m(t') = C_{m_0}(t') + C_{m_1}(t') + \frac{1-2a}{2}C_{L_1}(t') - 3\left(1-\frac{4a}{3}\right)\mu^{1/2}A\int_0^{\infty} s'^{1/2} g(s',t')\,ds'$$
$$+ 4\mu b - 2\mu^{3/2} A \frac{d}{dt'}\int_0^{\infty} s'^{3/2} g(s',t')\,ds' + 4\mu^{3/2} A\int_0^{\infty} s'^{1/2} g(s',t')\,ds'\,. \tag{6-25}$$

The function $B(x,t)$ can be determined in each case so that the factor $\left(\frac{x}{c}\right)^{1/2} = (1+\mu x)^{1/2}$ no longer appears in the terms $\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right]$ and $\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_1(x,t) - \frac{DB(x,t)}{Dt}\right]$ of (6-16) and (6-17). This gives a first-order partial differential equation for $B(x,t)$ which can be integrated subject to the conditions that

$$Y(c-,t) - B(c+,t) = 0$$

and

$$\frac{\partial Y(c-,t)}{\partial x} - \frac{\partial B(c+,t)}{\partial x} = 0\,,$$

or

$$w_0(c+,t) - \frac{DB(c+,t)}{Dt} = 0\,,$$

which make the boundary conditions, (6-18) to (6-21), dependent only upon the jet deflection, $\tau_0\,\mathfrak{F}(t')$. The constant A is chosen to simplify the form of the final equations.

The equations given above are exact. For small μ, however, neglecting terms of $O(\mu)$ enables the equations to be greatly simplified. The chief simplifications are in the kernels of the integrals of the downwash equations, (6-16) and (6-17), and, as will be seen below for the particular problems of Chapter 4, in the inhomogeneous terms on the right-hand sides of these equations. These simplifications are equivalent, in the first approximation, to neglecting $\mu x'$ with respect to 1, which is valid only near $x' = 0$, i.e., within a "boundary layer" near the trailing edge, and to neglecting $\mu t'$ with respect to 1, valid only near $t' = 0$, i.e., within a "boundary layer" near the time origin. Higher approximations in

μ may be obtained in the unsteady problem by expanding the solution in terms of μ as Spence did for the steady problem in II, as mentioned in Section 5.4.

The equations of the various problems of Chapter 4 will now be written in the transformed coordinates.

Jet-Deflection Problem: The input to this problem is the boundary condition on the jet slope at the trailing edge. Therefore, choosing

$$A = \tau_0 \tag{6-26}$$

and

$$B(x,t) = 0 \, , \tag{6-27}$$

the important boundary conditions (6-19), (6-20) or (6-21) become

$$\frac{\partial h(0+,t')}{\partial x'} = k(0+,t') = -\lim_{x' \to 0+} \frac{1}{\pi} \oint_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{g(\xi',t')\,d\xi'}{\xi' - x'} = \mathcal{F}(t') . \tag{6-28}$$

This is the problem formulated by Spence in III.

Plunging-Aifroil Problem: Here, A and $B(x,t)$ are found to be

$$A = -\frac{h_0}{2\mu^{3/2}} \tag{6-29}$$

and

$$B(x,t) = h_0 c\, F(t).$$

(6-30)

The important terms are, then, in the downwash equations (6-16) and (6-17),

$$\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right] = 2\mu\, x'^{1/2}\, \frac{d\mathcal{F}(t')}{dt'}\,, \quad 0 < x' < \infty$$

(6-31)

and

$$\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_1(x,t) - \frac{DB(x,t)}{Dt}\right] = \frac{\mathcal{F}(t')}{x'^{1/2}} + 2\mu\, x'^{1/2}\, \frac{d\mathcal{F}(t')}{dt'}\,, \quad 0 < x' < \infty;$$

(6-32)

in the boundary conditions, (6-18) to (6-21),

$$h(0+,t') = \frac{\partial h(0+,t')}{\partial x'} = k(0+,t') = -\lim_{x' \to 0+} \frac{1}{\pi}\oint_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{g(\xi',t')\,d\xi'}{\xi' - x'} = 0 \;;$$

(6-33)

and in the Wagner integral, (6-22),

$$\int_0^\infty \frac{g(\xi',t')\,d\xi'}{\xi'^{1/2}} = \pi\, \frac{d\mathcal{F}(t')}{dt'}\,.$$

(6-34)

The leading inhomogeneous term in μ in the equations is the right-hand side of the Wagner integral condition if the downwash equation, (6-16) with (6-31), is used, or $\mathcal{F}(t')/x'^{1/2}$ if the form of the downwash equation incorporating the Wagner condition, (6-17) with (6-32), is used. The second term in (6-32), i.e., the leading term in (6-31), is proportional to μ because it is a higher power of x'. It would be necessary, then, to carry the solution to the second approximation in μ to account completely for the airfoil-quasi-steady downwash. The first approximation will give, however, the leading term in μ for the lift and pitching-moment coefficients.

<u>Pitching-Airfoil Problem</u>: Here it is found that

$$A = \begin{cases} -\dfrac{(\frac{3}{4}-e)\alpha_0}{2\mu^{3/2}}, & e \neq 3/4 \\[2ex] -\dfrac{\alpha_0}{2\mu^{1/2}}, & e = 3/4 \end{cases} \tag{6-35}$$

and

$$B(x,t) = \alpha_0 (x - ec) F(t). \tag{6-36}$$

The important downwash terms are, from (6-16) and (6-17),

$$\frac{1}{U_0 A}\left(\frac{x'}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right] = \begin{cases} \mu\, 2\dfrac{(\frac{3}{2}-e)}{(\frac{3}{4}-e)}\, x'^{1/2}\dfrac{d\mathcal{F}(t')}{dt'} + \mu^2 \dfrac{2}{(\frac{3}{4}-e)}\left[x'^{1/2}\mathcal{F}(t') + x'^{3/2}\dfrac{d\mathcal{F}(t')}{dt'}\right], & e \neq 3/4 \\ & 0 < x' < \infty, \\[2ex] \dfrac{3}{2}\, x'^{1/2}\dfrac{d\mathcal{F}(t')}{dt'} + \mu\, 2\left[x'^{1/2}\mathcal{F}(t') + x'^{3/2}\dfrac{d\mathcal{F}(t')}{dt'}\right], & e = 3/4, \end{cases} \tag{6-37}$$

and

$$\frac{1}{U_0 A}\left(\frac{x'}{c}\right)^{1/2}\left[w_1(x,t)-\frac{\partial B(x,t)}{\partial t}\right]=\begin{cases}\dfrac{1}{x'^{1/2}}\dfrac{d\mathcal{F}(t')}{dt'}+\dfrac{\mu}{(\frac{3}{4}-e)}\left[\dfrac{\mathcal{F}(t')}{x'^{1/2}}+2(\tfrac{3}{4}-e)x'^{1/2}\dfrac{d\mathcal{F}(t')}{dt'}\right] \\ \qquad +\mu^2\dfrac{2}{(\frac{3}{4}-e)}\left[x'^{1/2}\mathcal{F}(t')+x'^{3/2}\dfrac{d\mathcal{F}(t')}{dt'}\right],\ e\neq 3/4 \\ \qquad\qquad\qquad\qquad 0<x'<\infty; \\ \left[\dfrac{\mathcal{F}(t')}{x'^{1/2}}+\dfrac{3x'^{1/2}}{2}\dfrac{d\mathcal{F}(t')}{dt'}\right]+\mu^2\left[x'^{1/2}\mathcal{F}(t')+x'^{3/2}\dfrac{d\mathcal{F}(t')}{dt'}\right],\ e=3/4\end{cases}$$

(6-38)

the boundary conditions, (6-18) to (6-21), are

$$h(0+,t')=\frac{\partial h(0+,t')}{\partial x'}=k(0+,t')=-\lim_{x'\to 0+}\frac{1}{\pi}\int_0^\infty\left(\frac{x'}{\xi'}\right)^{1/2}\frac{g(\xi',t')d\xi'}{\xi'-x'}=0\ ;$$

(6-39)

and the Wagner integral condition, (6-22), is

$$\int_0^\infty\frac{g(\xi',t')d\xi'}{\xi'^{1/2}}=\begin{cases}\pi\dfrac{d\mathcal{F}(t')}{dt'}+\mu\dfrac{\pi}{(\frac{3}{4}-e)}\mathcal{F}(t'),\ e\neq 3/4 \\ \\ \pi\,\mathcal{F}(t'),\ e=3/4.\end{cases}$$

(6-40)

The equations of the first approximation in μ are thus identical to those for the plunging airfoil, if $e\neq 3/4$. The higher approximations in μ satisfy different equations, however, including the $e=3/4$ case which is really a second approximation. Here the analysis must be carried to the third approximation in μ to account for all the airfoil-quasi-steady terms. The first approximation is now clearly restricted to very small t', since certain terms in (6-37) and (6-38) are higher-order in μ because $F(t)$ is higher-order in μ than $dF(t)/dt$.

Blown-Flap Problem: As the downwash equations (4-50) and (4-51) stand for this case, they are inconvenient for calculation; however the arctangent may be expanded like

$$\tan^{-1}\left[\tan\frac{\chi}{2}\left(\frac{x}{x-c}\right)^{1/2}\right] = \frac{\pi}{2} - \cot\frac{\chi}{2}\left(\frac{x-c}{x}\right)^{1/2} + O\left[\cot\frac{\chi}{2}\left(\frac{x-c}{x}\right)^{1/2}\right]^3, \tag{6-41}$$

valid if $\cot\frac{\chi}{2} << \left(\frac{x}{x-c}\right)^{1/2} \simeq \frac{1}{\mu^{1/2} x'^{1/2}}$. Provided χ is not too small, use of this expansion is consistent with the neglect of $\mu x'$ compared to 1 in the first approximation. Therefore a first approximation should be obtainable in this manner. Furthermore, when $\chi = \pi$ the equations agree with the pitching problem for $e = 0$. In this approximation,

$$A = -\frac{\beta_0\left[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\right]}{8\mu^{1/2}\pi} \tag{6-42}$$

and

$$B(x,t) = \beta_0\,(x - c\cos^2\frac{\chi}{2})\,F(t). \tag{6-43}$$

The downwash terms including $O(\mu)$ only are, from (6-16) and (6-17),

$$\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right] = \mu\,\frac{8\left[\chi + \frac{3}{2}\sin\chi - \frac{1}{2}\chi\cos\chi\right]x'^{1/2}}{\left[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\right]}\,\frac{d\mathcal{F}(t')}{dt'}, \quad 0 < x' < \infty, \tag{6-44}$$

and

$$\frac{1}{U_0 A}\left(\frac{x'}{c}\right)^{1/2}\left[w_1(x,t) - \frac{DB(x,t)}{Dt}\right] = \frac{1}{x'^{1/2}}\frac{d\mathcal{F}(t')}{dt'} + \mu\frac{4}{\left[\lambda(1-2\cos\lambda)+\sin\lambda(2-\cos\lambda)\right]}\left\{\frac{\lambda+\sin\lambda}{x'^{1/2}}\mathcal{F}(t')\right.$$

$$\left. + 2\left(\lambda+\tfrac{3}{2}\sin\lambda-\tfrac{1}{2}\lambda\cos\lambda\right)x'^{1/2}\,\frac{d\mathcal{F}(t')}{dt'}\right\}, \quad 0 < x' < \infty \quad ; \tag{6-45}$$

the boundary conditions, (6-18) to (6-21), are

$$h(0+,t') = \frac{\partial h(0+,t')}{\partial x'} = k(0+,t') = -\lim_{x'\to 0+}\frac{1}{\pi}\oint_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2}\frac{g(\xi',t')\,d\xi'}{\xi'-x'} = 0 \quad ; \tag{6-46}$$

and the Wagner integral condition, (6-22), becomes

$$\int_0^\infty \frac{g(\xi',t')\,d\xi'}{\xi'^{1/2}} = \pi\frac{d\mathcal{F}(t')}{dt'} + \mu\frac{4\pi(\lambda+\sin\lambda)\mathcal{F}(t')}{\left[\lambda(1-2\cos\lambda)+\sin\lambda(2-\cos\lambda)\right]} + o(\mu). \tag{6-47}$$

The equations for the first approximation in μ are again identical with those of the plunging and pitching airfoils. For the first approximation alone, the expansion (6-41) is not used, since the leading term in (6-45) comes from the airfoil-quasi-steady circulation which is exact. Again the approximations involved clearly limit the validity of the first approximation to very small t' .

<u>Sharp-Edged-Gust Problem:</u> The downwash equations (4-67) and (4-69) are also not in convenient form as they stand. An expansion for small t , and small $x-c$, may be made. From such an expansion, it is found that

$$A = -\frac{2\mu \overline{W}}{3\pi U_0 \lambda^{3/2}} \tag{6-48}$$

and

$$B(x,t) = 0 \quad . \tag{6-49}$$

The downwash terms including $O(\mu)$ only are, from (6-16) and (6-17),

$$\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_0(x,t) - \frac{DB(x,t)}{Dt}\right] = -\mu x'^{1/2} t'^{3/2} + O(\mu^2), \quad 0 < x' < \infty , \tag{6-50}$$

and

$$\frac{1}{U_0 A}\left(\frac{x}{c}\right)^{1/2}\left[w_1(x,t) - \frac{DB(x,t)}{Dt}\right] = \frac{t'^{3/2}}{x'^{1/2}} + \mu\left[\frac{3t'^{5/2}}{10\lambda x'^{1/2}} - x'^{1/2} t'^{3/2}\right] + O(\mu^2), \; 0 < x' < \infty; \tag{6-51}$$

the boundary conditions are, (6-18)-(6-21),

$$h(0+,t') = \frac{\partial h(0+,t')}{\partial x'} = k(0+,t') = -\lim_{x' \to 0+} \frac{1}{\pi} \oint_0^\infty \left(\frac{x'}{\xi'}\right)^{1/2} \frac{g(\xi',t')d\xi'}{\xi' - x'} = 0 \quad ; \tag{6-52}$$

and the Wagner integral condition, (6-22), is

$$\int_0^\infty \frac{g(\xi',t')d\xi'}{\xi'^{1/2}} = \pi t'^{3/2} + \mu \frac{3\pi t'^{5/2}}{10\lambda} + O(\mu^2) . \tag{6-53}$$

The equations for the first approximation here are the same as for the above three cases if their time dependence is explicitly made to be $\mathcal{F}(t') = t'^{3/2}$. The approximations have the same implications as to their validity as above.

From consideration of the above five problems, it is clear that in the first approximation for small μ and small t', there are two fundamental unsteady jet-flap problems to be solved.

The first of these is the jet-deflection problem first treated by Spence in III. The problem, in terms of the downwash, $k(x',t')$, and the jet vortex distribution, $g(x',t')$, for example, is governed, using (6-4) to (6-23) and dropping the primes for convenience, by

$$\frac{\partial^2 g(x,t)}{\partial t^2} + 2\frac{\partial^2 g(x,t)}{\partial t\,\partial x} + \frac{\partial^2 g(x,t)}{\partial x^2} = -\frac{\partial^3 k(x,t)}{\partial x^3}, \quad 0 < x < \infty, \tag{6-54}$$

and

$$k(x,t) = \frac{\partial h(x,t)}{\partial t} + \frac{\partial h(x,t)}{\partial x} = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x}, \quad 0 < x < \infty, \tag{6-55}$$

or

$$k(x,t) = \frac{\partial h(x,t)}{\partial t} + \frac{\partial h(x,t)}{\partial x} = -\frac{1}{\pi}\oint_0^\infty \left(\frac{\xi}{x}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x}, \quad 0 < x < \infty, \tag{6-56}$$

with either

$$h(0+,t) = \mathcal{F}(t), \tag{6-57}$$

or

$$\lim_{x\to 0+} \frac{1}{\pi} \oint_0^{\infty} \left(\frac{x}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x} = -\mathcal{F}(t) \; ,$$

(6-58)

and

$$\int_0^{\infty} \frac{g(\xi,t)\,d\xi}{\xi^{1/2}} = 0 \; .$$

(6-59)

The problem can also be formulated in terms of the jet ordinate, $h(x,t)$, and the potential difference, $f(x,t)$, using the first approximations to (6-9) and (6-10),

$$k(x,t) = \frac{\partial h(x,t)}{\partial t} + \frac{\partial h(x,t)}{\partial x}$$

(6-60)

and

$$g(x,t) = \frac{\partial f(x,t)}{\partial x} \; ,$$

(6-61)

and the appropriate equations from (6-11)-(6-23).

To the first approximation in μ, then, the lift and pitching-moment coefficients, (6-24) and (6-25), for the jet-deflection problem become

$$C_L(t) = -4\mu^{1/2}\tau_0 \frac{d}{dt}\int_0^{\infty} \xi^{1/2} g(\xi,t)\,d\xi$$

(6-62)

and

$$C_M(t) = -3\left(1-\frac{4a}{3}\right)\mu^{1/2}\tau_0 \frac{d}{dt}\int_0^\infty \xi^{1/2} g(\xi,t)\,d\xi \, .$$

(6-63)

It immediately follows that, to this order, the center of pressure is at the three-quarter-chord point, $a = 3/4$. For steady flow Spence found numerically in I (and unpublished calculations using the results of II confirm it) that the center of pressure, to first order in μ, is at the half-chord point. The validity of the "boundary-layer" transformation for small time only is pointed up here, since the center of pressure remains at the three-quarter chord point for all time in this approximation.

The second fundamental problem in the first approximation, corresponding to the airfoil-motion problems and the sharp-edged gust problem (if $\mathcal{F}(t) = t^{3/2}$) is, in terms of $k(x,t)$ and $g(x,t)$,

$$\frac{\partial^2 g(x,t)}{\partial t^2} + 2\frac{\partial^2 g(x,t)}{\partial t\,\partial x} + \frac{\partial^2 g(x,t)}{\partial x^2} = -\frac{\partial^3 k(x,t)}{\partial x^3}, \quad 0 < x < \infty,$$

(6-64)

and

$$k(x,t) = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x}, \quad 0 < x < \infty,$$

(6-65)

or

$$k(x,t) = -\frac{1}{\pi}\oint_0^\infty \left(\frac{\xi}{x}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x} + \frac{1}{x^{1/2}}\frac{d\mathcal{F}(t)}{dt}, \quad 0 < x < \infty,$$

(6-66)

with either

$$k(0+,t) = 0,$$

(6-67)

or

$$\lim_{\chi \to 0^+} \frac{1}{\pi} \int_0^\infty \left(\frac{\chi}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - \chi} = 0 ,$$

(6-68)

and

$$\int_0^\infty \frac{g(\xi,t)\,d\xi}{\xi^{1/2}} = \pi \frac{d\,\mathcal{F}(t)}{dt} .$$

(6-69)

The lift and pitching-moment coefficients, (6-24) and (6-25), are

$$C_L(t) = C_{L_1}(t) - 4\mu^{1/2} A \frac{d}{dt} \int_0^\infty \xi^{1/2} g(\xi,t)\,d\xi ,$$

(6-70)

and

$$C_M(t) = C_{M_0}(t) + C_{M_1}(t) + \frac{1-2a}{2} C_{L_1}(t) - 3\left(1 - \frac{4a}{3}\right)\mu^{1/2} A \frac{d}{dt} \int_0^\infty \xi^{1/2} g(\xi,t)\,d\xi ,$$

(6-71)

where $C_{L_1}(t)$, $C_{M_0}(t)$, and $C_{M_1}(t)$ are appropriate to the particular problem.

The solutions of the two fundamental, small $-\mu, -t$, problems are closely related. To explore this relationship fully in the transient case, it is necessary to consider Laplace transforms taken on the time variable. Using the definition of the Laplace transform, (5-17), and the initial conditions that the flow is undisturbed prior to initiation of the transient motion, expressed by

$$g(x,0) = \frac{\partial g(x,0)}{\partial t} = \frac{\partial g(x,0)}{\partial x} = \mathcal{F}(0) = 0 ,$$

(6-72)

the Laplace-transformed equations of the jet-deflection and airfoil-motion problems, (6-54) to (6-59) and (6-64) to (6-69), are

$$p^2 \bar{g}^\tau(x;p) + 2p\frac{\partial \bar{g}^\tau(x;p)}{\partial x} + \frac{\partial^2 \bar{g}^\tau(x;p)}{\partial x^2} = -\frac{\partial^3 \bar{k}^\tau(x;p)}{\partial x^3} , \quad 0 < x < \infty ,$$

(6-73)

$$\bar{k}^\tau(x;p) = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{\bar{g}^\tau(\xi;p)d\xi}{\xi - x} , \quad 0 < x < \infty ,$$

(6-74)

$$\bar{k}^\tau(0+;p) = -\lim_{x\to 0+}\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{\bar{g}^\tau(\xi;p)}{\xi - x}d\xi = \bar{\mathcal{F}}(p) ,$$

(6-75)

$$\int_0^\infty \xi^{-1/2}\bar{g}^\tau(\xi;p)d\xi = 0 ,$$

(6-76)

and

$$p^2 \bar{g}^a(x;p) + 2p\frac{\partial \bar{g}^a(x;p)}{\partial x} + \frac{\partial^2 \bar{g}^a(x;p)}{\partial x^2} = -\frac{\partial^3 \bar{k}^a(x;p)}{\partial x^3} , \quad 0 < x < \infty ,$$

(6-77)

$$\bar{k}^a(x;p) = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{\bar{g}^a(\xi;p)d\xi}{\xi - x} , \quad 0 < x < \infty ,$$

(6-78)

$$\bar{k}^a(0;p) = -\lim_{x\to 0+}\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{\bar{g}^a(\xi;p)d\xi}{\xi - x} = 0 ,$$

(6-79)

$$\int_0^\infty \xi^{-1/2}\bar{g}^a(\xi;p)\,d\xi = \pi p \bar{\mathfrak{F}}(p) \, ,$$

(6-80)

where $\bar{g}^\tau(x;p)$ and $\bar{k}^\tau(x;p)$ are the transformed vortex distribution and downwash for jet deflection; $\bar{g}^a(x;p)$ and $\bar{k}^a(x;p)$ those for airfoil motion, and $\bar{\mathfrak{F}}(p)$ the transform of $\mathfrak{F}(t)$.

Differentiating (6-77) and (6-78) with respect to x gives

$$p^2\frac{\partial\bar{g}^a(x;p)}{\partial x} + 2p\frac{\partial^2\bar{g}^a(x;p)}{\partial x^2} + \frac{\partial^3\bar{g}^a(x;p)}{\partial x^3} = -\frac{\partial^4\bar{k}^a(x;p)}{\partial x^4}, \quad 0<x<\infty,$$

(6-81)

and

$$\frac{\partial\bar{k}^a(x;p)}{\partial x} = -\frac{1}{\pi}\oint_0^\infty\left(\frac{\xi}{x}\right)^{1/2}\frac{\partial\bar{g}^a(\xi;p)}{\partial\xi}\frac{d\xi}{\xi-x}$$

$$= -\frac{1}{\pi}\oint_0^\infty\left(\frac{x}{\xi}\right)^{1/2}\frac{\partial\bar{g}^a(\xi;p)}{\partial\xi}\frac{d\xi}{\xi-x} - \frac{x^{1/2}}{\pi}\int_0^\infty\xi^{-1/2}\frac{\partial\bar{g}^a(\xi;p)}{\partial\xi}\,d\xi, \quad 0<x<\infty,$$

(6-82)

where the identity $\frac{1}{\xi-x} = \frac{x}{\xi(\xi-x)} + \frac{1}{\xi}$ has been used. By inspection of (6-81) and (6-82), the deflection solution of (6-73), (6-74) and (6-76) suggests that the airfoil-motion solution may be written as

$$\frac{\partial\bar{g}^a(x;p)}{\partial x} = A(p)\,\bar{g}^\tau(x;p)$$

(6-83)

and

$$\frac{\partial \bar{k}^a(x;p)}{\partial x} = A(p)\,\bar{k}^\tau(x;p) .$$

(6-84)

Integrating these with respect to x, and using the trailing-edge boundary condition, (6-79), and the condition that the vortex distribution vanishes at infinity downstream gives

$$\bar{g}^a(x;p) = \frac{\pi}{2}\, p\bar{\mathfrak{F}}(p) \int_x^\infty \bar{g}^\tau(\xi;p)\,d\xi \Big/ \int_0^\infty \xi^{1/2}\bar{g}^\tau(\xi;p)\,d\xi$$

(6-85)

and

$$\bar{k}^a(x;p) = -\frac{\pi}{2}\, p\bar{\mathfrak{F}}(p) \int_0^x \bar{k}^\tau(\xi;p)\,d\xi \Big/ \int_0^\infty \xi^{1/2}\bar{g}^\tau(\xi;p)\,d\xi ,$$

(6-86)

where $A(p)$ has been evaluated from the Wagner integral condition, (6-80), upon integration by parts and use of (6-83). To complete the solution, the integral required for the lift and pitching-moment coefficients, (6-70) and (6-71), is, upon integration by parts,

$$\int_0^\infty \xi^{1/2}\bar{g}^a(\xi;p)\,d\xi = \frac{\pi}{3}\, p\bar{\mathfrak{F}}(p) \int_0^\infty \xi^{3/2}\bar{g}^\tau(\xi;p)\,d\xi \Big/ \int_0^\infty \xi^{1/2}\bar{g}^\tau(\xi;p)\,d\xi .$$

(6-87)

Therefore, once the jet-deflection solution has been found, the airfoil-motion solution follows.

The Laplace-transformed equations of the transient problems also hold for the problems of steady-state oscillations, i.e., where

$$g(x,t) = \hat{g}(x;\nu)\, e^{i\nu t}$$

(6-88)

and

$$k(x,t) = \hat{k}(x;\nu)\, e^{i\nu t} ,$$

(6-89)

provided the transform variable p is replaced by $(i\nu)$ and $\bar{\mathcal{F}}(p)$ by 1. Thus the relations (6-85) to (6-87) with the proper modifications, also hold for steady-state oscillations in jet deflection and airfoil motion.

CHAPTER 7 - CRITIQUE OF ATTEMPTED SOLUTIONS IN "BOUNDARY-LAYER" COORDINATES

7.1 Critique of Spence's Solution of the Jet-Deflection Problem for Small Time.

Spence, in Section 3 of III, approximated to the equations of the transient jet-deflection problem for small time after a unit-step-function input, and then obtained a "solution" of the simplified equations. In the present section it will be shown that this "solution" is incorrect, and that no valid solution of Spence's approximate equations can be found.

The approximation of III is to neglect, for small time, the derivatives with respect to x compared to the derivatives with respect to t in the convective derivatives which appear in the problem; i.e., it is assumed that

$$\frac{\partial}{\partial x} = o\left(\frac{\partial}{\partial t}\right), \tag{7-1}$$

which is valid for

$$t = o(x) \tag{7-2}$$

This holds, for small time, everywhere except so near the trailing edge, $x = 0$, that $t = O(x)$ and $\frac{\partial}{\partial x} = O\left(\frac{\partial}{\partial t}\right)$. Since the important boundary condition must be evaluated at the trailing edge, it must be satisfied there in a sense consistent with (7-2). The small-time solution, if found, clearly will be non-uniformly valid near the trailing edge. Very near the trailing edge, on the other hand, if $x = o(t)$ and hence $\frac{\partial}{\partial t} = o\left(\frac{\partial}{\partial x}\right)$, the equations reduce to those for the steady jet-deflection problem treated in II (and discussed briefly above in Section 5.4). The steady solution is valid near the trailing edge in a region growing with time.

Instead of the downwash, Spence treated the jet ordinate, $h(x,t)$, as the unknown function, along with $g(x,t)$. Cross-differentiation of the small-time approximation of the appropriate equations, (6-13) and (6-54), and elimination of $g_t(x,t)$ gives the equation in $h(x,t)$ alone, denoting by subscripts $(\ )_t$ and $(\ )_x$ partial differentiation with respect to those variables,

$$h_{tt}(x,t) = \frac{1}{\pi} \int_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{h_{\xi\xi\xi}(\xi,t)\,d\xi}{\xi - x}, \quad 0 < x < \infty. \tag{7-3}$$

The form of (7-3) implies that

$$\frac{\partial^3}{\partial x^3} = O\left(\frac{\partial^2}{\partial t^2}\right), \tag{7-4}$$

i.e., that

$$x = O(t^{2/3}). \tag{7-5}$$

which from (7-2) implies that

$$t = o(t^{2/3}). \tag{7-6}$$

Instead of solving (7-3) for $h(x,t)$ directly in terms of a similarity variable, $\zeta = x/t^{2/3}$, as in III, it is more instructive to retain x and t explicitly, and to solve simultaneously for $h(x,t)$ and $g(x,t)$ while retaining the similarity approach. Improper treatment of the unit-step function appearing in the problem is one reason for the incorrect result of III.

If $\mathcal{F}(t) = 1(t)$, the equations (6-13), (6-55), (6-18), (6-19), (6-58) and (6-59) are, in this approximation,

$$g_t(x,t) = -h_{xxx}(x,t), \quad 0 < x < \infty \tag{7-7}$$

and

$$h_t(x,t) = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x}, \quad 0 < x < \infty, \tag{7-8}$$

with either

$$h(0+,t) = 0 \tag{7-9}$$

and

$$h_x(0+,t) = 1(t), \tag{7-10}$$

or

$$\lim_{x \to 0+} \frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x} = -1(t), \tag{7-11}$$

and

$$\int_0^\infty \xi^{-1/2} g(\xi,t)\,d\xi = 0 .$$

(7-12)

Following III, but in terms of x and t instead of ξ , define the Mellin-transform pairs

$$h(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s+1)}\, 1(t)\, H(s)\, ds \qquad (7\text{-}13)$$

$$1(t)\, t^{\frac{2}{3}(s+1)} H(s) = \int_0^\infty x^{s-1}\, h(x,t)\, dx , \qquad (7\text{-}14)$$

and

$$g(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\, G(s)\, ds \qquad (7\text{-}15)$$

$$1(t)\, t^{\frac{2}{3}(s-\frac{1}{2})} G(s) = \int_0^\infty x^{s-1}\, g(x,t)\, dx , \qquad (7\text{-}16)$$

where c is, as yet, an undetermined real constant.

From the form of (7-13) and (7-15), the assumption of an $x/t^{2/3}$ similarity is evident. For $x/t^{2/3} < 1$, the functions $h(x,t)$ and $g(x,t)$ can be evaluated in series in increasing powers of $x/t^{2/3}$, $(x/t^{2/3})\ln(x/t^{2/3})$, etc. by moving the line of integration, $s = c$, to the left past the singularities of $H(s)$ and $G(s)$, respectively, in the s- plane. Because of (7-2), such series would be valid in the limited range where $t \ll x \ll t^{2/3}$, i.e., where

$$t = o(x) = o\left[o(t^{2/3})\right].$$

(7-17)

In a similar manner for $x/t^{2/3} > 1$, $h(x,t)$ and $g(x,t)$ can be evaluated in series in increasing powers of $(x/t^{2/3})^{-1}$, $(x/t^{2/3})^{-1} \ln(x/t^{2/3})$, etc. by moving the line of integration, $s = c$, to the right past the singularities of $H(s)$ and $G(s)$, respectively, in the s- plane. These expansions are valid for

$$t^{2/3} = o(x).$$

(7-18)

By successive differentiation of (7-13) with respect to x,

$$h_{xxx}(x,t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-3}\, t^{\frac{2}{3}(s+1)}\, 1(t)\, s(s+1)(s+2)\, H(s)\, ds,$$

and, provided $s(s+1)(s+2)\,H(s)$ is regular in the infinite strip, $c-3 < \mathcal{R}(s) < c$, the line of integration can be moved to the left to $s = c-3$. Writing $s = \bar{s} - 3$ and dropping the bars then gives

$$h_{xxx}(x,t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s-2)}\, 1(t)\,(s-3)(s-2)(s-1)\, H(s-3)\, ds.$$

(7-19)

Substituting (7-15) into the integral operator in (7-8), interchanging the order of integration, and using the integral given in Section 3.1 of III (where the factor z^{-s} was inadvertently omitted), namely,

$$\frac{1}{\pi}\int_0^\infty \left(\frac{\chi}{\xi}\right)^{1/2} \frac{\xi^{-s}\,d\xi}{\xi-\chi} = -\chi^{-s}\tan\pi s\,, \quad |\mathcal{R}s| < \tfrac{1}{2}, \tag{7-20}$$

gives

$$\frac{1}{\pi}\int_0^\infty \left(\frac{\chi}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi-\chi} = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi^{-s}\, t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\tan\pi s\, G(s)\,ds, \tag{7-21}$$

if $|c| < \frac{1}{2}$.

Next, differentiating (7-13) with respect to t gives

$$h_t(\chi,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi^{-s}\left\{ t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\left(\frac{2+2s}{3}\right) + t^{\frac{2}{3}(s+1)}\,\delta(t)\right\} H(s)\,ds, \tag{7-22}$$

where $\delta(t)$ is the Dirac delta function, (5-10). For $\mathcal{R}(s) < c$, $t \ll \chi \ll t^{2/3}$ by (7-17) and t is necessarily greater than zero. Therefore the $\delta(t)$ term may be neglected since it is zero for $t > 0$. For $\mathcal{R}(s) > c$, $t^{2/3} \ll \chi$ by (7-18) and t is not necessarily greater than zero. Thus the $\delta(t)$ term must be considered. By the definition of $\delta(t)$, however, it follows that if $\epsilon > 0$, $t^\epsilon\,\delta(t) = 0$. Therefore the $\delta(t)$ term may be neglected provided the first pole of $H(s)$ to the right of $\mathcal{R}(s) = c$ occurs for $\mathcal{R}(s) > -1$. This is clearly satisfied here since $|c| < \frac{1}{2}$, so

$$h_t(\chi,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi^{-s}\, t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\left(\frac{2+2s}{3}\right) H(s)\,ds. \tag{7-23}$$

Differentiating (7-15) with respect to t gives

$$g_t(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\left\{t^{\frac{2}{3}(s-2)}\mathbf{1}(t)\left(\frac{-1+2s}{3}\right) + t^{\frac{2}{3}(s-\frac{1}{2})}\delta(t)\right\}G(s)\,ds. \tag{7-24}$$

By the same arguments given for $h_t(x,t)$, the $\delta(t)$ term may be dropped if the first pole of $G(s)$ to the right of $\mathcal{R}(s)=c$ occurs for $\mathcal{R}(s) > \frac{1}{2}$. Assuming this to be satisfied gives

$$g_t(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s-2)}\mathbf{1}(t)\left(\frac{-1+2s}{3}\right)G(s)\,ds. \tag{7-25}$$

Finally, differentiating (7-13) and (7-15) with respect to x gives

$$h_x(x,t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1}\, t^{\frac{2}{3}(s+1)}\mathbf{1}(t)\, s\, H(s)\,ds, \tag{7-26}$$

and

$$g_x(x,t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1}\, t^{\frac{2}{3}(s-\frac{1}{2})}\mathbf{1}(t)\, s\, G(s)\,ds. \tag{7-27}$$

Substituting (7-19), (7-21), (7-23) and (7-25) into the equations (7-7) and (7-8) gives

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} t^{\frac{2}{3}(s-2)}\mathbf{1}(t)\left\{\left(\frac{-1+2s}{3}\right)G(s) - (s-3)(s-2)(s-1)H(s-3)\right\}ds = 0 \tag{7-28}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s-2)}\, 1(t)\left\{-\tan\pi s\, G(s) + \left(\frac{2+2s}{3}\right) H(s)\right\} ds = 0 .$$

(7-29)

These will be satisfied for arbitrary x and t provided

$$G(s) = \left(\frac{2+2s}{3}\right)\cot\pi s\, H(s) = \frac{(s-3)(s-2)(s-1)}{\left(\frac{-1+2s}{3}\right)} H(s-3) .$$

(7-30)

From (7-13), it follows that the trailing-edge boundary condition, (7-9), will be satisfied if $0 < c < \frac{1}{2}$ and $H(s)$ is regular at $s = 0$. The input trailing-edge boundary condition, (7-10), will be satisfied, using (7-26), if $H(s)$ has a simple pole with unit residue at $s = -1$.

By inspection of (7-16), the Wagner integral condition, (7-12), will be satisfied provided $G(\frac{1}{2}) = 0$.

Summarizing, (7-30) must be solved for $H(s)$ and $G(s)$ subject to the conditions that

$s(s+1)(s+2)H(s)$ is regular in the infinite strip,

$$-3 < \mathcal{R}(s) < \frac{1}{2},$$

(7-31)

$G(s)$ is regular in the infinite strip, $0 < \mathcal{R}(s) \leq \frac{1}{2}$,

(7-32)

$H(s)$ is regular at $s = 0$,

(7-33)

$$H(s) = \frac{1}{s+1} \quad \text{near} \quad s = -1 \quad , \tag{7-34}$$

and

$$G(\tfrac{1}{2}) = 0 . \tag{7-35}$$

The difference equation, (7-30), for $H(s)$ will be satisfied if

$$H(s) = \frac{(s-1)! \sin \frac{\pi s}{3}}{\left(\frac{2+2s}{3}\right)!} M(s) \psi(s) , \tag{7-36}$$

which implies that

$$G(s) = \frac{(s-1)! \sin \frac{\pi s}{3}}{\left(\frac{-1+2s}{3}\right)!} \cot \pi s \, M(s) \psi(s) , \tag{7-37}$$

where $M(s)$ is some function of period three, and $\psi(s)$ is the function introduced and discussed by Spence in III.

The Spence function, $\psi(s)$, satisfies the difference equation

$$\psi(s) + \tan \pi s \, \psi(s-3) = 0 , \tag{7-38}$$

and may be written as

$$\psi(s) = G_0\left(\frac{s-1}{3}\right) G_0\left(\frac{s}{3}\right) G_0\left(\frac{s+1}{3}\right) , \tag{7-39}$$

where $G_0(s)$ is the Lighthill function discussed by Spence in II. This latter function satisfies

$$G_0(s) - \tan \pi s \, G_0(s-1) = 0 \tag{7-40}$$

and is represented by the infinite product,

$$G_0(s) = G_0(-\tfrac{1}{2}-s) = \prod_{n=1}^{\infty} \left\{ \frac{(1-\frac{s}{n})(1+\frac{s}{n+\frac{1}{2}})}{(1+\frac{s}{n})(1-\frac{s}{n-\frac{1}{2}})} \right\}^n , \tag{7-41}$$

i.e., $G_0(s)$ has poles of order n at $s = -n$ and at $s = n - \frac{1}{2}$, zeroes of order n at $s = n$ and at $s = -n - \frac{1}{2}$, and equals unity at $s = 0$ and $s = -\frac{1}{2}$. Therefore, in the region of primary interest, say $-\frac{9}{2} < R(s) < \frac{7}{2}$, $\psi(s)$ has simple poles at $s = -4, -3, -2, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and simple zeroes at $s = -\frac{7}{2}, 2, 3$.

Using these properties, $\frac{(s-1)! \sin\frac{\pi s}{3}}{(\frac{2+2s}{3})!} \psi(s)$ is seen to be regular in $-3 < R(s) < \frac{1}{2}$, except for the required simple pole at $s = -1$, and a double pole at $s = -2$, where only a simple pole is permitted. Also, $\frac{(s-1)! \sin\frac{\pi s}{3}}{(\frac{-1+2s}{3})!} \cot \pi s \, \psi(s)$ is regular in $0 < R(s) < 1$, but $G(\frac{1}{2}) \neq 0$. Therefore the function $M(s)$ must have a zero at $s = -2$ to satisfy the regularity condition, (7-31), and a zero at $s = \frac{1}{2}$ to satisfy the Wagner integral condition, (7-35).

To insure the existence of the Mellin transforms, (7-13) and (7-15), it is necessary that $(x/t^{2/3})^{-s} H(s)$ and $(x/t^{2/3})^{-s} G(s)$ be integrable as $|\mathcal{I}(s)| \to \infty$ along $R(s) = c$. In this limit $(x/t^{2/3})^{-s}$ is bounded, as is $\cot \pi s$. Using the appropriate

form of Stirling's formula, $(s-1)! \sim |\mathcal{I}(s)|^{\mathcal{R}(s)-\frac{1}{2}}\, e^{-\frac{\pi}{2}|\mathcal{I}(s)|}$,
$\left[\left(\frac{2+2s}{3}\right)!\right]^{-1} \sim |\mathcal{I}(s)|^{-\frac{2}{3}\mathcal{R}(s)-\frac{7}{6}}\, e^{\frac{\pi}{3}|\mathcal{I}(s)|}$, and
$\left[\left(\frac{-1+2s}{3}\right)!\right]^{-1} \sim |\mathcal{I}(s)|^{-\frac{2}{3}\mathcal{R}(s)-\frac{1}{6}}\, e^{\frac{\pi}{3}|\mathcal{I}(s)|}$. Spence found in II and III that $\Psi(s) \sim e^{-\frac{\pi}{3}|\mathcal{I}(s)|}$, while $\sin\frac{\pi s}{3} \sim e^{\frac{\pi}{3}|\mathcal{I}(s)|}$. Therefore,
$(x/t^{2/3})^{-s} H(s) \sim (x/t^{2/3})^{-s} G(s) \sim e^{-\frac{\pi}{3}|\mathcal{I}(s)|}$, which is integrable.

The requirement that $M(s)$ introduce zeroes at $s=-2$, $s=\frac{1}{2}$, could be met in only two ways. In the first of these it would be at the expense of introducing corresponding poles into $H(s)$ and $G(s)$ in violation either of the regularity condition, (7-31), or of the regularity condition, (7-32). The second would be at the expense of introducing exponential factors making $H(s)$ and $G(s)$ fail to be integrable along $\mathcal{R}(s)=c$. Therefore, no function $M(s)$ with the desired properties can be found. Thus no solution of the approximate set of equations, (7-7) to (7-12), can be found. Discussion of reasons for this will follow the details of Spence's "solution" of III.

Spence obtained his "solution" in III by choosing $M(s)$ as

$$M(s) = \frac{4}{3G_0(-\frac{2}{3})G_0(-\frac{1}{3})}\,\frac{\sin\pi\frac{s-1}{3}}{\sin\pi\frac{s-1/2}{3}}, \qquad (7\text{-}42)$$

where the numerical factor is chosen to give the pole of $H(s)$ at $s=-1$ unit residue. The zero required at $s=-2$ is introduced at the expense of a simple pole of $M(s)$

at $s = \frac{1}{2}$, $s = -\frac{5}{2}$, etc. While the simple pole at $s = -\frac{5}{2}$ does not violate the regularity of $s(s+1)(s+2)H(s)$ in $-3 < \mathcal{R}(s) < \frac{1}{2}$, (7-31), since $\frac{(s-1)!\,\sin\frac{\pi s}{3}}{\left(\frac{2+4s}{3}\right)!}\,\psi(s)$ has a simple zero there, the condition on the regularity of $G(s)$ in $0 < \mathcal{R}(s) \le \frac{1}{2}$, (7-32), is violated by the simple pole of $G(s)$ introduced at $s = \frac{1}{2}$ by $M(s)$. Therefore, the neglect of the $\delta(t)$ term in (7-24) was not justified. Furthermore, of course, the Wagner integral condition, (7-35), is violated; in fact the integral, (7-12), does not even exist with the $G(s)$ found here. On these two grounds, the "solution" is clearly invalid.

Investigation of the properties of this invalid "solution" helps indicate the failure of this attempt to approximate the full equations for small time, and is thus worthwhile. Since the behavior of $H(s)$ and $G(s)$ in the whole s - plane, in particular their poles, is known from (7-36), (7-37) and (7-42), series expansions for $h(x,t)$ and $g(x,t)$ may be found to be

$$h(x,t) = \begin{cases} x\,1(t) - \dfrac{2x^2}{3^{1/2}(\frac{1}{3})!\,t^{2/3}} + \dfrac{2x^3}{27(-\frac{1}{3})!\,t^{4/3}} + O\left(\dfrac{x^3}{t^{4/3}}\right), & x < t^{2/3} \\[2ex] \dfrac{3t\,1(t)}{\pi^{1/2}\,x^{1/2}} \ln\left(x/t^{2/3}\right) + O\left(\dfrac{t\,1(t)}{x^{1/2}}\right), & x > t^{2/3} \end{cases} \qquad (7\text{-}43)$$

and

$$g(x,t) = \begin{cases} \dfrac{4}{3(-\frac{1}{3})!\, t^{1/3}} + \dfrac{2}{3\pi}\dfrac{x}{t} + o\left(\dfrac{x}{t}\right), & x < t^{2/3} \\[2ex] \dfrac{3\,\mathbb{1}(t)}{\sqrt{\pi}\, x^{1/2}} + \dfrac{45}{16\pi^{3/2}}\dfrac{t^2\,\mathbb{1}(t)}{x^{7/2}} \ln(x/t^{2/3}) + O\left(\dfrac{t^2}{x^{7/2}}\right), & x > t^{2/3} \end{cases}$$

(7-44)

(The third term in the small - $x/t^{2/3}$ expansion for $h(x,t)$ has the sign corrected from equation (38) of III.) Similarly, the first terms in the expansions for the important derivatives and integrals are, from (7-19), (7-21), (7-23), (7-24), (7-26) and (7-27),

$$h_t(x,t) = \begin{cases} \dfrac{4x^2}{9\sqrt{3}\,(\frac{1}{3})!\, t^{5/3}} + o\left(\dfrac{x^2}{t^{5/3}}\right), & x < t^{2/3} \\[2ex] \dfrac{3\,\mathbb{1}(t)}{\pi^{3/2} x^{1/2}} \ln(x/t^{2/3}) + o\left(\dfrac{\ln(x/t^{2/3})}{x^{1/2}}\right), & x > t^{2/3}, \end{cases}$$

(7-45)

$$h_x(x,t) = \begin{cases} \mathbb{1}(t) + o(1), & x < t^{2/3} \\[2ex] -\dfrac{3t\,\mathbb{1}(t)}{\pi^{3/2} x^{3/2}} \ln(x/t^{2/3}) + o\left(\dfrac{t \ln(x/t^{2/3})}{x^{3/2}}\right), & x > t^{2/3}, \end{cases}$$

(7-46)

$$h_{xxt}(x,t) = \begin{cases} \dfrac{4}{9(-\frac{1}{3})!\, t^{4/3}} + o(t^{-4/3}), & x < t^{2/3} \\[2ex] -\dfrac{45\, t\, 1(t)}{8\pi^{3/2} x^{9/2}} \ln(x/t^{2/3}) + o\left(\dfrac{t \ln(x/t^{2/3})}{x^{9/2}}\right), & x > t^{2/3}, \end{cases} \tag{7-47}$$

$$g_t(x,t) = \begin{cases} -\dfrac{4}{9(-\frac{1}{3})!\, t^{7/3}} + O\left(\dfrac{x}{t^{3}}\right), & x < t^{2/3} \\[2ex] \dfrac{3\,\delta(t)}{\pi^{1/2} x^{3/2}} + \dfrac{45\, t\, 1(t)}{8\pi^{3/2} x^{7/2}} \ln(x/t^{2/3}) + O\left(\dfrac{t}{x^{7/2}}\right), & x > t^{2/3}, \end{cases} \tag{7-48}$$

$$g_x(x,t) = \begin{cases} \dfrac{2}{3\pi t} + o\left(\dfrac{1}{t}\right), & x < t^{2/3} \\[2ex] -\dfrac{3\, 1(t)}{\pi^{1/2} x^{3/2}} + O\left[x^{-7/2} t^2 \ln(x/t^{2/3})\right], & x > t^{2/3}. \end{cases} \tag{7-49}$$

and

$$\frac{1}{\pi}\int_0^{\infty} \left(\frac{x}{\xi}\right)^{1/2} \frac{g(\xi,t)\,d\xi}{\xi - x} = \begin{cases} -\dfrac{4x^2}{9\sqrt{2}\,(\frac{1}{3})!\, t^{8/3}} + o(x^2 t^{-8/3}), & x < t^{2/3} \\[2ex] -\dfrac{3\, 1(t)}{\pi^{1/2} x^{1/2}} \ln(x/t^{2/3}) + o\left[x^{-1/2} \ln(x/t^{2/3})\right], & x > t^{2/3}. \end{cases} \tag{7-50}$$

From (7-45) and (7-50) it is seen that the approximate equation, (7-8), is satisfied; from (7-47) and (7-48) it is seen that the approximate equation, (7-7), is not satisfied for large $x/t^{2/3}$, the extraneous $\delta(t)$ term in (7-48) arising since it was incorrectly neglected in (7-24). The boundary conditions at the trailing edge, (7-9) and (7-10), are satisfied, considering (7-43) and (7-46), however, the alternate statement of these conditions in terms of the vortex distribution, (7-11), is not satisfied, considering (7-50).

The lift coefficient was evaluated in III using (6-62), which is

$$C_L(t) = -4\mu^{1/2}\tau_0 \frac{d}{dt}\int_0^\infty \xi^{1/2} g(\xi,t)\,d\xi . \tag{7-51}$$

It is important that in the derivation of this form of the lift coefficient, satisfaction of the Wagner integral condition was required. Since this condition is violated here, (7-51) should not have been used. Furthermore, (7-44) shows that the integral in (7-51) does not even exist. In III, the calculation was actually carried out by taking $\partial/\partial t$ inside the integral in (7-51) and using (7-7), the ensuing integral existing only because, as seen above, (7-7) is not satisfied by the "solution" of III.

As a final property of the "solution," the assumptions that $g_x(x,t) = o[g_t(x,t)]$ and $h_x(x,t) = o[h_t(x,t)]$ will be checked. For $x/t^{2/3} < 1$, the expansions are valid in the range $t \ll x \ll t^{2/3}$, (7-17). If

$$\left.\begin{aligned} t &= O(K^{-6}) \\ x &= O(K^{-5}) \end{aligned}\right\},$$

and

(7-52)

where $K \gg 1$ is some constant, this inequality is met. Furthermore, the inequality $t \ll t^{2/3}$, (7-6), is also satisfied. Substituting (7-52) into (7-45), (7-46), (7-48) and (7-49) indicates that

$$\left.\begin{aligned} h_t(x,t) &= O(K^{0}) \\ h_x(x,t) &= O(K^{0}) \end{aligned}\right\}$$

(7-53)

and

$$\left.\begin{aligned} g_t(x,t) &= O(K^{8}) \\ g_x(x,t) &= O(K^{6}) \end{aligned}\right\}. \qquad (7\text{-}54)$$

Therefore $g_t(x,t) \gg g_x(x,t)$ as assumed, but $h_x(x,t)$ is of the same order as $h_t(x,t)$, inconsistent with the assumption that $h_t(x,t) \gg h_x(x,t)$. In a similar fashion for $x/t^{2/3} > 1$, the expansions are valid for $x \gg t^{2/3}$, (7-18), such that the inequalities (7-2) and (7-6) are also satisfied. These are if

$$\left.\begin{aligned} t &= O(K^{-3}) \\ x &= O(K^{-1}) \end{aligned}\right\},$$

(7-55)

where $K >> 1$ is again some constant. Substituting (7-55) into (7-45), (7-46), (7-48) and (7-49) indicates that

$$\left.\begin{aligned} h_t(x,t) &= O(K^{1/2}\ln K) \\ h_x(x,t) &= O(K^{-3/2}\ln K) \end{aligned}\right\} \tag{7-56}$$

and

$$\left.\begin{aligned} g_t(x,t) &= \begin{cases} O(K^{7/2}) &, t=0 \\ O(K^{1/2}\ln K), & t>0 \end{cases} \\ g_x(x,t) &= O(K^{3/2}) \end{aligned}\right\}. \tag{7-57}$$

Therefore $h_t(x,t) >> h_x(x,t)$ as assumed. At $t=0$, $g_t(x,t) >> g_x(x,t)$ as assumed; however the solution should also be valid for $t>0$ provided (7-6) is satisfied, but $g_t(x,t) << g_x(x,t)$, and the assumption breaks down.

The question arises whether the unit-step-function time dependence assumed here, being discontinuous at $t=0$, is too singular to permit a solution. However, if the general input, $\mathfrak{F}(t) = \mu^n t^n 1(t)$, $n>0$, is treated in precisely the above fashion, again no solution can be found that satisfies the regularity condition on $s(s+1)(s+2)H(s)$, satisfies the regularity condition on the new $G(s)$ and satisfies the Wagner integral condition.

This approach to the solution of the small-time jet-deflection problem clearly is inadequate. Consideration of the result, (7-53), indicated that $h_x(x,t)$ came out to be of the same order as $h_t(x,t)$, indicating that it should not have been neglected. This does not seem surprising in retrospect, since the important input to the problem is the boundary condition, (7-10), on $h_x(x,t)$. Neglecting $h_x(x,t)$ with respect to $h_t(x,t)$ in the downwash equation, yet requiring satisfaction of the boundary condition on $h_x(x,t)$ seems inconsistent. This suggests, as an alternative, formulation in terms of the downwash, $k(x,t)$, and the vortex distribution, $g(x,t)$, still making the approximation $\frac{\partial}{\partial t} >> \frac{\partial}{\partial x}$ for small time. The next section will deal with this formulation and its subsequent failure to give a valid approximation to the complete set of equations for small time.

7.2 Further Critique of Small-Time Approach: Jet-Deflection and Airfoil-Motion Problems.

Formulation in terms of the downwash on the jet, $k(x,t)$, and the vortex distribution representing the jet, $g(x,t)$, by equations (6-54) to (6-59) in the jet-deflection problem, and in the airfoil-motion problem by (6-64) to (6-69) eliminates the contradiction of satisfying the input boundary condition on $h_x(x,t)$ while neglecting it with respect to $h_t(x,t)$ in the equations. The small-time approximation of neglecting x- derivatives with respect to t- derivatives, wherever convective derivatives appear, will be retained. Therefore, the various order-of-magnitude restrictions, (7-1), (7-2) and (7-4) to (7-6) still hold.

The approximate small-time jet-deflection equations are then, from (6-54) to (6-59), denoting this approximation by a subscript $(\ )_0$,

$$g^{\tau}_{0_{tt}}(x,t) = -k^{\tau}_{0_{xxx}}(x,t)\,, \quad 0<x<\infty\,,$$

(7-58)

and

$$k^{\tau}_0(x,t) = -\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g^{\tau}_0(\xi,t)\,d\xi}{\xi - x}\,, \quad 0<x<\infty,$$

(7-59)

with

$$k^{\tau}_0(0+,t) = 1(t) = -\lim_{x\to 0+}\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g^{\tau}_0(\xi,t)\,d\xi}{\xi - x}\,,$$

(7-60)

and

$$\int_0^\infty \xi^{-1/2} g^{\tau}_0(\xi,t)\,d\xi = 0\,.$$

(7-61)

Equation (7-59) is exact, so only in (7-58) has it been necessary to approximate for small time.

To treat the problem, again define the Mellin-transform pairs

$$\left.\begin{aligned} k^{\tau}_0(x,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} t^{\frac{2s}{3}}\, 1(t)\, \mathcal{K}^{\tau}_0(s)\,ds \\ t^{\frac{2s}{3}}\, 1(t)\, \mathcal{K}^{\tau}_0(s) &= \int_0^\infty x^{s-1} k^{\tau}_0(x,t)\,dx \end{aligned}\right\}$$

(7-62)

(7-63)

and

$$g_0^\tau(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} t^{\frac{2s}{3}} 1(t)\, \mathcal{G}_0^\tau(s)\, ds$$

$$t^{\frac{2s}{3}} 1(t)\, \mathcal{G}_0^\tau(s) = \int_0^\infty x^{s-1} g_0^\tau(x,t)\, dx \qquad (7\text{-}64)$$

(7-65)

where c is, as yet, an undetermined real constant.

Successive differentiations of (7-62) with respect to x give

$$k_{0_{xxx}}^\tau(x,t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3} t^{\frac{2s}{3}} 1(t)\, s(s+1)(s+2)\, \mathcal{X}_0^\tau(s)\, ds\,. \qquad (7\text{-}66)$$

Substituting (7-64) into the integral operator of (7-59) and evaluating it as in Section 7.1 gives

$$\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g_0^\tau(\xi,t)\,d\xi}{\xi - x} = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} t^{\frac{2s}{3}} 1(t) \tan \pi s\, \mathcal{G}_0^\tau(s)\, ds\,, \qquad (7\text{-}67)$$

if $|c| < \frac{1}{2}$. Differentiating (7-64) twice with respect to t gives

$$g_{0_{tt}}^\tau(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Big\{ t^{\frac{2}{3}(s-3)} 1(t)\, \tfrac{4}{9}\, s(s-\tfrac{3}{2}) + \tfrac{4}{3}\, s\, t^{\frac{1}{3}(2s-3)} \delta(t)$$

$$+\, t^{\frac{2s}{3}} \frac{d\,\delta(t)}{dt}\Big\}\, \mathcal{G}_0^\tau(s)\, ds\,,$$

but, provided $\mathcal{B}_0^\tau(s)$ is regular in the infinite strip, $c < \mathcal{R}(s) < c+3$, the arguments of Section 7.1 justify neglect of the $\delta(t)$ and $d\delta(t)/dt$ terms, so

$$g_{0_{tt}}^{\tau}(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\, t^{\frac{2s}{3}}\, 1(t)\left(2+\tfrac{2s}{3}\right)\left(1+\tfrac{2s}{3}\right)\mathcal{B}_0^\tau(s+3)\,ds. \tag{7-68}$$

Substitution of (7-62), (7-66), (7-67) and (7-68) into equations (7-58) and (7-59) gives

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2s}{3}}\, 1(t)\left\{-s(s+1)(s+2)\,\mathcal{K}_0^\tau(s) + \left(2+\tfrac{2s}{3}\right)\left(1+\tfrac{2s}{3}\right)\mathcal{B}_0^\tau(s+3)\right\}ds = 0, \tag{7-69}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\, t^{\frac{2s}{3}}\, 1(t)\left\{\mathcal{K}_0^\tau(s) - \tan\pi s\,\mathcal{B}_0^\tau(s)\right\}ds = 0, \tag{7-70}$$

which will be satisfied if

$$\mathcal{K}_0^\tau(s) = \tan\pi s\,\mathcal{B}_0^\tau(s) = \frac{\left(2+\frac{2s}{3}\right)\left(1+\frac{2s}{3}\right)}{s(s+1)(s+2)}\,\mathcal{B}_0^\tau(s+3). \tag{7-71}$$

From (7-67) it is seen that the trailing-edge boundary condition, (7-60), will be satisfied if $0 < c < \frac{1}{2}$ and $\mathcal{B}_0^\tau(s)$ has a double pole at the origin with coefficient $\frac{1}{\pi}$. The Wagner integral condition, (7-61), will be satisfied, using (7-65), if $\mathcal{B}_0^\tau(\frac{1}{2}) = 0$.

Summarizing, (7-71) may be considered a difference equation for $\mathcal{B}_0^\tau(s)$ which must be solved subject to the conditions that

$\mathcal{G}_0^\tau(s)$ is regular in the infinite strip, $0 < \mathcal{R}(s) < \frac{7}{2}$,

(7-72)

$$\mathcal{G}_0^\tau(s) = \frac{1}{\pi s^2} \quad \text{near} \quad s = 0,$$

(7-73)

and

$$\mathcal{G}_0^\tau(\tfrac{1}{2}) = 0.$$

(7-74)

The difference equation, (7-71), will be satisfied if

$$\mathcal{G}_0^\tau(s) = \frac{2}{\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})} \; \frac{(s-1)!\,\sin\pi\frac{s-1}{3}}{\left(\frac{2s}{3}\right)!} N(s)\,\psi(s-3),$$

(7-75)

which implies that

$$\mathcal{K}_0^\tau(s) = -\frac{2}{\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})} \; \frac{(s-1)!\,\sin\pi\frac{s-1}{3}}{\left(\frac{2s}{3}\right)!} N(s)\,\psi(s),$$

(7-76)

where $N(s)$ is again some function of period three, and $\psi(s)$ is Spence's function, (7-39). For $N(s) \equiv 1$, the conditions (7-72) and (7-73) are both satisfied. However, the requirement, (7-74), to satisfy the Wagner integral condition is not met. As in Section 7.1, if $N(s) \equiv 1$, $\mathcal{G}_0^\tau(s) \sim e^{-\frac{\pi}{3}|\mathcal{J}(s)|}$ and is integrable as $|\mathcal{J}(s)| \to \infty$ along $\mathcal{R}(s) = C$. Introduction of a zero at $s = \frac{1}{2}$ by a function $N(s)$ of period three would either introduce a pole into the strip $0 < \mathcal{R}(s) < \frac{7}{2}$ in violation of the regularity condition on $\mathcal{G}_0^\tau(s)$, or would introduce an exponential factor making the Mellin transform of $\mathcal{G}_0^\tau(s)$ fail to converge as $|\mathcal{D}(s)| \to \infty$ along $\mathcal{R}(s) = c$. Therefore, a solution satisfying all the conditions cannot be found. Evaluating the integral, (7-61), for $N(s) \equiv 1$, using (7-65), (7-75), (7-40) and (7-41) gives

$$\int_0^\infty \xi^{-1/2} g_0^\tau(\xi, t)\, d\xi = \frac{\sqrt{\pi}}{(-\frac{1}{3})!}\, t^{1/6}\, 1(t), \tag{7-77}$$

clearly greater than zero. If the order-of-magnitude considerations of the previous section are observed, the assumptions of neglecting $g^\tau_{0_{tx}}(x,t)$ and $g^\tau_{0_{xx}}(x,t)$ with respect to $g^\tau_{0_{tt}}(x,t)$ is satisfied. Therefore, failure to obtain a solution here results solely from the inability to satisfy the Wagner integral condition in this approximation.

The approximations $g_0^\tau(x,t)$ and $k_0^\tau(x,t)$ may be considered as the first terms in an expansion of the full equations, (6-54) to (6-59). This expansion may be written in terms of the Mellin transforms as

$$g^\tau(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\left\{t^{\frac{2s}{3}}\, 1(t)\, \mathcal{G}_0^\tau(s) + t^{\frac{2}{3}(s+\frac{1}{2})}\, 1(t)\, \mathcal{G}_1^\tau(s) + O\left(t^{\frac{2}{3}(s+1)}\right)\right\} ds, \tag{7-78}$$

and

$$k^\tau(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\left\{t^{\frac{2s}{3}}\, 1(t)\, \mathcal{K}_0^\tau(s) + t^{\frac{2}{3}(s+\frac{1}{2})}\, 1(t)\, \mathcal{K}_1^\tau(s) + O\left(t^{\frac{2}{3}(s+1)}\right)\right\} ds. \tag{7-79}$$

The second approximations, $\mathcal{G}_1^\tau(s)$ and $\mathcal{K}_1^\tau(s)$, are the transforms of $g_1^\tau(x,t)$ and $k_1^\tau(x,t)$, say, which satisfy

$$g^\tau_{1_{tt}}(x,t) + 2g^\tau_{0_{tx}}(x,t) = -k^\tau_{1_{xx}}(x,t)\ , \quad 0<x<\infty \tag{7-80}$$

and

$$k_1^\tau(x,t) = -\frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g_1^\tau(\xi,t)\,d\xi}{\xi-x}\ , \quad 0<x<\infty, \tag{7-81}$$

with

$$k_1^\tau(0+,t) = -\lim_{x\to 0+}\frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g_1^\tau(\xi,t)\,d\xi}{\xi-x} = 0 \tag{7-82}$$

and

$$\int_0^\infty \xi^{-1/2} g_1^\tau(\xi,t)\,d\xi = 0. \tag{7-83}$$

Operating on the Mellin transforms (7-78) and (7-79) for $g_1^\tau(x,t)$ and $k_1^\tau(x,t)$ as for $g_0^\tau(x,t)$ and $k_0^\tau(x,t)$, the inhomogeneous difference equation,

$$\mathcal{K}_1^\tau(s) = \tan\pi s\, \mathcal{G}_1^\tau(s) = \frac{\left(\frac{7+2s}{3}\right)\left(\frac{4+2s}{3}\right)}{s(s+1)(s+2)}\mathcal{G}_1^\tau(s+3) - \frac{2\left(\frac{4+2s}{3}\right)}{s(s+1)}\mathcal{G}_0^\tau(s+2), \tag{7-84}$$

must be solved, with $0<c<\frac{1}{2}$, $\mathcal{G}_1^\tau(s)$ regular in $0<\mathcal{R}(s)<\frac{7}{2}$, $\mathcal{G}_1^\tau(s)$ having no stronger a singularity at the origin than a simple pole in order to satisfy the downwash condition, (7-82), and $\mathcal{B}_1^{\tau}(\frac{1}{2}) = 0$ to satisfy the Wagner integral condition. The solution, which again satisfies everything but the Wagner integral condition, is

$$\mathcal{B}_1^{\tau}(s) = -\frac{4}{3\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})}\,\frac{\sin\pi\frac{s+1}{3}}{\left(\frac{1+2s}{3}\right)!}\left\{(s-\tfrac{1}{2})\,\psi(s-4) + \frac{1}{2\sqrt{3}}\,\psi(s-3)\right\}, \tag{7-85}$$

and the corresponding $\mathcal{K}_1^{\tau}(s)$. Evaluating the integral, (7-83), using (7-78) and (7-85) gives

$$\int_0^{\infty} \xi^{-1/2} g_1^{\tau}(\xi,t)\,d\xi = \frac{2\sqrt{3\pi}}{9\,(\frac{2}{3})!}\,t^{2/3}\,\mathbb{1}(t). \tag{7-86}$$

Therefore the second approximation in t not only fails to correct the error in the first approximation of failing to satisfy the Wagner integral condition, but introduces further error.

To complete the picture, the small-time approximation will be applied to the airfoil-motion problem, as defined by (6-64) to (6-69), and the solution attempted. The time dependence here will be assumed to be $\mathcal{F}(t) = \mu t\,\mathbb{1}(t)$ to avoid the difficulties with the Dirac delta function as discussed in Section 5.2. With the same small-time approximations of (7-1), (7-2) and (7-4) to (7-6), the equations simplify to

$$g^{a}_{0\,tt}(x,t) = -k^{a}_{0\,xxx}(x,t)\,, \quad 0 < x < \infty \tag{7-87}$$

and

$$k_0^{a}(x,t) = -\frac{1}{\pi}\oint_0^{\infty}\left(\frac{x}{\xi}\right)^{1/2}\frac{g_0^{a}(\xi,t)\,d\xi}{\xi - x}\,, \quad 0 < x < \infty, \tag{7-88}$$

with

$$\dot{k}_0^a(0+,t) = -\lim_{x\to 0+}\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2}\frac{g_0^a(\xi,t)\,d\xi}{\xi - x} = 0$$

(7-89)

and

$$\int_0^\infty \xi^{-1/2} g_0^a(\xi,t)\,d\xi = \pi\mu\, 1(t)$$

(7-90)

The important input to the problem is the Wagner integral condition, (7-90), as contrasted to the trailing-edge boundary condition, (7-60), in the jet-deflection problem.

In view of the relation of the airfoil-motion solution to the jet-deflection solution as pointed out in Chapter 6, it is impossible that an airfoil-motion solution of this type will be found. However, it is instructive to carry through the Mellin-transform approach to see exactly how it fails.

Assuming a similarity solution in $x/t^{2/3}$, the Mellin-transform pairs are

$$g_0^a(x,t) = \frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\, \mathcal{G}_0^a(s)\,ds$$

(7-91)

$$\mu\, t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\, \mathcal{G}_0^a(s) = \int_0^\infty x^{s-1} g_0^a(x,t)\,dx$$

(7-92)

and

$$k_0^a(x,t) = \frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, t^{\frac{2}{3}(s-\frac{1}{2})}\, 1(t)\, \mathcal{K}_0^a(s)\,ds$$

(7-93)

$$\mu t^{\frac{2}{3}(s-\frac{1}{2})} 1(t)\, \chi_0^a(s) = \int_0^\infty x^{s-1} k_0^a(x,t)\, dx \; . \tag{7-94}$$

Three x- derivatives of (7-93) give

$$k^a_{0\,xxx}(x,t) = -\frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3} t^{\frac{2}{3}(s-\frac{1}{2})} 1(t)\, s(s+1)(s+2)\, \chi_0^a(s)\, ds \, . \tag{7-95}$$

Substitution of (7-91) into the integral operator in (7-88) and evaluating it as usual yields

$$\frac{1}{\pi}\oint_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{g_0^a(\xi,t)\,d\xi}{\xi - x} = -\frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} t^{\frac{2}{3}(s-\frac{1}{2})} 1(t)\, \tan\pi s\, \mathcal{G}_0^a(s)\, ds \, , \tag{7-96}$$

if $|c| < \frac{1}{2}$. Differentiating (7-91) twice with respect to t , and dropping the $\delta(t)$ and $d\delta(t)/dt$ terms by the arguments of Section 7.1, provided $\mathcal{G}_0^a(s)$ is regular in the infinite strip, $c < \mathcal{R}(s) < c+3$, gives

$$g^a_{0\,tt}(x,t) = \frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3} t^{\frac{2}{3}(s-\frac{1}{2})} 1(t) \left(\frac{5+2s}{3}\right)\left(\frac{2+2s}{3}\right) \mathcal{G}_0^a(s+3)\, ds . \tag{7-97}$$

The equations (7-87) and (7-88) are, upon substitution of (7-93) and (7-95) to (7-97),

$$\frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3} t^{\frac{2}{3}(s-\frac{1}{2})} 1(t) \left\{ -s(s+1)(s+2)\chi_0^a(s) + \left(\frac{5+2s}{3}\right)\left(\frac{2+2s}{3}\right) \mathcal{G}_0^a(s+3) \right\} ds = 0 \tag{7-98}$$

and

$$\frac{\mu}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} t^{\frac{2}{3}(s-\frac{1}{2})} 1(t) \left\{ \chi_0^a(s) - \tan\pi s\, \mathcal{G}_0^a(s) \right\} ds = 0 \, , \tag{7-99}$$

which will be satisfied if

$$\mathcal{K}_0^a(s) = \tan\pi s\, \mathcal{G}_0^a(s) = \frac{\left(\frac{5+2s}{3}\right)\left(\frac{2+2s}{3}\right)}{s(s+1)(s+2)}\, \mathcal{G}_0^a(s+3).$$

(7-100)

The trailing-edge boundary condition, (7-89), will be satisfied if $0 < c < \frac{1}{2}$ and if $\mathcal{G}_0^a(s)$ has a singularity at the origin no stronger than a simple pole. The Wagner integral condition, (7-90), is satisfied, using (7-92), if $\mathcal{G}_0^a(\frac{1}{2}) = \pi$.

In summary, the difference equation, (7-100), must be solved for $\mathcal{G}_0^a(s)$ subject to the conditions that

$$s\,\mathcal{G}_0^a(s) \text{ is regular in the infinite strip, } 0 \leq \Re(s) < \tfrac{7}{2},$$

(7-101)

and

$$\mathcal{G}_0^a(\tfrac{1}{2}) = \pi .$$

(7-102)

The difference equation, (7-100), will be satisfied if

$$\mathcal{G}_0^a(s) = \frac{2\sqrt{\pi}}{\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})}\ \frac{(s-1)!\sin\pi\frac{s-1}{3}}{\left(\frac{-1+2s}{3}\right)!}\ P(s)\,\psi(s-3)$$

(7-103)

and

$$\mathcal{K}_0^a(s) = -\frac{2\sqrt{\pi}}{\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})}\ \frac{(s-1)!\sin\pi\frac{s-1}{3}}{\left(\frac{-1+2s}{3}\right)!}\ P(s)\,\psi(s) ,$$

(7-104)

where $P(s)$ is a function of period three and $\psi(s)$ is Spence's function, (7-39). With $P(s) \equiv 1$, $s\,\mathcal{G}_0^a(s)$ is regular in the strip, $0 < \mathcal{R}(s) < \frac{7}{2}$, and $\mathcal{G}_0^a(\frac{1}{2}) = \pi$. However, $\mathcal{G}_0^a(s)$ has a double pole at the origin, removal of which by a suitable function, $P(s)$, would again either introduce a pole into the strip $0 < \mathcal{R}(s) < \frac{7}{2}$, violating the regularity of $\mathcal{G}_0^a(s)$, or would add an exponential factor making $\mathcal{G}_0^a(s)$ fail to converge as $|\mathcal{I}(s)| \to \infty$ along $\mathcal{R}(s) = c$. Thus, as expected, a solution satisfying all the conditions of the problem cannot be found. The double pole at the origin of $\mathcal{G}_0^a(s)$ gives, in terms of $k_0^a(x,t)$, from (7-93) and (7-104)

$$k_0^a(0+,t) = \frac{\sqrt{\pi}\,\mu}{(-\frac{1}{2})!}\,t^{-1/2}, \tag{7-105}$$

which is not zero, but in fact singular, at $t = 0$.

Introduction of a continuous time dependence at $t = 0$ by $\mathcal{F}(t) = \mu^n t^n \mathbb{1}(t)$ for both the jet-deflection and airfoil-motion problems lead to the same failure as above to yield solutions. Therefore, it is not a question of the motion being too singular at $t = 0$.

It has been seen that in both the jet-deflection and airfoil-motion problems, no solution using the present small-time approximation can be found. For the jet-deflection problem, the input boundary condition at the trailing edge was satisfied, but not the Wagner integral condition. On the other hand for the airfoil-motion problem, the input through the Wagner integral condition was satisfied, but not the trailing-edge boundary condition.

The condition left unsatisfied in each instance is an important physical condition on the problem, and a "solution" which fails to satisfy it is meaningless. Moreover, the approximation does not change the order of the equations, since the highest x- derivative, $k_{xxx}(x,t)$ (or if $k(x,t)$ is eliminated, $g_{xxx}(x,t)$) is retained, so there is no justification for failure to satisfy one of the conditions. Failure to get a solution in this small-time approximation, i.e., the failure to get a similarity solution to the full equations for small time, must arise from the inability of this approach to give a valid result near the trailing edge. As discussed in the beginning of this section, any solution foundwould have been non-uniformly valid in time near the trailing edge. Furthermore, as expressed by (7-17), a similarity solution for small t would be limited in validity, because of the small-time approximation, to the range where $t \ll x \ll t^{2/3}$. This restriction apparently prohibits satisfaction of the equations at $x = 0$, and results in the failure of the present approach.

A recent private communication from Dr. D. A. Spence indicates that he has solved the small-μ jet-deflection problem exactly, using Laplace transforms on both the x and t variables. The exact leading lift term for small time has been found. His results also indicate that the solution near the trailing edge is not expressible in terms of the $x/t^{2/3}$ similarity. These results will be reported shortly by Dr. Spence.

7.3 High-Frequency Steady-State Oscillations; Jet Deflection and Airfoil Motion Problems.

No similarity solution in $x/t^{2/3}$ could be found for small time in the previous sections, yet the same approach for high frequencies of steady-state oscillations appears to lead to an $x\nu^{2/3}$ similarity solution satisfying the equations and all the conditions, still approximating by neglecting x-derivatives with respect to t-derivatives. In Section 5.2 of III, Spence found high-frequency "solutions" to the jet-deflection problem in terms of the jet ordinate and vortex strength. Although these "solutions" do not, of course, encounter the step-function difficulty, they do fail to satisfy the Wagner integral condition, hence are incorrect.

The equations for the jet-deflection problem in steady oscillations, with $\mathcal{I}(\nu) \leq 0$, are, from (6-73) to (6-76), using (6-88) and (6-89) and the remarks following them,

$$(i\nu)^2 \hat{g}^\tau(x;\nu) + 2(i\nu)\hat{g}^\tau_x(x;\nu) + \hat{g}^\tau_{xx}(x;\nu) = -\hat{k}^\tau_{xxx}(x;\nu)\,, \quad 0 < x < \infty \tag{7-106}$$

and

$$\hat{k}^\tau(x;\nu) = -\frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{\hat{g}^\tau(\xi;\nu)\,d\xi}{\xi - x}\,, \quad 0 < x < \infty, \tag{7-107}$$

with

$$\hat{k}^\tau(0+;\nu) = -\lim_{x\to 0+}\frac{1}{\pi}\int_0^\infty \left(\frac{x}{\xi}\right)^{1/2} \frac{\hat{g}^\tau(\xi;\nu)\,d\xi}{\xi - x} = 1\,, \tag{7-108}$$

and

$$\int_0^{\infty} \xi^{-1/2}\, \hat{g}^{\tau}(\xi;\vartheta)\, d\xi = 0 \,.$$

(7-109)

In full generality, Mellin transforms may be defined by

$$\left.\begin{aligned} \hat{k}^{\tau}(x;\vartheta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, \hat{K}^{\tau}(s;\vartheta)\, ds \\ \hat{K}^{\tau}(s;\vartheta) &= \int_0^{\infty} x^{s-1}\, \hat{k}^{\tau}(x;\vartheta)\, dx \end{aligned}\right\} \quad (7\text{-}110)$$

(7-111)

and

$$\left.\begin{aligned} \hat{g}^{\tau}(x;\vartheta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, \hat{G}^{\tau}(s;\vartheta)\, ds \\ \hat{G}^{\tau}(s;\vartheta) &= \int_0^{\infty} x^{s-1}\, \hat{g}^{\tau}(x;\vartheta)\, dx \end{aligned}\right\} \quad (7\text{-}112)$$

(7-113)

Using the techniques of the preceding sections, the various derivatives and integrals are

$$\hat{k}^{\tau}_{xxx}(x;\vartheta) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\, s(s+1)(s+2)\, \hat{K}^{\tau}(s;\vartheta)\, ds \;;$$

(7-114)

and, provided $\hat{G}^{\tau}(s;\vartheta)$ is a regular function of s in the infinite strip, $c < \Re(s) < c+3$,

$$\hat{g}^{\tau}(x;\vartheta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\, \hat{G}^{\tau}(s+3;\vartheta)\, ds \,,$$

(7-115)

$$\hat{g}^{\tau}_{x}(x;\nu) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\,(s+2)\,\hat{G}^{\tau}(s+2;\nu)\,ds\,,$$

(7-116)

and

$$\hat{g}^{\tau}_{xx}(x;\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\,(s+1)(s+2)\,\hat{G}^{\tau}(s+1;\nu)\,ds\,;$$

(7-117)

and, if $|c| < \frac{1}{2}$,

$$\frac{1}{\pi}\oint_{0}^{\infty}\left(\frac{x}{\xi}\right)^{1/2}\frac{\hat{g}^{\tau}(\xi;\nu)\,d\xi}{\xi - x} = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\tan\pi s\,\hat{G}^{\tau}(s;\nu)\,ds\,.$$

(7-118)

Substituting these into the governing equations, (7-106) and (7-107), gives

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-3}\Big\{(i\nu)^2\hat{G}^{\tau}(s+3;\nu) - 2(i\nu)(s+2)\hat{G}^{\tau}(s+2;\nu) + (s+1)(s+2)\hat{G}^{\tau}(s+1;\nu)$$
$$- s(s+1)(s+2)\,\hat{K}^{\tau}(s;\nu)\Big\}\,ds = 0$$

(7-119)

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Big\{\hat{K}^{\tau}(s;\nu) - \tan\pi s\,\hat{G}^{\tau}(s;\nu)\Big\}\,ds = 0\,,$$

(7-120)

which will be satisfied, if

$$\hat{K}^{\tau}(s;\nu) = \tan\pi s\,\hat{G}^{\tau}(s;\nu) = \frac{(i\nu)^2}{s(s+1)(s+2)}\,\hat{G}^{\tau}(s+3;\nu) - \frac{2(i\nu)}{s(s+1)}\,\hat{G}^{\tau}(s+2;\nu) + \frac{1}{s}\,\hat{G}^{\tau}(s+1;\nu).$$

(7-121)

The trailing-edge boundary condition, (7-108), is satisfied if $0 < c < \frac{1}{2}$ and $\hat{G}^{\tau}(s;\nu)$ has a double pole with coefficient $\frac{1}{\pi}$ at $s = 0$. The Wagner integral condition, (7-109), is satisfied if $\hat{G}^{\tau}(\frac{1}{2};\nu) = 0$.

Summarizing, the difference equation, (7-121), must be solved for $\hat{G}^{\tau}(s;\nu)$ subject to the conditions that

$$\hat{G}^{\tau}(s;\nu) \text{ is regular in the infinite strip, } 0 < \Re(s) < \tfrac{3}{2}, \tag{7-122}$$

$$\hat{G}^{\tau}(s;\nu) = \frac{1}{\pi s^2} \text{ near } s = 0, \tag{7-123}$$

and

$$\hat{G}^{\tau}(\tfrac{1}{2};\nu) = 0. \tag{7-124}$$

Examination of (7-121) indicates that an approximation scheme in ν, for large ν, may be made by expanding $\hat{K}^{\tau}(s;\nu)$ and $\hat{G}^{\tau}(s;\nu)$ in terms of ν by

$$\hat{K}^{\tau}(s;\nu) = (s-1)!\,\nu^{-\frac{2s}{3}}\left[K_0^{\tau}(s) + i\,\nu^{-1/3} K_1^{\tau}(s) + O(\nu^{-2/3})\right] \tag{7-125}$$

and

$$\hat{G}^{\tau}(s;\nu) = (s-1)!\,\nu^{-\frac{2s}{3}}\left[G_0^{\tau}(s) + i\,\nu^{-1/3} G_1^{\tau}(s) + O(\nu^{-2/3})\right]. \tag{7-126}$$

This approximation is precisely the same as that of the previous sections; i.e., it is assumed that (7-1) is, here,

$$\hat{g}^{\tau}_{\chi\chi}(\chi;\nu) << (i\nu)\hat{g}^{\tau}_{\chi}(\chi;\nu) << (i\nu)^2\hat{g}^{\tau}(\chi;\nu). \tag{7-127}$$

The same limitations in the regions of validity of the solution hold here, (7-2), (7-5), (7-6), (7-17) and (7-18), with $1/\nu$ replacing t. Equations (7-121) to (7-124) may thus be written

$$K_0^{\tau}(s) + i\nu^{-1/3}K_1^{\tau}(s) + o(\nu^{-2/3}) = \tan\pi s\, G_0^{\tau}(s) + i\nu^{-1/3}\tan\pi s\, G_1^{\tau}(s)$$

$$+ o(\nu^{-2/3}) = -G_0^{\tau}(s+3) - i\nu^{-1/3}\left[G_1^{\tau}(s+3) + 2G_0^{\tau}(s+2)\right] + o(\nu^{-2/3}), \tag{7-128}$$

with

$$G_0^{\tau}(s) + i\nu^{-1/3}G_1^{\tau}(s) + o(\nu^{-2/3}) \qquad \text{regular in} \qquad 0 < \Re(s) < \tfrac{7}{2}, \tag{7-129}$$

$$\frac{1}{s}\left[G_0^{\tau}(s) + i\nu^{-1/3}G_1^{\tau}(s) + o(\nu^{-2/3})\right] = \frac{1}{\pi s^2} \quad \text{near} \quad s = 0, \tag{7-130}$$

and

$$\sqrt{\pi}\,\nu^{-1/3}\left[G_0^{\tau}(\tfrac{1}{2}) + i\nu^{-1/3}G_1^{\tau}(\tfrac{1}{2}) + o(\nu^{-2/3})\right] = 0. \tag{7-131}$$

Equating like powers of $\nu^{-1/3}$ then gives the iteration scheme for successive approximations in ν.

The first approximation requires solution of

$$K_0^\tau(s) = \tan\pi s\, G_0^\tau(s) = -G_0^\tau(s+3), \tag{7-132}$$

subject to the conditions that

$G_0^\tau(s)$ is regular in the infinite strip, $0 < \mathcal{R}(s) < \frac{7}{2}$,

(7-133)

$$G_0^\tau(s) = \frac{1}{\pi s^2} \text{ near } s=0, \tag{7-134}$$

and

$$G_0^\tau(\tfrac{1}{2}) = 0; \tag{7-135}$$

the second approximation requires solution of

$$K_1^\tau(s) = \tan\pi s\, G_1^\tau(s) = -G_1^\tau(s+3) - 2\,G_0^\tau(s+2), \tag{7-136}$$

subject to the conditions that

$s\,G_1^\tau(s)$ is regular in the infinite strip, $0 \le \mathcal{R}(s) < \frac{7}{2}$,

(7-137)

and

$$G_1^\tau(\tfrac{1}{2}) = 0. \tag{7-138}$$

The difference equation, (7-132), of the first approximation is satisfied if

$$G_0^\tau(s) = -\frac{4}{\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})} \sin\pi\frac{s-1}{3} \sin\pi\frac{s-\frac{1}{2}}{3}\,\psi(s-3). \tag{7-139}$$

Furthermore, all the conditions, (7-133) to (7-135), are satisfied, including the Wagner integral condition. Using the results of Section 7.1, $(s-1)!\,G_0^\tau(s) \sim |g(s)|^{\mathcal{R}(s)-\frac{1}{2}} e^{-\frac{\pi}{3}|g(s)|}$ as $|g(s)| \to \infty$ along $\mathcal{R}(s)=c$. The complex ν may be written $\nu = |\nu| e^{i\arg\nu}$, and since $\nu^{-\frac{2s}{3}} \sim e^{\frac{2}{3}|\arg\nu||g(s)|}$ as $|g(s)| \to \infty$ along $\mathcal{R}(s)=c$, $\hat{G}^\tau(s;\nu) \sim |g(s)|^{c-\frac{1}{2}} e^{-\frac{\pi}{3}[1-\frac{2|\arg\nu|}{\pi}]|g(s)|}$ as $|g(s)| \to \infty$ along $\mathcal{R}(s)=c$. Therefore, since the Mellin transforms, (7-110) and (7-112), must be integrable along $\mathcal{R}(s)=c$ for $0<c<\frac{1}{2}$, it is necessary to restrict $|\arg\nu| < \frac{\pi}{2}$. This is a mild restriction for the case of steady-state oscillations, eliminating only purely divergent motion.

In a like manner, the inhomogeneous difference equation, (7-136), of the second approximation is satisfied if

$$G_1^\tau(s) = \frac{8}{3\sqrt{3}\,G_0(-\frac{1}{3})G_0(\frac{1}{3})}(s-1)\sin\pi\frac{s+1}{3}\sin\pi\frac{s+\frac{3}{2}}{3}\,\psi(s-4)$$
$$+\frac{16}{9\,G_0(-\frac{1}{3})G_0(\frac{1}{3})}\sin\pi\frac{s-1}{3}\sin\pi\frac{s-\frac{1}{2}}{3}\,\psi(s-3), \tag{7-140}$$

where the first term is the particular solution to the inhomogeneous equation, (7-136), and the second term is the general solution of the corresponding homogeneous equation. Both conditions, (7-137) and (7-138), are satisfied and the Mellin transform is again integrable if $|\arg\nu| < \frac{\pi}{2}$.

Higher approximations $G_2^\tau(s)$, etc. could also be found, but are not treatedhere. Also, higher approximations in μ could be found following the technique of Spence in II for the steady problem, but these, too, are omitted here.

The lift and pitching-moment coefficients can be evaluated, using (6-62), (6-63), (7-113), (7-126), (7-139), (7-140), (7-39) and (7-40), giving

$$\hat{C}_L = 2\sqrt{\pi\mu}\,\tau_0\left\{i + \frac{4}{3\sqrt{3}\,\nu^{1/3}} + O(\nu^{-2/3})\right\} \tag{7-141}$$

and

$$\hat{C}_M = \left(\tfrac{3}{4} - a\right)\hat{C}_L. \tag{7-142}$$

As derived in Chapter 6, the solution for airfoil motion is related to that for jet deflection by (6-85) to (6-87). Here, in particular, using (7-113), (7-126), (7-139), (7-140), (7-39) and (7-40),

$$\hat{g}^a(x;\nu) = -i\sqrt{\pi}\,\nu^2\left[1 + i\,\frac{4}{3\sqrt{3}}\,\nu^{-1/3} + O(\nu^{-2/3})\right]\int_x^\infty \hat{g}^\tau(\xi;\nu)\,d\xi, \quad 0<x<\infty \tag{7-143}$$

and

$$\hat{k}^a(x;\nu) = i\sqrt{\pi}\,\nu^2\left[1 + i\,\frac{4}{3\sqrt{3}}\,\nu^{-1/3} + O(\nu^{-2/3})\right]\int_0^x \hat{k}^\tau(\xi;\nu)\,d\xi, \quad 0<x<\infty, \tag{7-144}$$

while the important integral in the lift and pitching-moment coefficients is

$$\int_0^\infty \xi^{1/2}\hat{g}^a(\xi;\nu)\,d\xi = i\,\frac{\pi\nu^{1/3}}{\sqrt{3}}\left[1 - i\,\frac{\sqrt{3}}{2}\,\nu^{-1/3} + O(\nu^{-2/3})\right]. \tag{7-145}$$

These results check, of course, with those found by a direct Mellin-transform solution of the airfoil-motion problem.

For the plunging airfoil, from (6-70), (6-71), (4-23), (4-24), (4-25) and (7-145),

$$\hat{C}_L = \frac{\pi h_0}{2}\left(\frac{i\nu}{\mu}\right)^2\left[1 + \frac{4\mu}{\sqrt{3}\,\nu^{2/3}} - i\,\frac{2\mu}{\nu} + O(\mu\nu^{-4/3})\right] \tag{7-146}$$

and

$$\hat{C}_M = \frac{\pi h_0}{2}\left(\frac{i\nu}{\mu}\right)^2\left[\frac{1-2a}{2} + \sqrt{3}\left(1-\frac{4a}{3}\right)\frac{\mu}{\nu^{2/3}} - i\,\frac{\mu(1-4a)}{2\nu} + O(\mu\nu^{-4/3})\right]. \tag{7-147}$$

The leading term in both lift and pitching-moment coefficients is the apparent-mass contribution and is lowest-order in both ν and μ. The next term is circulatory in nature and is the leading jet effect. The last term, also circulatory, may be identified as precisely the leading circulatory term for the classical, $\mu = 0$, limit. That is, the leading jet-induced term, proportional to $\nu^{4/3}/\mu$, is lower-order than the leading classical circulatory term, which is recovered, however, in this approximate solution.

With like interpretation, the lift and pitching-moment coefficients for the pitching and blown-flap airfoils may be written. For the pitching airfoil they are

$$\hat{C}_L = 2\pi a_0\left(\frac{i\nu}{\mu}\right)^2\left[\frac{1-2e}{8} + \frac{\sqrt{3}}{4}\left(1-\frac{4e}{3}\right)\frac{\mu}{\nu^{2/3}} - i\,\frac{5}{8}\left(1-\frac{4e}{5}\right)\frac{\mu}{\nu} + O(\mu\nu^{-4/3})\right] \tag{7-148}$$

and

$$\hat{C}_M = 2\pi\alpha_0\left(\frac{i\nu}{\mu}\right)^2\left\{\frac{1+8(1-2a)(1-2e)}{128} + \frac{3\sqrt{3}}{16}\left(1-\frac{4a}{3}\right)\left(1-\frac{4e}{3}\right)\frac{\mu}{\nu^{2/3}}\right.$$

$$\left. - i\,\frac{8(e-a)+9\left(1-\frac{4a}{3}\right)\left(1-\frac{4e}{3}\right)\mu}{32\nu} + O(\mu\nu^{-4/3})\right\};$$

(7-149)

for the blown-flap airfoil,

$$\hat{C}_L = \frac{\beta_0}{2}\left(\frac{i\nu}{\mu}\right)^2\left\{\left[-\frac{\chi\cos\chi}{2} + \frac{\sin\chi}{3} + \frac{\sin\chi\cos^2\chi}{6}\right]\right.$$

$$+ \frac{1}{\sqrt{3}}\left[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\right]\frac{\mu}{\nu^{2/3}} - i\left[\frac{3}{2}(\chi - \sin\chi\cos\chi) + \sin\chi - \chi\cos\chi\right]\frac{\mu}{\nu}$$

$$\left. + O(\mu\nu^{-4/3})\right\}.$$

(7-150)

and

$$\hat{C}_M = \frac{\beta_0}{2}\left(\frac{i\nu}{\mu}\right)^2\left\{\frac{1}{32}\left[\chi - \frac{\sin\chi}{6}(10\cos\chi - 4\cos^3\chi)\right] + \frac{1-2a}{2}\left[-\frac{\chi\cos\chi}{2} + \frac{\sin\chi}{3} + \frac{\sin\chi\cos^2\chi}{6}\right]\right.$$

$$+ \frac{\sqrt{3}}{4}\left(1-\frac{4a}{3}\right)\left[\chi(1-2\cos\chi) + \sin\chi(2-\cos\chi)\right]\frac{\mu}{\nu^{2/3}} + i\left[-\frac{\chi\cos\chi}{2} + \frac{\sin\chi}{2} + \frac{\sin^3\chi}{3}\right.$$

$$\left. - \frac{(1-2a)}{2}(\chi - \sin\chi\cos\chi) - \frac{3}{8}\left(1-\frac{4a}{3}\right)\chi(1-2\cos\chi) - \frac{3}{8}\left(1-\frac{4a}{3}\right)\sin\chi(2-\cos\chi)\right]\frac{\mu}{\nu}$$

$$\left. + O(\mu\nu^{-4/3})\right\}.$$

(7-151)

It appears then, that in the sense that $(i\nu)\,\hat{g}_x(x;\nu)$ and $\hat{g}_{tt}(x;\nu)$ are neglected in (7-106), a solution of the problem of steady-state oscillations can be found satisfying all the equations of the problem, including both the trailing-edge boundary condition and the Wagner integral condition. Whereas there was failure to solve the transient problem for small time, solution to the oscillating problem has been found. The restriction, $|\arg \nu| < \frac{\pi}{2}$, imposed to insure existence of the Mellin transforms, is interesting in view of the Laplace-transform approach to the transient problem. As mentioned above, replacement of $(i\nu)$ by a real p, and multiplication of the result by $\bar{\mathfrak{F}}(p)$ should give the Laplace transform of the transient solution. Such an identification is excluded by the restriction $|\arg \nu| = |\arg(-ip)| < \frac{\pi}{2}$. This inability to extend the solutions to real p, hence failure to get a Laplace-transform solution of the transient problem, is consistent with the failure in Section 7.2 to get a similarity solution in $x/t^{2/3}$, for small time. However, these circumstances lead to the remarkable conclusion that, contrary to the usual experience in such problems (in particular the classical unsteady solution discussed in Sections 5.2 and 5.3), the transient response for small time is not related through its Laplace transform to the high-frequency response to the corresponding problem of steady-state oscillations. Finally, it should be remarked that Spence's latest approach to the transient problem, if correct, must be made to tie in consistently with the above solution for steady oscillations. It must be concluded, then, that the solutions given in this section for steady oscillations must be regarded as tentative, subject to further study, both of them and Spence's new results.

CHAPTER 8 - CONCLUSIONS

By extension of the existing steady, jet-flapped, thin-airfoil and classical, unsteady, thin-airfoil theories, a model for unsteady motions of jet-flapped thin airfoils has been formulated in the first four chapters. Study of these existing theories has clarified certain features of the unsteady jet-flapped-airfoil behavior. Such studies have suggested the feasibility of a "boundary-layer" transformation, which amplifies the region near the trailing edge of the airfoil for small jet-momentum strengths, μ , and for either small times after initiation of transient motions or for high-frequency steady-state oscillations.

The invalidity of the small-time "solution" found by Spence in III for the jet-deflection problem has been pointed out and discussed in detail. No correct solutions using Spence's approach could be found, however. Likewise Spence's error in III for the high-frequency response to steady-state oscillations in jet deflection has been pointed out. For this problem a tentative solution, as yet not fully understood, has been put forth. It satisfies the equations and all conditions of the problem in an apparently consistent sense, and can be extended to the corresponding airfoil-motion problem.

Spence's long-time solution in III, i.e., the approach to the steady solution after a transient jet deflection or airfoil motion, is the only established result remaining for this problem. Although not discussed in this report, this solution is found from the full equations by examining the neighborhood $x = c + U_0 t$ and showing that in the vicinity of this point for very large times there is concentrated an amount of circulation equal in magnitude but opposite in sign to the total circulation of airfoil plus jet in the steady solutions of I and II. Considering the interaction of this "starting circulation" and the "steady-flow circulation," the lift coefficient for approach to the steady solution was found to be

$$\frac{C_L(t)}{C_L(\infty)} = 1 - \frac{1}{4\pi}\left[\frac{\partial C_L(\infty)}{\partial \alpha} - C_J\right]\left(\frac{c}{U_0 t}\right) + o\left(\frac{c}{U_0 t}\right), \tag{8-1}$$

where $\partial C_L(\infty)/\partial\alpha$ is the lift-curve slope in the steady, incidence, jet-flap solution, say equation (65) of II, and t is the physical time.

The surface has only been scratched in finding the lift and pitching-moment responses to unsteady motions of jet-flapped airfoils. The approach through the "boundary-layer" transformation, if solved would only give results in a limited range of μ and t or ω. Nevertheless these equations have the strength of their relative simplicity and give some hope for further analytic attempts to solve them. It remains, as mentioned in the previous chapter, to investigate fully Spence's new, as yet unpublished, solution for the transient case, as well as the tentative high-frequency solutions for steady oscillations given in that chapter.

With the resolution and understanding of these limiting results, digital computation of the full equations would probably be necessary to give solutions for all intermediate times between small and large, and for all frequencies up to the high ones. As briefly discussed in Section 5.4, Spence in I obtained rapid convergence to the steady-state solution using a numerical collocation scheme, since the functions being sought had a monotonic behavior in x . A collocation scheme should also be applicable in the unsteady problems, although convergence would by no means be as rapid, due to the much more complicated behavior, in x and t or ω , of the unsteady solution. Furthermore, the collocation points would have to be chosen in a fashion to adequately handle the important effects in the immediate vicinity of the trailing edge, $x = c$, and those in the vicinity of $x = c + U_0 t$ or $x = c + U_0 (t - \frac{\lambda c}{U_0})$, say.

Finally, and probably most important, is the need for definitive experiments to test the validity of the model formulated here, and any solutions which might be found. Only in this way can the ultimate value of this theory be established.

It is strongly felt that further research along the above lines is worthwhile, not only to obtain results of use to the design engineer, but also to understand the interesting mathematical and physical points raised by the present model and its equations.

REFERENCES

Brocher, E. F. (1961) "The Jet-Flap Compressor Cascade" Journal of Basic Engineering, Vol. 83, Series D, no. 3, September 1961, pp. 401-407.

Carleman, T. (1922) "Sur la résolution de certaines equations intégrales" Arkiv För Matematik, Astronomi Och Fysik, Band 16, No. 26, May 1922, pp. 1-19.

Carrier, G.F. (1953) "Boundary Layer Problems in Applied Mechanics" in Advances in Applied Mechanics, Vol. III, Academic Press, Inc., N.Y.C., 1953.

Cheng, H.K. and Rott, N. (1954) "Generalization of the Inversion Formula of Thin Airfoil Theory" Journal of Rational Mechanics and Analysis, Vol. 3, no. 3, May 1954, pp. 357-382.

Cicala, P. (1941) "Present State of Development in Nonsteady Motion of a Lifting Surface" L'Aerotecnica, Vol. 21, no. 9-10 September-October 1941 (in Italian). For translation, see NACA Technical Memorandum No. 1277, October 1951.

Clark, E. L., Jr., and Ordway, D.E. (1959) "An Experimental Study of Jet-Flap Compressor Blades" Journal of the Aero/Space Sciences, Vol. 26, no. 11, November 1959, pp. 698-702.

Davidson, I.M. (1956) "The Jet Flap" Journal of the Royal Aeronautical Society, Vol. 60, no. 1, January 1956, pp. 25-50.

Dengler, M.A., Goland, M. and Luke, Y.L. (1952) "Notes on the Calculation of the Response of Stable Aerodynamic Systems" Journal of the Aeronautical Sciences, vol. 19, no. 3, March 1952, pp. 213-214.

Dorand, R. (1959) "The Application of the Jet Flap to Helicopter Rotor Control" Journal of the Helicopter Association of Great Britain, vol. 13, no. 6, December 1959, pp. 323-367.

Dwight, H.B. (1947) Tables of Integrals and other Mathematical Data. MacMillan Company, Revised Edition, 1947.

Friedrichs, K.O. (1955) "Asymptotic Phenomena in Mathematical Physics" Bulletin of the American Mathematical Society, Vol. 61, 1955, pp. 485-504.

Glauert, H. (1927) "Theoretical Relationships for an Aerofoil with Hinged Flap" A.R.C. Reports and Memoranda No. 1095, April 1927.

Glauert, H. (1943) "The Elements of Aerofoil and Airscrew Theory" Cambridge Univ. Press, American Edition, 1943.

Heaslet, M.A. and Lomax, H. (1954) "Supersonic and Transonic Small Perturbation Theory" Article D of "General Theory of High Speed Aerodynamics," Volume VI of High Speed Aerodynamics and Jet Propulsion, Princeton University Press, 1954.

Helmbold, H.B. (1955) "The Lift of a Blowing Wing in a Parallel Stream" Journal of the Aeronautical Sciences, Vol. 22, No. 5, May 1955, pp. 341-342.

Ho, H.T. (1961) "The Linearized Theory of a Supercavitating Hydrofoil with a Jet Flap" Hydronautics, Inc. Technical Report 119-2, June 1961.

Hobbs, N.P. (1957) "The Transient Downwash Resulting from the Encounter of an Airfoil with a Moving Gust Field" Journal of the Aeronautical Sciences, vol. 24, no. 10, October 1957, pp. 731-740, 754.

Jahnke, E. and Emde, F. (1945) Tables of Functions with Formulae and Curves, Dover Publications, Fourth Edition, 1945.

Jones, W.P. (1952) "The Generalized Theodorsen Function" Journal of the Aeronautical Sciences, Vol. 19, No. 3, March 1952, p. 213.

von Kármán, Th. and Sears, W. R. (1938) "Airfoil Theory for Non-Uniform Motion" Journal of the Aeronautical Sciences, vol. 5, no. 10, August 1938, pp. 379-390.

Kemp, N. (1952) "On the Lift and Circulation of Airfoils in some Unsteady-Flow Problems" Journal of the Aeronautical Sciences, vol. 19, no. 10, October 1952, pp. 713-714.

Korbacher, G. K. and Sridhar, K. (1960) "A Review of the Jet Flap" University of Toronto Institute of Aerophysics Review No. 14, May 1960.

Küssner, H.G. (1936) "Zusammenfassender Bericht über den instationären Auftrieb von Flügeln" Luftfahrtforschung, Band 13, Nr. 12, 20 Dezember 1936, pp. 410-424.

Laitone, E.V. (1952) "Theodorsen's Circulation Function for Generalized Motion" Journal of the Aeronautical Sciences, vol. 19, no. 3, March 1952, pp. 211-213.

Lapin, E. Crookshanks, R. and Hunter, H.F. (1952) "Downwash behind a Two-Dimensional Wing Oscillating in Plunging Motion" Journal of the Aeronautical Sciences, vol. 19, no. 7, July 1952, pp. 447-450, 458.

Luke, Y.L. and Dengler, M.A. (1951) "Tables of the Theodorsen Function for Generalized Motion" Journal of the Aeronautical Sciences, vol. 18, no. 7, July 1951, pp. 478-483.

Malavard, L.C. (1957) "Recent Developments in the Method of the Rheoelectric Analogy Applied to Aerodynamics" Journal of the Aeronautical Sciences, vol. 24, no. 5, May 1957, pp. 321-331.

Malavard, L., Poisson-Quinton, Ph. and Jousserandot, P. (1956) "Recherches théoretiques et experimentale sur le contrôle de circulation par soufflage appliqué aux ailes d'avions" ONERA Note Technique No. 37, June 1956. For translation, see either Princeton University Report No. 358, July 1956, or Aero Digest, vol. 73, no. 3, September, 1956, pp. 21-27; no. 4 October 1956, pp. 46-48, 50-59; no. 5, November 1956, pp. 34,36,38-46.

Miles, J.W. (1956) "The Aerodynamic Force on an Airfoil in a Moving Gust" Journal of the Aeronautical Sciences, Vol. 23, no. 11, November 1956, pp. 1044-1050.

Paulon, J. (1959) "Application du Soufflage au Bord de Fuite aux Contrôle de la Déflexion des Grilles d'Aubes" La Recherche Aeronautique, Numero 73, November-December 1959, pp. 17-20.

Prandtl, L. (1904) "Über Flüssigkeitsbewegung bei sehr kleiner Reibung" Proceedings III International Mathematical Congress, Heidelberg (1904). For translation, see NACA Technical Memorandum No. 452, 1928.

Prandtl, L. and Tietjens, O.G. (1934) Applied Hydro-and Aeromechanics Dover Publications, Inc. 1957.

Richards, E.J. and Jones, J.P. (1956) "The Application of the Jet Flap to Helicopter Rotors" Journal of the Helicopter Association of Great Britain, Vol. 9, no. 3, January 1956, pp. 414-423

Robinson, A. and Laurmann, J.A. (1956) "Wing Theory," Cambridge University Press, 1956.

Schwarz, L. (1943) "Untersuchen Einiger mit den Zylinderfunktionen Nullter Ordnung Verwandter Funktionen" Luftfahrtforschung, Band 20, Lfg. 12 8 Feb. 1944, pp. 341-372.

Sears, W. R. (1940) "Operational Methods in the Theory of Airfoils in Non-Uniform Motion" Journal of the Franklin Institute, vol. 230, no. 1, July-December 1940, pp. 95-111.

Sears, W. R. (1941) "Some Aspects of Non-Stationary Airfoil Theory and its Practical Application" Journal of the Aeronautical Sciences, vol. 8, no. 3, January 1941, pp. 104-108.

Sears, W. R. (1954) Theoretical Aerodynamics Part 1, Introduction to Theoretical Hydrodynamics, Ithaca, N.Y. Second Revised Edition, 1954.

Spence, D.A. (1956) - referred to as I - "The Lift Coefficient of a Thin, Jet-Flapped Wing" Proceedings of the Royal Society, A, vol. 238, no. 121, 4 December 1956, pp. 46-68.

Spence, D.A. (1958) "The Lift on a Thin Aerofoil with a Jet-Augmented Flap" The Aeronautical Quarterly, vol. 9, part 3, August 1958, pp. 287-299.

Spence, D.A. (1961A) - referred to as II - "The Lift Coefficient of a Thin, Jet-Flapped Wing II. A Solution of the Integro-Differential Equation for the Slope of the Jet" Proceedings of the Royal Society, A, vol. 261, no. 1304, 11 April 1961, pp. 97-118.

Spence, D.A. (1961B) - referred to as III - "The Theory of the Jet-Flap for Unsteady Motion" Journal of Fluid Mechanics, Vol. 10, part 2, 1961, pp. 237-258.

Stratford, B.S. (1956) "Early Thoughts on the Jet-Flap" The Aeronautical Quarterly, Vol. 7, part 1, February 1956, pp. 45-59.

Theodorsen, Th. (1934) "General Theory of Aerodynamic Instability and the Mechanism of Flutter" NACA Technical Report No. 496, 1934.

Van de Vooren, A.I. (1952) "Generalization of the Theodorsen Function to Stable Oscillations" Journal of the Aeronautical Sciences, Vol. 19, no. 3, March 1952, pp. 209-211.

Wagner, H. (1925) "Dynamischer Auftrieb von Tragflügeln" Zeitschrift für Angewandte Mathematik und Mechanik, Band 5, Hefl. February 1925, pp. 17-35.

Webster, A.G. (1955) "Partial Differential Equations of Mathematical Physics" Second Corrected Edition, Dover Publications, Inc. 1955.

Yen, K. T. (1960) "On the Thrust Hypothesis for the Jet Flap Including Jet-Mixing Effects" Journal of the Aerospace Sciences, vol. 27, no. 8, August 1960, pp. 607-614.

APPENDIX A

Evaluation of Certain Integrals

Although many of the integrals to be evaluated in this Appendix may be found elsewhere, it is convenient to treat them in general and collect their results here for the particular applications required in the text.

The first type of integral to be treated is defined by

$$I_n^o(\alpha,\beta) \equiv \frac{1}{\pi}\int_\alpha^\beta \left(\frac{\xi}{c-\xi}\right)^{1/2} \xi^n \, d\xi \, , \tag{A-1}$$

for $0 \leq \alpha < \beta \leq c$, with integer values of n, such that $n \geq -1$. The superscript, o, refers to this type of integral and the subscript, n, to the exponent of ξ. The integral is readily evaluated by making the transformation,

$$\left.\begin{aligned} \xi &= c\sin^2\theta \\ d\xi &= 2c\sin\theta\cos\theta\, d\theta \end{aligned}\right\} \tag{A-2}$$

which gives

$$I_n^o(\alpha,\beta) = \frac{2c^{n+1}}{\pi}\int_{\sin^{-1}[(\alpha/c)^{1/2}]}^{\sin^{-1}[(\beta/c)^{1/2}]} \sin^{2(n+1)}\theta \, d\theta \, . \tag{A-3}$$

Using the trigonometric identity given by extension of equation 404 of Dwight (1947), namely

$$\sin^{2(n+1)}\theta \equiv \binom{2n+2}{n+1}\frac{1}{2^{2(n+1)}} + \frac{1}{2^{2n+1}}\sum_{k=1}^{n+1}(-1)^k\binom{2n+2}{n+k+1}\cos 2k\theta\,, \tag{A-4}$$

(A-3) may be immediately integrated to give

$$\begin{aligned} I_n^0(\alpha,\beta) = {} & \frac{2}{\pi}\left(\frac{c}{4}\right)^{n+1}\binom{2n+2}{n+1}\left\{\sin^{-1}\left[\left(\frac{\beta}{c}\right)^{1/2}\right] - \sin^{-1}\left[\left(\frac{\alpha}{c}\right)^{1/2}\right]\right\} \\ & + \frac{2}{\pi}\left(\frac{c}{4}\right)^{n+1}\sum_{k=1}^{n+1}(-1)^k\frac{1}{k}\binom{2n+2}{n+k+1}\left\{\sin\left[2k\sin^{-1}\left(\frac{\beta}{c}\right)^{1/2}\right] - \sin\left[2k\sin^{-1}\left(\frac{\alpha}{c}\right)^{1/2}\right]\right\}. \end{aligned} \tag{A-5}$$

A particular case of this is

$$I_n^0(0,c) = \left(\frac{c}{4}\right)^{n+1}\binom{2n+2}{n+1}. \tag{A-6}$$

The examples of (A-5) and (A-6) which are required in the text are

$$I_0^0\left(0,\frac{U_0t}{\lambda}\right) = \frac{c}{\pi}\sin^{-1}\left[\left(\frac{U_0t}{\lambda c}\right)^{1/2}\right] - \frac{1}{\pi}\,\frac{(U_0t)^{1/2}(\lambda c-U_0t)^{1/2}}{\lambda}\,, \tag{A-7}$$

$$I_0^0\left(c\cos^2\frac{\chi}{2},c\right) = \frac{c(\chi+\sin\chi)}{2\pi}\,, \tag{A-8}$$

$$I_0^0(0,c) = \frac{c}{2}\,, \tag{A-9}$$

$$I_1^{\circ}(c\cos^2\tfrac{\chi}{2}, c) = \frac{c^2}{8\pi}(3\chi + 4\sin\chi + \sin\chi\cos\chi), \tag{A-10}$$

and

$$I_1^{\circ}(0, c) = \frac{3c^2}{8}. \tag{A-11}$$

The second type of integral is

$$I_n^{1}(\alpha,\beta) = \frac{1}{\pi}\int_{\alpha}^{\beta}\left(\frac{c-\xi}{\xi}\right)^{1/2}\xi^n\, d\xi, \tag{A-12}$$

where $0 \leq \alpha < \beta \leq c$ with integer values of n, such that $n \geq 0$. Although this is readily evaluable using the transformation (A-2), it may be related directly to the previous integral, $I_n^{\circ}(\alpha,\beta)$. With the identity

$$\left(\frac{c-\xi}{\xi}\right)^{1/2} \equiv \left(\frac{\xi}{c-\xi}\right)^{1/2}\left(\frac{c}{\xi}-1\right), \tag{A-13}$$

it is seen from (A-1) that

$$I_n^{1}(\alpha,\beta) = c\, I_{n-1}^{\circ}(\alpha,\beta) - I_n^{\circ}(\alpha,\beta). \tag{A-14}$$

A particular case is

$$I_n^{1}(0,c) = \frac{2}{n+1}\left(\frac{c}{4}\right)^{n+1}\binom{2n}{n}. \tag{A-15}$$

The required examples are

$$I_1'\left(0, \frac{U_0 t}{\lambda}\right) = \frac{c^2}{4\pi} \sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right] - \frac{c}{4\pi} \frac{(U_0 t)^{1/2}(\lambda c - U_0 t)^{1/2}}{\lambda} + \frac{1}{2\pi} \frac{(U_0 t)^{3/2}(\lambda c - U_0 t)^{1/2}}{\lambda^2}, \tag{A-16}$$

$$I_1'\left(c\cos^2\frac{\chi}{2}, c\right) = \frac{c^2}{8\pi}\left(\chi - \sin\chi\cos\chi\right), \tag{A-17}$$

$$I_1'(0, c) = \frac{c^2}{8}, \tag{A-18}$$

$$I_2'\left(0, \frac{U_0 t}{\lambda}\right) = \frac{c^3}{8\pi} \sin^{-1}\left[\left(\frac{U_0 t}{\lambda c}\right)^{1/2}\right] - \frac{c^2}{8\pi} \frac{(U_0 t)^{1/2}(\lambda c - U_0 t)^{1/2}}{\lambda}$$
$$- \frac{c}{12\pi} \frac{(U_0 t)^{3/2}(\lambda c - U_0 t)^{1/2}}{\lambda^2} + \frac{1}{3\pi} \frac{(U_0 t)^{5/2}(\lambda c - U_0 t)^{1/2}}{\lambda^3}, \tag{A-19}$$

$$I_2'\left(c\cos^2\frac{\chi}{2}, c\right) = \frac{c^3\chi}{16\pi} + \frac{c^3\sin\chi}{24\pi} - \frac{c^3}{16\pi}\sin\chi\cos\chi - \frac{c^3}{24\pi}\sin\chi\cos^2\chi, \tag{A-20}$$

$$I_2'(0, c) = \frac{c^3}{16}, \tag{A-21}$$

$$I_3'\left(c\cos^2\frac{\chi}{2}, c\right) = \frac{5c^4}{128\pi}\chi + \frac{c^4}{384\pi}\sin\chi\left[16 - 9\cos\chi - 16\cos^2\chi - 6\cos^3\chi\right], \tag{A-22}$$

and

$$I_3'(0, c) = \frac{5c^4}{128}. \tag{A-23}$$

The third type of integral is

$$I_n^2(\alpha,\beta,\chi) \equiv \frac{1}{\pi}\int_{\alpha}^{\beta}\left(\frac{\xi}{c-\xi}\right)^{1/2}\frac{\xi^n d\xi}{\xi-\chi} \quad , \tag{A-24}$$

where $0 \leq \alpha < \beta \leq c$ and n is an integer such that $n \geq -1$. The integral exists for $0 < \chi < \alpha$, $\beta < \chi < c$ and $\chi > c$, and in the sense of the Cauchy Principal Value for $\alpha < \chi < \beta$.

Substitution of (A-2) makes it

$$I_n^2(\alpha,\beta,\chi) = \frac{2c^n}{\pi}\int_{\sin^{-1}[(\alpha/c)^{1/2}]}^{\sin^{-1}[(\beta/c)^{1/2}]}\frac{\sin^{2(n+1)}\theta\, d\theta}{\sin^2\theta - \frac{\chi}{c}} \quad . \tag{A-25}$$

The integrand may be rewritten, using the identity

$$\frac{\sin^{2(n+1)}\theta}{\sin^2\theta - \frac{\chi}{c}} \equiv \sum_{k=0}^{n}\left(\frac{\chi}{c}\right)^{n-k}\sin^{2k}\theta - \left(\frac{\chi}{c}\right)^{n+1}\frac{1}{\frac{\chi}{c}-\sin^2\theta} \quad , \tag{A-26}$$

and the definition of $I_n^0(\alpha,\beta)$, (A-1), as

$$I_n^2(\alpha,\beta,\chi) = \sum_{k=0}^{n}\chi^{n-k}I_{k-1}^0(\alpha,\beta) - \frac{2\chi^{n+1}}{\pi c}\int_{\sin^{-1}[(\alpha/c)^{1/2}]}^{\sin^{-1}[(\beta/c)^{1/2}]}\frac{d\theta}{\frac{\chi}{c}-\sin^2\theta} \quad . \tag{A-27}$$

The remaining integral may be evaluated using equation 436-7 of Dwight (1947), as

$$\int_{\sin^{-1}[(\alpha/c)^{1/2}]}^{\sin^{-1}[(\beta/c)^{1/2}]} \frac{d\theta}{\frac{x}{c}-\sin^2\theta} = \begin{cases} \dfrac{c}{2x^{1/2}(c-x)^{1/2}}\left\{\ln\left|\dfrac{\left(\frac{c-\beta}{\beta}\right)^{1/2}+\left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{c-\beta}{\beta}\right)^{1/2}-\left(\frac{c-x}{x}\right)^{1/2}}\right| - \ln\left|\dfrac{\left(\frac{c-\alpha}{\alpha}\right)^{1/2}+\left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{c-\alpha}{\alpha}\right)^{1/2}-\left(\frac{c-x}{x}\right)^{1/2}}\right|\right\}, & 0<x<c \quad \text{(A-28)} \\ \\ \dfrac{c}{x^{1/2}(x-c)^{1/2}}\left\{\tan^{-1}\left[\left(\frac{\beta}{c-\beta}\right)^{1/2}\left(\frac{x-c}{x}\right)^{1/2}\right] - \tan^{-1}\left[\left(\frac{\alpha}{c-\alpha}\right)^{1/2}\left(\frac{x-c}{x}\right)^{1/2}\right]\right\}, & c<x<\infty. \end{cases}$$

(A-29)

The final result is, in general,

$$I_n^2(\alpha,\beta,x) = \begin{cases} \displaystyle\sum_{k=0}^{n} x^{n-k} I_{k-1}^0(\alpha,\beta) - \frac{x^n}{\pi}\left(\frac{x}{c-x}\right)^{1/2}\left\{\ln\left|\dfrac{\left(\frac{c-\beta}{\beta}\right)^{1/2}+\left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{c-\beta}{\beta}\right)^{1/2}-\left(\frac{c-x}{x}\right)^{1/2}}\right| - \ln\left|\dfrac{\left(\frac{c-\alpha}{\alpha}\right)^{1/2}+\left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{c-\alpha}{\alpha}\right)^{1/2}-\left(\frac{c-x}{x}\right)^{1/2}}\right|\right\}, \\ \qquad 0<x<c \quad \text{(A-30)} \\ \\ \displaystyle\sum_{k=0}^{n} x^{n-k} I_{k-1}^0(\alpha,\beta) - \frac{2x^n}{\pi}\left(\frac{x}{x-c}\right)^{1/2}\left\{\tan^{-1}\left[\left(\frac{\beta}{c-\beta}\right)^{1/2}\left(\frac{x-c}{x}\right)^{1/2}\right] - \tan^{-1}\left[\left(\frac{\alpha}{c-\alpha}\right)^{1/2}\left(\frac{x-c}{x}\right)^{1/2}\right]\right\}, \\ \qquad c<x<\infty. \end{cases}$$

(A-31)

A particular case is

$$I_n^2(0,c,x) = \begin{cases} \displaystyle\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{c}{4}\right)^k x^{n-k}, & 0<x<c \\[2ex] \displaystyle\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{c}{4}\right)^k x^{n-k} - x^n\left(\frac{x}{x-c}\right)^{1/2}, & c<x<\infty . \end{cases} \tag{A-32}$$

(A-33)

The particular integrals necessary to the text are

$$I_0^2\left(0,\frac{U_0t}{\lambda},x\right) = \begin{cases} \dfrac{2}{\pi}\sin^{-1}\left[\left(\dfrac{U_0t}{\lambda c}\right)^{1/2}\right] - \dfrac{1}{\pi}\left(\dfrac{x}{c-x}\right)^{1/2} \ln\left|\dfrac{\left(\frac{\lambda c-U_0t}{U_0t}\right)^{1/2}+\left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{\lambda c-U_0t}{U_0t}\right)^{1/2}-\left(\frac{c-x}{x}\right)^{1/2}}\right|, & 0<x<c \\[2ex] \dfrac{2}{\pi}\sin^{-1}\left[\left(\dfrac{U_0t}{\lambda c}\right)^{1/2}\right] - \dfrac{2}{\pi}\left(\dfrac{x}{x-c}\right)^{1/2}\tan^{-1}\left[\left(\dfrac{U_0t}{\lambda c-U_0t}\right)^{1/2}\left(\dfrac{x-c}{x}\right)^{1/2}\right], & c<x<\infty, \end{cases} \tag{A-34}$$

(A-35)

$$I_0^2\left(c\cos^2\frac{\chi}{2},c,x\right) = \begin{cases} \dfrac{\chi}{\pi} + \dfrac{1}{\pi}\left(\dfrac{x}{c-x}\right)^{1/2}\ln\left|\dfrac{\tan\frac{\chi}{2}+\left(\frac{c-x}{x}\right)^{1/2}}{\tan\frac{\chi}{2}-\left(\frac{c-x}{x}\right)^{1/2}}\right|, & 0<x<c \\[2ex] \dfrac{\chi}{\pi} - \dfrac{2}{\pi}\left(\dfrac{x}{x-c}\right)^{1/2}\tan^{-1}\left[\left(\dfrac{x}{x-c}\right)^{1/2}\tan\dfrac{\chi}{2}\right], & c<x<\infty, \end{cases} \tag{A-36}$$

(A-37)

$$I_0^2(0,c,x) = \begin{cases} 1\,, & 0<x<c \\ 1-\left(\dfrac{x}{x-c}\right)^{1/2}, & c<x<\infty\,, \end{cases} \tag{A-38}$$

(A-39)

$$I_1^2\left(c\cos^2\tfrac{\chi}{2},c,x\right) = \begin{cases} \dfrac{\chi}{\pi}\left(x+\dfrac{c}{2}\right)+\dfrac{c}{2\pi}\sin\chi+\dfrac{x}{\pi}\left(\dfrac{x}{c-x}\right)^{1/2}\ln\left|\dfrac{\tan\frac{\chi}{2}+\left(\frac{c-x}{x}\right)^{1/2}}{\tan\frac{\chi}{2}-\left(\frac{c-x}{x}\right)^{1/2}}\right|, & 0<x<c \quad \text{(A-40)} \\ \dfrac{\chi}{\pi}\left(x+\dfrac{c}{2}\right)+\dfrac{c}{2\pi}\sin\chi-\dfrac{2x}{\pi}\left(\dfrac{x}{x-c}\right)^{1/2}\tan^{-1}\left[\tan\dfrac{\chi}{2}\left(\dfrac{x}{x-c}\right)^{1/2}\right], & c<x<\infty\,, \quad \text{(A-41)} \end{cases}$$

and

$$I_1^2(0,c,x) = \begin{cases} x+\dfrac{c}{2}\,, & 0<x<c \\ x+\dfrac{c}{2}-x\left(\dfrac{x}{x-c}\right)^{1/2}, & c<x<\infty\,. \end{cases} \tag{A-42}$$

(A-43)

The final type of integral is

$$I_n^3(\alpha,\beta,x) \equiv \frac{1}{\pi}\int_{\alpha}^{\beta}\left(\frac{c-\xi}{\xi}\right)^{1/2}\frac{\xi^n\,d\xi}{\xi-x}\,,$$

(A-44)

where $0 \le \alpha < \beta \le c$ and n is an integer such that $n \ge 0$. As for $I_n^3(\alpha,\beta,x)$, it exists for $0 < x < \alpha$, $\beta < x < c$ and $x > c$, and in the sense of the Cauchy Principal Value for $\alpha < x < \beta$. Use of the identity, (A-13), reduces it to

$$I_n^3(\alpha,\beta,x) = c\, I_{n-1}^2(\alpha,\beta,x) - I_n^2(\alpha,\beta,x) . \tag{A-45}$$

A particular case of interest is

$$I_n^3(0,c,x) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \left(\frac{c}{4}\right)^{k+1} \binom{2k}{k} \frac{2}{k+1}\, x^{n-1-k} - x^n , & 0 < x < c \quad \text{(A-46)} \\[2ex] \displaystyle\sum_{k=0}^{n-1} \left(\frac{c}{4}\right)^{k+1} \binom{2k}{k} \frac{2}{k+1}\, x^{n-1-k} - x^n + x^n \left(\frac{x-c}{x}\right)^{1/2} , & c < x < \infty , \quad \text{(A-47)} \end{cases}$$

The integrals required in the text are

$$I_0^3(0,\beta,x) = \begin{cases} -\dfrac{2}{\pi} \sin^{-1}\left[\left(\dfrac{\beta}{c}\right)^{1/2}\right] - \dfrac{1}{\pi}\left(\dfrac{c-x}{x}\right)^{1/2} \ln \left| \dfrac{\left(\frac{c-\beta}{\beta}\right)^{1/2} + \left(\frac{c-x}{x}\right)^{1/2}}{\left(\frac{c-\beta}{\beta}\right)^{1/2} - \left(\frac{c-x}{x}\right)^{1/2}} \right| , & 0 < x < c \quad \text{(A-48)} \\[2ex] -\dfrac{2}{\pi} \sin^{-1}\left[\left(\dfrac{\beta}{c}\right)^{1/2}\right] + \dfrac{2}{\pi}\left(\dfrac{x-c}{x}\right)^{1/2} \cot^{-1}\left[\left(\dfrac{c-\beta}{\beta}\right)^{1/2}\left(\dfrac{x}{x-c}\right)^{1/2}\right] , & c < x < \infty \quad \text{(A-49)} \end{cases}$$

$$I_0^3(0,c,x) = \begin{cases} -1, & 0 < x < c \\ -1 + \left(\frac{x-c}{x}\right)^{1/2}, & c < x < \infty, \end{cases}$$ (A-50)

(A-51)

$$I_1^3(0,c,x) = \begin{cases} -x + \frac{c}{2}, & 0 < x < c \\ -x + \frac{c}{2} + x\left(\frac{x-c}{x}\right)^{1/2}, & c < x < \infty, \end{cases}$$ (A-52)

(A-53)

and

$$I_2^3(0,c,x) = \begin{cases} -x^2 + \frac{cx}{2} + \frac{c^2}{8}, & 0 < x < c \\ -x^2 + \frac{cx}{2} + \frac{c^2}{8} + x^2\left(\frac{x-c}{x}\right)^{1/2}, & c < x < \infty. \end{cases}$$ (A-54)

(A-55)