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INVARIANT IMBEDDING AND TIME-DEPENDENT SCATTERING OF LIGHT IN A ONE-DIMENSIONAL MEDIUM

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SANTA MONICA • CALIFORNIA

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PREFACE

In this RAND Memorandum the authors present further mathematical results in the problem of radiative transfer in a one-dimensional medium. This subject has important implications for meteorology, astrophysics, and the detection of nuclear blasts.

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SUMMARY

In the present Memorandum, by means of the invariant-imbedding technique, the integral equations for the reflection and transmission coefficients of radiation in a one-dimensional medium are obtained, allowing for the release of absorbed energy with a random time delay. Furthermore, the reflected and transmitted intensities for the fluorescence problem in a one-dimensional medium are expressed in terms of these coefficients, assuming no distribution of emitting sources in the medium.

CONTENTS

INVARIANT IMBEDDING AND TIME-DEPENDENT SCATTERING OF LIGHT IN A ONE-DIMENSIONAL MEDIUM

1. INTRODUCTION

In a recent series of papers, based on the principle of invariant imbedding, which stems from the invariance principles of Ambarzumian and Chandrasekhar, various kinds of time-dependent neutron-transport problems in a fixed rod of finite length made of fissionable material were exactly treated by Bellman and Kalaba [1] and Wing [12]. It was shown that the Laplace transform of the integral equation for the reflected flux derived by them provides an analytic expression of the solution because of the convolution form of the integral term [10]. Furthermore, the technique was extended [2] to the derivation of a functional equation governing the reflected neutron flux from a rod of varying length; the analytical study of the solution of the partial differential integral equation was made with the aid of an iterative procedure [11].

In the theory of radiative transfer, Miss Busbridge [5] used the Laplace transform for reducing the non-stationary transfer equation to the stationary one. The invariant-imbedding technique and the principle-of-invariance method were applied by Bellman, Kalaba, and Ueno [3], [9] to time-dependent diffuse reflection problems of parallel rays by an inhomogeneous flat layer.

On the other hand, time-dependent scattering problems of light in a one-dimensional medium were discussed by several Russian astrophysicists. Introducing the duration of temporal capture t_1 and the mean free time t_2 in formulation of the transfer equation, Sobolev [8] extended the probabilistic method to some transient scattering problems. For simplicity, he derived an exact solution in a semi-infinite homogeneous medium when $t_1 \gg t_2$. With the aid of the Laplace transform method, Minin [7] obtained an explicit expression of the quantum reflection probability and of the quantum emergence probability at any level from the semi-infinite homogeneous medium, allowing for both time parameters, t_1 and t_2. Recently, Kaplan [6] also extended Sobolev's procedure to the three-dimensional nonsteady case and discussed the functional equation for the reflection coefficients in a semi-infinite homogeneous medium when $t_1 \gg t_2$.

In a series of papers, making use of the probabilistic method, Kaplan and others [6] considered the scattering of light in a one-dimensional semi-infinite homogeneous medium with a moving boundary. Biberman and Veklenko [4] also extended their probabilistic approach to the derivation of relationships for the generalized reciprocity principle for noncoherent scattering in radiative transfer. Lists of many other papers attempting to solve the time-dependent transfer process will be found in the references of our preceding papers [3], [9].

In the present Memorandum, the invariant-imbedding technique is used to derive the integral equations for the coefficients of reflection and transmission of radiation by a one-dimensional homogeneous medium of finite optical thickness, allowing for two characteristic time-scales, t_1 and t_2. Furthermore, the reflected and transmitted intensities from this medium, illuminated by a unit step-function time-dependent pencil of radiation from outside, are expressed in terms of these coefficients, assuming no distribution of emitting sources in the medium.

2. THE EQUATION OF TRANSFER

Consider a one-dimensional homogeneous medium of optical thickness $\tau_1 > 0$, illuminated by radiation of time-dependent specific intensity I_0 incident on the right-hand boundary $\tau = \tau_1$ (see Fig. 1). Scattering of light in either direction is assumed to be equally probable.

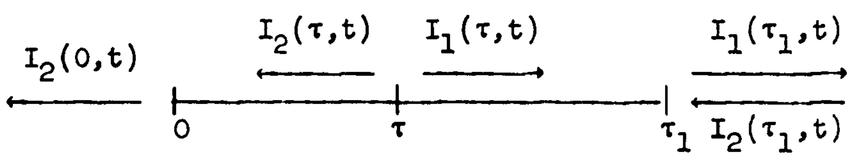

Fig. 1

A Time-dependent Transport Process

Let $I_1(\tau,t)$ and $I_2(\tau,t)$ denote respectively the specific intensities of radiation at the level τ at time t, directed toward the boundaries $\tau = \tau_1$ and $\tau = 0$.

The equation of transfer appropriate to the present case takes the form

$$(2.1) \qquad \frac{\partial I_1}{\partial \tau} + \frac{1}{c\ell}\frac{\partial I_1}{\partial t} + I_1 = B(\tau,t),$$

$$(2.2) \qquad -\frac{\partial I_2}{\partial \tau} + \frac{1}{c\ell}\frac{\partial I_2}{\partial t} + I_2 = B(\tau,t),$$

where c is the speed of light, ℓ is the attenuation coefficient, and $B(\tau,t)$ is the source function.

We assume that the duration of temporal capture, which corresponds to the mean molecular interaction time in the kinetic theory of dilute gases, is equal to t_1, and the probability of emission during the interval of time $(t, t + dt)$ is given by $\exp(-t/t_1)dt/t_1$.

Assuming the above emission probability in the successive scattering processes, consisting of absorption and emission, we see that the source function $B(\tau,t)$ can be written in the form

$$(2.3) \qquad B(\tau,t) = \frac{a}{2}\int_{-\infty}^{t} [I_1(\tau,t') + I_2(\tau,t')]\, e^{-(t-t')/t_1}\, \frac{dt'}{t_1},$$

where a is the albedo for single scattering, i.e., the probability of quantum survival. Under some limited conditions Sobolev [8] discussed the solution of (2.1) – (2.2).

If we write

$$c\ell = 1/t_2, \tag{2.4}$$

where t_2 is the mean free time, the equations of transfer (2.1) – (2.2) become

$$\frac{\partial I_1}{\partial \tau} + t_2 \frac{\partial I_1}{\partial t} = - I_1 + B(\tau,t), \tag{2.5}$$

$$-\frac{\partial I_2}{\partial \tau} + t_2 \frac{\partial I_2}{\partial t} = - I_2 + B(\tau,t), \tag{2.6}$$

together with the boundary and initial conditions

$$I_1(0,t) = 0, \quad I_2(\tau_1,t) = I_0(t), \tag{2.7}$$

$$I_1(\tau,0) = 0, \quad I_2(\tau,0) = 0 \qquad (0 \leq \tau \leq \tau_1), \tag{2.8}$$

where $I_1(\tau_1,t)$ and $I_2(0,t)$ are called respectively the <u>reflected</u> and <u>transmitted intensities</u>, and $I_2(\tau_1,t)$ is the intensity incident on the boundary $\tau = \tau_1$ at time t.

3. THE REFLECTED INTENSITY

Let $R(\tau_1,t)$ denote the coefficient of reflection. Consider that at time $t = t_0$ $(0 \leq t_0 \leq t)$ the incident radiation falls on the boundary $\tau = \tau_1$. Because the optical properties of the medium are constant with respect to time, for convenience we can put t_0 equal to zero without loss of generality. Then, by definition,

$$(3.1)\qquad I_1(\tau_1,t) = \int_{-\infty}^{t} R(\tau_1,t-t')I_2(\tau_1,t')dt',$$

where $I_2(\tau_1,t)$ is given by (2.7).

We shall seek an integral equation for the reflection coefficient $R(\tau_1,t)$, making use of the invariant-imbedding technique.

Imbedding the one-dimensional medium of the optical thickness τ_1 in position and time, we have

$$(3.2)\qquad I_1(\tau_1+\Delta, t+t_2\Delta)$$
$$= I_1(\tau_1,t) + \Delta\cdot\{-I_1(\tau_1,t) + B(\tau_1,t)\} + o(\Delta),$$

where $o(\Delta)$ is of the order of magnitude of the infinitesimal Δ^2. From (2.6) we get

$$(3.3)\qquad \hat{I}_2(\tau_1,t) = I_2(\tau_1+\Delta, t-t_2\Delta)$$
$$- I_2(\tau_1,t)\Delta + B(\tau_1,t)\Delta + o(\Delta).$$

By initial condition (2.8), (3.3) becomes

$$(3.4)\qquad \hat{I}_2(\tau_1,t) = I_0(t-t_2\Delta) - I_0(t)\Delta$$
$$+ \frac{a}{2}\,\Delta\int_{-\infty}^{t} I_1(\tau_1,t')e^{-(t-t')/t_1}\,\frac{dt'}{t_1}$$
$$+ \frac{a}{2}\,\Delta\int_{-\infty}^{t} I_0(t')e^{-(t-t')/t_1}\,\frac{dt'}{t_1}.$$

Making use of (3.1) and (3.4), we obtain

$$(3.5)\qquad I_1(\tau_1,t) = \int_{-\infty}^{t} R(\tau_1, t - t')\hat{I}_2(\tau_1,t')dt'$$

$$= \int_{-\infty}^{t} R(\tau_1, t - t')I_0(t' - t_2\Delta)dt'$$

$$- \Delta\int_{-\infty}^{t} R(\tau_1, t - t')I_0(t')dt'$$

$$+ \frac{a}{2}\Delta\int_{-\infty}^{t} R(\tau_1, t - t')dt'$$

$$\cdot \int_{-\infty}^{t'} I_1(\tau_1, t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}$$

$$+ \frac{a}{2}\Delta\int_{-\infty}^{t} R(\tau_1, t - t')dt'$$

$$\cdot \int_{-\infty}^{t'} I_0(t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}\ .$$

On the other hand, we have

$$(3.6)\qquad I_1(\tau_1 + \Delta, t + t_2\Delta)$$

$$= \int_{-\infty}^{t+t_2\Delta} R(\tau_1 + \Delta, t + t_2\Delta - t')I_0(t')dt'.$$

With the aid of (3.5) and (3.6), (3.2) becomes

$$(3.7)\qquad \int_{-\infty}^{t+t_2\Delta} R(\tau_1 + \Delta, t + t_2\Delta - t')I_0(t')dt'$$

$$= \int_{-\infty}^{t} R(\tau_1, t - t')I_0(t' - t_2\Delta)dt'$$

$$- 2\Delta \int_{-\infty}^{t} R(\tau_1, t - t')I_0(t')dt'$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} R(\tau_1, t - t')dt' \int_{-\infty}^{t'} I_1(\tau_1, t'')e^{-(t'-t'')/t_1} \frac{dt''}{t_1}$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} R(\tau_1, t - t')dt' \int_{-\infty}^{t'} I_0(t'')e^{-(t'-t'')/t_1} \frac{dt''}{t_1}$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} I_1(\tau_1, t')e^{-(t-t')/t_1} \frac{dt'}{t_1}$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} I_0(t')e^{-(t-t')/t_1} \frac{dt'}{t_1} .$$

<u>The Dirac delta-function time-dependent case</u>. In what follows, we shall treat the case of the Dirac delta time-dependent function, $I_0(t) = F\delta(t)$, where F is a constant and δ is the Dirac delta-function. The substitution of $I_0(t) = F\delta(t)$ into (3.7) provides

$$(3.8)\qquad R(\tau_1 + \Delta, t + t_2\Delta) = R(\tau_1, t - t_2\Delta) - 2\Delta R(\tau_1, t)$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} R(\tau_1, t - t')dt' \int_{-\infty}^{t'} R(\tau_1, t'')e^{-(t'-t'')/t_1} \frac{dt''}{t_1}$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} R(\tau_1, t - t')e^{-t'/t_1} \frac{dt'}{t_1}$$

$$+ \frac{a}{2}\Delta \int_{-\infty}^{t} R(\tau_1, t')e^{-(t-t')/t_1} \frac{dt'}{t_1} + \frac{a}{2}\Delta \frac{e^{-t/t_1}}{t_1} .$$

Hence, letting $\Delta \to 0$, we get

$$(3.9)\qquad \frac{\partial R}{\partial \tau_1} + 2t_2\frac{\partial R}{\partial t} + 2R$$
$$= a\left[\frac{e^{-t/t_1}}{2t_1} + \int_{-\infty}^{t} R(\tau_1, t - t')e^{-t'/t_1}\frac{dt'}{t_1}\right.$$
$$\left. + \frac{1}{2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} R(\tau_1, t - t')R(\tau_1, t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}\right].$$

The conditions imposed on R are

$$(3.10)\qquad R(\tau_1, t) = 0 \text{ for } 0 \geq t; \quad R(0, t) = 0 \text{ for } t \geq 0.$$

Equation (3.9) is the requisite integral equation governing the reflection coefficient $R(\tau_1, t)$.

The unit step-function time-dependent case. Consider a fluorescence problem for which the diffusely reflected light decreases for a long time after the sudden switching-off of the external radiation incident on the boundary, assuming no emitting sources in the medium. In the time interval $(-\infty, 0)$, let the radiation of constant intensity F fall on the boundary $\tau = \tau_1$, and at time $t = 0$ let the incident radiation be suddenly quenched.

Writing

$$(3.11)\qquad I_2(\tau_1, t) = FH^*(t),$$

where

$$H^*(t) = \begin{cases} 1, & t < 0, \\ 0, & t > 0, \end{cases} \tag{3.12}$$

we find that the requisite intensity $I_1(\tau_1,t)$, reflected by the end $\tau = \tau_1$ at time t, is given by

$$\begin{aligned} I_1(\tau_1,t) &= F\int_{-\infty}^{t} R(\tau_1,t-t')H^*(t')dt' \\ &= F\int_{t}^{\infty} R(\tau_1,u)du, \end{aligned} \tag{3.13}$$

where $R_1(\tau_1,t)$ is governed by (3.9).

In the next place, consider a Heaviside unit-function time-dependent case, in which from the instant $t = 0$ the radiation of constant intensity F falls continuously on the boundary $\tau = \tau_1$ from the outside. We ask for the gradual increase of the intensity reflected from the medium at time $t > 0$. Writing

$$I_2(\tau_1,t) = FH(t), \tag{3.14}$$

where

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases} \tag{3.15}$$

we obtain

$$\begin{aligned} I_1(\tau_1,t) &= F\int_{-\infty}^{t} R(\tau_1,t-t')H(t')dt' \\ &= F\int_{0}^{t} R(\tau_1,u)du, \end{aligned} \tag{3.16}$$

which provides the required reflected intensity in this case.

4. THE TRANSMITTED INTENSITY

Let $T(\tau_1,t)$ denote the coefficient of transmission. Then we have

$$(4.1) \qquad I_2(0,t) = \int_{-\infty}^{t} T(\tau_1,t-t')I_2(\tau_1,t')dt'.$$

We inquire into an integral equation for $T(\tau_1,t)$. In a manner similar to that used in the preceding section, we have

$$(4.2) \qquad I_2(0,t+t_2\Delta) = I_2(0,t) + \Delta\{-I_2(0,t) + B(0,t)\} + o(\Delta).$$

From (4.1), we obtain

$$(4.3) \qquad I_2(0,t+t_2\Delta) = \int_{-\infty}^{t+t_2\Delta} T(\ _1+\Delta,t+t_2\Delta-t')I_0(t')dt'.$$

On the other hand, using (3.4), we see that the transmitted intensity $I_2(0,t)$ is given by

$$(4.4)\quad I_2(0,t) = \int_{-\infty}^{t} T(\tau_1,t-t')\hat{I}_2(\tau_1,t')dt'$$

$$= \int_{-\infty}^{t} T(\tau_1,t-t')I_0(t'-t_2\Delta)dt'$$

$$- \Delta\int_{-\infty}^{t} T(\tau_1,t-t')I_0(t')dt'$$

$$+ \frac{a}{2}\Delta\int_{-\infty}^{t} T(\tau_1,t-t')dt$$

$$\cdot \int_{-\infty}^{t'} I_1(\tau_1,t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}$$

$$+ \frac{a}{2}\Delta\int_{-\infty}^{t} T(\tau_1,t-t')dt'$$

$$\cdot \int_{-\infty}^{t'} I_0(t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}.$$

Then, recalling (2.8), (4.3), and (4.4), we find that (4.2) becomes

$$(4.5)\quad \int_{-\infty}^{t+t_2\Delta} T(\tau_1+\Delta,t+t_2\Delta-t')I_0(t')dt'$$

$$= \int_{-\infty}^{t} T(\tau_1,t-t')I_0(t'-t_2\Delta)dt' - \Delta\int_{-\infty}^{t} T(\tau_1,t-t')I_0(t')dt'$$

$$+ \frac{a}{2}\int_{-\infty}^{t} T(\tau_1,t-t')dt' \int_{-\infty}^{t'} I_1(\tau_1,t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}$$

$$+ \frac{a}{2}\Delta\int_{-\infty}^{t} T(\tau_1,t-t')dt' \int_{-\infty}^{t} I_0(t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}$$

$$+ \Delta\left\{- I_2(0,t) + \frac{a}{2}\int_{-\infty}^{t} I_2(0,t')e^{-(t-t')/t_1}\frac{dt'}{t_1}\right\} + o(\Delta).$$

<u>The Dirac delta-function time-dependent case</u>. Inserting $I_2(\tau_1,t) = F\delta(t)$ into (4.5) and letting $\Delta \to 0$, we have

$$\frac{\partial P}{\partial \tau_1} + 2t_2 \frac{\partial P}{\partial t} + 2T = a\left[\int_{-\infty}^{t} T(\tau_1,t-t')e^{-t'/t_1}\frac{dt'}{t_1} + \frac{1}{2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} T(\tau_1,t-t')R(\tau_1,t'')e^{-(t'-t'')/t_1}\frac{dt''}{t_1}\right], \tag{4.6}$$

along with the boundary and initial conditions

$$T(\tau_1,t) = 0 \quad (\tau_1 > 0,\ \ 0 \geq t \ \text{ or } \ t < \tau_1 t_\tau), \tag{4.7}$$
$$T(0,t) = F\delta(t) \quad (t \geq 0).$$

<u>The unit step-function time-dependent case</u>. To begin with, we consider the same quenching fluorescence problem as that treated in the preceding section.

Under the incident intensity $I_2(\tau_1,t)$, given by (3.11), the requisite intensity transmitted from the boundary $\tau = 0$ at time t is provided by

$$I_2(0,t) = F\int_{-\infty}^{t} T(\tau_1,t-t')H^*(t')dt' = F\int_{t}^{\infty} T(\tau_1,u)du, \tag{4.8}$$

when $T(\tau_1,t)$ satisfies (4.6).

Next, we seek the transmitted intensity when the boundary $\tau = \tau_1$ is illuminated by a pencil of external radiation of intensity $I_2(\tau_1,t)$, given by (3.13).

Then we have

$$(4.9) \qquad I_2(0,t) = F\int_{-\infty}^{t} P(\tau_1, t-t')H(t')dt'$$
$$= F\int_{0}^{t} T(\tau_1, u)du.$$

5. DISCUSSION

In later papers, the present approach will be applied to problems of light scattering in a finite one-dimensional medium with a moving boundary. Furthermore, the fluorescence problem in which the distribution of emitting sources in the one-dimensional medium is a function of τ and the external light source is distinguished at time zero will be considered. An analytical and computational study of the solutions of the integral equations for the R- and T-functions will also be made.

REFERENCES

1. Bellman, R., R. Kalaba, and G. M. Wing, "Invariant Imbedding and Mathematical Physics—I: Particle Processes," J. Math. Phys., Vol. 1, 1960, pp. 280–308.
2. ———, "Invariant Imbedding and Neutron Transport in a Rod of Varying Length," Proc. Nat. Acad. Sci. USA, Vol. 46, 1960, pp. 128–130.
3. Bellman, R., R. Kalaba, and S. Ueno, Invariant Imbedding and Time-dependent Diffuse Reflection by a Finite Inhomogeneous Atmosphere, The RAND Corporation, RM-2969-ARPA, February 1962; also published in ICARUS, Vol. 1, 1962, pp. 191–199.
4. Biberman, L. M., and B. A. Veklenko, "Generalized Reciprocity Principle," J. Exptl. Theoret. Phys. USSR, Vol. 39, 1960, pp. 88–93.
5. Busbridge, I. W., The Mathematics of Radiative Transfer, Cambridge Tract No. 50, London, 1960.
6. Kaplan, S. A., "On the Theory of Light Scattering in a Nonsteady State Medium," Astron. J. USSR, Vol. 39, 1962, pp. 702–709.
7. Minin, I. N., "On the Solution of a Non-stationary Problem in Radiative Transfer," Vestnik Leningrad University, No. 13, 1959, pp. 1–5.
8. Sobolev, V. V., Transfer of Radiation Energy in the Atmospheres of Stars and Planets, Moscow, 1956.
9. Ueno, S., "On the Time-dependent Principle of Invariance in a Semi-infinite Medium," J. Math. Anal. Appl., Vol. 4, 1962, pp. 1–8.
10. Wing, G. M., "Solution of a Time-dependent One-dimensional Neutron Transport Problem," J. Math. and Mech., Vol. 7, 1958, pp. 757–766.
11. ———, "Analysis of a Problem of Neutron Transport in a Changing Medium," J. Math. and Mech., Vol. 11, 1962, pp. 21–34.
12. ———, An Introduction to Transport Theory, John Wiley and Sons, London, 1962.