NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
THE ATYPICAL ZEROS OF A CLASS
OF ENTIRE FUNCTIONS

P. M. Anselone
and
R. P. Boas, Jr.

MRC Technical Summary Report #394
April 1963

Madison, Wisconsin
ABSTRACT

A homogeneous differential-difference-integral equation with constant coefficients and a convolution integral has an exponential solution \( \varphi(t) = e^{zt} \) if and only if \( z \) is the zero of a certain entire function. This paper is concerned with the asymptotic distribution of the zeros of such entire functions.
THE ATYPICAL ZEROS OF A CLASS OF ENTIRE FUNCTIONS

P. M. Anselone and R. P. Boas, Jr.

This note is concerned with the distribution of the zeros of certain entire functions which appear in the study of difference-integral and differential-difference-integral equations (cf. [1], [2], [4]). We shall deal first with a typical problem and generalize later. The equation

$$\varphi(t) - \varphi(t - 2) = \int_1^t K(t - s) \varphi(s) \, ds = \int_0^1 K(s) \varphi(t - s) \, ds,$$

with $K(s)$ integrable, has an exponential solution $\varphi(t) = e^{zt}$ if and only if $z$ is a zero of the characteristic function

$$\bar{\psi}(z) = e^{-z} - e^{-2z} - \int_0^1 e^{-zs} K(s) \, ds = e^{-z}[1 - e^{-z} - e^z \int_0^1 e^{-zs} K(s) \, ds].$$

Exponential solutions are important, for example, in a Laplace transform treatment of the equation.

Since $\bar{\psi}(z)$ is an entire function of exponential type and $\bar{\psi}(iy)$ is bounded, the zeros $r_n e^{i\theta_n}$, $n = 1, 2, \ldots$, of $\bar{\psi}(z)$ satisfy $\sum r_n^{-1} |\cos \theta_n| < \infty$ (cf., e.g., [3], ch. 8). This suggests that the zeros are concentrated near the imaginary axis. In particular, by the Riemann-Lebesgue lemma and Rouche's

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.
theorem, $e^z \psi(z)$, and hence $\psi(z)$, have zeros $2\pi n + o(1)$ as $n \to \pm \infty$.

It may, therefore, seem surprising that for a rather extensive class of functions $K(s)$ there are zeros of $\psi(z)$ of arbitrarily large real part.

We shall prove the following theorem.

Theorem. Let $K(s)$ be absolutely continuous, $K(0) = y \neq 0$, and $K'(s)$ essentially bounded. Let

$$\psi(z) = e^{-z}[1 + h(z) - e^z \int_0^1 e^{-zs} K(s) \, ds],$$

where

$$h(z) \to 0 \text{ uniformly in } y \text{ as } x \to \infty.$$

Then $\psi(z)$ has zeros $z_n$, $n = \pm N, \pm (N + 1), \ldots$, such that

$$z_n = \log |n| + 2\pi n + O(1).$$

(After proving the theorem we shall indicate how the condition $K(0) \neq 0$ can be relaxed.)

We cannot apply the usual Abelian theorems for Laplace transforms since we require an estimate valid near the imaginary axis. Instead, the proof is based on the
Lemma. If $A \neq 0$, then $z - Ae^z$ has zeros $\zeta_n$, $n = \pm 1, \pm 2, \ldots$, such that

$$\zeta_n = \log |n| + 2\pi i n + O(1).$$

If $z - Ae^z = 0$ and $z = -w$, then $w^w + A = 0$. The asymptotic distribution of the zeros of $w^w + A$ (in a form more precise than we require) is given by Wright [5, p. 199]. The lemma follows.

We now establish the theorem. Integrate by parts to obtain

$$\int_0^1 e^{-zs}K(s)\,ds = z^{-1}[\gamma - e^{-z}K(1) + B(z)]$$

where $\gamma = K(0)$ and

$$B(z) = \int_0^1 e^{-zs}K'(s)\,ds = O(1/x), \quad x > 0.$$  

Therefore,

$$ze^z w(z) = z - \gamma e^z + zh(z) + K(1) - e^z B(z).$$

Let $\zeta_n = \gamma e^n$ as in Lemma 1 with $A = \gamma$. Let $z = \zeta_n + \delta$, where $\delta$ is a fixed small complex number. Then
\[ z - ye^z = \zeta_n + \delta - ye^n = \zeta_n + \delta - \zeta_n \epsilon^\delta = \zeta_n (1 - \epsilon^\delta) + \delta, \]

\[ zh(z) = (\zeta_n + \delta) o(1) = o(1 | \zeta_n |) = o( | z - ye^z |), \]

\[ e^z B(z) = e^{\zeta_n + \delta} \xi_n B(\zeta_n + \delta) = \zeta_n o(1) = o(1 | \zeta_n |) = o( | z - ye^z |). \]

Therefore,

\[ ze^{\psi}(z) = z - ye^z + o( | z - ye^z |). \]

Consequently, by Rouché's theorem, \( ze^{\psi}(z) \) and \( z - ye^z \) have the same number of zeros with \( | z - \zeta_n | < | \delta | \), namely one, for each sufficiently large \( | n | \).

This establishes the theorem.

We can relax the requirement that \( K(0) \neq 0 \) to some extent. For a fixed positive integer \( m \), let \( K^{(m-1)}(s) \) be absolutely continuous, \( K(0) = \ldots = K^{(m-2)}(0) = 0 \), \( K^{(m-1)}(0) = \gamma \neq 0 \), and \( K^{(m)}(s) \) essentially bounded. Then \( m \) integrations by parts yield

\[ \frac{1}{0} e^{-z s} K(s) ds = \gamma z^{-m} e^{-z} [z^{-1} K(1) + \ldots + z^{-m} K^{(m-1)}(1)] + z^{-m} B(z), \]

where \( B(z) = O(1/x), \quad x > 0, \) as above. Therefore
\[ z^m e^z \varPsi(z) = z^m - y e^z + z^m h(z) + \left[ z^{m-1} K(1) + \cdots + K^{(m-1)}(1) \right] - e^z B(z). \]

Consider \( z^m - y e^z \). If \( z = m \xi \) and \( z = m \xi \), then \( z^m = A^m e^{m \xi} \) and \( z = (mA)^m e^z \). Let \( A = m^{-1} \gamma^{1/m} \) with the principal \( m \)th root to obtain \( z^m = y e^z \). Therefore, \( z^m - y e^z \) has zeros \( \xi_n \), \( n = \pm 1, \pm 2, \ldots \), where the \( \xi_n \) are as in the Lemma.

Essentially the same argument as in the proof of the theorem shows that \( \varPsi(z) \) has zeros \( \xi_n \), \( n = \pm N, \pm (N + 1), \ldots \), such that

\[ z^m = m \log |n| + 2\pi i mn + O(1). \]

Again \( \varPsi(z) \) has zeros of arbitrarily large real part.

We now consider briefly a general differential-difference-integral equation

of the form

\[ \sum_{k=0}^{q} \sum_{j=0}^{t} a_{j} \varphi^{(k)}(t-t) = \int_{t-c}^{t} K(s) \varphi(t-s) ds, \]

where \( K(s) \) is integrable, \( t_1 < \ldots < t_q \) and, without loss of generality, \( a_{j} \neq 0 \) for \( j = 0, 1, \ldots, p \). An exponential function \( \varphi(t) = e^{zt} \) satisfies the equation if and only if \( z \) is a zero of the characteristic function.
\[ \Psi(z) = \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk} z^j e^{-t_j z} - \int_{0}^{c} e^{-zs} K(s) \, ds. \]

Note that
\[ z^{-q_0} e^{0z} \Psi(z) = a_0 q_0 + h(z) - z^{-q_0} e^{0z} \int_{0}^{c} e^{-zs} K(s) \, ds, \]

where \( a_0 q_0 \neq 0 \) and
\[ h(z) \to 0 \text{ uniformly in } y \text{ as } x \to \infty. \]

If \( t_0 \leq 0 \), then \( z^{-q_0} e^{0z} \Psi(z) \to a_0 q_0 \) uniformly in \( y \) as \( x \to \infty \), so that the zeros of \( \Psi(z) \) are bounded above in real part. If \( t_0 > 0 \) then, under the conditions on \( K(s) \) assumed above, \( \Psi(z) \) has zeros of arbitrarily large real part.

The proof is essentially the same as before.

Similarly, if \( t_p \geq c \), then the zeros of \( \Psi(z) \) are bounded below in real part. If \( t_p < c \) and \( K(s) \) satisfies the above conditions with \( s = 0 \) replaced by \( s = c \), then \( \Psi(z) \) has zeros of arbitrarily small (negative, of large absolute value) real part.

With the foregoing results we are able to give a complete description of the asymptotic distribution of the zeros of \( \Psi(z) \) for a large class of kernels \( K(s) \).
REFERENCES


