AFCRL - 63 - 259

63 3 4

MONITORING AGENCY DOCUMENT Nº

ASTIA DOCUMENT Nº

UNDAMPED WAVES IN A COLLISION FREE ELECTRON PLASMA

Donald Mc Lean, Observatoire de Meudon (France)

TECHNICAL NOTE Nº3

CONTRACT Nº AF 61 (052) - 432

July, 10th, 1962.

"The research reported in this document has been sponsored in part by the
AIR FORCE CAMBRIDGE CENTER
of the AIR RESEARCH AND DEVELOPMENT COMMAND, UNITED STATES AIR FORCE,
through its European Office".

Attachment # 2

UNDAMPED WAVES IN A COLLISION FREE ELECTRON PLASMA

ooo

By Donald Mc LEAN

INTRODUCTION

Recently Bernstein, Green and Kruskal [1] have shown by a non-linear theory that there can exist undamped longitudinal waves in a "collision free" plasma, and further, they claim that these waves do not satisfy a dispersion relation.

This is in complete contradiction to the results of Landau [2] and of Van Kampen [3], both based on the same linear approximation to the Boltzmann equation, and apparently equivalent, which predict that all sufficiently smooth initial distribution functions give rise to damped potential waves, the minimum damping rate being given by Landau. It is true that, if the distr bution functions giving rise to the B.G.K., undamped waves are replaced by singular distributions, one can demonstrate undamped waves in the linear theory but since the distribution functions corresponding to the B.G.K. equilibria are not in general singular, the linear and non-linear theories must be considered to be in contradiction.

Landau and Van Kampen did find a dispersion relation in the sense that the most slowly damped waves are those which satisfy the relation, previously given by Vlasov [4].

This dispersion relation is basically the same as that found by Bohm and Gross [5], by assuming that there are no particles near the wave velocity. In the same paper Bohm and Gross showed that the presence of trapped electrons would modify their results, but argued, intuitively, that only those waves satisfying their dispersion relation would be strongly excited.

Recently Denisse [6], in a paper largely aimed at criticizing Landau's results, has derived an expression for k, the wave number, which must be satisfied by small amplitude, stable waves and which is in form identical with the dispersion relation of Vlasov, but which is to be interpreted slightly differently, due to its different derivation. Denisse's approach is basically similar to that of Bernstein, Greene and Kruskal and his linear approximation is more satisfactory than that of the linearised Boltzmann equation in that he only neglects small quantities.

Most of this report is concerned with an examination of the properties of such small amplitude, stable waves based on Denisse's results, which are quoted in Section III.

Before this in Section II, a short discussion is given of the bearing of the paper of Bernstein, Greene and Kreskal on the theory of Landau. In particular an example of a wave, for which the calculations can be completed exactly, and which is undamped, is presented.

In Section III, after the presentation of Denisse's results the interpretation and significance is discussed.

In Section IV this discussion is continued to demonstrate the absence of a true dispersion relation, although for a different reason from that given by Bernstein, Greene and Kruskal.

In Section V the results of section IV are used to criticize the results of Landau.

Finally in Section VI an attempt is made to consider the effect of collisions within the framework of the theory of stable solutions. Such an approach is of course, impossible to follow up rigorously, but the somewhat intuitive results seem sufficiently interesting to be worth presenting, as thay lead to a new interpretation of the dispersion relation.

SECTION II

Very briefly, Bernstein, Greene and Kruskal's method consisted of assuming a stationary wave and finding a differential equation which the potential, φ, must satisfy this assumption (collisions are being neglected). This equation can equally well be interpreted as an integral equation for the distribution function of the trapped electrons as a function of energy - if $\varphi(x)$ and the distribution functions for protons and untrapped electrons are specified. Bernstein, Greene and Kruskal have presented the solution of the equation in this form. It is not however completely clear that these solutions, the existence of which is the essential point of their argument, can be physically meaningful. Montgomery and Gorman [7] for example, question the analyticity of these solutions. It is therefore interesting to demonstrate a particular case of a stationary wave for which the distribution function is an analytic function of the position x and the velocity v (for x and v real). In the Appendix A, it is demonstrated that at least one such solution exists. The electron distribution function considered is :

$$f(x, v) = \frac{n_{oe}}{\pi} \frac{V}{V^2 + v^2 + \frac{q_e}{m}\varphi(x)}$$

where n_{oe} is the electron density at the potential maximum, at which point the potential $\varphi(x)$ is zero, V is an arbitrary positive constant, q_e and m are respectively the electronic charge and mass and the potential $\varphi(x)$ is an analytic function of x, the inverse of which is derived analytically in the appendix.

As will be seen in the Appendix A this is not a particularly interesting example physically : not only is the temperature of the plasma infinite, but the potential $\varphi(x)$ tends to $-\infty$ exponentially when x tends to $\pm\infty$. Nevertheless it proves that not all the B.G.K. equilibria have singular distribution functions (as a function of x and v).

It has been shown by Bernstein, Greene and Kruskal that their non-linear solutions correspond to singular approximate distribution functions in the linear approximation but this does not "salvage the conventional theory" since this implies that before performing the linear calculations we must replace the true initial distribution function by another modified distribution, but we only know how to do this (or even that it is possible) for the case of the B.G.K. equilibria.

The paper of Bernstein, Greene and Kruskal does not attempt to treat the general problem, treated by Landau, of describing the evolution in time of a plasma which is perturbed at the time $t = 0$. More exactly Landau assumed the initial distribution $f(x, v, 0)$ known and sought to derive the distribution $f(x, v, t)$ subject to the condition that $f(x, v, 0)$ represents a small perturbation from the equilibrium conditions. He found that the electric field $E(x, t) \rightarrow 0$ when $t \rightarrow \infty$. Effectively Bernstein _et al_. have produced a counter example. If we take the special case in which $f(x, v, 0)$ is the distribution corresponding to an undamped wave we must have $f(x, v, t) = f(x - ut, v, 0)$. Whence $E(x, t) = E(x - ut, 0) \not\rightarrow 0$. (u is the velocity of the wave relative to the frame of reference).

Hence, subject to the assumption that the equations used have a unique solution (which has apparently been proven by Iordanskii [8]). Landau's results are disproven by those of Bernstein et al.

The rest of this report is concerned with the properties of stable plasma waves - the existence of which is not further questioned, except for a short demonstration in Section V that Landau's results depend critically on his assumptions.

SECTION III

Those of Denisse's results which interest us especially here can be collected into the following three equations :

$$- \frac{d^2\varphi}{dx^2} = \frac{n_i \; q_i + n_{oe} \; q_e}{k_o} + k^2 \varphi + O(\varphi^{3/2}) \tag{1}$$

where
$$k^2 = \frac{q_e^{\;2}}{k_o m} \; P \int_{-\infty}^{\infty} \frac{1}{v_o} \frac{\partial f_o(v_o)}{\partial v_o} \, dv_o \tag{2}$$

and
$$n_{oe} = \int_{-\infty}^{\infty} f_o(v_o) \, dv_o \tag{3}$$

Here, x is the distance coordinate in the frame of reference in which the wave is stationary, parallel to the direction of propagation, $\varphi(x)$ is the electric potential, n_i the ion density (assumed constant) q_i and q_e the ionic and electronic charge respectively, m the mass of an electron. The potential is assumed to have a maximum which can be taken as zero; v_o is then the velocity of an electron along the x-axis at this maximum and $f_o(v_o)$ the electron distribution function at this point (which is the only point through which all electrons pass). (k_o is the permittivity of free space).

These equations have been derived on the assumption that all quantities are independent of time in some frame of reference, and have no sense otherwise.

Further, it must be assumed that $\frac{\partial f_o}{\partial v_o}$ is continuous for $v_o = 0$, otherwise the Cauchy principal value of the integral indicated in equation (2) does not exist. This does not appear to be an unreasonable assumption physically.

We can at once write the solution of equation (1), valid for $|\varphi| < \varphi_{max}$. where φ_{max} is some arbitrary level above which the higher order terms in φ cannot be neglected.

We have for $k^2 > 0$

$$\varphi(x) = \frac{\rho_o}{k^2} \left[\cos k(x - x_o) - 1 \right] \tag{4}$$

and for $k^2 = -K^2 < 0$

$$\varphi(x) = - \frac{\rho_o}{K^2} \left[\cosh K(x - x_o) - 1 \right] \tag{5}$$

where $\rho_o = n_{oe} \quad q_e + n_i \quad q_i$ (6)

The assumption that $\varphi \leq 0$ requires that $\rho_o \geqslant 0$

ie $n_i \geqslant n_{oe}$

Clearly, in equation (4), n_i can be chosen sufficiently close to n_{oe} that $\frac{\rho_o}{k^2} < \varphi_{max}$ (the limit of validity for the linearized form of equation (1), and in this case equation (4) will be a good approximation to the potential which could be predicted by Bernstein, Greene and Kruskal's non-linear theory, if it were calculable. Hence if k^2, derived from equation (2) is positive, it yields the wave number, for small (potential) amplitude waves. On the other hand, since cosh x is unbounded above, equation (5) can only ever be a good approximation for a small range of x. In fact, in this case, the potential is not even necessarily periodic and so knowledge of the value of K is of very little value.

When the right hand side of equation (2) is negative there are no waves of arbitrary small amplitude - but beyond this equation (2) cannot be considered as giving any useful information. In particular it should not be interpreted as indicating evanescent waves, such waves being explicitely excluded from Denisse's calculations.

On the other hand, when the right hand side of equation (2) is positive, equation (2) has the sense of a dispersion relation, except that we have not yet specified f_o. Such a specification requires a detailed description the manner in which the wave is excited . Such a description is of course beyond the scope of any theory so far developed. The only logical procedure is to consider f_o as an arbitrary function, the different functions f_o corresponding to different mechanisms of excitation. This question is taken up in detail in the next section.

It should be noted however, that physically we must exclude distribution functions, f_o , which are not symmetric about zero for all velocities less than that necessary for an electron to "escape" from the potential trough. In the limit of small amplitudes, in which we are interested here, this interval of velocities vanishes, and it is not unreasonable to ignore the restriction, provided we retain the symbol for the principal value of the integral in equation (2). Even if we were to extend the theory to waves of non-zero amplitude, we can formally avoid this restriction because the contribution of an electron to the charge density (through the time it spends in each part of the wave) depends only on its speed relative to the wave, and not on the direction of its velocity, we find that there is no change to the wave if we reverse the velocity of a group of particles. This is true, both in the non-linear theory of Bernstein et al. and in Denisse's linear approximation. Consequently (as can be easily verified by substitution in equation (2) , the results obtained for a given $f_o(v_o)$ are identical with those obtained for any distribution which can be constructed by adding an odd function to $f_o(v_o)$ - including of course a set of functions which are symmetric in any required range. Hende, when, in what follows, we calculate results for functions $f_o(v_o)$ which do not satisfy the symmetry conditions in any range, we are really calculating, for all those functions, symmetric in the required range, which could be constructed by the addition of an odd function. In the limit of small amplitudes, in which we are interested here the difference is very small.

SECTION IV

In this section we discuss what significance can be given to equation (2) in the light of the point, made in the last section, that $f_o (v_o)$ must be considered as an arbitrary function, unless the excitation mechanism can be described.

Bernstein, Greene and Kruskal argue that no dispersion relation can exist, due to the fact that the velocity (relative to the wave) of groups of untrapped electrons can be reversed without modifying the potential. This has already been pointed out in the last section and is represented in their calculations by the fact that it is only necessary to know the distribution of the particle energies (measured in the wave frame) to calculate the potential. Using this fact the velocity of the plasma relative to the wave or the wave velocity relative to the plasma, can be modified by reversing the velocity relative to the wave of some of the electrons without in any way modifying the potential wave, and so there is no relation between wave velocity and wavelength, or between frequency and wave number.

However, in seeking a dispersion relation, one expects such a relation to reflect the nature of the unperturbed plasma - for instance Bohm and Gross derived on approximate dispersion relation involving the density and temperature of the plasma. If however the disturbance of the plasma is very great, the temperature, for instance, may be greatly modified, in which case the dispersion relation calculated using the "equilibrium" temperature is no longer obeyed. Hence we can only expect a dispersion relation to have a sense if the deviation from equilibrium is small, and it is not surprising that Bernstein *et al.* did not find a dispersion relation without imposing such a restriction.

In this section we consider the existence of a dispersion relation for stable waves subject to the additional condition (which clearly limits the argument used by Bernstein *et al*. that:

$$f\,(x,\,v) \approx f_{eq}\,(v - v_p) \tag{7}$$

is satisfied for all x and v : v_p is effectively the phase velocity of the wave, and is considered as a parameter, f_{eq} is the Maxwell distribution for the velocity component parallel to the direction of propagation of the wave defined by equation (8) :

$$f_{eq}(v) = n_e \sqrt{\frac{m}{2\pi kT}}\; e^{-\frac{m}{2kT} v^2} \tag{8}$$

(k : Boltzmann's constant; T: electron temperature; n_e : equilibrium electron density).

From equation (7) we deduce that the amplitude of the potential must be small, since otherwise the density varies greatly from its equilibrium value and (7) is certainly not satisfied. Hence we may apply equation (2). In addition we shall make use of a special case of equation (7) :

$$f_o(v_o) \approx f_{eq}(v_o - v_p) \tag{7'}$$

which we rewrite $f_o(v_o) = \alpha f_{eq}(v_o - v_p) + \delta f(v_o)$ (7")

It should be noted that $f_{eq}(v)$ is the equilibrium distribution in the rest frame of the plasma but $f_o(v_o)$ is the distribution at a point in a particular wave, the velocities being measured in the frame in which the wave is stationary.

Here α is approximately unity and $\delta f(v_o)$ is small compared with $f_{eq}(v_o - v_p)$.

It is possible to examine a number of special cases analytically, in order to study the behaviour of equation (2) subject to the conditions of equation (7').

The first is the obvious case $\delta f = 0$.

$$f_o(v_o) = \alpha f_{eq}(v_o - v_p) \tag{9}$$

Combining equations (2), (8) and (9), we obtain :

$$k^2 = -\frac{\omega_p^2}{2\sqrt{\pi}} \frac{1}{V^2} g_1\left(\frac{v_p}{\sqrt{2}\,V}\right) \tag{10}$$

where $\omega_p^2 = \frac{\alpha n_e q_e^2}{K_o m} \approx \frac{n_e q_e^2}{K_o m}$, the square of the plasma frequency of the equilibrium distribution,

$$V^2 = \frac{kT}{m}$$

and

$$g_1(y) = P\int_{-\infty}^{\infty} \frac{x\,e^{-x^2}}{x-y}\,dx \tag{11}$$

In Appendix B it is shown that the function $g_1(y)$, defined by equation (11), is given by :

$$g_1(y) = -\sqrt{\pi}\left\{2y\,e^{-y^2}\int_0^y e^{u^2}\,du - 1\right\} \tag{12}$$

This function is tabulated is various places (see reference 8). A rough sketch of k^2 given by equation (10), deduced from various approximations which can readily be obtained from equation (12) is shown in Fig. 1.

In Appendix C it is shown that $g_1(y)$ can be expanded in an asymptotic series, of which the first two terms are :

$$g_1 (y) \sim \sqrt{\pi} \left(\frac{1}{2y^2} + \frac{3}{4} \frac{1}{y^4} \right) \tag{13}$$

Hence $$\omega^2 = k^2 v_p^2 \approx \omega_p^2 + 3 V^2 \frac{\omega_p^2}{\omega^2} k^2$$

In the region for which this expansion is useful $\frac{\omega_p^2}{\omega^2} \approx 1$ and so we find for the case of small k

$$\omega^2 \approx \omega_p^2 + \frac{3 kT}{m} k^2 \tag{13'}$$

which is the Bohm and Gross dispersion relation.

The approximation to k^2, obtained by substituting (13) in (10) is presented as a broken curve of figure 1.

We now consider the second special case

$$\delta f(v_o) = \frac{\delta n}{\sqrt{2\pi} W} e^{-1/2 \left(\frac{v_o - w}{W} \right)^2} \tag{14}$$

The resulting distribution function $f_o(v_o)$ is shown in figure 2.

The correction $\delta(k^2)$, which must be made to k^2 which we have already deduced from equation (10) is simply :

$$\delta(k^2) = \frac{\omega_{p}^2}{n_e} \cdot \frac{\delta n}{\sqrt{\pi} W} \cdot \frac{1}{W} \cdot g_1 \left(\frac{w}{\sqrt{2} W} \right) \tag{15}$$

This correction is plotted in figure 3 as a function of w for several values of W It is of course the same curve as figure 2 on a different scale, with different coordinates.

An examination of this curve brings out the following points. If W is large $\delta(k^2)$ is small for all w. If w is large, $\delta(k^2)$ is small, that is to say small modifications to $f_o(v_o)$ do not affect k in the region where $v_o \gg 0$. More precisely :

$$\delta(k^2) \ll k^2$$

$$\text{if} \quad \frac{\delta n}{n_e} \quad \frac{k^2 \omega_p^2}{W^2} \ll 1 \tag{16}$$

Since this inequality does not involve the form of δf, it is not un reasonable to suppose that it holds for localized modifications to f_o other than gaussian ones.

On the other hand we see that for $w \approx 0$, $\delta(k^2)$ may differ greatly from zero. Using the properties of $g_1(x)$ $(g_1(0) = \sqrt{\pi}$ $g_1(x) \lesssim \sqrt{\pi}$, we see from equation (15) that

$$\left|\delta(k^2)\right|_{max} = \frac{\omega_p^2}{n_{oe}} \frac{\delta n}{\sqrt{\pi} W} \cdot \frac{\sqrt{\pi}}{W} \tag{17}$$

The condition (7') (i.e. $f_o \approx f_{eq}$) requires that $\frac{\delta n}{\sqrt{\pi} W}$ be small but this condition can be satisfied for any W by choosing δn. Hence $\left|\delta(k^2)\right|_{max}$ can be made as large as desired by reducing W in equation (17) at the same time reducing δn so that $\frac{\delta n}{W}$ remains constant.

We see at once that the condition of equation (7) $(f(x, v) \approx f_{eq}(v-v_p))$ is not sufficient to give to equation (2) the meaning of a true dispersion relation.

We also see another important point. That is that a crude knowledge of the distribution of those electrons which move very rapidly relative to the wave is sufficient to estimate their contribution to the wave - but that the wavelength (and in fact the form) of the wave is very sensitive to the precise details of the distribution of those electrons (trapped or almost trapped) which move slowly relative to the wave.

This is a very important point because all approximate calculations based on the "linearized" Boltzmann equation, as well as the approximate calculations of Bohm and Gross - give an accurate description of those particles moving rapidly relative to the wave (for which we find that a crude description suffices) and give a quite false description of those particles with velocities close to the wave velocity (to which, as we have just seen, the wave is very sensitive).

It is possible to argue that the absence of a dispersion relation can be demonstrated from Bernstein, Greene and Kruskal's results, since given the distribution for the untrapped electrons, it is possible to derive a distribution of trapped electrons, which together with the untrapped electrons, give rise to any desired potential wave (subject to certain symmetry conditions when we ignore the ion motion).

However it does not seem possible to be sure that the derived distribution function will not be very great or, worse, negative, for an arbitrary potential wave. It therefore seems best to argue from Denisse's results.

To summarize this section, if we substitute $f_{eq}(v_o - v_p)$ for $f_o(v_o)$ in equation (2) we find effectively the same dispersion relation as Bohm and Gross or (except for the damping) Landau. It does not however seem worth while discussing the curve representing this dispersion relation in too much detail, because as we have shown using the special case of the sum of several gaussian functions, we can choose $f_o(v_o)$ to differ from $f_{eq}(v_o - v_p)$ by as little as we please, and yet k^2 differs from the dispersion relation by as much as we please. In other words, the conditions we have imposed on $f_o(v_o)$ are not sufficient to ensure the existence of a dispersion relation. The question remains as to whether physically meaningful additional conditions cannot be imposed on f_o - which lead to a dispersion relation in some sense. This problem is taken up in Section VI.

SECTION V

In this section the value of a theory which is based on a linear approximation to the Boltzmann equation is questione ' (in particular Landau's theory and, since it..is equivalent, that of Van Kampen).

We have already seen in Section II that Bernstein, Greene and Kruskal's non-linear stationary solutions serve as counter examples to Landau's theory.

As pointed out in Section IV for the case of stable waves the linear Boltzmann equation describes the untrapped particles quite well but the trapped particles very badly. In fact Landau's method amounts to treating all particles according to an approximation which is sound for those moving very rapidly relative to the wave. On the contrary Denisse [6] has been able to make the distinction between these two groups of particles, in that a range of velocities, corresponding to some of the trapped particles, are always excluded from his integrals. He has also shown that if one removes this distinction between trapped and untrapped particles by (quite incorrectly) integrating over all velocities the extra term which results in the "dispersion" equation has exactly the same form as Landau's damping term. Since this extra term corresponds to the lack of distinction between trapped and untrapped electrons, inherent in Landau's approach there is a very strong suggestion that it is from this aspect of his approximation that the erroneous damping term arises.

In Section IV we have seen that the Landau approximation is crude where it needs to be precise (trapped particles) and precises where it can afford to be crude (untrapped particles) which in itself is a rather severe criticism.

Finally, in the rest of this section we present a new argument against Landau's theory by re-examining his assumptions. Although he did not write it all down as explicitely, his argument starts as follows (The notation is necessarily different from Landau's).

The equations to be solved are :

$$\frac{\partial f(x,v,t)}{\partial t} + v \frac{\partial f(x,v,t)}{\partial x} + \frac{q_e}{m} E(x,t) \frac{\partial f(x,v,t)}{\partial v} = 0 \qquad (18)$$

$$K_o \frac{\partial E(x,t)}{\partial x} = q_e \int dv \quad f(x,v,t) + n_i q_i \qquad (19)$$

In order to develop a perturbation theory one writes :

$$f(x,v,t) = F_o(v) + f_1 (x,v,t) \qquad (20)$$

subject to the assumptions that $|f_1| \ll F_o$ and $f \approx f_{eq}$ which implies :

$$F_o(v) \approx f_{eq}(v) \tag{21}$$

Landau straight away took $F_o(v) = f_{eq}(v)$, which is of course the obvious choice, but not the only one valid. We shall see however that the results of his calculations depend critically on the method in which the division between the zero and first order distributions is carried out in equation (20). It is of course just as valid to carry out the calculations for an arbitrary F_o subject to the condition (21). We shall however add here the condition that

$$\int F_o(v)\, dv = \int f_{eq}(v)\, dv = n_i$$

which gives for equation (19)

$$\frac{\partial E}{\partial x} = \frac{q_e}{K_o m} \int_{-\infty}^{\infty} f_1(x,v,t)\, dv \tag{19'}$$

This is convenient in that the remaining calculations are identical to Landau's and we may simply take over his results, with f_{eq} replaced by F_o.

Landau then "approximates" to equation (18) by

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{q_e}{m} E \frac{\partial F_o}{\partial v} = 0 \tag{18'}$$

the term $\frac{q_e}{m} E \frac{\partial f_1}{\partial v}$ being dropped on the grounds that E and $\frac{\partial f_1}{\partial v}$ are both first order quantities. Since neither the integral nor the derivative of a small quantity is necessarily small this approximation is immediately questionable and should have been justified a posteriori. In fact Landau avoided the justification by taking the equation (18') and (19') (with $F_o = f_{eq}$) as his starting point. After this starting point his procedure appears to be valid, as is born out by Van Kampen. We can therefore use his results to demonstrate that his implicit assumptions, stated explicitely above, are unsound. It is only necessary to replace f_{eq} by F_o in all his equations and note the effect of small variations of F_o on ω (the wave frequency) and γ (the damping rate).

If γ is very small ω is given approximately by the root or roots of :

$$k^2 = \omega_p^2 \quad P\int \frac{F_o'(v)}{v - \omega/k}\, dv \tag{22}$$

and γ by

$$\gamma \approx \frac{\pi}{2}\, \omega^2 \quad F_o'(\omega/k) \frac{d}{dk}\left(\frac{\omega}{k}\right) \tag{23}$$

where k is the wave number of the initial perturbation.(See for example Weenink [9] for a derivation of these results without assuming $F_o = f_{eq}$). The equation (23) clearly depends on the assumed value of F_o, but it is not immediately clear what the effect on γ of small modification to F_o will be. However it is shown in Appendix D, that we can choose F_o to satisfy the following condition :

$$\left| F_o(v) - f_{eq}(v) \right| < \varepsilon \tag{24}$$

for any $\varepsilon > 0$, and such that Landau's formulae predict waves for which the damping rate is identically zero.

Further it has already been shown in Section IV that equation (22) does not relate ω/k (or ω) and k if F_o is only subject to the condition (21). (Compare (21) and (22) with (2) and (7').)

Landau's results therefore depend critically on his assumption that $F_o(v) = f_{eq}$, but this assumption is quite arbitrary.

To conclude this section, we see that the following arguments can be made against Landau's paper.

1) Bernstein, Greene and Kruskal have produced a counter example which contradicts Landau. This in itself is a conclusive point which continues to be ignored.

2) In the case of stable, or almost stable, longitudinal waves, the linear Boltzmann equation gives a very inaccurate description of the trapped particles, which are just the particles which are most significant in determining the wavelength and form of the waves.

3) In Denisse's theory one can reproduce Landau's damping term by introducing an error which is equivalent to the error inherent in a linearized Boltzmann equation.

4) The dispersion relation derived by Landau (within the sense he gave it) depends very critically on the zero order distribution function on which the solution is based. This demonstrates that it is not a sound perturbation calculation, since a first order change in the zero order function should not have an enormous effect on the results.

Any one of these points is sufficient to raise serious doubts as to the validity of the results of Landau and of all equivalent calculations.

SECTION VI

This report has mostly been based on Denisse's results, which in turn are based on the assumptions, generally made, that collisions can be neglected, and that the behaviour of a(large) set of "point" particles can be adequately described by a distribution function.

However it is known from a very general theorem (Chapman and Cowling [10]) that in the presence of collisions all distributions tend towards a uniform Maxwellian distribution and therefore that all waves are damped. It is generally assumed, as justification for neglecting the effect of collisions, that the damping rate due to collisions will be of the order of one of the relaxation times for a "test particle" in a plasma calculated by Spitzer (for example) [11] and therefore negligible.

In this section it is planned to show that this is not necessarily so. It is of course impossible to study rigourously, within the framework of a theory of stationary modes, what the effect of collisions will be. However the calculations which will be presented here do give an acceptable physical picture of the significance of collisions.

Before proceeding with this, however, it should be noted that the representation of a plasma by a distribution function needs careful justification - and the properties of a distribution function, if such can be adequately defined, should be carefully examined.

That such an examination is necessary can be seen from Van Kampen's [3] classic paper on the subject of plasma oscillations in the absence of collisions. In the section in which he demonstrates the existence of normal modes, he asserts that a distribution function is sufficiently well defined and physically acceptable if only it can be used to calculate averages. He then demonstretes his set of modes, which are singular. However, in a later section, when he decomposes an "arbitrary" initial distribution function which is of the form $g_0(v)\, e^{ikx}$ into these modes, he assumes that the function

$$g_{0+}(u) = \tfrac{1}{2}\, g_0(u) + \frac{1}{2\ i}\ p \qquad \frac{g_0(u')}{u - u'}\, du'$$

does not include any delta functions. In fact this excludes singular distributions, quite contrary to the argument that a distribution function is valid as long as it can be used to calculate averages.

Turning now to the effect of collisions we consider only the case in which the time of interest is much shorter that the time between collisions, that is the probability of a given electron suffering a close collision and we are only interested in the effect of the large number of "distant collisions", which any electron suffers is even a very short interval of time.

We shall further limit the argument by considering only those distribution functions, of the form considered in Section IV and defined by equations (7") and (14), i.e. :

$$f_o(v_o) = \alpha f_{eq}(v_o - v_p) + \delta f(v_o) \tag{7"}$$

$$\delta f(v_o) = \frac{\delta n}{\sqrt{2\pi}\, w}\, e^{-\frac{1}{2}\left(\frac{v_o - w}{W}\right)^2} \tag{14}$$

In this case the effect of collisions on the function f_{eq} will be negligible at least if the wave amplitude is sufficiently small since this is just the function which is not affected by collisions (in a uniform plasma).

We are left with the problem of the effect of collisions on δf. For this it is convenient to replace the velocity v of a particle by the velocity v_o which this particle would have at its next passage through a potential maximum, if there were no collisions. In the absence of collisions v_o would be a constant for a given particle by definition. In fact v_o will vary in time due to collisions, in a manner which it is impossible to determine. It is however possible to estimate the probability that a particle which has a velocity v_o at time t will have a velocity $v_o + \delta v_o$ at time $t + \delta\tau$. In fact if $\delta\tau$ is much smaller than "time between collisions", the change in v_o will be almost entirely due to the large number of small changes due to distant collisions, the probability of a close collision being extremely small. The net result of a large number of small effects generally has an approximately Gaussian distribution and so we shall assume further a Gaussian distribution for δv_o. That is we assume that if :

$$\delta f(v_o,t) = \delta(v_o - V_o) = h\,(V_o,0) \quad \text{say} \qquad (25)$$

Then $$\delta f(v_o,t+\tau) = \frac{1}{\sqrt{\pi F(\tau)}}\; e^{(v_o - V_o)^2/F(\tau)} = h\,(V_o,\tau) \text{ say } (26).$$

where the "width" $\sqrt{F(\tau)}$ has yet to be determined. In fact $F(\tau)$ will depend also on V_o but since we are only considering the effect of collisions on a localized distribution ($\delta f(v_o)$ of equatinn (14) we can ignore the dependance of F on V_o.

We now note that a physically real distribution $\delta f(v_o,t)$ can be decomposed into the set of "modes" h defined in equation (25) :

$$\delta f(v_o,t) = \delta f(v_o,t) * h(v_o,0) \qquad (27)$$

From equation (26) we then see that

$$f(v_o,t+\tau) = f(v_o,t) * h(v_o,\tau) \qquad (28)$$

(Provided we neglect collisions between pairs of the electrons "belonging" to δf which are negligibly few composed with those between an electron from δf an electron of f_{eq} .

Applying this result twice we see easily that

$$\begin{aligned} f(v_o, t+2\tau) &= f(v_o,t+\tau) * h(v_o,\tau) \\ &= f(v_o,t) * h(v_o,\tau) * h(v_o,\tau) \\ &= f(v_o,t) * h(v_o,2\tau) \end{aligned}$$

Whence $h(v_o, 2\tau) = h(v_o,\tau) * h(v_o,\tau)$

The right hand side is readily calculable and we obtain finally

$$F\,(2\tau) = 2\,F(\tau)$$

In the same way we can obtain

$$F\,(n\tau) = n\,F(\tau)$$

and so $F(\tau)$ must be of the form

$$F(\tau) = 2\alpha\tau$$

where is a constant which we shall leave undetermined. This form for $\Gamma($ $)$ is a direct consequence of the assumption of the Gaussian distribution for δv_0. In fact α is just the $\langle(\Delta w_{\parallel})^2\rangle$ of Spitzer [12], who gives formulae for its calculation. Typical values in the solar corona ($n_e \sim 10^7$ cm^{-3}, $T_e \sim 10^6$ °C, $V_0 \sim 2.5$V) would be $\alpha \sim 3.10^{15}$ cm^2 sec^{-3}. Increasing V_0 by a factor of 4 reduces α by almost 2 orders of magnitude (Indicating that the assumed Gaussian spread can only be true for very narrow peaks).

If we now substitute $\delta f(v_0)$ from equation (14) for $\delta f(v_0,t)$ in equation (28) the result is :

$$f(v_0, t + \tau) = \frac{\delta n}{\sqrt{2\pi(\alpha\tau + W^2)}}\, e^{-\frac{1}{2}\frac{(v_0 - w)^2}{\alpha\tau + W^2}} \tag{29}$$

In other words our assumptions are such that if δf is initially gaussian it will remain gaussian, but it will spread steadily. Intuitively we know that this is qualitatively correct except that the mean velocity will change slowly as well. Also if the amplitude of the wave is sufficiently large that the number of trapped particles is large the rather complicated exchange process between trapped and untrapped particles will provide an extra source of instability which we shall avoid by considering only small amplitude waves.

With the reserves in mind we can investigate what will be the consequences of these collisions. The equation (2) which we have interpreted as a dispersion relation can equally well be interpreted as necessary condition on f_0 for the stability of a wave of given k. If the wave is stable, k will not change, and if initially $f_0(v_0,0) = \alpha f_{eq} + \delta f(v_0,0)$ then at time τ later we have seen that $f_0(v_0,\tau) = \alpha f_{eq} + \delta f(v_0,\tau)$

If $k^2(\tau)$ calculated from equation (2) by replacing $\delta f(v_0)$ by $\delta f(v_0,\tau)$ is not equal to the original k^2, ($k^2(0)$), then the wave is unstable. This will almost always be the case. However Montgomery [12] has shown that, at least in some cases, the solutions of B.G.K. are "asymptotically" stable - and so we can hope that if $k^2(\tau) \approx k^2(0)$ then the waves will remain fairly stable. On the other hand there can be little doubt that if $k^2(\tau)$ is very different from $k^2(0)$ the wave will be very unstable, perhaps evolving towards a more stable state or damping out completely.

We have already seen that if the maximum of δf is very far from the wave velocity (more precisely if $w^2 \gg \frac{\delta n}{n_e} k^2 \omega_p^2$) the width of the gaussian curve does not enter the approximate expression for k^2 and so the spreading of the gaussian curve will probably not lead to instabilities; in this case the waves approximately satisfy the dispersion relation obtained by setting $f_o = f_{eq}$. On the other hand when w and W are both small (corresponding to a sharp peak in the velocity distribution near the wave velocity) $\delta k^2 (\tau)$ (calculated on the assumption that the wave is stable) will change very rapidly, leaving little doubt that the wave is unstable.

For example if $w = 0$

$$\delta k^2 (\tau) = - \frac{\omega_p^2}{n_e} \quad \frac{\delta n}{W^2 + \alpha\tau} \tag{30}$$

which clearly changes very rapidly if W^2 / α is small.

These unstable waves are just those which differ greatly from the dispersion relation obtained by setting $f_o = f_{eq}$.

We can summarize, and at the same time generalize these results as follows. If $f_o \approx f_{eq}$, this does not imply that $k^2 \approx k^2_{eq}$ (k^2_{eq} that given by setting $f_o = f_{eq}$). However the difference in k^2 is only due to very high first (or second, or higher ?) derivatives of f_o in the close neighbourhood of the wave velocity. If we did not insist that f_o be continuous, discontinuities of f_o could also introduce large differences between k^2 and k^2_{eq}. However it is easy to see, even without making the above calculations, that collisions will smooth all such "roughness" of f_o very rapidly, and since k^2 is so very sensitive to these details, the wave will become unstable in a time very much shorter than the time between collisions. Hence only those waves for which f_o is very "smooth" in the region of the wave velocity can conceivably be stable, and those waves approximately satisfy a dispersion relation.

We shall now try to make this statement a little more precise, at least for the case of δf gaussian. It should be emphasized however that what follows depends to a large extent on the additional assumptions made.

For instance, to make more precise our condition that $f_o \approx f_{eq}$ we shall write :

$$\left| \delta f \right|_{max} = \frac{\delta n}{W} \lesssim b \, f_{eq} \, (w - v_p) \tag{31}$$

This corresponds to the case where the initial disturbance effects all the electrons more or less equally, i.e. there is not selective acceleration of a small group of electrons (making up δf) but just a sort of "stirring up" of the distribution function which results in the "roughness" which we have idealized by a narrow gaussian peak. Once again, the validity of such an assumption can only be discussed in terms of an excitation process.

For the case of small w, which is the only case we need consider, we shall rewrite (31) :

$$\frac{\delta n}{W} \leqslant b \quad \frac{n_e}{V} \; e^{-\frac{1}{2} v_p^2/V^2}$$

where b is some constant, less than 1.

As a condition for stability we shall arbitrarily suppose that it is necessary that :

$$\frac{\partial}{\partial \tau} \left(\delta \; k^2 (\tau) \right) < B \tag{32}$$

$\frac{\partial}{\partial \tau} \delta k^2 (\tau)$ is a complicated function of W, w, τ etc. However it can be shown that the average value of this function, over a short period $(0, \tau)$, for small values of w, is approximately :

$$\left\langle \frac{\partial}{\partial \tau} \; \delta \; k^2 (\tau) \right\rangle \approx \omega_p^2 \quad \frac{\delta n}{n_e} \quad \frac{\alpha}{W^4}$$

Hence we require that :

$$\frac{\delta n}{W^4} < B \; \frac{n_e}{\omega_p^2 \; \alpha} \tag{32'}$$

Bearing in mind that :

$$\left| \delta (k^2) \right|_{max} = \frac{\omega_p^2}{n_{oe}} \quad \frac{\delta n}{W^2} \tag{17}$$

we can readily, verify by studying figure 6, that (31') and (32') impose an upper limit of $\left| \delta (k^2) \right|_{max}$ (The "max" refers to the maximum obtained by varying w holding δn and W fixed : here we are interested in the effect of varying δn and W as well, subject to the restraints (31') and (32')). In figure 6 the regions of the (δn, W) plane which do not satisfy (31') or (32') are shaded out. The lines of constant $\left| \delta (k^2) \right|_{max}$ ($\frac{\delta n}{W^2}$ = constant) are drawn on the same plane and it will be seen that above a critical value for $\left| \delta (k^2) \right|_{max}$ the curves are contained entirely in the forbidden (shaded) region. This critical value is of course our upper limit and so we have

$$\left|\delta(k^2)\right|_{max} < A \quad \exp\left(-1/3\ v_p^2\ /V^2\right)$$

where A is a parameter depending on ω_p^2 and V and on the assumed values of b ($|\delta f_1| \ll b\ f_{eq}$) and B ($\frac{\partial}{\partial \tau}\ \delta^p\ (k^2) < B$).

In figure (4) the curves k^2 and $k^2 \pm |\delta k^2|_{max}$ are sketched as functions of v_p which is almost exactly equal to the wave velocity. In figure (5) these curves are re-drawn in terms of the coordinate used by Denisse and Delcroix when discussing dispersion relations. We see that a dispersion relation exists for those waves which travel much faster than the thermal velocity and, as we have already shown in section IV this dispersion relation is well represented by approximate dispersion relation of Bohm and Gross

$$\omega^2 \approx \omega_p^2 + \frac{3\,\kappa\,T}{m}\,k^2 \qquad (13')$$

It should be noted however that this dispersion relation now only describes those waves which are relatively stable in the presence of collisions. On the other hand there appears to be no relation between ω and $v_p = \omega/k$ when v_p is of the order of V the thermal velocity. In this respect it should be noted that in fact Denisse's equations only describe the potential in the region between two consecutive potential minima. To describe a periodic wave it is necessary to assume that the trapped electrons have the same distribution between each pair of potential minima, an assumption which is of little consequence in the case of waves travelling faster than V, but which needs to be reconsidered for waves travelling slower than V (the thermal velocity). In fact it is not hard to see that in a given stable wave travelling slower than V, the distance between potential minima may vary from one minimum to the next and the potential observed at a point fixed in the plasma will vary more or less at random. There is therefore no real reason to distinguish between such waves and thermal fluctuations - the stable waves of low velocity being like a special case of thermal fluctuations.

All the above calculations have been made for the very special case of a distribution made up of two Gaussian functions. The calculations can be immediately extended to distribution functions made up of a finite number of gaussian functions. It is possible that the calculations can be extended to the case of a general distribution function. This has not as yet been attempted as it is doubtful if the model used for the collision effects justifies an elaborate mathematical theory. At the same time the physical picture is clear : for those small perturbations of a plasma which propagate as stable waves in the absences of collisions we can suppose that $f_o\ (v_o) \approx f_{eq}(v_o - v_p)$.

. This however is not a strong enough condition to determine $\underline{k}$. However those f_o (v_o) which give a $\underline{k}$ greatly different from that given by the case $f_o(v_o) = f_{eq}(v_o - v_p)$ are characterized by large values of $\frac{\partial f_o}{\partial v_o}$ in the neighbourhood of $v_o = 0$.

If we take the characteristic effect of collision to be the introduction of a spread in the velocities we see that such critically steep slopes cannot exist for any length of time. On the other hand, if the collisionless theory has any meaning at all, it is in the case where the distribution function is smooth, and so we come back to the dispersion relation found in linear theories - at least when the wave velocity is large - but without the damping.

The points are : 1 Collision damping, or rather collision induced instability may occur in a time much shorter than normally assumed.

2 The most stable waves are probably those which satisfy the linear dispersion relation.

SECTION VII - CONCLUSION AND DISCUSSION

This report is largely concerned with the nature of stable, small amplitude, longitudinal waves in a collision free electron plasma. However, since the results of Landau and Van Kampen are generally taken as meaning that no such waves can be stable, a considerable part of the report has been taken up by arguments concerning the existence of non-damped waves.

After this the approximate results, obtained by Denisse, are used to examine the existence of a dispersion relation for these stable waves, subject to the assumption that the distribution function in the wave is everywhere approximately equal to the equilibrium distribution. Such an assumption excludes the interesting case of a wave excited by a beam of particles injected into the plasma but since we have as yet no description of possible excitation processes this appears to be a necessary exclusion. Even subject to all these restrictions (stable waves, $f \approx f_{eq}$) it turns out that there is no dispersion relation, i.e. any wavelength can correspond to any wave velocity (at least for wave velocities less than the velocity of light).

However in Section VI, by a rather intuitive considerations, it has been shown that the effect of collisions is probably to render highly unstable those waves which do not satisfy approximately the normally obtained dispersion relation.

The work described is of course strictly limited, for example the general problem of the evolution of a plasma after an initial time, attempted by Landau, has not been attempted, Nor is it clear how it can be approached. It has been argued by Bernstein, Greene and Kruskal and by Weenink that the B G K "modes" can be identified with Van Kampen modes. However the Van Kampen modes are independent in that they satisfy a linear differential equation. On the other hand there is every reason to suppose that the stable waves which can exist in a plasma cannot in general be considered to propagate independently.

Bohm and Gross have already pointed out that it is a good approximation to consider two waves propagating in opposite directions (or more generally with highly different velocities) to be independent. On the other hand the implication is that waves with almost the same velocity will interact strongly. That this is so is easily seen : the properties of the wave, such as the wavelength, depend strongly on the distribution of electrons near the wave velocity. On the other hand it is just these electrons which will be most strongly perturbed by another wave with almost the same velocity. Hence the two waves certainly interact through those electrons close to both wave velocities.

The identification of the B G K equilibria with the Van Kampen modes does not therefore automatically justify the application of Van Kampen's analysis for which the independance of these modes is essential.

It is possible therefore that the analysis of stationary waves (including that describes here) will not be directly applicable to a study of the more general problem. On the other hand the concepts exposed here and elsewhere should be of some value in the development of the more general theory.

REFERENCES

1 Bernstein, I., Greene, J.M., and Kruskal, M.D., Phys. Rev. 108 , 546, (1957)

2 Landau, L.; J. Phys. (U.S.S.R.) 10 , 25, (1946)

3 Van Kampen, N.G.; Physica 21, 949, (1955)

4 Vlasov, A.; J. Phys. (U.S.S.R.) 9 , 25 (1945)

5 Bohm, D., and Gross, E.P. ; Phys. Rev. 75, 1851, (1949)

6 Denisse, J.F. ; C.R.A.S. 253, 1539 (1961)

7 Montgomery, D. and Gorman, D. ; Phys. Rev. 124, 1309, (1961)

8 Iordanskii, S. ; Doklady Akad. Nauk. U.S.S.R. 127, 509, (1959)

9 Lebedev, A.V., and Fedorova, R.M. ; " A guide to mathematical Tables", Pergamon, London, (1960)

10 Weenink, M.P.H. ; "General Theory of Longitudinal Waves in a Plasma", Rÿnhuizen Report 61-05 (1961)

11 Chapman, S. and Cowling, T.G. ; "The Mathematical Theory of Non-Uniform Gases", Cambridge University Press (2nd ed.) (1960)

12 Spitzer, L. ; Jnr. ; "Physics of Fully Ionized Gases", Interscience, London, (1956)

13 Montgomery, D. ; Phys. Fluids 3, 275 (1960)

14 Denisse, J.F. et Delcroix, J.L. ; "Théorie des Ondes dans les Plasmas", Dunod (Monographie n°33) Paris, (1961)

APPENDIX A

Consider the distribution function :

$$f(x,v) = \frac{n_{oe}}{\pi} \quad \frac{V}{V^2 + v^2 + a^2} \qquad a^2 = \frac{q_e}{m} \varphi(x)$$

Then the electron density is :

$$n_e = \int_{-\infty}^{\infty} f(x,v)\, dv = \frac{n_{oe}\ V}{\pi} \int_{-\infty}^{\infty} \frac{dv}{(V^2 + a^2) + v^2}$$

$$= \frac{n_{oe}\ V}{\pi} \quad \frac{1}{\sqrt{V^2 + a^2}} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

$$= n_{oe} \frac{V}{\sqrt{V^2 + a^2}}$$

Poissons equation becomes

$$- K_o \frac{d^2\varphi}{dx^2} = n_i q_i + n_e q_e$$

$$\text{i.e. } \frac{d^2\varphi}{dx^2} = \frac{n_i q_i}{K_o} \left\{ \frac{\alpha}{1 - A} - 1 \right\} \qquad \alpha = n_{oe}/n_i$$

$$A = - \frac{\bar{q}_e}{m} \cdot \frac{1}{V^2} > 0$$

$$\left(\frac{d\varphi}{dx}\right)^2 = - \frac{2n_i q_i}{K_o} \left\{ \frac{2\alpha}{A} \sqrt{1 - A\varphi} + \varphi \right\} + C$$

When we determine C from the conditions $\frac{d\varphi}{dx} = 0$ for $\varphi = 0$ this becomes

$$\frac{d\varphi}{dx}^2 = \frac{2n_i q_i}{K_o\ A} \left\{ 2\alpha (1 - \sqrt{1 - A\varphi}) - A\varphi \right\}$$

$$\frac{d\varphi}{dx} = \pm \sqrt{\frac{2\ n_i q_i}{K_o\ A}} \left\{ 2\ \alpha\ (1 - \sqrt{1 - A\varphi}) - A\ \varphi \right\}^{\frac{1}{2}}$$

$$\frac{d\ \varphi}{\sqrt{2\ \alpha\ (1 - \sqrt{1 - A\varphi}) - A\ \varphi}} = \pm \sqrt{\frac{2\ n_i q_i}{K_o\ A}}\ dx$$

Set $I = \int \dfrac{d\varphi}{\sqrt{2\alpha(1-\sqrt{1-A\varphi})-A\varphi}}$ and $z^2 = 1 - A\varphi \;(z = +\sqrt{1-A\varphi})$

$A\varphi = 1 - z^2$

$2 z\, dz = - A\, d\varphi$

$$= \int \frac{-\dfrac{2z}{A}\, dz}{\sqrt{2\alpha(1-z) - 1 + z^2}}$$

$$= \frac{-2}{A}\int \frac{z\, dz}{\sqrt{(z-\alpha)^2 - (1-\alpha)^2}}$$

Now set $z - \alpha = (1-\alpha)\cosh\mu$ where we can suppose $\mu > 0$

$$(z-\alpha)^2 - (1-\alpha)^2 = (1-\alpha)^2 \;(\cosh^2\mu - 1)$$
$$= (1-\alpha)^2 \;\sinh^2\mu$$

$$dz = (1-\alpha)\sinh\mu \;\; d\mu$$

$$I = \frac{-2}{A}\int\left\{\alpha + (1-\alpha)\cosh\mu\right\} d\mu$$
$$= \frac{-2}{A}\left[\alpha\mu + (1-\alpha)\sinh\mu\right]$$
$$= \frac{-2}{A}\left[\alpha \cosh^{-1}\frac{z-\alpha}{1-\alpha} + \sqrt{(z-\alpha)^2 - (1-\alpha)^2}\right]$$

Hence $$\sqrt{\frac{2 n_i q_i}{K_o A}}\,(x - x_o) = \pm \frac{2}{A}\left[\alpha \cosh^{-1}\frac{z-\alpha}{1-\alpha} + \sqrt{(z-\alpha)^2 - (1-\alpha)^2}\right]$$

where our choice of $\mu > 0$ imposes the condition $\cosh^{-1} \dfrac{z-\alpha}{1-\alpha} > 0$.

If we had chosen $\mu < 0$ this would have changed the sign of both terms in $[\;\;]$

Here $z^2 = 1 - A\varphi$

and x_o is a constant of integration.

We can readily plot $(x - x_o)$ as a function of z, or by distorting the scale, φ. In other words we have x as a function of φ the inverse function of φ (x). We can easily see that φ is an analytic function of x. It is true that we have ignored the motion of the ions, but there is no reason to suppose that this introduces any fundamental difference.

APPENDIX B

We wish to study the function :

$$g_1(y) = p \int_{-\infty}^{\infty} \frac{x\, e^{-x^2}}{x - y}\, dx$$

By definition the Hilbert transform of the function $f(x)$ is proportional (réf) :

$$\mathcal{H}(f(x)) = p \int \frac{f(x)}{x - y}\, dx$$

We shall therefore make use of the general rules concerniog this transform, which (except for the first) can readily be justified from the definition

$$\mathcal{H}(f(x)) = g(y) \Rightarrow \begin{cases} \mathcal{H}(f'(x)) = g'(y) & \text{(B 1)} \\ \mathcal{H}(xf(x)) = y\, g(y) + \int f(x)\, dx & \text{(B 2)} \end{cases}$$

and $\mathcal{H}(f_1(x) + f_2(x)) = \mathcal{H}(f_1(x)) + \mathcal{H}(f_2(x))$ (B 3)

Consider now $f_o(x) = e^{-x^2}$; $g_o(y) = \mathcal{H}(f_o(x))$ (B 4)

Then $2x\, f_o + f'_o = 0$

$$\mathcal{H}(2x\, f_o + f'_o) = 2y\, g_o(y) + g'_o(y) + 2\sqrt{\pi} = 0$$

from (B1) (B2) and (B 3)

which is a differential equation for $g_o(y)$

It can readily be verified that the general solution is

$$g_o(y) = -2\sqrt{\pi}\, e^{-y^2} \int_0^y e^{u^2}\, du + e^{-y^2}\, C$$

and since $g_o(0) = p \int \frac{e^{-x^2}}{x}\, dx = 0$

$$g_o(y) = -2 \sqrt{\pi} \; e^{-y^2} \int_o^y e^{u^2} \, du \qquad \text{(B 5)}$$

If we now define $\quad f_1(x) = x\,e^{-x^2}$

we have $\quad g_1(y) = \mathfrak{H}(f_1(x)) \qquad \text{(B 6)}$

But $\quad f_1(x) = -\frac{1}{2} f_o'(x)$

$$g_1(y) = -\tfrac{1}{2} g_o'(y) \qquad \text{(eq. B 1)}$$

$$= -\sqrt{\pi}\left\{2\,y\,e^{-y^2}\int_o^y e^{u^2}\,du - 1\right\} \qquad \text{(B 7)}$$

which is the required result (equation (12)).

APPENDIX C

To find an asymptotic expansion for $g_1(y)$ we first note that :

$$1 + \frac{x}{y} + \frac{x^2}{y^2} + \ldots\ldots \frac{x^{n-1}}{y^{n-1}} = \frac{1 - \frac{x^n}{y^n}}{1 - \frac{x}{y}} = \frac{-y}{x-y} + \frac{x^n/y^{n-1}}{x-y}$$

$$\text{i.e.} \quad \frac{1}{(x-y)} = -\frac{1}{y}\sum_{r=0}^{n-1}\frac{x^r}{y^r} + \frac{x^n/y^n}{x-y}$$

$$\text{Then } g_1(y) = p\int_{-\infty}^{\infty}\frac{x\,e^{-x^2}}{x-y}\,dx$$

$$= -\sum_{r=1}^{n}\left\{\frac{1}{y^r}\int x^r e^{-x^2}\,dx\right\} - \frac{1}{y^n}\,p\int_{-\infty}^{\infty}\frac{x^{n+1}e^{-x^2}}{x-y}\,dx$$

$$= -\sum_{r=1}^{m}\frac{\Gamma(r+\frac{1}{2})}{y^{2r}} + R_{2m}(y) \qquad \text{(C 1)}$$

$$(2m = n \text{ or } n-1)$$

$$\text{where} \quad R_{2m} = -\frac{1}{y^{2m+1}}\,p\int_{-\infty}^{\infty}\frac{x^{2m+2}e^{-x^2}}{x-y}\,dx \qquad \text{(C 2)}$$

$$\text{Since} \int_{-\infty}^{\infty}x^r e^{-x^2}\,dx = 0 \quad \text{if } r \text{ is odd}$$

$$= \int_0^{\infty}e^{-t}\,t^{\frac{r-1}{2}}\,dt \quad \text{if } r \text{ is even.}$$

Since the n^{th} term of the series in equation (C1) is greater than the $(n-1)^{th}$ term if $n > y^2 + \frac{1}{2}$, it is clear that the series diverges, and therefore that

$$R_{2m}(y) \xrightarrow[m\to\infty]{} +\infty$$

We intend to show that the expansion of equation (C1) is nevertheless useful as on asymptotic series.

xxxxxxxxxxxxxx

If we write

$$f_n(x) = x^n e^{-x^2}$$

and

$$g_n(y) = \mathcal{H}(f_n(x)) \tag{C 3}$$

then

$$R_{2(m-1)}(y) = -\frac{1}{y^{2m-1}} g_{2m}(y) \tag{C 4}$$

Proceeding exactly as in appendix B it is easy to show that

$$g_{2m}(y) = 2\Gamma(m+\tfrac{1}{2})\; y^{2m} e^{-y^2}\left\{ \int_a^y e^{u^2} u^{-2m}\, du + C \right\} \tag{C 5}$$

where $a > 0$ (since we are interested in the case, $y > 0$) and in principle C can be determined for given a. For example it can be seen intuitively that $g_{2m}(y) = 0$ for some $y = y_{2m} \sim \sqrt{m}$; we could write $a = y_{2m}$ and $C = 0$.

It suffices however, to note that $\{\ \} \xrightarrow[y \to +\infty]{} +\infty$ monotonically and is therefore always positive for some $y > Y(m)$ which we need not evaluate.

Hence $R_{2m}(y) < 0$ for $y > Y(m)$ (C 6)

But from (C 1)

$$R_{2m}(y) - R_{2m-2}(y) = + \frac{\Gamma(m+\frac{1}{2})}{y^{2m}} > 0$$

i.e. $$R_{2m}(y) > R_{2m-2}(y) \tag{C 7}$$

Combining (C 6) and (C 7) we see that for $y > Y(m)$

$$|R_{2m}(y)| < |R_{2m-2}(y)|$$

If we can invert the relation $y > Y(m)$ to read

$$m < M(y) \tag{C 8}$$

then $M(y)$ gives us the number of terms to take for the best approximation to $g_1(y)$.

If the intuitive result mentioned above

$$Y(m) \sim \sqrt{m}$$

is correct, then $y > Y(m) \Longrightarrow m < y^2$ (approx.)

i.e. $M(y) \approx y^2$

(which corresponds approximately to the smallest term of the series).

It can be shown that $R_{2m}(y) \xrightarrow[y \to \infty]{} 0$

Choose $a > \sqrt{m}$ so that $e^{u_1^2} u_1^{-2m} > e^{u_2^2} u_2^{-2m}$ for all $u_1 > u_2 > a$

Then
$$y^{2m} e^{-y^2} \left\{ \int_a^y e^{u^2} u^{-2m} du + C \right\}$$
$$= y^{2m} e^{-y^2} C + y^{2m} e^{-y^2} \int_a^{Ay} e^{u^2} u^{-2m} du$$
$$+ y^{2m} e^{-y^2} \int_{Ay}^{y} e^{u^2} u^{-2m} du$$

The first term tends to zero when $y \to \infty$ $(0 < A < 1)$

When $y > a/A$ the second term is :

$$y^{2m} e^{-y^2} \int_a^{Ay} e^{u^2} u^{-2m} du \leqslant y^{2m} e^{-y^2} \int_a^{Ay} e^{(Ay)^2} (Ay)^{-2m} du$$
$$= \frac{(Ay - a)}{A^{2m}} e^{-(1 - A^2) y^2}$$
$$\to 0 \qquad (y \to \infty)$$

The third term is
$$y^{2m} e^{-y^2} \int_{Ay}^{y} e^{u^2} u^{-2m} du \leqslant \frac{y^{2m}}{(Ay)^{2m+1}} e^{-y^2} \int_{Ay}^{y} e^{u^2} u \, du$$
$$= \frac{1}{A^{2m+1}} \frac{e^{-y^2}}{y} \Big[e^{u^2} \Big]_{Ay}^{y} \to 0 \qquad (y \to \infty)$$

Clearly the $R_{2m}(y) \to 0$ $(R_{2m} \sim 1/y^{2m})$ $(y \to \infty)$.

The functions $R_{2m}(y)$ are sketched in figure (C1).

APPENDIX D

Landau's result is that asymptotic behaviour in time of a plasma is characterized by a potential wave

$$\varphi \sim \varphi_0 \, e^{i(kx - \omega t)} \tag{D 1}$$

where k(real) and ω (complex) satisfy

$$k^2 = \omega_p^2 \int_C \frac{F_0'(v)}{v - \omega/k} \, dv \tag{D 2}$$

where the path of integration, C, is that indicated in figure (D 1), which passes below the pole ($v_0 = \omega/k$) of the integrand. The function F_0 is supposed to have been continued analytically into the whole complex velocity plane. We have used F_0 in (D 2) rather than Landau's f_0, to distinguish from Denisse's f_0

We shall show here that we can choose F_0 to satisfy equation (24), such that there exist real ω and k satisfying equation (D 2).

Suppose that ω,k (both real) and F_0 satisfy equation (D 2). In this case we can replace $\int_C$ by $\int_\Gamma + \int_\gamma$ where Γ, γ are the contours shown in figure (D 2)

The assumption that F_0' can be analytically continued into the complex plane means that on the semi-circle $v = \omega/k + r\,e^{i\theta}$ which the contour γ definies we have

$$F_0'(\omega/k + r\,e^{i\theta}) = F_0'(\omega/k) + r\,e^{i\theta}\,F_0''(\omega/k) + O(r^2)$$

$$v = \omega/k + r\,e^{i\theta} \qquad \frac{dv}{d\theta} = i\,r\,e^{i\theta}$$

$$\frac{F_0'(v)}{v - \omega/k} = \frac{F_0'(\omega/k + r\,e^{i\theta})}{\omega/k + r\,e^{i\theta} - \omega/k} = \frac{F_0'(\omega/k)}{r\,e^{i\theta}} + F_0''(\omega/k) + O(r)$$

$$\int_\gamma \frac{F_0'(v)}{v - \omega/k} \, dv = \int_\pi^{2\pi} i\,r\,e^{i\theta} \, d\theta \left\{ \frac{F_0'(\omega/k)}{e^{i\theta}\,r} + F_0''(\omega/k) + O(r) \right\}$$

$$= i\,F_0'(\omega/k)\int_{\pi}^{2\pi} d\theta \;+ O(r)$$

$$= i\pi F_0'(\omega/k) + O(r)$$

Hence

$$\int_C \frac{F_0'(v)}{v-\omega/k}\,dv = \left[\int_{-\infty}^{\omega/k - r} + \int_{\omega/k+r}^{\infty}\right]\frac{F_0'(v)}{v-\omega/k}\,dv + i\,F_0'(\omega/k) + O(r)$$

Since the left hand side is independent of r, and the equation is true for any $r > 0$, we may take the limit as $r \rightarrow +0$.

$$\int_C \frac{F_0'(v)}{v-\omega/k}\,dv = p\int_{-\infty}^{\infty}\frac{F_0'(v)}{v-\omega/k}\,dv + i(F_0'(v))_{v=\omega/k} \qquad \text{(D 3)}$$

This equation is only exact if ω/k is real.

We now see the condition that k^2 be real requires that $(F_0'(v))_{v=\omega/k}$ be zero.

The problem is therefore to choose d n $F_0(v)$, which can be continued analytically into the whole complex plane, which satisfies $F_0'(v) = 0$ for some v and which satisfies relation (24), i.e.

$$\left| F_0(v) - f_{eq}(v) \right| < \varepsilon \qquad (24)$$

These conditions can clearly be satisfied by writing :

$$F_0(v) = f_{eq}(v) + \delta f(v)$$

where $$f(v) = \alpha\, e^{-\left(\frac{v-V}{\beta}\right)^2}$$

subject to the condition that $|\alpha| < \varepsilon$

Then β can always be chosen small enough that $F_0'(v)$ will have a zero for some $v = v_0$.

To ensure that $\int F_0(v)\,dv = \int f_{eq}(v)\,dv$ we must replace $\delta f(v)$ by $\frac{1}{2}\delta f(v) - \frac{1}{2}\delta f(2v_0 - v)$ which is an odd funtion of $v - v_0$. Clearly this does not change the derivative : $F_0'(v_0)$.

This suffices to prove that there exist a real ω and a real k which satisfy equation (D 2), subject to the relation (24), for <u>any</u> $\varepsilon > 0$.

$$\left(k^2 = \omega_p^2 \quad P \int \frac{F_0'(v)}{v - v_0} \, dv \; ; \quad \omega = v_0 k \right).$$

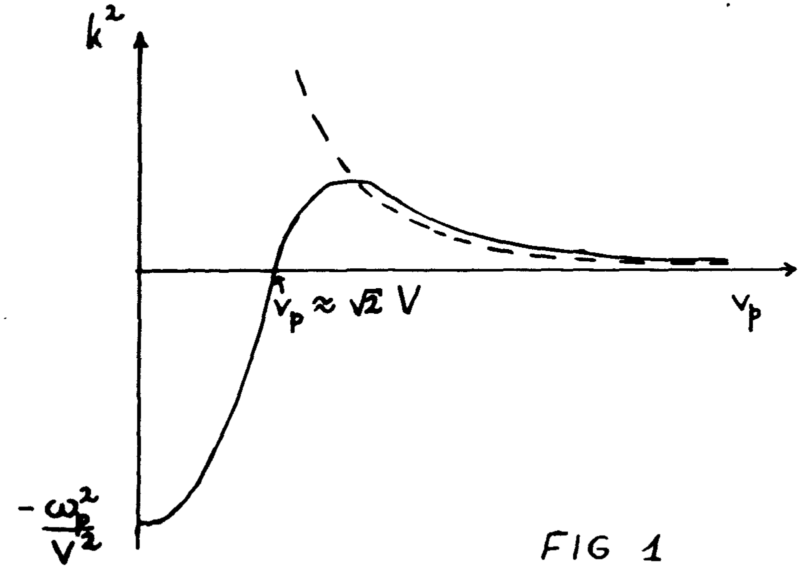

FIG 1

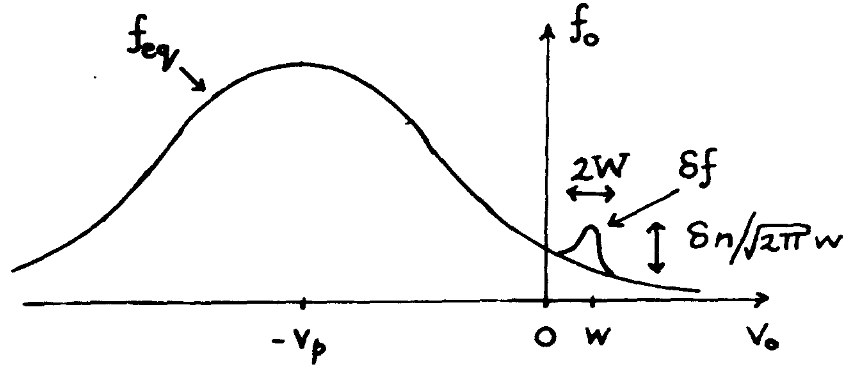

FIG 2

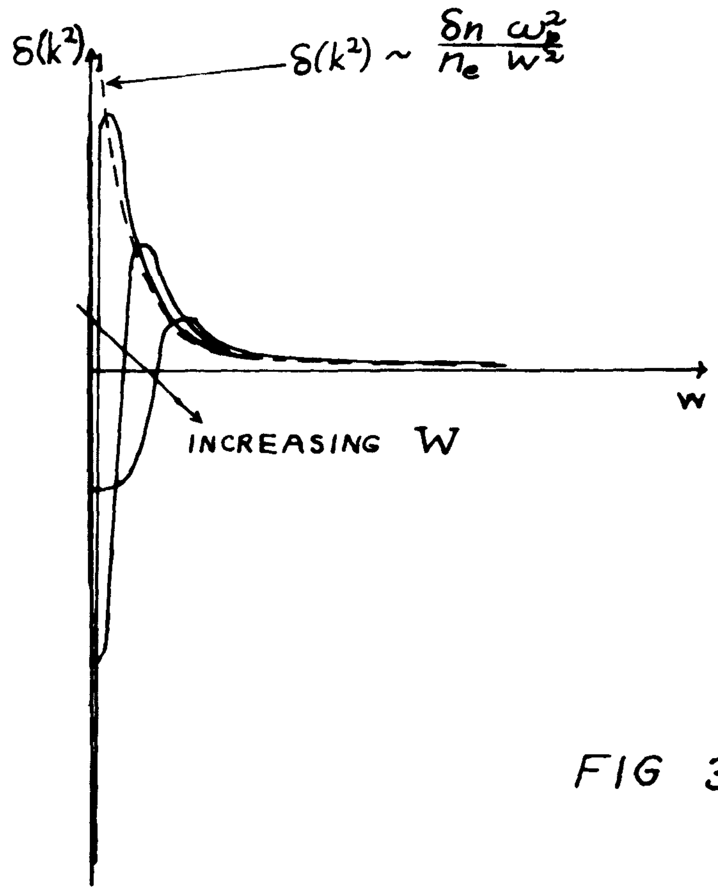

FIG 3

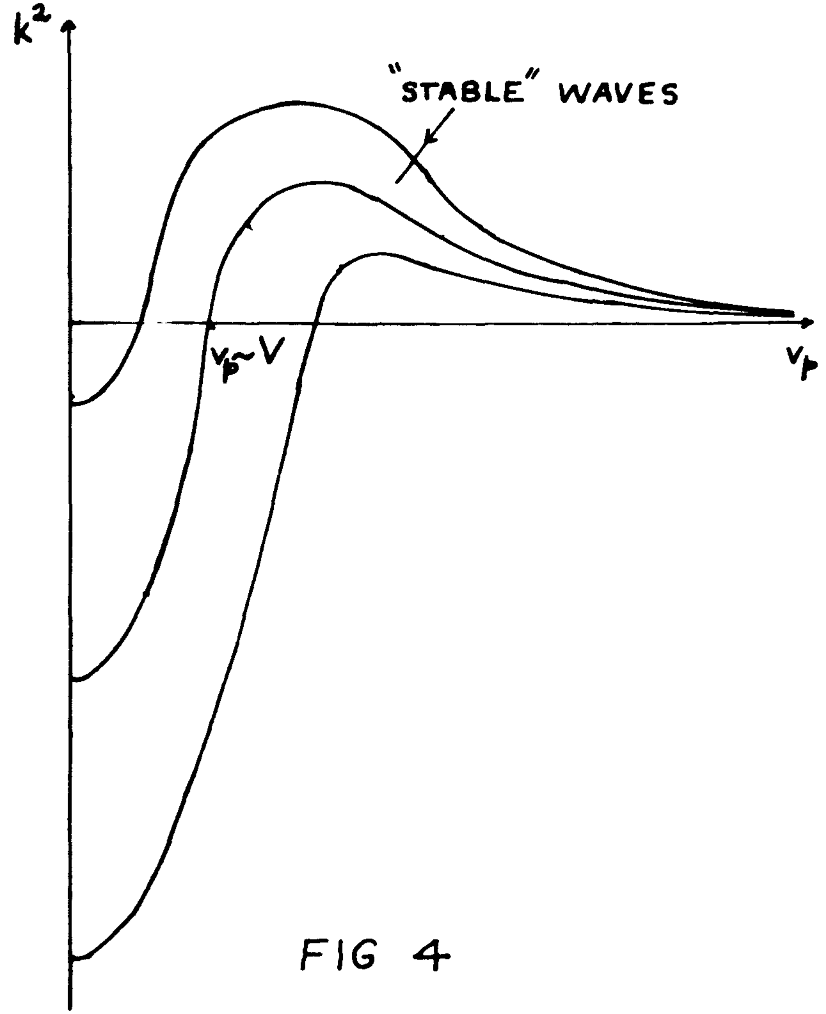
k^2
"STABLE" WAVES
$v_p \sim V$
v_p

FIG 4

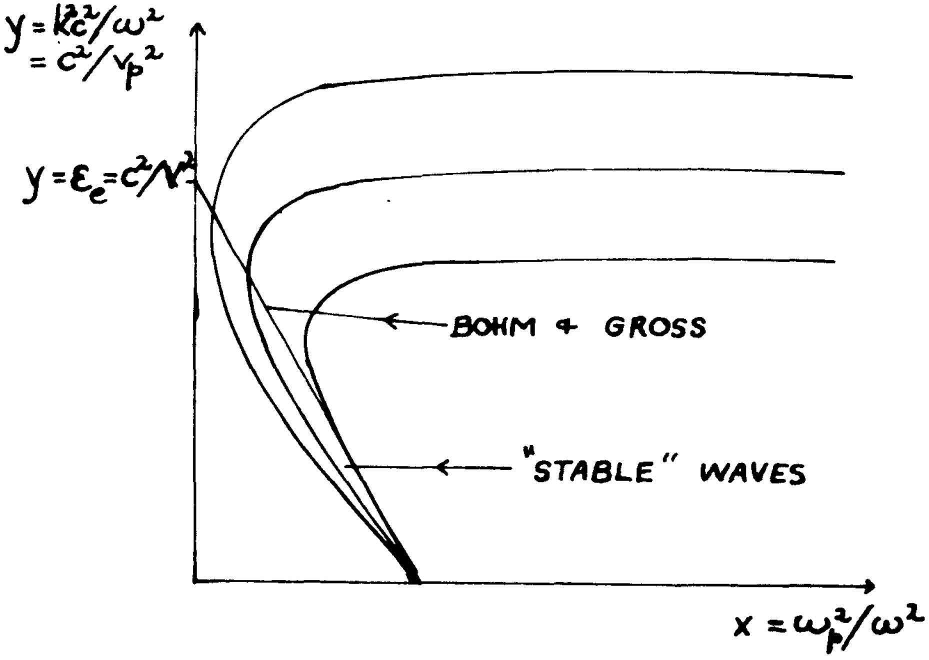
$y = k^2c^2/\omega^2$
$= c^2/v_p^2$
$y = \varepsilon_e = c^2/V^2$
BOHM + GROSS
"STABLE" WAVES
$x = \omega_p^2/\omega^2$

FIG 5

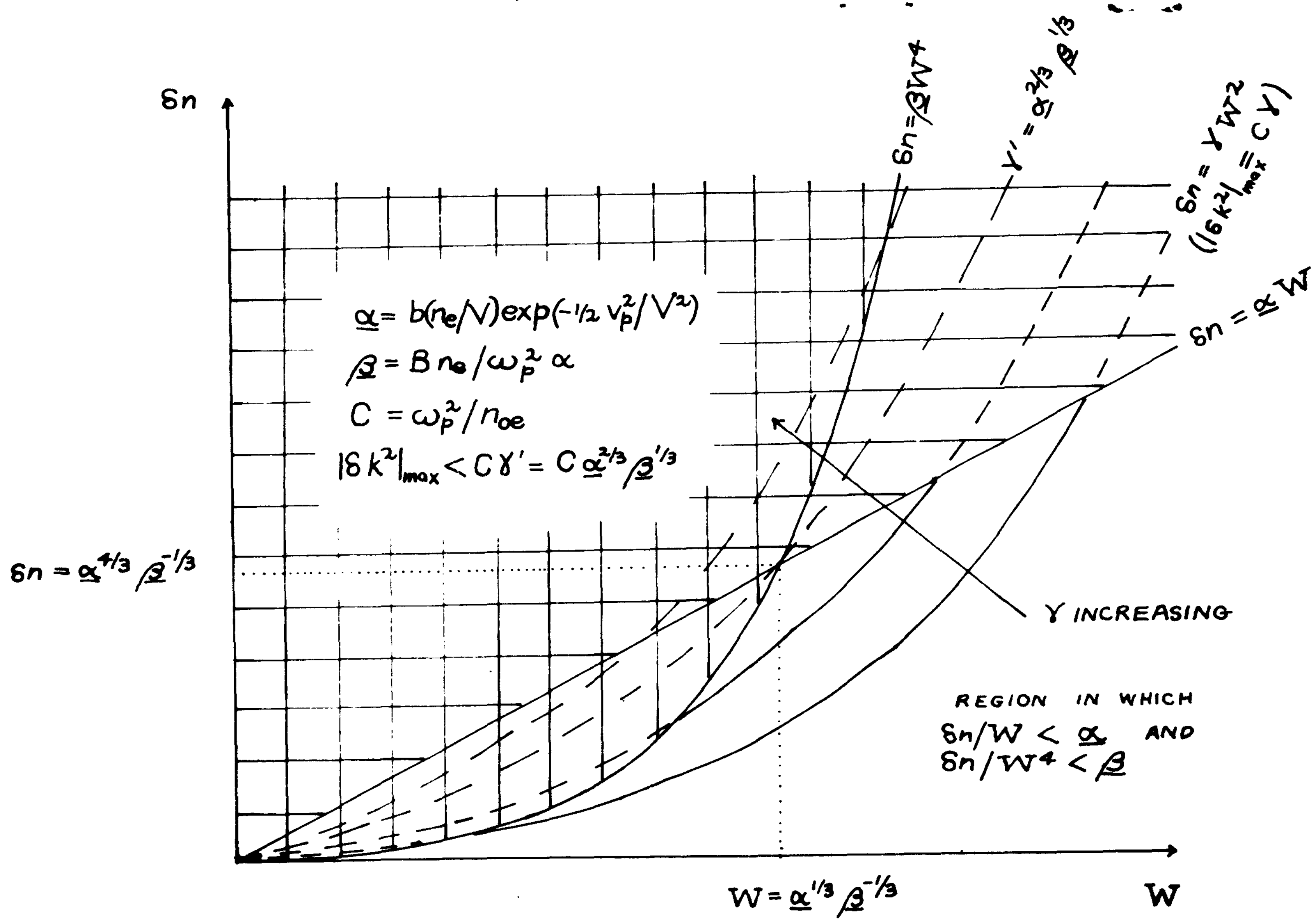
δn
$\underline{\alpha} = b(n_e/V)\exp(-1/2\, v_p^2/V^2)$
$\underline{\beta} = B n_e/\omega_p^2 \alpha$
$C = \omega_p^2/n_{oe}$
$|\delta k^2|_{max} < C\gamma' = C\underline{\alpha}^{2/3}\underline{\beta}^{1/3}$
$\delta n = \underline{\beta} W^4$
$\gamma' = \underline{\alpha}^{2/3}\underline{\beta}^{1/3}$
$\delta n = \gamma W^2$
$(|\delta k^2|_{max} = C\gamma)$
$\delta n = \underline{\alpha} W$
$\delta n = \underline{\alpha}^{4/3}\underline{\beta}^{-1/3}$
γ INCREASING
REGION IN WHICH
$\delta n/W < \underline{\alpha}$ AND
$\delta n/W^4 < \underline{\beta}$
$W = \underline{\alpha}^{1/3}\underline{\beta}^{-1/3}$
W

FIG 6

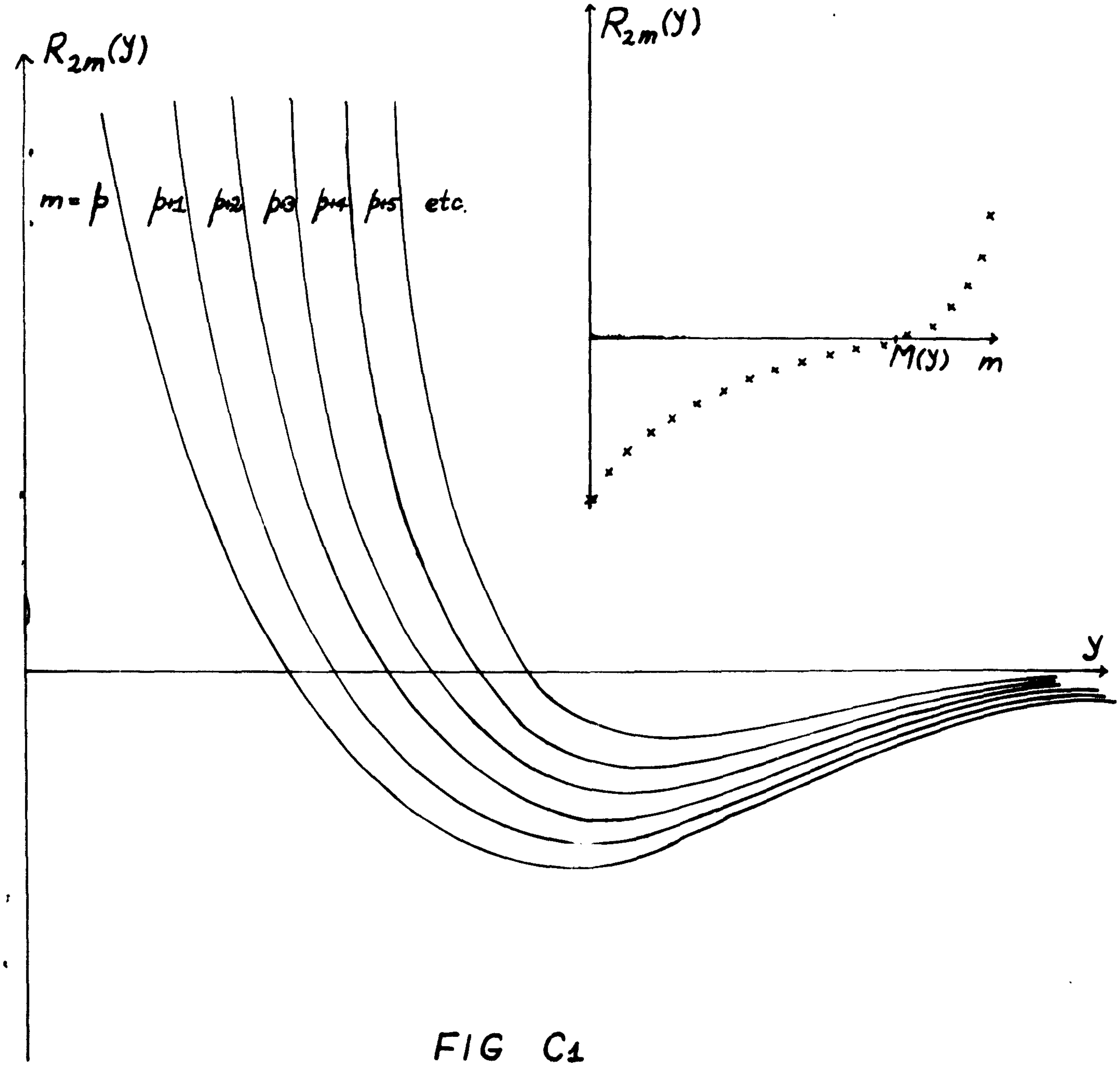

FIG C1

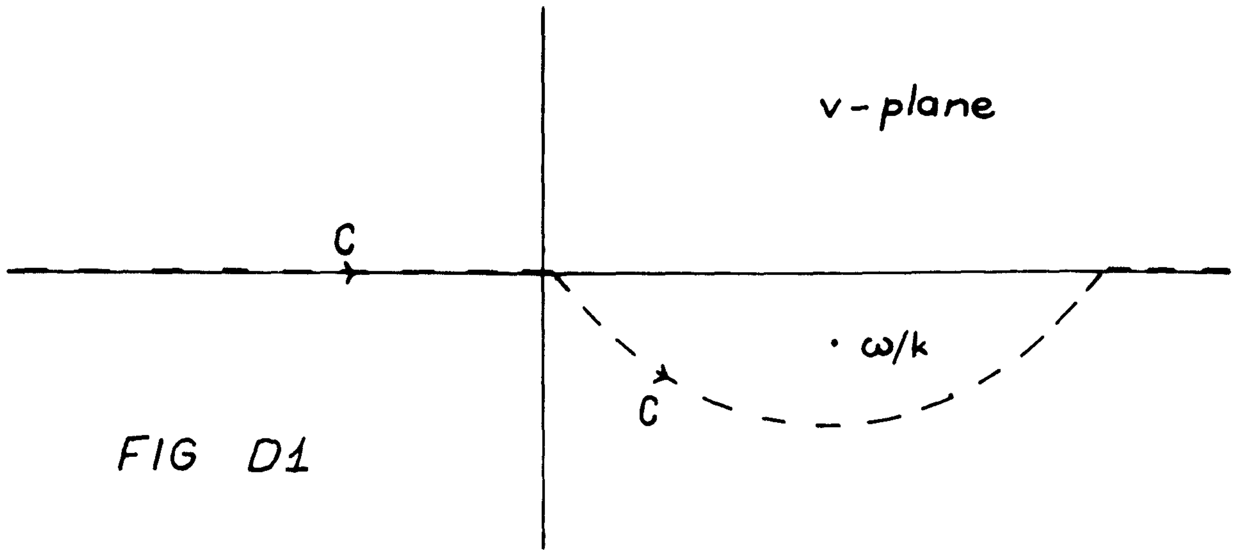

FIG D1

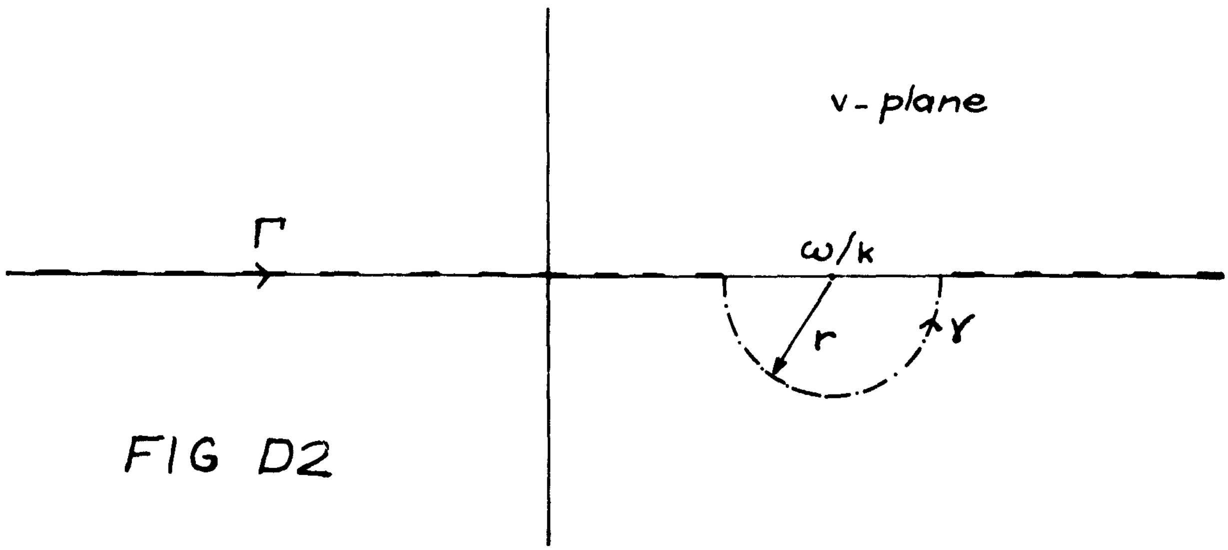

FIG D2